Stochastic Heat and Burgers Equations and Their Singularities — Geometrical and Analytical Properties (The Fish and the Butterfly, and Why.)

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Abstract

Arnol'd and Thom's beautiful classification of caustics (shockwaves for Burgers equation) suggests a similar one for the corresponding wavefronts for the heat equation. For instance, the wavefront corresponding to the simplest semi-cubical parabolic Cusp is the Tricorn and that corresponding to the Butterfly is the Fish. The Tricorn meets the semi-cubical parabola in three cusps, the Fish meets the Butterfly in three cusped curves and touches it along a straight line. We give here a general theorem for Hamiltonian systems characterising how the level surfaces of Hamiliton's principal function meet the caustic explaining the way that the Butterfly and Fish meet and a myriad of similar results. We show how these results can be applied to the stochastic Burgers equation by using earlier results of Truman and Zhao. We also explain how the characterisation of caustics and wavefronts carries over to the stochastic case and how our results generalise to this situation.

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1 Introduction

Consider the stochastic viscous Burgers equation for the velocity field $v^\mu = v^\mu(x,t)$, $x \in \mathbb{R}^d$, $t > 0$,

$$\frac{\partial v^\mu}{\partial t} + (v^\mu \cdot \nabla) v^\mu = \frac{\mu^2}{2} \Delta v^\mu - \nabla c(x) - \epsilon \nabla k(x,t) \dot{W}_t, \tag{1.1}$$

with initial velocity $v^\mu(x,0) = \nabla S_0(x)$ where μ^2 is the coefficient of viscosity. Here c and k are C^2 functions and W_t is a Wiener process on the probability space $\{\Omega, \mathcal{F}, P\}$. We shall be interested in the 'blow-up' of $v^0(x,t) = \lim_{\mu \to 0} v^\mu(x,t)$.

The corresponding heat equation for $u^\mu = u^\mu(x,t)$ is

$$\begin{aligned} \frac{\partial u^\mu}{\partial t} &= \frac{\mu^2}{2} \Delta u^\mu + \frac{1}{\mu^2} c(x) u^\mu + \frac{\epsilon}{\mu^2} k(x,t) u^\mu \circ \dot{W}_t, \\ u^\mu(x,0) &= \exp\left(-S_0(x)/\mu^2\right), \end{aligned} \tag{1.2}$$

the connection between u^μ and v^μ being the Hopf-Cole logarithmic transformation $v^\mu = -\mu^2 \nabla \ln u^\mu$.

Following Donsker, Freidlin et al [17, 19] we expect as $\mu \to 0$,

$$-\mu^2 \ln u^\mu(x,t) \to \inf_{X(0)} \left[A(X(0),x,t) + S_0(X(0))\right] = S(x,t), \tag{1.3}$$

with

$$A(X(0),x,t) = \inf_{\substack{X(s) \\ X(t)=x}} A[X], \tag{1.4}$$

where $A[X]$ is the stochastic action

$$A(X(0),x,t) = \frac{1}{2} \int_0^t \dot{X}^2(s) - \int_0^t c(X(s)) ds - \epsilon \int_0^t k(X(s),s) dW_s. \tag{1.5}$$

$S(x,t)$ is the solution of the Hamilton-Jacobi equation

$$dS_t + \frac{|\nabla S|^2}{2} dt + c(x)\, dt + \epsilon k(x,t)\, dW_t = 0, \quad S(x,0) = S_0(x), \tag{1.6}$$

and so $S(x,t)$ is Hamilton's principal function for a stochastic mechanical path. To see this observe that if $\dot{X}(s)$ is a continuous process with the integration by parts property (ibp^2) : for all $u \in C^1(0,t)$

$$\int_0^t \dot{u}(s) \dot{X}(s)\, ds = u(t)\dot{X}(t) - u(0)\dot{X}(0) - \int_0^t u(s)\, d\dot{X}(s),$$

setting $u(t) = 0$, $\mathcal{A}[X] = A[X] + S_0(X(0))$, we easily obtain for $S_0 \in C^1$, $c \in C^1$, $k \in C^{1,0}$

$$
\begin{aligned}
&- \frac{d}{d\eta}\bigg|_{\eta=0} \mathcal{A}[X + \eta u] \\
&= \int_0^t u(s) \left[d\dot{X}(s) + \nabla c(X(s))\, ds + \epsilon \nabla k(X(s), s)\, dW_s \right] \\
&\quad + u(0) \left(\dot{X}(0) - \nabla S_0(X(0)) \right).
\end{aligned}
$$

Hence necessary conditions for the extremizer are :

$$
d\dot{X}(s) + \nabla c(X(s))\, ds + \epsilon \nabla k(X(s), s)\, dW_s = 0, \qquad \dot{X}(0) = \nabla S_0(X(0)). \tag{1.7}
$$

As $\mu \to 0$, u^μ switches from being exponentially large to exponentially small across the level surface

$$
S(x, t) = 0. \qquad (\textit{ wavefront})
$$

If we demand that $X(t) = x$, for fixed t and x, $X(s)$ above may not be unique. Hence, we expect that shockwaves for v arise from precaustics (in (x_0, t) variables) where infinitely many of these classical mechanical paths from x_0 and a neighbourhood focus in a set of zero volume centred on x, i.e. for paths starting from x_0 focusing at point x at time t,

$$
\mathrm{Det} \left(\frac{\partial X(t)}{\partial x_0} \right) = 0.
$$

The image of the precaustic under Φ_t is called the caustic (in (x, t) variables, where $x = \Phi_t x_0$.)
Define the random map $\Phi_s : \mathbb{R}^d \to \mathbb{R}^d$ corresponding to the classical flow by

$$
d_s \dot{\Phi}_s = -\nabla c(\Phi_s)\, ds - \nabla k(\Phi_s, s)\, dW_s, \tag{1.8}
$$

with $\Phi_0 = I$ and $\dot{\Phi}_s = \nabla S_0$ so that $X(s) = \Phi_s \Phi_t^{-1} x$, where we accept that $x_0(x, t) = \Phi_t^{-1} x$ is not necessarily unique. Given some regularity, the global inverse function theorem gives a caustic time $T(\omega)$ such that for $s < T(\omega)$, Φ_s is a random diffeomorphism [34]. Therefore,

$$
v^0(x, t) = \dot{\Phi}_t \Phi_t^{-1} x = \nabla S(x, t) \tag{1.9}
$$

is a formal solution of Burgers equation with $\mu = 0$, which is well defined up to the caustic time because $x_0(x, t)$ is unique. Thereafter, for polynomial

S_0, $x_0(x,t)$ will have finite multiplicity. As long as the minimising $x_0(x,t)$ is unique equation (1.9) can be assumed to be true beyond the caustic time if we take that part of the level surface of Hamilton's principal function corresponding to the minimising $x_0(x,t)$. Needless to say the main contribution to $\lim_{\mu\to 0} v^\mu$ will come from the minimising $x_0(x,t)$. In what follows we assume that this minimising $x_0(x,t)$ is unique. When there are a finite number of minimising x_0's the solution v^0 is more complicated [10].

We expect non-uniqueness of $x_0(x,t)$ to be associated with the appearance of discontinuities in $v^0(x,t)$. An obvious way for these to arise is if infinitely many paths $X(s)$ focus in zero volume centred at x,

$$\mathrm{Det}\left(\frac{\partial X(t)}{\partial x_0}\right) = 0 \qquad (\textit{precaustic}), \tag{1.10}$$

or eliminating x_0 by using $x = \Phi_t x_0$ in,

$$\mathrm{Det}\left(\frac{\partial X(t)}{\partial x_0}\right) = 0 \qquad (\textit{caustic}). \tag{1.11}$$

A simple example in one space dimension with $c = k \equiv 0$ illustrates this if we use the method of characteristics. Here the point is that the convected derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla,$$

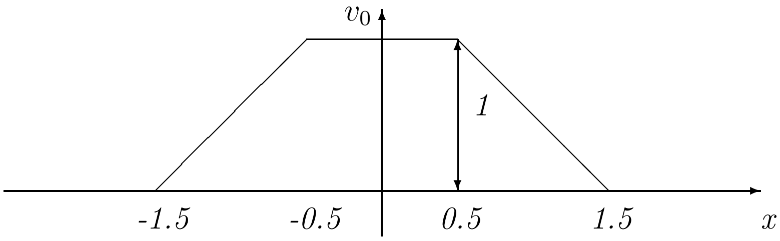
so

$$\frac{Dv^{\mu=0}}{Dt} = 0 \Longrightarrow v^{\mu=0}(x,t) = v_0(x_0(x,t)),$$

$x_0(x,t)$ being chosen so that the infimum of the action is achieved.

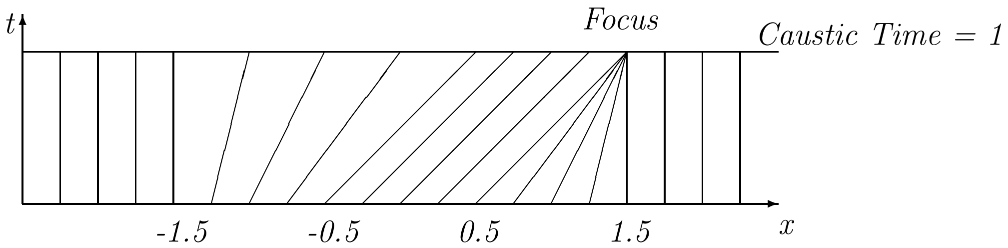
Example 1.1. *Consider the inviscid Burgers equation in 1-dimension with the initial velocity field v_0 graphed below.*

Figure A



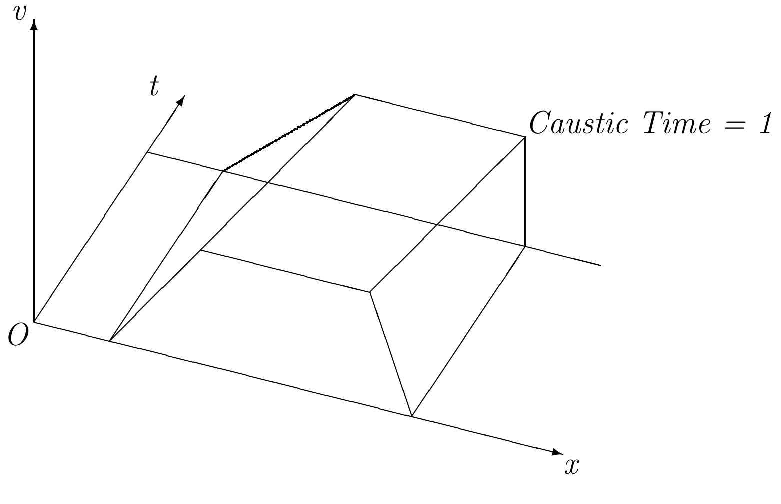
The characteristics of the inviscid Burgers equation can be drawn as shown below.

