Generalized fractal dimensions of compactly supported measures on the negative axis

François Germinet
UMR 8524 CNRS, UFR de Mathématiques,
Université de Lille 1, F-59655 Villeneuve d'Ascq Cédex, France
E-mail: germinet@agat.univ-lille1.fr

Serguei Tcheremchantsev
UMR 6628 - MAPMO,
Université d'Orléans, B.P. 6759, F-45067 Orléans Cédex, France
E-mail: serguei.tcherem@labomath.univ-orleans.fr

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Abstract

We study generalized fractal dimensions of measures, called the Hentschel-Procaccia dimensions and the generalized Rényi dimensions. We consider compactly supported Borel measures with finite total mass on a complete separable metric space. More precisely we discuss in great generality finiteness and equality of the different dimensions for negative values of their argument q. In particular we do not assume that the measure satisfies to the so called "doubling condition". A key tool in our analysis is, given a measure μ, the function $g(\varepsilon)$, $\varepsilon > 0$, defined as the infimum over all points x in the support of μ of the quantity $\mu(B(x,\varepsilon))$, where $B(x,\varepsilon)$ is the ball centered at x and of radius ε.

Contents

1 Introduction and presentation of the results

1.1 Introduction

Generalized fractal dimensions (or multifractal dimensions) of finite Borel measures are some families of real numbers taking value in $[0, +\infty]$ and indexed by real parameter $q \in \mathbb{R}$. Two important families of generalized fractal dimensions are the Hentschel-Procaccia dimensions ($D^+(q)$ for the upper dimension, $D^-(q)$ for the lower dimension) and the generalized Rényi dimensions, which can be seen as discretized versions of the Hentschel-Procaccia dimensions. In this article we study these two families of dimensions for *negative* values of their argument: $q \leq 0$. More precisely we investigate the finiteness of these dimensions and the equivalence of the different definitions. As for the setting, we work with positive and regular Borel measures of finite mass with compact support on a complete separable metric space (X, ϱ). No further condition on the measure is assumed. In particular we do not suppose that μ satisfies to the so-called "doubling condition". Note also that we do not resort to the Besicovitch covering Theorem (which is only valid on more specific spaces X). The case of non compactly supported measures will be treated in the companion paper [GT].

Interest in such families of fractal dimensions, also called generalized entropies, goes back, at least to the late 50's with the work of Rényi on information theory [R]. For twentysome years generalized fractal dimensions are also known in theoretical physics to enter the game of many interesting phenomena, and specially dimensions of measures defined on attractors in dynamical systems [HP] [GP] [HJKPS]. Multifractal dimensions of Gibbs measures of dynamical systems appear rigorously in numerous works, e.g. [GV] [PW1] [PW2] [TV1] [TV2] [LPV]. A fruitful multifractal formalism has been developped in this context, e.g. [BMP] [Ri] [O1]. In astronomy and biology researchers also came accross these dimensions, e.g. [VBP] [Be].

In quantum dynamics, numerical computations first suggested that the Hentschel-Procaccia dimensions should play a non trivial role in the phenomenon of anamolous transport [Ma]. It is recently that these dimensions have been rigorously proved to enter the game of the transport properties of wave-packets in quantum dynamics, [BGT1] [GSB1] [T1] (see also [BSB]). In these works the generalized fractal dimensions are indeed shown to influence directly the speed of the electronic transport. If $q \in (0, 1)$ is the main regime reached by the results of [BGT1][GSB1][T1], dimensions for negative values of q appeared in [Ma] [BSB], and they are rigorously proved to enter the play under some assumptions [BGT1][GSB1] [T1].

However the mathematical study of these objects is fairly recent and actually not so much is known about them, except for some particular examples. To our best knowledge most of what is known for $q \neq 1$ can be found in Olsen [O1], Pesin [P], Lau and Ngai [LN], and the recent paper of the two present authors and Barbaroux [BGT2]. As for the particular point $q = 1$ we refer to the work of Heurteaux [He] (and reference therein), and also to [O2][BGT2].