Figure B



The region for $x \in [0.5, 1.5]$ *corresponds to the precaustic with the minimiser* $x_0(x,t)$ *jumping from* .5 *to* 1.5 *at* $t = 1$ *and* $x = 1.5$. *It is evident that* $\frac{\partial X(t)}{\partial x_0} = 0$ *for* $x_0 \in (0.5, 1.5)$. *The appearance of the shockwave for* $v = v(x,t)$ *can be seen in the following picture.*

Figure C



Note here $v(x,t)$ *is discontinuous at* $x = 1.5, t = 1$.

Choose $S_0(3/2) = 0$, $S_0' \in C^1$ with $S_0'(x) = v_0(x)$. Then

$$S_0(x) = \theta(x+3/2)\left(\mathcal{X}_{[-\frac{3}{2},-\frac{1}{2})}\frac{(x+\frac{3}{2})^2}{2} + \mathcal{X}_{[-\frac{1}{2},\frac{1}{2})}(1+x) + \mathcal{X}_{[\frac{1}{2},\frac{3}{2})}\left(2 - \frac{(x-\frac{3}{2})^2}{2}\right) + 2\theta(x-3/2)\right) - 2.$$

For $t<1$ when $-\frac{1}{2}+t<x<\frac{1}{2}+t$, from Figure B we have $x=x_0(x,t)+t$ so the minimiser x_0 satisfies $-\frac{1}{2}<x_0<\frac{1}{2}$. Hence using the explicit heat kernel

$$\begin{aligned} u^\mu(x,t) &\sim (2\pi\mu^2 t)^{-1/2}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\frac{(x-x_0)^2}{2\mu^2 t}-\frac{(1+x_0)}{\mu^2}+\frac{2}{\mu^2}}\,dx_0 \\ &\sim e^{\left(\frac{t}{2}+1-x\right)/\mu^2}. \end{aligned}$$

The wavefront $S(x,t)=0$ corresponds to $x=1+\frac{t}{2}$. We see that u^μ is exponentially large for $x<1+\frac{t}{2}$ and exponentially small for $x>1+\frac{t}{2}$ due to the wavefront being the single point $1+\frac{t}{2}$. Our choice of $S_0(3/2)=0$ has led to the wavefront and caustic point coinciding at $t=1$. The fluid density here $\rho(x,t)=\frac{\partial x_0(x,t)}{\partial x}$ 'blows up' at the caustic $x=\frac{3}{2}$ at the caustic time $t=1$ and $v^0(x,t)$ has a discontinuity.
In more than one dimension, unlike the last example, for $S_0\in C^2$ there is a robust geometrical relationship between wavefronts and caustics even in the presence of noise. Our main theorem concerning this geometric relationship for the above stochastic Hamiltonian system, modulo some regularity assumptions, is for fixed t:

Theorem. *Consider any level surface H_t of Hamilton's principal function. This will meet the caustic surface C_t in a fixed curve a_t. Any smooth curve on H_t can only have a generalised cusp at a point α if $\alpha\in a_t$ and α has at least one inverse image point $\Phi_t^{-1}\alpha$ on the intersection of both precaustic $\Phi_t^{-1}C_t$ and pre-level-surface $\Phi_t^{-1}H_t$. Moreover, as long as $\Phi_t^{-1}\{\alpha\}$ contains at least one point of $\Phi_t^{-1}H_t\cap\Phi_t^{-1}C_t$, generically there will be a cusp at α.*

This result explains the bizarre pictures of the Fish and the Butterfly (Figures 9, 10, 11, 12) and similar phenomena. We also develop its applications to Burgers equation by showing how we can divide up caustic surfaces into *hot* and *cool* parts. In the next section we recapitulate our main results for Burgers equation. This can be skipped if the reader is primarily interested in the geometrical result. An outline proof of our main geometrical result is given in Section 3. Section 4 contains examples mostly related to the Harmonic Oscillator as well as an outline proof of the stochastic version of our geometrical result. Section 5 treats general potentials with small noise.

2 Solutions of inviscid Burgers and Heat Equations

Here let $v^\mu = v^\mu(x,t)$ satisfy for $t > 0$, $x \in \mathbb{R}^d$,

$$dv^\mu + (v^\mu \cdot \nabla) v^\mu \, dt + \nabla c \, dt + \epsilon \nabla k \, dW_t = \frac{\mu^2}{2} \Delta v^\mu \, dt,$$

with $v^\mu(x,0) = \nabla S_0(x)$, where $c = c(x,t)$, $k = k(x,t)$. Later we assume $\frac{\partial c}{\partial t} = 0$.
Since the convected derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla,$$

the above Burgers equation describes a fluid where particles pass through each other subject to a body force $(-\nabla c(x,t) - \epsilon \nabla k(x,t) \dot{W}_t)$ with $\dot{W}_t$ being white noise. (In fact infinitely many pass through each other at points on the caustic and to this extent our continuation of the solution is unphysical.)
Let $v^\mu = \nabla S^\mu$. Then S^μ satisfies the stochastic Hamilton-Jacobi equation

$$dS^\mu + \frac{1}{2} |\nabla S^\mu|^2 \, dt + c \, dt + \epsilon k \, dW_t = \frac{1}{2} \mu^2 \Delta S^\mu \, dt.$$

Formally, if $S^\mu \sim \sum_{j=0}^{\infty} \mu^{2j} S_j$, we obtain the stochastic Hamilton-Jacobi continuity equations

$$\frac{\partial S_j}{\partial t} + \frac{1}{2} \sum_{\substack{i_1, i_2 \geq 0 \\ i_1 + i_2 = j}} \nabla S_{i_1} \cdot \nabla S_{i_2} = \frac{1}{2} \Delta S_{j-1},$$

for $j = 0, 1, 2, \ldots$, with the convention $\frac{1}{2} \Delta S_{-1} = -c - \epsilon k \dot{W}_t$, $\dot{W}_t$ being white noise.
The corresponding stochastic mechanical flow Φ_s satisfies

$$d\dot{\Phi}_s = -\nabla c(\Phi_s, s) \, ds - \epsilon \nabla k(\Phi_s, s) \, dW_s,$$

with $\Phi_0(x) = x$, $\dot{\Phi}_0(x) = \nabla S(x,0) = \nabla S_0(x)$. Now assume $\Phi_s(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism for $0 \leq s < T(\omega)$. Define

$$\tilde{S}_0(y,t) = \frac{1}{2} \int_0^t |\dot{\Phi}_s(y)|^2 \, ds + S(y,0) - \int_0^t c(\Phi_s y, s) \, ds - \epsilon \int_0^t k(\Phi_s y, s) \, dW_s$$

and

$$S_0(x,t) = \tilde{S}_0(\Phi_t^{-1} x, t).$$

The main result of stochastic Hamilton-Jacobi theory is :

Proposition 2.1. *Assume* $c \in C^2(\mathbb{R}^d \times \mathbb{R})$, $k \in C^2(\mathbb{R}^d \times \mathbb{R})$ *and* $S_0(\cdot, 0) \in C^2(\mathbb{R}^d)$. *Then*
(i) For a.e. $\omega \in \Omega$, $0 \le t < T(\omega)$,

$$\dot{\Phi}_t = \nabla S_0(\Phi_t, t)$$

and S_0 *satisfies the stochastic Hamilton-Jacobi equation*

$$dS_0(x,t) + \frac{1}{2}|\nabla S_0(x,t)|^2\,dt + c(x,t)\,dt + \epsilon k(x,t)\,dW_t = 0.$$

(ii) Define $\rho(x,t) = |\det \frac{\partial}{\partial x}\Phi_t^{-1}(x)|$. *Then for a.e.* $\omega \in \Omega$, *any* $x \in \mathbb{R}^d$, $0 \le t < T(\omega)$, ρ *satisfies the continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla S_0) = 0.$$

Suppose now that $T_0 : \mathbb{R}^d \to \mathbb{R}$ is a smooth positive function (eventually we will set $T_0 \equiv 1$). Define

$$T_0(y,t) = T_0(y)$$

and for $j = 1, 2, \ldots,$

$$T_j(y,t) = \int_0^t \rho^{-\frac{1}{2}}(\cdot, s)\Delta_{\bullet}\left(\rho^{\frac{1}{2}}(\cdot, s)T_{j-1}(\Phi_s^{-1}\cdot, s)\right)\Big|_{\Phi_s y}\,ds.$$

Then for $j = 0, 1, 2, \ldots,$ define

$$\psi_j(x,t) = T_j(\Phi_t^{-1}x, t)\rho^{\frac{1}{2}}(x,t).$$

Then we have:

Lemma 2.1.

$$\frac{\partial \psi_j}{\partial t} + \nabla \psi_j \cdot \nabla S_0 = -\frac{1}{2}\psi_j \Delta S_0 + \Delta \psi_{j-1}$$

for $j = 0, 1, 2, \ldots,$ *with the convention* $\psi_{-1} \equiv 0$.

We have now assembled the main tools we need. We now explain how to solve the stochastic Hamilton-Jacobi continuity equations.

Proposition 2.2. *Assume that* $c \in C^2(\mathbb{R}^d \times \mathbb{R})$, $k \in C^2(\mathbb{R}^d \times \mathbb{R})$, $S_0 \in C^2(\mathbb{R}^d)$ *and* Φ_t *satisfies a no caustic condition. Then for a.e.* $\omega \in \Omega$ *solutions of the Hamilton-Jacobi continuity equations*

$$\frac{\partial S_j}{\partial t} + \frac{1}{2}\sum_{\substack{i_1, i_2 \ge 0 \\ i_1 + i_2 = j}} \nabla S_{i_1} \cdot \nabla S_{i_2} = \frac{1}{2}\Delta S_{j-1}, \qquad j \ge 0,$$

(with the convention $\frac{1}{2}\Delta S_{-1} = -c - \epsilon k \dot{W}_t$) are given by

$$S_1(x,t) = -\ln \psi_0(x,t)$$

and for $j \geq 2$

$$S_j(x,t) = 2^{1-j}\left(-\frac{\psi_{j-1}}{\psi_0} + \sum_{\substack{i_1,i_2\geq 1\\ i_1+i_2=j-1}} \frac{\psi_{i_1}\psi_{i_2}}{2\psi_0^2} - \right.$$
$$\left. - \sum_{\substack{i_1,i_2,i_3\geq 1\\ i_1+i_2+i_3=j-1}} \frac{\psi_{i_1}\psi_{i_2}\psi_{i_3}}{3\psi_0^3} + \cdots + \frac{(-1)^{j-1}\psi_1^{j-1}}{(j-1)\psi_0^{j-1}}\right)(x,t).$$

There is an important Corollary:

Corollary 2.1.

$$\frac{\partial}{\partial t}\left(\sum_{j=0}^{m} \mu^{2j} S_j\right) + \frac{1}{2}\sum_{j=0}^{m}\mu^{2j}\left(\sum_{\substack{i_1,i_2\geq 0\\ i_1+i_2=j}} \nabla S_{i_1}\cdot\nabla S_{i_2}\right) + c + k\dot{W}_t = \frac{\mu^2}{2}\Delta \sum_{j=0}^{m-1}\mu^{2j} S_j.$$

An all important role is played by the Nelson diffusion process y_s^μ with drift given by $-\nabla\left(\sum_{j=0}^{m}\mu^{2j}S_j(y_s^\mu, t-s)\right)$

$$dy_s^\mu = \mu\, dB(s) - \nabla\sum_{j=0}^{m}\mu^{2j}S_j(y_s^\mu, t-s)\, ds,$$
$$y_0^\mu = x.$$

Consider the stochastic heat equation of Stratonovich type

$$du_t^\mu(x) = \left[\frac{\mu^2}{2}\Delta u_t^\mu(x) + \mu^{-2}c(x,t)u_t^\mu(x)\right] dt + \epsilon\mu^{-2}k(x,t)u_t^\mu(x)\circ dW_t,$$
$$u_0^\mu(x) = T_0(x)\exp\left(-S_0(x)/\mu^2\right).$$

Proposition 2.3. *For each $m \geq 0$,*

$$u_t^\mu(x) =$$
$$\exp\left\{-\mu^{-2}\sum_{j=0}^{m}\mu^{2j}S_j(x,t)\right\}\mathbb{E}\exp\left\{-\frac{\mu^{2m}}{2}\int_0^t \Delta S_m(y_s^\mu, t-s)\, ds\right.$$
$$\left. + \frac{1}{2}\sum_{j=m+1}^{2m}\mu^{2(j-1)}\sum_{\substack{0\leq i_1,i_2\leq m\\ i_1+i_2=j}}\int_0^t \nabla S_{i_1}\cdot\nabla S_{i_2}(y_s^\mu, t-s)\, ds\right\}.$$

Note that the second factor is of the form $(1+O(\mu^{2m}))$ and the first factor gives the expansion up to μ^{2m-2} for each m.

Our main result is:

Proposition 2.4. *The solution of the viscous stochastic Burgers equation is for each $m \geq 0$*

$$
\begin{aligned}
v^{\mu}(x,t) = {} & \sum_{j=0}^{m} \mu^{2j} v_j(x,t) \\
& - \mu^2 \nabla \ln \mathbb{E}\Bigg\{ \exp\Bigg\{ -\frac{\mu^{2m}}{2} \int_0^t \nabla \cdot v_m(y_s^{\mu}, t-s)\, ds \\
& \qquad + \frac{1}{2} \sum_{j=m+1}^{2m} \mu^{2(j-1)} \sum_{\substack{0 \leq i_1, i_2 \leq m \\ i_1+i_2=j}} \int_0^t v_{i_1} \cdot v_{i_2}(y_s^{\mu}, t-s)\, ds \Bigg\}\Bigg\}
\end{aligned}
$$

where $v_j(x,t) = \nabla S_j(x,t)$.

Remark 2.1. *The fluid density ρ is simply given by*

$$\rho^{1/2}(x,t) = e^{-S_1(x,t)}. \tag{2.1}$$

3 Singularities of Heat and Burgers Equations

We study a local geometric relationship between curves on wavefronts and caustics. In two dimensions the curves are wavefronts. In three dimensions one can think of these curves as being on planar cross sections.

Definition 3.1. *A curve $x = x(\gamma)$ is said to have a generalised cusp if*

$$\frac{dx(\gamma)}{d\gamma} = 0,$$

γ being an intrinsic parameter such as arc-length.

3.1 A Local Geometric Result

Let $\mathcal{A} = \mathcal{A}(x_0, x, t)$ be the action for the initial momentum ∇S_0,

$$\mathcal{A}(x_0, x, t) = A(x_0, x, t) + S_0(x_0), \tag{3.1}$$

$A(x_0, x, t)$ being the classical action for the classical mechanical paths starting from x_0 getting to x at time t in some deterministic external potential $c(x)$. We tacitly assume $\mathcal{A}(x_0, x, t)$ is C^2 in space and C^1 in time.
Defining a diffeomorphism Φ_t corresponding to the classical mechanical flow now reduces to

$$\frac{\partial}{\partial x_0}\mathcal{A}(x_0, x, t) = 0, \tag{3.2}$$

i.e.

$$\frac{\partial}{\partial x_0}A(x_0, x, t) + \frac{\partial}{\partial x_0}S_0(x_0) = 0. \tag{3.3}$$

This gives

$$x = \Phi_t x_0, \text{ and } x_0 = \Phi_t^{-1}x = x_0(x, t), \tag{3.4}$$

where x_0 is possibly non-unique. Consider

$$\mathcal{A}(x_0, x, t) = 0, \quad \frac{\partial}{\partial x_0}\mathcal{A}(x_0, x, t) = 0. \tag{3.5}$$

Then we obtain the pre-wavefront by eliminating x in the above equation, the wavefront by eliminating x_0.
Imitating the above, equations of precaustics and caustics are

$$\frac{\partial}{\partial x_0}\mathcal{A}(x_0, x, t) = 0, \quad \text{Det}\left(\frac{\partial^2}{\partial x_0^2}\mathcal{A}(x_0, x, t)\right) = 0, \tag{3.6}$$

where for the precaustic we eliminate x and for the caustic we eliminate x_0. Different parts of the caustic can correspond to different $x_0^i(x,t)$ in $\Phi_t^{-1}\{x\}$. Let $x \to \alpha$, a point on the caustic, then since we expect the multiplicity of the x_0^i to change at α let us assume that $x_0^j(x,t) \to x_0^k(x,t)$ as $x \to \alpha$. Then $S_j(\alpha,t) = S_k(\alpha,t) = c$ and so $S_j(\alpha,t) = c$ and $S_k(\alpha,t) = c$ meet at α. This point α could be a cusp or a simple cross-over point.
We now come to one of our main results. We consider a differentiable curve on the prewavefront $x_0(\gamma)$ and its image under Φ_t, the classical mechanical flow, $x(\gamma) = \Phi_t x_0(\gamma)$. We need to assume $x(\gamma)$ is differentiable in γ at γ_0 where $x_0(\gamma_0)$ is on the precaustic.

Theorem 3.1. *Any differentiable image of a differentiable curve on the pre-wavefront meeting the precaustic at $x_0(\gamma_0)$ is a curve on the wavefront with a generalised cusp on the caustic at the image point $\Phi_t x_0(\gamma_0)$, as long as*

$$\mathrm{Det}\left(\frac{\partial^2}{\partial x_0^2}\mathcal{A}(x_0,x,t)\right)\Bigg|_{\gamma=\gamma_0} \neq 0.$$

The key lemma is:

Lemma 3.1. *On the precaustic, the tangent plane to the pre-wavefront at x_0 is spanned by* $\mathrm{Ker}\left(\frac{\partial^2}{\partial {x_0}^2}\mathcal{A}(x_0,x,t)\right)$, *if $x = \Phi_t x_0$ is C^1 differentiable in x_0, the point on the precaustic.*

A special case is when $c = k \equiv 0$ yielding $\Phi_t x_0 = x_0 + t\nabla S_0(x_0)$. The equation of the prewavefront is the eikonal equation $\frac{t}{2}|\nabla S_0(x_0)|^2 + S_0(x_0) = 0$ and of the precaustic is $\mathrm{Det}(1 + tS_0''(x_0)) = 0$. We trivially have

$$\nabla_{x_0}\left(\frac{t}{2}|\nabla S_0(x_0)|^2 + S_0(x_0)\right) = (1 + tS_0''(x_0))\nabla S_0(x_0)$$

and note that the normal to the prewavefront is in the direction of the non-zero eigenspace of $(1 + tS_0''(x_0)$.

Proof. (Outline Proof of Theorem 3.1)
The equation of the prewavefront is

$$\mathcal{A}(x_0,x,t) = 0, \qquad \frac{\partial}{\partial x_0}\mathcal{A}(x_0,x,t) = 0, \tag{3.7}$$

where $x = x(\gamma) = \Phi_t x_0(\gamma)$, $x_0 = x_0(\gamma)$. We assume $\mathcal{A}$ is C^2 in space variables.