For the purpose of this presentation we briefly recall the definition of the generalized fractal dimensions we consider in this article. We shall come back to these definitions in the next section. Set

$$I(q,\varepsilon) = \int_{\mathrm{supp}\mu} \mu(B(x,\varepsilon))^{q-1} \mathrm{d}\mu(x) \ , \ q \in \mathbb{R}, \ \varepsilon > 0,$$

with $B(x,\varepsilon) = \{y \in X, \varrho(x,y) \leq \varepsilon\}$. The lower and upper Hentschel-Procaccia dimensions are then defined for $q \neq 1$ as

$$D^-(q) = \liminf_{\varepsilon \downarrow 0} \frac{\log I(q,\varepsilon)}{(q-1)\log \varepsilon} \quad \text{and} \quad D^+(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log I(q,\varepsilon)}{(q-1)\log \varepsilon}. \tag{1.1}$$

We note that in general upper dimensions $D^+(q)$ and lower dimensions $D^-(q)$ are distinct (for an example in quantum transport see [T2]). So that in the present paper we shall distinct them.

Discrete analogous of the Hentschel-Procaccia dimensions are the generalized Rényi dimensions, which can be computed by summing either over coverings or over packings of $\mathrm{supp}\,\mu$. For a set of closed balls of radius ε, say $u = B(x_i,\varepsilon)_{i \in I}$, set

$$S(u,q,\varepsilon) = \sum_{i \in I} \mu(B(x_i,\varepsilon))^q \tag{1.2}$$

(implicitely the summation is over i's such that $\mu(B(x_i,\varepsilon)) > 0$). We assume that I is finite or countable. The set $u = (B(x_i,\varepsilon))_{i \in I}$ is called centered if $x_i \in \mathrm{supp}\mu$ for all $i \in I$. We denote by $\mathcal{C}^{(\varepsilon)}$ and $\mathcal{C}_c^{(\varepsilon)}$ the set of ε-coverings and centered ε-coverings of $\mathrm{supp}\,\mu$ respectively. Similarly, we denote by $\mathcal{P}^{(\varepsilon)}$ and $\mathcal{P}_c^{(\varepsilon)}$ the set of ε-packings and centered ε-packings of $\mathrm{supp}\,\mu$ respectively (see Section 2.3 for precise definitions). We then define for $q \neq 1$ the centered covering Rényi dimensions as

$$C_cD^-(q) = \liminf_{\varepsilon \downarrow 0} \frac{\log C_c(q,\varepsilon)}{(q-1)\log \varepsilon} \quad \text{and} \quad C_cD^+(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log C_c(q,\varepsilon)}{(q-1)\log \varepsilon}. \tag{1.3}$$

and the centered packing Rényi dimensions as

$$P_cD^-(q) = \liminf_{\varepsilon \downarrow 0} \frac{\log P_c(q,\varepsilon)}{(q-1)\log \varepsilon} \quad \text{and} \quad P_cD^+(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log P_c(q,\varepsilon)}{(q-1)\log \varepsilon}. \tag{1.4}$$

where

$$C_c(q,\varepsilon) = \inf_{u \in \mathcal{C}_c^{(\varepsilon)}} S(u,q,\varepsilon), \quad P_c(q,\varepsilon) = \sup_{u \in \mathcal{P}_c^{(\varepsilon)}} S(u,q,\varepsilon). \tag{1.5}$$

Similarly, one can define the covering and packing Rényi dimensions (denoted as $CD^\pm(q)$ and $PD^\pm(q)$), taking in (1.5) infimum or supremum over all ε-coverings or all ε-packings respectively. One can observe that all dimensions defined above are positive decreasing functions of their argument q and lower dimensions (with liminf in the definition) are always not bigger than the upper dimensions.