Differentiate the above equations with respect to γ and set $\gamma = \gamma_0$. This yields

$$\frac{\partial \mathcal{A}(x_0, x, t)}{\partial x}\frac{dx}{d\gamma} = 0, \tag{3.8}$$

$$\frac{\partial^2 \mathcal{A}(x_0, x, t)}{\partial {x_0}^2}\frac{dx_0}{d\gamma} + \frac{\partial^2 \mathcal{A}(x_0, x, t)}{\partial x_0 \partial x}\frac{dx}{d\gamma} = 0. \tag{3.9}$$

The first term of equation (3.9) is zero by the above lemma since $x_0(\gamma)$ is a curve on the prewavefront. The last two conditions force $\frac{dx(\gamma)}{d\gamma} = 0$ if $\mathrm{Det}\left(\frac{\partial^2 \mathcal{A}(x_0,x,t)}{\partial x_0 \partial x}\right) \neq 0$. Hence, in these circumstances we must have a generalised cusp. $\square$

The above also shows that if $\frac{dx(\gamma)}{d\gamma} = 0$ then we are compelled to assume in addition to equations (3.7)

$$\frac{\partial^2 \mathcal{A}(x_0, x, t)}{\partial {x_0}^2}\frac{dx_0}{d\gamma} = 0 \tag{3.10}$$

and so x_0, a point on the prewavefront, must also be on the precaustic.

Remark 3.1. *The above is still valid if we replace $S_0(x)$ by $S_0(x) + c$, c a real constant. This leaves the caustic invariant.*

Theorem 3.2. *Consider any level surface H_t of Hamilton's principal function. This will meet the caustic surface C_t in a fixed curve a_t. Any smooth curve on H_t can only have a generalised cusp at a point α if $\alpha \in a_t$ and α has at least one inverse image point $\Phi_t^{-1}\alpha$ on the intersection of both precaustic $\Phi_t^{-1}C_t$ and pre-level-surface $\Phi_t^{-1}H_t$. Moreover, there will be a cusp at α if the tangent to the curve is not in* $\mathrm{Ker}\left(\frac{\partial^2 \mathcal{A}(\Phi_t^{-1}\alpha,\alpha,t)}{\partial x_0 \partial x}\right)$ *which is true generically.*

Remark 3.2. *The point is that*

$$\Phi_t^{-1}C_t \cap \Phi_t^{-1}H_t \subset \Phi_t^{-1}(C_t \cap H_t), \tag{3.11}$$

but the opposite inclusion does not hold. Images of points in $\Phi_t^{-1}(C_t \cap H_t) \setminus (\Phi_t^{-1}C_t \cap \Phi_t^{-1}H_t)$ give rise to cross-over points.

Theorem 3.3. *Let $\tau(a)$ be the first time such that there exist minimisers $y_1 \neq y_2$, $y_1, y_2 \in \mathbb{R}^d$ such that $\Phi_\tau(y_1) = \Phi_\tau(y_2) = a$ and $\dot{\Phi}_\tau(y_1) \neq \dot{\Phi}_\tau(y_2)$, i.e. τ is a caustic time at point a. Then $v^0(x,t)$ is discontinuous at (τ, a).*

For polynomial S_0, in two dimensions, we can divide one side of the caustic into *hot* and *cool* parts. We consider points of intersection α with the level surface $S(x,t) = S(\alpha,t)$ cusped at α: if the points $\Phi_t^{-1}\alpha$ common to the precaustic and the pre-level-surface include the minimising $x_0(x,t)$ then $|v^0(x,t)| \to 0$ as $x \to \alpha$ from the cusped side, because $\nabla S(x,t) \to \nabla S(\alpha,t) = 0$. We call such points α cool because the Burgers fluid has zero velocity on one side of the caustic. The whole of one side of the generic semi-cubical parabolic cusp is cool. Not all of one side of the two dimensional Swallowtail is cool. Similar results hold in three dimensions.

4 Examples

Consider S_0 given in the following table for two and three dimensions respectively,

Shockwave	Wavefront	S_0
Cusp	Tricorn	$x^2y/2$
Swallowtail	Plough	$x^5 + \lvert x\rvert^{3/2}y$
Butterfly	Fish	$x^3y/3 + x^2z/2$
Swallowtail	Seagull	$\lvert x\rvert^5 + \lvert x\rvert^{1+c}y + \lvert x\rvert^c z$

where $\frac{1}{2} < c < 1$. The nature of the singularity, of the shockwave, is given in the first column and the corresponding wavefronts in the second. The wavefront corresponding to the semicubical parabolic Cusp is the Tricorn, which has three cusps on the semi-cubical parabola, one of which coincides with the original cusp of the shockwave, and a line-pair. The wavefront corresponding to the Butterfly is the Fish which meets the Butterfly in four curves (one of which is a straight line on which they touch).

4.1 A two dimensional example

For $S_0(x_0, y_0) = x_0^2 y_0/2$ the prewavefront has equation

$$x_0^2\left(\frac{(y_0+\frac{1}{2t})^2}{(\frac{1}{2t})^2} + \frac{x_0^2}{(\frac{1}{t})^2} - 1\right) = 0,$$

and the precaustic has equation

$$1 + ty_0 = t^2 x_0^2.$$

The line-pair $x_0^2 = 0$ has been omitted for clarity. We now map $(x, y) = \Phi_t(x_0, y_0)$ to obtain the cusp and hypocycloid tricorn

$$x = \frac{\cos\theta(1+\sin\theta)}{2t}, \qquad y = \frac{\sin\theta(1-\sin\theta)}{2t},$$

for $0 \le \theta < 2\pi$, as the wavefront at time t for the heat equation and semi-cubical parabolic cusp

$$8\,(yt+1)^3 = 27t^2x^2,$$

as shockwave at time t for the corresponding Burgers velocity field.

The level surface $S_0(x_0, y_0) = 0$.

Figure 1 : Precaustic and Prewavefront

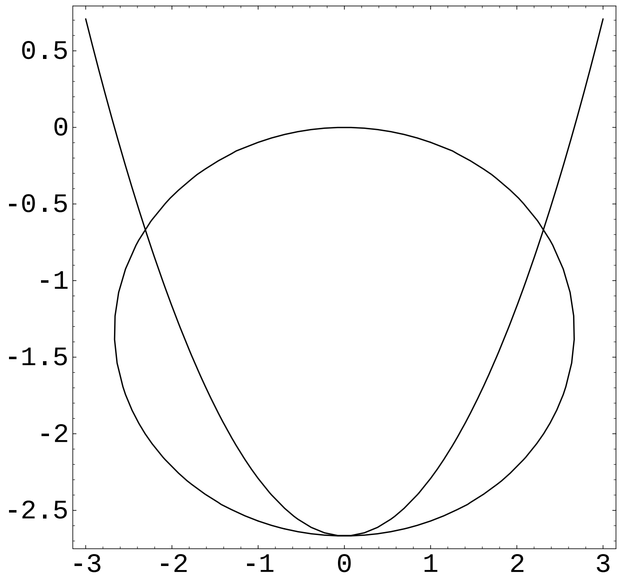
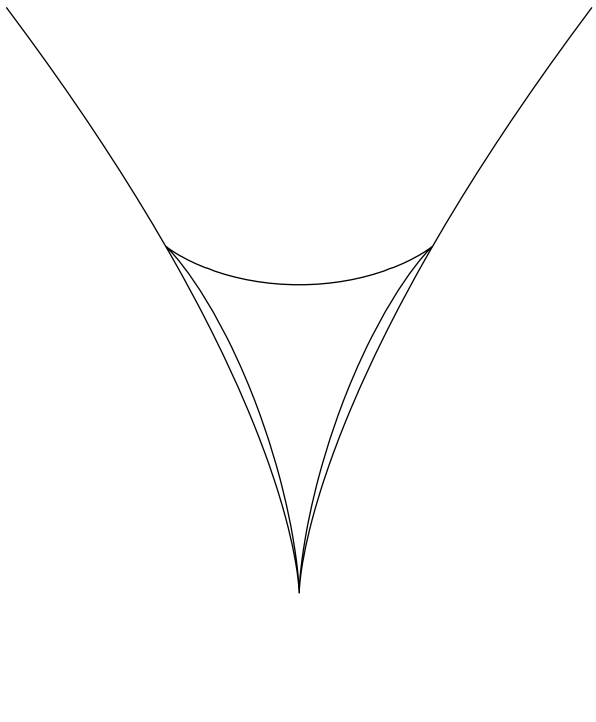


Figure 2 : Cusp and Tricorn



The level surface $S_0(x_0, y_0) = 2$

Figure 3 : Precaustic (Parabola) and Prewavefront (Kneecap)

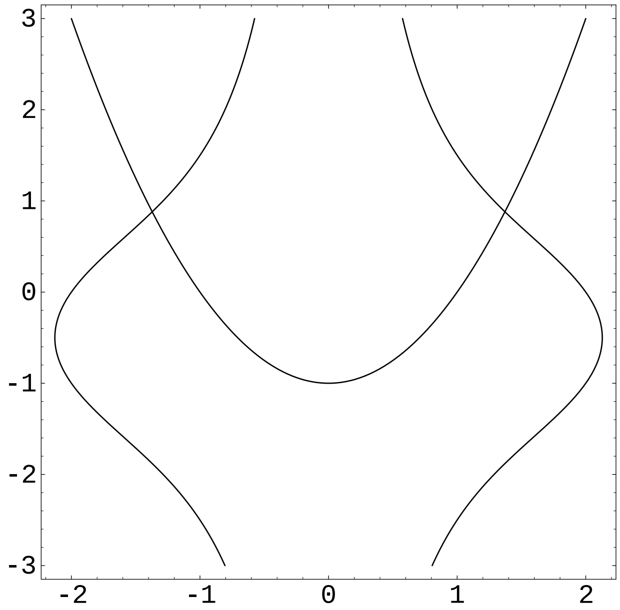
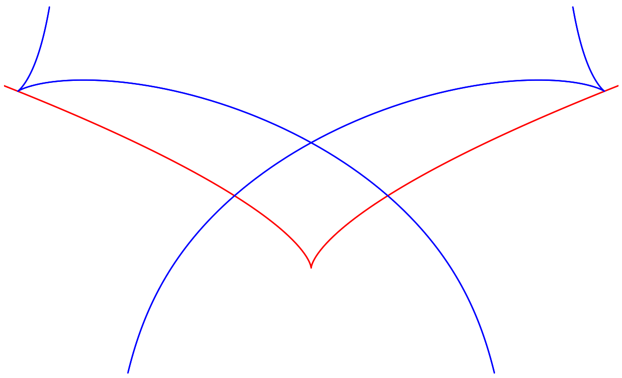


Figure 4 : Caustic (Cusp) and Wavefront (Neckline)



The level surface $S_0(x_0, y_0) = -2^{-7}$

Figure 5 : Precaustic (Parabola) and Prewavefront (Pebble Pair)

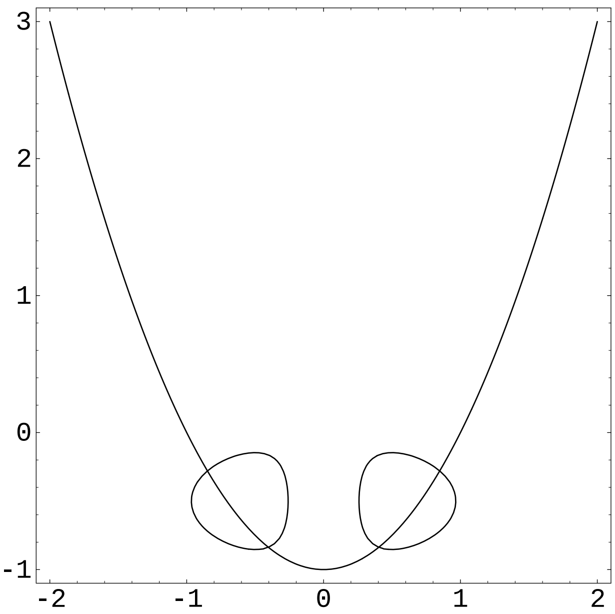
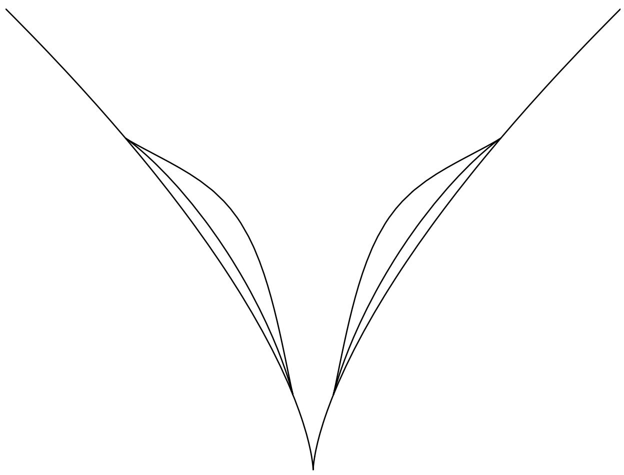


Figure 6 : Caustic (Cusp) and Wavefront (Lapel Pair)



We have verified numerically that the caustic surfaces are indeed discontinuities for the Burgers velocity field as shown below. Further, it is not difficult to see that the whole of one side of this Cusp surface is cool [11, 12].

Figure 7 : $|v^{\mu}|^2$ for small μ

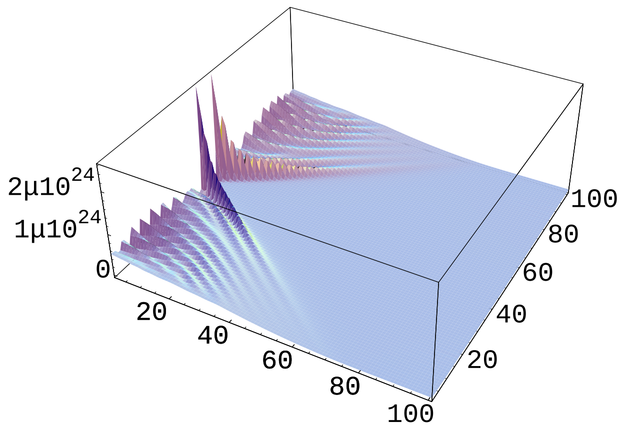
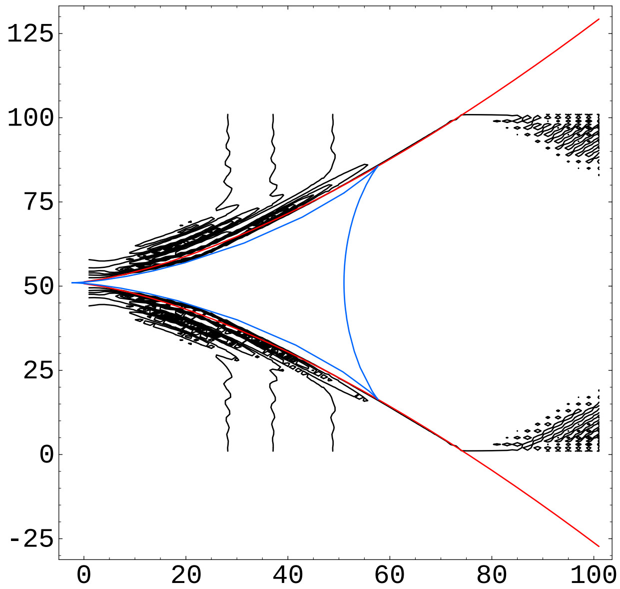


Figure 8 : Contour Plot of $|v^{\mu}|^2$ with Cusp and Tricorn



4.2 A three dimensional example

For $S_0(x_0, y_0, z_0) = x_0^3 y_0/3 + x_0^2 z_0/2$ we can quickly see the close relationship with the previous example by means of the cross-sections shown below.

Figure 9 : Pre - Butterfly and Fish

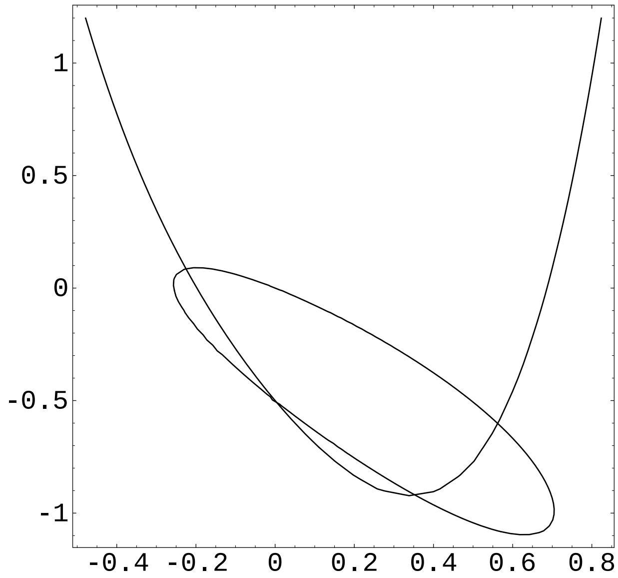
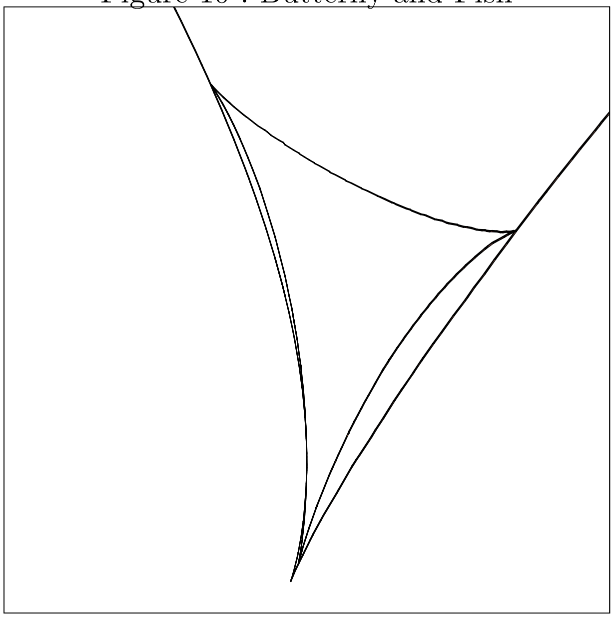


Figure 10 : Butterfly and Fish



To see where the Butterfly and Fish meet, consider the classical mechanics

$$\begin{aligned} x &= x_0 + t(x_0^2 y_0 + x_0 z_0), \\ y &= y_0 + \frac{t}{3} x_0^3, \\ z &= z_0 + \frac{t}{2} x_0^2. \end{aligned}$$

Solving the precaustic equation leads to

$$z_0 = -\frac{1 - t^2 x_0^2 - t^2 x_0^4 + 2t x_0 y_0}{t}$$

and with this we simplify the pre-wavefront equation, one solution of which is $x_0 = 0$ leaving a quadratic equation for y_0 with roots.

$$y_0 = \frac{-2 + 6t^2 x_0^2 + 6t^2 x_0^4 - \sqrt{4 + 3t^2 x_0^2 + 8t^2 x_0^4}}{6t x_0},$$

and

$$y_0 = \frac{-2 + 6t^2 x_0^2 + 6t^2 x_0^4 + \sqrt{4 + 3t^2 x_0^2 + 8t^2 x_0^4}}{6t x_0}.$$