In the particular case of measures on $\mathbb{R}^d$, there exists an alternative definition of Rényi dimensions with summation over grids [P] [LN] [BGT2]: we shall denote it by $GD^{\pm}(q)$ (for grids dimensions). This definition in its spirit is close to $PD^{\pm}(q)$, for it uses disjoint cubes that may have an arbitrary small intersection with the support of the measure. Because of that, for instance, the Lebesgue measure on $\mathbb{R}$ with support $[0,1]$, has infinite dimensions $GD^{+}(q)$ and $PD^{\pm}(q)$ for any $q<0$ [LN] [BGT2], while for such a measure $D^{\pm}(q)=P_cD^{\pm}(q)=C_cD^{\pm}(q)=CD^{\pm}(q)=1$ for any $q\in\mathbb{R}$. This illustrates why dimensions $GD^{\pm}(q)$ and $PD^{\pm}(q)$ are not relevant objects to consider in the regime of negative q's, and we will not discuss these dimensions in the present paper. However let us mention how usefull the grids dimensions $GD^{\pm}(q)$ turn out to be in the regime of positive q's for measures on $\mathbb{R}^d$, mainly because they are defined in a simple way, avoiding the supremum or infimum over particular families of balls. This has been crucial in many results presented in [BGT2].

At this stage we note that if $q\leq 0$, the equality $CD^{\pm}(q)=C_cD^{\pm}(q)$ holds for any finite Borel measure on X. It is an easy derivation that we prove in Proposition 2.1. As a consequence of the above considerations, in the present paper we shall focuss our attention on the following three dimensions: $D^{\pm}(q)$, $C_cD^{\pm}(q)$ and $P_cD^{\pm}(q)$, as far as negative q's are concerned.

Recall that all the dimensions defined previously are decreasing functions of their argument q. Basic questions about these dimensions are the following:
a) For which q's are these dimensions finite?
b) What are the regularity property of these dimensions (continuity and differentiability) and their asymptotic behaviour as q goes to $\pm\infty$?
c) Are these definitions equivalent? In other terms, do the different definitions give rise to the same dimensions?

In this paper we treat questions a) and c) for compactly supported measures with finite mass and negative q's. In [GT], we shall discuss a) and c) for non compactly supported measures with finite mass and negative q's. We briefly list below what can be found (to our best knowledge) in the litterature about these points.
a) Note that $P_cD^{\pm}(0)=C_cD^{\pm}(0)=\dim^{\pm}_B(\operatorname{supp}\mu)$ (e.g [M]), so that in $\mathbb{R}^d$, $P_cD^{\pm}(q)$ and $C_cD^{\pm}(q)$ are finite for $q\geq 0$ and compactly supported measures. If $q\in(1,+\infty)$ the dimensions $D^{\pm}(q)$ and $GD^{\pm}(q)$, are known to be finite (actually non bigger than d) for any measure on $\mathbb{R}^d$ (see e.g. [BGT2]). In Lau and Ngai [LN] the finiteness of $P_cD^{\pm}(q)$ for $q<0$ and compactly supported measures is discussed. They show that $P_cD^{+}(q)$ is either defined (i.e. finite) on $\mathbb{R}$ or on $\mathbb{R}^{+}$, depending on the finitness of the quantity we call g^{+} in this paper (see (1.8) below), and moreover $P_cD^{+}(-\infty)=g^{+}$. Finiteness of $D^{\pm}(q)$ for non compactly supported measure is discussed in [BGT2] for $q>0$ and in [GT] for $q<0$.
b) On the domain $D^{+}(q)<\infty$, continuity of $D^{+}(q)$ and differentiability everywhere except maybe at a countable set of points derive from general arguments about convex functions. On the same domain $D^{+}(q)<\infty$, Lipschitz continuity of $D^{-}(q)$ and thus differentiability Lebesgue a.e. of the latter is proved in [BGT2] (the proof made for measures on $\mathbb{R}$ extends to the general setting of the present paper).
c) We first mention the general results:

- For $q<0$, $P_cD^{\pm}(q)=C_cD^{\pm}(q)$ is proved by Olsen [O1] on spaces X where a Besicovitch covering theorem is available (it is done on $\mathbb{R}^d$). We will see that it is actually a general property that holds on any separable complete metric space.
- For $q>0$, $P_cD^+(q)=GD^+(q)$ is proved in [LN], and in [O1] it is shown that $C_cD^{\pm}(q)\leq P_cD^{\pm}(q)$ for $q\in(0,1)$ and $P_cD^{\pm}(q)\leq C_cD^{\pm}(q)$ for $q>1$.
- For $q>1$, the equality $D^{\pm}(q)=GD^{\pm}(q)$ is rather immediate and it can be found in [GY] [P] [BGT2].
- For $q\in(0,1)$ and measures on $X=\mathbb{R}^d$, equivalence between the Hentschel-Procaccia dimensions $D^{\pm}(q)$ and the grids dimensions $GD^{\pm}(q)$ was more difficult to obtain, requiring a substantially different proof (we briefly comment on that around Bound 1.7 below). It has recently been established in [BGT2], where $D^{\pm}(q)=GD^{\pm}(q)=CD^{\pm}(q)$ is proved. The use of the grid dimensions $GD^{\pm}(q)$ was playing a crucial simplifying role. The proofs of [LN] and [BGT2] readily extend to get $D^{\pm}(q)=GD^{\pm}(q)=CD^{\pm}(q)=C_cD^{\pm}(q)=P_cD^{\pm}(q)=PD^{\pm}(q)$ for any $q\in(0,1)$ and measures on $\mathbb{R}^d$.

To our best knowledge this is all as far as general result are concerned. We note that if $CD^{\pm}(q)=D^{\pm}(q)$, $q>1$ fails in $\mathbb{R}^2$ [GY], and if for $q<0$, $P_cD^+(q)=GD^+(q)$ and $D^+(q)=GD^+(q)$ fail in $\mathbb{R}^d$ [LN][BGT2], the intermediate regime $(0,1)$ turns out to be more stable under the change of definitions since the dimensions all coincide (at least for measures defined on $\mathbb{R}^d$).

Further results hold if one assumes a strong condition on the measure called the doubling condition. Let us recall the definition of a "doubling measure" or "diametrically regular" (we refer to Subsection 3.2 where this condition is discussed). A Borel measure on the metric space X is said to satisfy to a doubling condition if there exist two constants $K>1,\nu>0$ such that

$$\frac{\mu(B(x,2\varepsilon))}{\mu(B(x,\varepsilon))}\leq K \tag{1.6}$$

for all $x\in\operatorname{supp}\mu,\ 0<\varepsilon\leq\nu$. If a probability measure μ, on $\mathbb{R}^d$, satisfies to a doubling condition then as noticed in Olsen [O1] it is easy to get $D^{\pm}(q)=C_cD^{\pm}(q)=P_cD^{\pm}(q)$, for all $q\in\mathbb{R}$ (see below, and Proposition 3.3).

Comparison of the different Rényi dimensions ($\mathrm{C}_cD^{\pm}(q)$, $P_cD^{\pm}(q)$, $GD^{\pm}(q)$) mostly consists in geometric arguments, for it reduces in most cases to comparisons of coverings, packings and grids: all these dimensions are computed with the same sums $S(u,q,\varepsilon)$ defined in (1.2); only the way of cutting the support of the measure differs. Sometimes equalities are rather immediate to get, and in other cases one has to resort to a more involved geometric theorem (like the Besicovitch covering theorem). Comparison with Hentschel-Procaccia dimensions $D^{\pm}(q)$ for $q<1$ requires different considerations and is more delicate, for the weight that the measure gives to balls of different size is of the most importance.

We would like in the few lines below to give an idea of what the issue is. Pick $q<0$. As we will show $P_cD^{\pm}(q)=C_cD^{\pm}(q)$ in full generality, so that to prove the equality with the dimensions $D^{\pm}(q)$ it is enough to show 1) $D^{\pm}(q)\leq C_cD^{\pm}(q)$ and 2) $D^{\pm}(q)\geq P_cD^{\pm}(q)$. Pick $x\in B(y,\varepsilon)$, $y\in\operatorname{supp}\mu$, $\varepsilon>0$. It is then clear that $B(y,\varepsilon)\subset B(x,2\varepsilon)$, so that immediately, for any centered ε-covering $u=(B(y_i,\varepsilon))_{i\in I}$ and for any $q<1$,