When $x_0 = 0$ we have $z_0 = -\frac{1}{t}$. Therefore the wavefront and caustic meet on a straight line at

$$x = 0,\ z = -1/t,\ y \in \mathbb{R}.$$

For one value of y_0 , we obtain the parametric equations of the two curves of intersection ($x_0 > 0$ and $x_0 < 0$)

$$\begin{aligned} x &= \frac{1}{6} x_0 \left(2 + \sqrt{4 + t^2 x_0^2 (3 + 8x_0^2)} \right), \\ y &= -\frac{2 - 2t^2 x_0^2 (3 + 4x_0^2) + \sqrt{4 + t^2 x_0^2 (3 + 8t^2)}}{6t x_0}, \\ z &= \frac{-3t^2 x_0^2 (1 + 2x_0^2) + 2\left(-1 + \sqrt{4 + t^2 x_0^2 (3 + 8t^2)} \right)}{6t}. \end{aligned}$$

For the other value of y_0 we obtain the parametric equation of one additional curve

$$\begin{aligned} x &= -\frac{1}{6} x_0 \left(-2 + \sqrt{4 + t^2 x_0^2 (3 + 8x_0^2)} \right), \\ y &= \frac{-2 + 2t^2 x_0^2 (3 + 4x_0^2) + \sqrt{4 + t^2 x_0^2 (3 + 8t^2)}}{6t x_0}, \\ z &= -\frac{3t^2 x_0^2 (1 + 2x_0^2) + 2\left(1 + \sqrt{4 + t^2 x_0^2 (3 + 8t^2)} \right)}{6t}. \end{aligned}$$

Figure 11 : The Butterfly Singularity for Burgers Equation

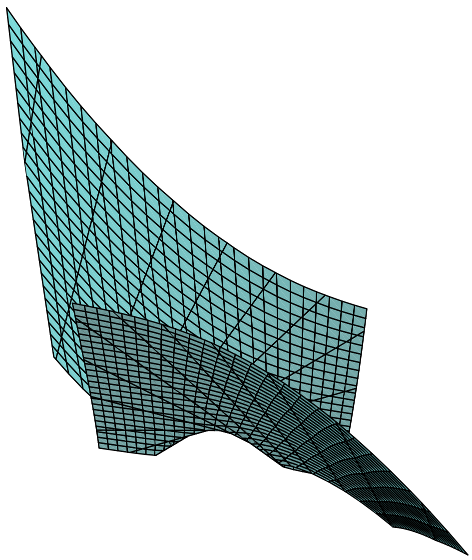
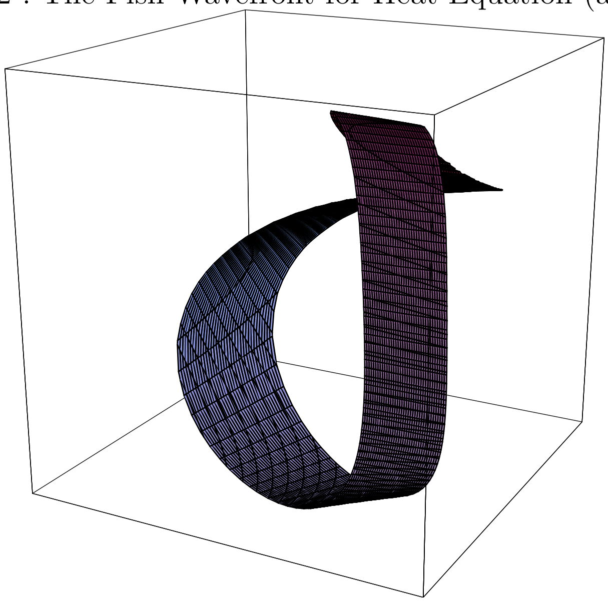


Figure 12 : The Fish Wavefront for Heat Equation (a section)



4.3 Swallowtail

We briefly illustrate the caustic and wavefront for $S_0(x_0, y_0, z_0) = |x_0|^5 + |x_0|^{5/3}y + |x_0|^{2/3}z_0$.

Figure 13 : The Swallowtail Singularity for Burgers Equation

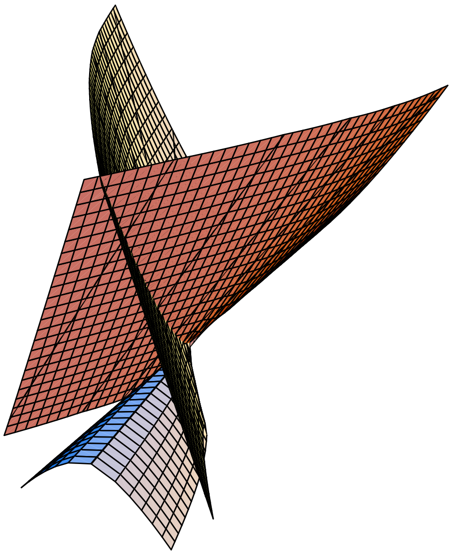
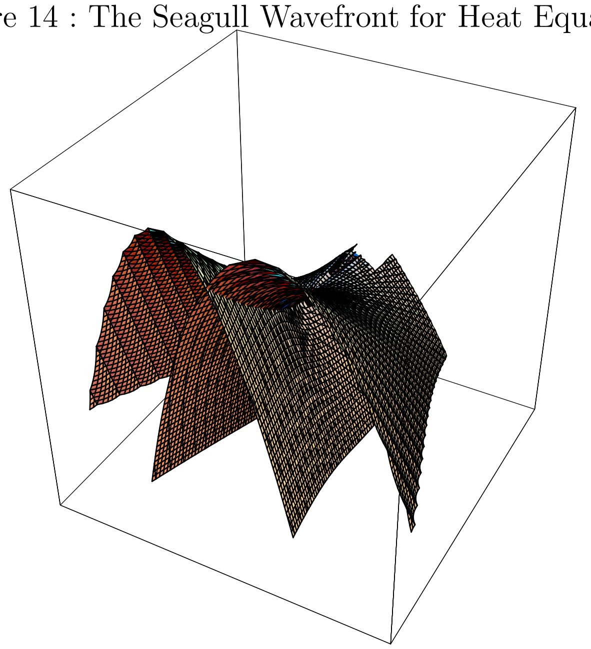


Figure 14 : The Seagull Wavefront for Heat Equation



4.4 Local Geometric Result – Stochastic Case

Here we have to assume for fixed time t, $A(x_0, x, t)$ is C^2 in space variables, almost surely. We seek to generalise the above results to include noise. Recall the ibp[2] for the continuous process $\dot{X}(s)$, the stochastic integral

$$\int_0^t u(s)\,d\dot{X}(s) = u(t)\dot{X}(t) - u(0)\dot{X}(0) - \int_0^t \dot{u}(s)\dot{X}(s)\,ds,$$

u being deterministic, $u \in C^1(0, t)$.
The above is also valid for random u, if the stochastic integral is of Itô's form and the corresponding Itô correction is zero. An important example is when u_s is the almost sure derivative

$$u_s = \frac{\partial X_s}{\partial x_0^\alpha} \qquad (\alpha = 1, 2, \ldots, n). \tag{4.12}$$

Then, if we assume ∇c, ∇k Lipschitz, with Hessians with respect to space variables $\nabla^2 c$, $\nabla^2 k$ and all second derivatives with respect to space variables of c and k being bounded, $\frac{\partial X_s}{\partial x_0^\alpha}$ satisfies

$$\begin{aligned} &d\left(\frac{\partial X_s}{\partial x_0^\alpha}\right) \\ &= \left(\frac{\partial \dot{X}_s}{\partial x_0^\alpha} - \int_0^s \left[\nabla^2 c(X(r))\frac{\partial X(r)}{\partial x_0^\alpha}\,dr + \epsilon\nabla^2 k(X(r), r)\frac{\partial X(r)}{\partial x_0^\alpha}\,dW_r\right]\right) ds, \end{aligned} \tag{4.13}$$

so $du_s\,d\dot{X}_s = 0$ (Kunita [23]).

Lemma 4.1. *Assume in addition S_0, $c \in C^2$ and $k \in C^{2,0}$. Then, almost surely, we have*

$$\frac{\partial A(x_0, x, t)}{\partial x_0} = -\dot{X}(0). \tag{4.14}$$

Theorem 4.1. *Modulo some differentiability of curves, curves on level surfaces of Hamilton's stochastic principal function meet the random caustic surface in generalised cusps as long as the point of intersection is the image under map Φ_t of at least one common point on precaustic and pre-level-surface.*

Here we illustrate the random caustic and wavefront for the case $S_0(x_0, y_0) = x_0^2 y_0/2$, $c(x) = 0$ and $k(x,t) = x$.

Figure 15 : Stochastic Precaustic and Prewavefront

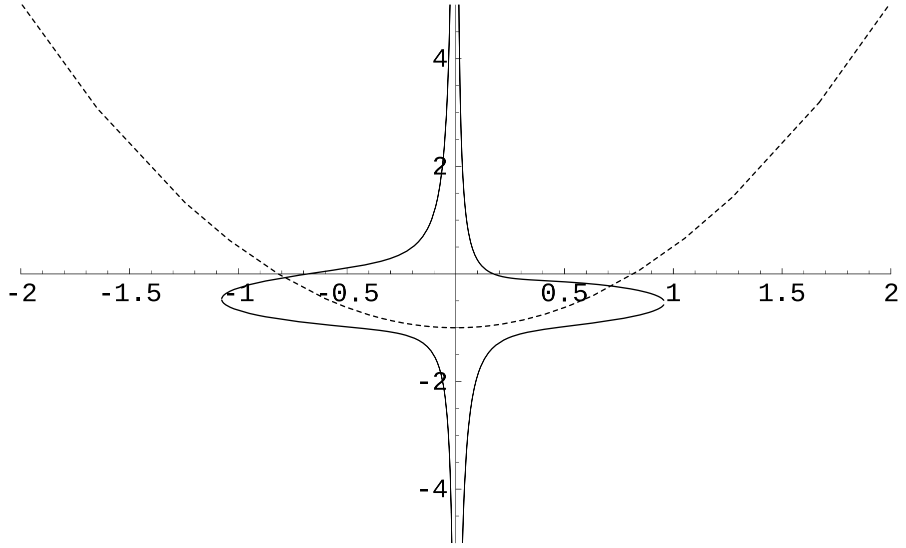
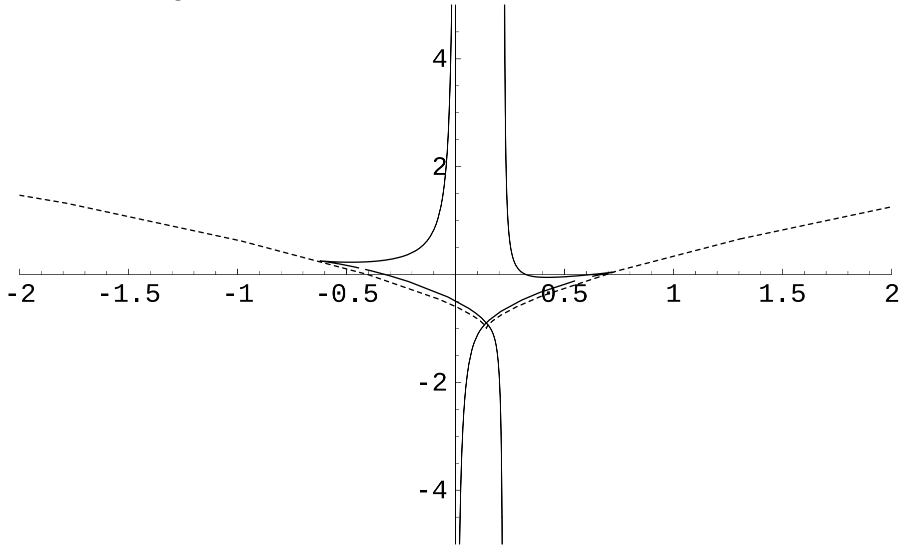


Figure 16 : Stochastic Caustic and Wavefront



4.5 Harmonic Oscillator with noise

Consider $v = v(x,t)$, $x \in \mathbb{R}^d$, $t \in \mathbb{R}^+$,

$$dv + (v \cdot \nabla)v\,dt = -\Omega^2 x\,dt - \epsilon \nabla k(x,t)\,dW_t$$

with $v(x,0) = \nabla S_0(x)$ and the corresponding stochastic mechanics

$$d_s \dot{X}^\epsilon(s) + \Omega^2 X^\epsilon(s)\,ds = -\epsilon \nabla k(X^\epsilon(s), s)\,dW_s.$$

Theorem 4.2. *Let* $\Delta_t^0(x_0) = \mathrm{Det}\left(\frac{\partial X^0(t)}{\partial x_0}\right) = \mathrm{Det}\left(\frac{\partial}{\partial x_0}\Phi_t x_0\right)$*, where* Φ_t *is a map defined by the harmonic oscillator without noise. Then, if* $\nabla k(x,t) = a(t)$ *is independent of* x*, the random shockwave has equation*

$$\Delta_t^0\left((\Phi_t^0)^{-1}(y + \epsilon B_t)\right) = 0,$$

where B_t *is a random vector* $i = 1, 2, \ldots, n$

$$(B_t)_i = W\left(\int_0^t \left(\sin(\Omega(t-s))\Omega^{-1}a(s)\right)_i^2\,ds\right).$$

Noise leaves the shape of the shockwaves unchanged merely displacing them by a random vector B_t.

We now illustrate the development of the semi-cubical parabolic shockwave with respect to time for the velocity field of a Burgers' fluid with initial velocity $(x_0 y_0, x_0^2/2)$ in a harmonic well with $V(x,y) = (x^2 + \omega^2 y^2)/2$. The effect of noise here is to superimpose on the deterministic movement of the cusp a brownian motion. The deterministic part is given by

$$8\left(y \csc \omega t + \cot t \cot \omega t\right)^3 \omega = 27x^2 \cot^2 \omega t \csc^2 t,$$

and we illustrate the motion in two cases $\omega = \frac{3}{2}$ and $\omega = \sqrt{2}$ below. We show the contour $\Delta_t^0\left((\Phi_t^0)^{-1}(y)\right) = 0$ and time progresses as one moves along the grid of images from left to right and top to bottom.

Figure 17 : Evolution of shockwave; $\omega = \frac{3}{2}$

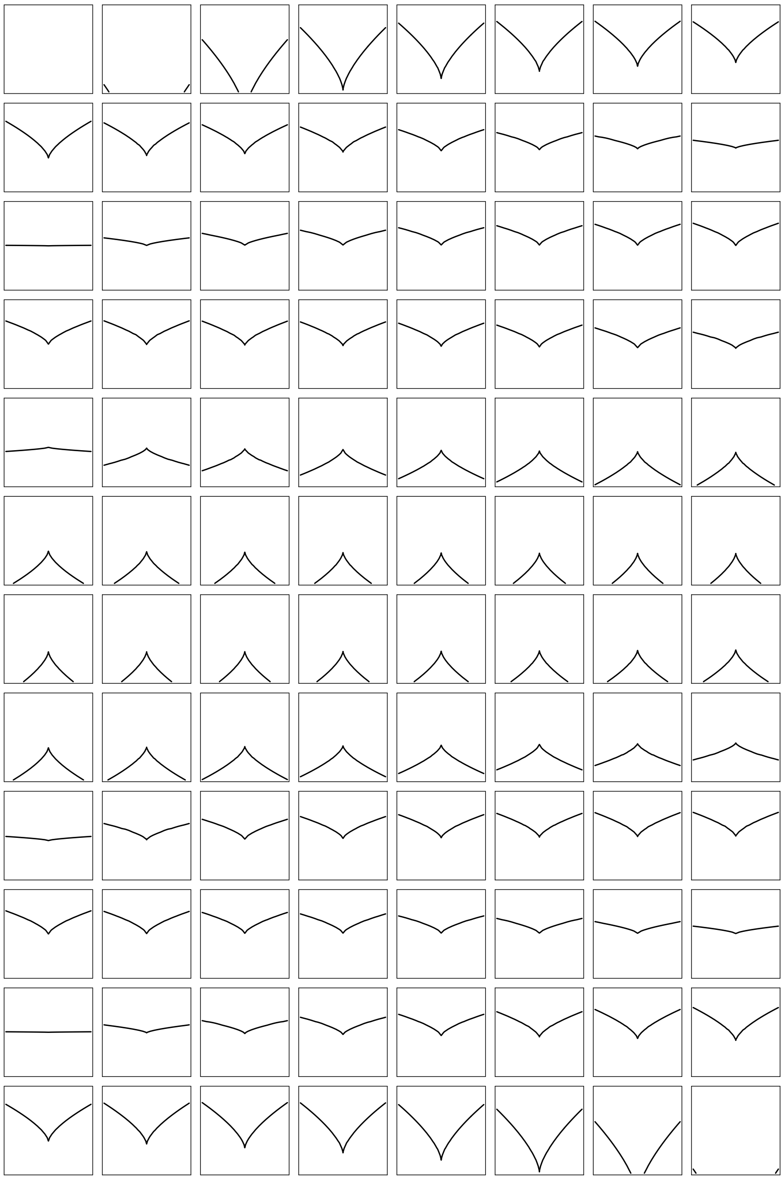
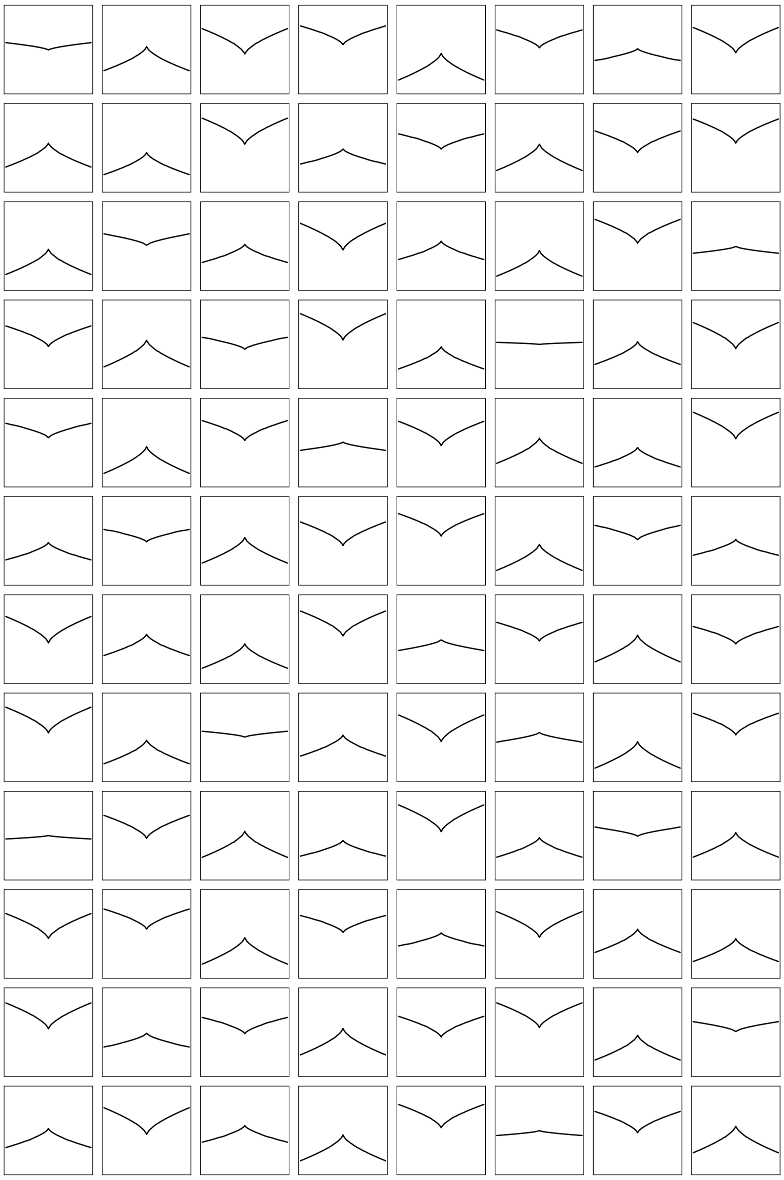