$$I(q,2\varepsilon)\leq\sum_{i\in I}\int_{B(y_i,\varepsilon)}\mu(B(x,2\varepsilon))^{q-1}\leq S(u,q,\varepsilon),$$

so that $D^{\pm}(q) \leq C_c D^{\pm}(q)$ follows for $q < 1$. This is the easy inequality. Difficulties arise with the converse one: is it true that $D^{\pm}(q) \geq P_c D^{\pm}(q)$ for $q < 1$? Pick again $x \in B(y,\varepsilon)$, then note that $B(x,\varepsilon) \subset B(y,2\varepsilon)$, so that for any centered ε-packing $v = (B(y_i,\varepsilon))_{i\in I}$, one has the following lower bound:

$$
\begin{aligned}
I(q,\varepsilon) &\geq \sum_{i\in I} \int_{B(y_i,\varepsilon)} \mu(B(x,\varepsilon))^{q-1} \geq \sum_{i\in I} \mu(B(y_i,\varepsilon))\mu(B(y_i,2\varepsilon))^{q-1} \\
&\geq \sum_{i\in I} \left(\frac{\mu(B(y_i,2\varepsilon))}{\mu(B(y_i,\varepsilon))}\right)^{q-1} \mu(B(y_i,\varepsilon))^q. \qquad (1.7)
\end{aligned}
$$

And one needs to prevent the ratio $\mu(B(y_i,2\varepsilon))/\mu(B(y_i,\varepsilon))$ from blowing up for some y_i and ε, which would thereby destroy the bound (1.7). Of course if the measure is assumed to be doubling, then trivially this ratio is bounded by the constant K given in (1.6), and $I(q,\varepsilon) \geq K^{q-1} P_c(q,\varepsilon)$ follows, leading to $D^{\pm}(q) \geq P_c D^{\pm}(q)$ for any $q < 1$. But besides this strong assumption on the measure that makes things trivial, what can we say? If one cannot avoid bad points (i.e. points where the ratio $\mu(B(y_i,2\varepsilon))/\mu(B(y_i,\varepsilon))$ becomes very big for some sequence of ε), at least one could try to show that, in some sense, the set of such bad points has a small enough mass. This is what has been achieved in the regime $q \in (0,1)$ in [BGT2] for any measure of finite mass in $\mathbb{R}^d$. Unfortunately this idea does not work at all in the regime $q < 0$ we want to investigate here, and the techniques dev! eloped in [BGT2] are totally unefficient. Indeed, even a single point with zero weight can have a destroying effect on (1.7). Consequences: 1) on a technical point of view, a more involved minoration than (1.7) will be necessary (see Subsection 4.2) and 2) the equality of the dimensions does not hold anymore for all measures of finite mass (like in the regime $q \in (0,1)$), and we shall exhibit the correct condition on the measure to ensure $D^{\pm}(q) = P_c D^{\pm}(q) = C_c D^{\pm}(q) < \infty$ for all $q < 0$. It will hold for the class of measures $\mathcal{P}_g(X)$ defined in (1.9) and that contains the doubling measures; we refer the reader to Subsection 3.2, Proposition 3.2 and Observation 3.1 for further discussions on the link between doubling measures and measures that belong to the class $\mathcal{P}_g(X)$.

1.2 Presentation of the results

We turn to the description of our results. In Proposition 2.1 we prove in full generality that for any $q \leq 0$,

$$P_c D^{\pm}(q) = C_c D^{\pm}(q) = C D^{\pm}(q) \ .$$

In the sequel we shall thus omit to refer again and again to the covering Rényi dimensions $C_c D^{\pm}(q)$ and $C D^{\pm}(q)$. We shall use $P_c D^{\pm}(q)$ as a representative of the generalized Rényi dimensions for $q \leq 0$.

A quantity of major interest in our analysis is the following function:

$$g(\varepsilon) = \inf_{x\in \operatorname{supp}\mu} \mu(B(x,\varepsilon)), \quad \varepsilon > 0.$$

Introduce then

$$g^- = \liminf_{\varepsilon\downarrow 0} \frac{\log g(\varepsilon)}{\log \varepsilon} \quad \text{and} \quad g^+ = \limsup_{\varepsilon\downarrow 0} \frac{\log g(\varepsilon)}{\log \varepsilon} \ . \qquad (1.8)$$

We note right away that as proved in Proposition 3.1, $g(\varepsilon) > 0$ for any $\varepsilon > 0$ for compactly supported measures, so that the previous expressions make sense.