Figure 18 : Evolution of shockwave; $\omega = \sqrt{2}$



5 More general potentials

Consider the physically interesting case $v = v(x,t)$, $x \in \mathbb{R}^d$, $t \in \mathbb{R}^+$,

$$dv + (v \cdot \nabla)v\, dt = -\nabla c(x)\, dt - \epsilon \nabla k(x)\, dW_t,$$

with the stochastic classical mechanics

$$d\dot{X}^\epsilon(x_0,s) = -\nabla c(X^\epsilon(x_0,s))ds - \epsilon\nabla k(X^\epsilon(x_0,s))dW_s,$$

with $X^\epsilon(x_0,0) = x_0$ and $\dot{X}^\epsilon(x_0,0) = \nabla S_0(x_0)$, for $0 < s < t$.
Let $X^0(x_0,s) = \Phi_s x_0$ satisfy the deterministic version

$$\ddot{X}^0(s) = -\nabla c(X^0(s)), \qquad 0 < s < t,$$

and let $\mathcal{G}$ be given by

$$\mathcal{G}_{ij} = \left\{X_i^0(u), X_j^0(s)\right\}\theta(s-u),$$

Lemma 5.1. *Let X^0 be defined as above. Then $\mathcal{G}$ satisfies the matrix Jacobi equation*

$$\left(\frac{d^2}{ds^2} + \nabla^2 c(X^0(x_0,s))\right)\mathcal{G}(x_0,s,u) = 0,$$

with boundary conditions

$$\mathcal{G}(x_0,s_+,s) = 0, \quad \frac{d\mathcal{G}}{ds}(x_0,s,u)\bigg|_{s=u_+} = I.$$

Theorem 5.1. *Assume (i) there exists a unique minimiser $x_0 = x_0(x,t)$ for fixed t and x such that $\Phi_t x_0 = x$, (ii) ∇c, $\nabla^2 c$, and ∇k, $\nabla^2 k$ are Lipschitz continuous with respect to x, the Hessians of c, k and S_0 are bounded and all the third derivatives of c and k and fourth derivatives of c are bounded, (iii) $\sup\limits_{x_0\in\mathbb{R}^d} \int_0^s (\mathcal{G}(s,u)\nabla k(\Phi_u x_0))^8 du < \infty$ for all $0 \le s \le t$. Define the linearised perturbed solution $\tilde{X}^\epsilon(x_0,s)$*

$$\tilde{X}^\epsilon(x_0,s) = \Phi_s x_0 - \epsilon\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\, dW_u,$$

for $s \in [0,t]$. Then there exists a constant $M > 0$ such that for any $\delta > 0$ and sufficiently small $\epsilon > 0$,

$$P\{\frac{1}{\epsilon^{\frac{3}{2}}}\sup_{x_0\in\mathbb{R}^d}|X^\epsilon(x_0,s) - \tilde{X}^\epsilon(x_0,s)| > \delta \text{ some } s \in [0,t]\} < \frac{M\epsilon^2}{\delta^4},$$

and

$$P\{\frac{1}{\epsilon^{\frac{3}{2}}}\sup_{x_0\in\mathbb{R}^d}|\nabla X^\epsilon(x_0,s)-\nabla\tilde{X}^\epsilon(x_0,s)|>\delta \text{ for some } s\in[0,t]\}<\frac{M\epsilon}{\delta^2}.$$

In particular, $X^\epsilon(x_0,s)-\tilde{X}^\epsilon(x_0,s)=\circ(\epsilon^{\frac{3}{2}})$ *and* $\nabla X^\epsilon(x_0,s)-\nabla\tilde{X}^\epsilon(x_0,s)=\circ(\epsilon^{\frac{3}{2}})$ *as* $\epsilon\to 0$ *in probability.*

Define for each $\omega\in\Omega_0$,

$$\mathcal{D}^\epsilon_{pre}=\{(t,x_0):\text{Det}\,\nabla_{x_0}X^\epsilon(x_0,t)\neq 0, 0\le t\le T\},$$

and

$$\mathcal{D}_{pre}=\{(t,x_0):\text{Det}\,\nabla_{x_0}\Phi_t x_0\neq 0, 0\le t\le T\}.$$

Lemma 5.2. *As* $\epsilon\to 0$, $[0,T]\times\mathbb{R}^d-\mathcal{D}^\epsilon_{pre}\to[0,T]\times\mathbb{R}^d-\mathcal{D}_{pre}$ *in probability for any given* $T>0$*. That is to say the precaustic surface of the stochastic mechanics converges to the precaustic surface of the classical mechanics as* $\epsilon\to 0$ *in probability.*

Theorem 5.2. *Denote for any given* $T>0$

$$\mathcal{D}^\epsilon=\{(t,x):\text{Det}(\nabla(X^\epsilon)^{-1}_t x)\neq 0, 0\le t\le T\},$$

and

$$\mathcal{D}=\{(t,x):\text{Det}(\nabla\Phi^{-1}_t x)\neq 0, 0\le t\le T\}.$$

Then as $\epsilon\to 0$, $[0,T]\times\mathbb{R}^d-\mathcal{D}^\epsilon\to[0,T]\times\mathbb{R}^d-\mathcal{D}$*. That is to say the caustic surface of the stochastic mechanics with noise converges in probability to the caustic surface of the classical mechanics without noise as* $\epsilon\to 0$.

Theorem 5.3. *Let* ϕ_s *be the minimizer of* $\frac{1}{2}\int_0^t|\dot{\phi}_s|^2ds+S_0(\phi_t)-\int_0^t c(\phi_s)ds$*, satisfying* $\phi_t=x$ *and* ϕ^ϵ_s *be the minimizer of* $\frac{1}{2}\int_0^t|\dot{\phi}^\epsilon_s|^2ds+S_0(\phi^\epsilon_t)-\int_0^t c(\phi^\epsilon_s)ds-\epsilon\int_0^t k(\phi^\epsilon_s)dW_s$*, satisfying* $\phi^\epsilon_t=x$ *for almost all* $\omega\in\Omega$*. Then we have for almost all* $\omega\in\Omega$

$$S(x,t)-\epsilon\int_0^t k(\phi^\epsilon_s)dW_s\le S^\epsilon(x,t)\le S(x,t)-\epsilon\int_0^t k(\phi_s)dW_s.$$

In particular, as $\epsilon\to 0$, $S^\epsilon(x,t)\to S(x,t)$ *a.s..*
Furthermore, for the classical mechanics , assume there exists a unique minimising x_0 *for fixed* t *and* x *such that* $\Phi_t x_0=x$*. The random wavefront for the heat equation has* (x,t) *equation*

$$S^0(x,t)=\epsilon\int_0^t k(\Phi_s x_0)\,dW_s+\circ(\epsilon),$$

where $S^0(x,t)$ *is Hamilton's principal function for the path* $X^0(x,t)(s)$*.*

6 Conclusion

Recall that our random map $\Phi_s(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$d_s\dot{\Phi}_s = -\nabla c(\Phi_s)\,ds - \epsilon\nabla k(\Phi_s)\,dW_s,$$

with $\Phi_0 = I$ and $\dot{\Phi}_0 = \nabla S_0$. The global inverse function theorem gives:

Proposition 6.1. *If c, k and S_0 are smooth with bounded second order partial derivatives, there exists $T(\omega) > 0$ such that $\Phi_s(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a random diffeomorphism $0 \leq s \leq T(\omega)$.*

It follows that :

Proposition 6.2. *The random Burgers velocity field in the limit of zero viscosity is*

$$v^0(x,t) = v^0(\Phi_t^{-1}x, 0) - \int_0^t \nabla c(\Phi_s\Phi_t^{-1}x)\,ds - \int_0^t \epsilon\nabla k(\Phi_s\Phi_t^{-1}x)\,dW_s,$$

with $v^0(x,0) = \nabla S_0(x)$. This v^0 is finite almost surely $0 \leq t \leq T(\omega)$. The corresponding solution of the continuity equation, the random density $\rho(x,t) = \mathrm{Det}(\nabla\Phi_t^{-1}x)$, is finite almost surely for $0 \leq t \leq T(\omega)$ but 'blows up' on the random caustics. Evidently $\nabla \wedge v^0 \equiv 0$, almost surely.

This 'blow-up' result should be compared with results for Euler's equation which is identical to the above save for the constraint $\rho \equiv 1$. In a future paper we hope to relax the assumption $\nabla \wedge v^0 \equiv 0$, when more realistic comparisons with Euler's equation can be made. Needless to say, the detailed nature of the geometry of the caustic surface and the wavefront will affect the value of $v^\mu(x,t)$ on the caustic. Our geometrical and analytical results are clearly relevant here for small noise in the way they relate the stochastic and deterministic situations. We believe that they will also be important in determining where the 'blow-up' occurs for the inviscid Burgers equation. Recall that the fluid density ρ is infinite at all points of the caustic but as we have seen at cool points on one side of the caustic the fluid is at rest.
As for a simple Arnol'd Thom classification of wavefronts, this is clearly not going to be possible here because these wavefronts correspond to zero contours of Hamilton's principal function and any perturbation changes the zero set. For instance, the simplest wavefront, the Tricorn plus the Line-pair fuse under the influence of a small positive deterministic potential to form a Neckline as in Figure 4 (see the Lapel Pair as in Figure 6 for the case of a small negative deterministic potential). However this Neckline is indeed

stable under perturbations by small bounded potentials as is the Lapel Pair. This is not surprising since they correspond to level surfaces of Hamilton's principal function $S(x,t)$ positive or negative, respectively. The deterministic analogue of Theorem 5.3 explains how this arises. Of course, given our main result, it is more natural to refer to their preimages or unfoldings – the Kneecap and the Pebble Pair (Figures 3 and 5). We therefore see that these are generic in the sense we have just described. Our many examples bear this out for small noise.

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