Define

$$\mathcal{P}_g(X) = \{\mu \in \mathcal{M}(X),\ \mu(X) < \infty \text{ with compact support, } g^+ < \infty\}, \tag{1.9}$$

where $\mathcal{M}(X)$ is the set of positive regular Borel measures on X. As shown in Proposition 3.2, the class $\mathcal{P}_g(X)$ contains all the doubling measures (i.e. measures satisfying to (1.6) above) with compact support. But not all measures in $\mathcal{P}_g(X)$ are doubling, as shown in Subsection 5.1. So $g^+ < \infty$ can be seen as a generalization of the doubling condition. Further generalizations are given by conditions (1.12) and (1.15) below. We refer to Subsection 3.2 for further discussions.

We first treat the case $q < 0$ and then consider the particular value $q = 0$ in Theorem 1.5. Our first result concerns the upper dimensions for which the picture is complete.

Theorem 1.1 *Let (X, ϱ) be a complete separable metric space, and $\mu \in \mathcal{M}(X)$ with compact support and finite mass.*
(i) For any $q < 0$, the upper dimensions coincide:

$$D^+(q) = P_c D^+(q)\ . \tag{1.10}$$

(ii) Moreover, the dimensions are either finite for all $q < 0$ or infinite for all $q < 0$.
(iii) $D^+(q) < +\infty$ for $q < 0$ iff $g^+ < +\infty$. In addition $D^+(-\infty) = g^+$.

Points (ii) and (iii) of Theorem 1.1 are shown in a rather immediate way for the packing dimensions $P_c D^+(q)$ in Theorem 3.2 (and similar results for $P_c D^-(q)$). That $D^+(q)$, resp. $D^-(q)$, is finite as soon as $g^+ < \infty$, resp. $g^- < \infty$, is immediate too and is given in Theorem 3.1. The “only if” part of (iii) is the non trivial part and is proven in Theorem 4.1. It follows that $D^+(q)$ and $P_c D^+(q)$ are simultaneously finite or infinite in the region $q < 0$, depending on g^+. It thus remains to prove that when finite the dimensions are equal. This follows from Theorem 4.3.

For some classes of measures it is known that $D^+(q) = D^-(q)$ for all $q \in \mathbb{R}$. Of course, for such measures Theorem 1.1 then provides the full picture. However it is possible that $D^-(q) < D^+(q)$ for some (or all) $q \in \mathbb{R}$. This is for instance what happens in quantum transport in some interesting cases [T2]. So one needs to treat the lower dimensions separately, and the situation is more complex. If, as proved by Theorem 1.1, equality of the upper dimensions is a general property for $q < 0$, it is not anymore the case for the lower dimensions. However if the upper dimensions are finite, then the picture of Theorem 1.1 can be completed: equality holds for lower dimensions as well. This is the content of Theorem 1.2 which follows from Theorem 4.3.

Theorem 1.2 *Let (X, ϱ) and μ as in Theorem 1.1. Suppose that the upper dimensions are finite ($D^+(q) < +\infty$ for $q < 0$), then the lower dimensions coincide too:*

$$D^-(q) = P_c D^-(q) < +\infty, \quad q < 0\ . \tag{1.11}$$

In addition $D^-(-\infty) = g^- \leq g^+ < +\infty$.

Of course by Theorem 1.1,

$$D^+(q) < +\infty \text{ for some } q < 0 \iff D^+(q) < +\infty \text{ for all } q < 0 \iff \mu \in \mathcal{P}_g(X).$$

As a consequence of Theorem 1.1 and Theorem 1.2, the set $\mathcal{P}_g(X)$ is the natural (and optimal) class of compactly supported measures where the generalized fractal dimensions behave nicely: upper dimensions are finite and equal, and lower dimensions as well (but of course upper and lower dimensions need not to coincide). We have (we also incorporate the result of Theorem 1.5 concerning the point $q = 0$):

Corollary 1.1 *Let (X, ϱ) be a complete separable metric space, and $\mu \in \mathcal{P}_g(X)$. Then*

$$D^{\pm}(q) = P_c D^{\pm}(q) < +\infty, \quad q \leq 0 \ ,$$

and $D^{\pm}(-\infty) = P_c D^{\pm}(-\infty) = g^{\pm} < +\infty$.

Harder is to study the lower dimensions $D^-(q)$ when $\mu \notin \mathcal{P}_g(X)$, i.e. when $g^+ = +\infty$. We note that concerning the finiteness of lower Rényi dimensions, the situation is quite clear: the finiteness of $P_c D^-(q)$ is equivalent to the one of g^-. However, $D^-(q)$ may be finite even if $g^- = +\infty$. It is also possible that $D^-(q) < P_c D^-(q) < +\infty$ if $g^+ = +\infty$, $g^- < +\infty$. Such examples are developped in Subsection 5.2.

We prove the following criterium for the dimensions $D^-(q)$:

Theorem 1.3 *Let (X, ϱ) and μ as in Theorem 1.1. Assume*

$$\limsup_{\varepsilon \downarrow 0} \frac{\log \log \log 1/g(\varepsilon)}{\log 1/\varepsilon} = 0 \ . \tag{1.12}$$

Then for any $q < 0$, $D^-(q) = P_c D^-(q) = C_c D^-(q)$, being finite or not (depending on the finiteness of g^-).

The result follows from Theorem 4.3, point (i). As shown in Subsection 5.2, this result is optimal in the following sense: for any $\delta > 0$ one can construct an example where the limsup in (1.12) is equal to δ and $D^-(q) < P_c D^-(q) < +\infty$ for some $q < 0$.

Note that if $g^+ < +\infty$ then (1.12) holds, so that Theorem 1.2 is actually a corollary of Theorem 1.3. However if the upper dimensions are infinite ($g^+ = +\infty$), then Theorem 1.3 still provides a criterium for the equivalence of the lower dimensions. Next Theorem presents more involved conditions that force $D^-(q)$ to be infinite. For instance if g^- is known to be infinite (and thus $P_c D^-(q) = +\infty$) then the hypothesis of Theorem 1.3 can be relaxed to (1.13) below. Theorem 1.4 is a rewriting of Theorem 4.2, point (i) and point (iii).

Theorem 1.4 *Let (X, ϱ) and μ as in Theorem 1.1.*
(i) Assume that $g^- = +\infty$ and that

$$\limsup_{\varepsilon \downarrow 0} \frac{\log \log \log 1/g(\varepsilon)}{\log 1/\varepsilon} < +\infty. \tag{1.13}$$

Then $D^-(q) = +\infty$ for all $q < 0$.
(ii) Assume that for some $p = 2, 3...$ the following condition is fulfilled:

$$\limsup_{\varepsilon \downarrow 0} \frac{\log_{p+2} 1/g(\varepsilon)}{\log 1/\varepsilon} < \liminf_{\varepsilon \downarrow 0} \frac{\log_p 1/g(\varepsilon)}{\log 1/\varepsilon}, \tag{1.14}$$

where $\log_p = \log \circ \cdots \circ \log$, p times. Then $D^-(q) = +\infty$ for all $q < 0$.

Note that point (i) is a particular case of point (ii) where $p = 1$ and the right quantity is equal to $+\infty$. In Subsection 5.2 Remark 5.1 we give an example of a measure where $g^- = +\infty$, the limit in (1.13) is equal to $+\infty$ and $D^-(q) < +\infty$ for all $q < 0$.

So far we were only considering the regime $q < 0$ (whereas [BGT2] was mostly interested by the regime $q > 0$). A little bit like $q = 1$, $q = 0$ is a special and delicate point. We recall that it is known (e.g. [M]) that $P_c D^\pm(0) = C_c D^\pm(0) = \dim_B^\pm(\operatorname{supp}\mu)$, where $\dim_B^\pm(S)$ is the lower and upper box counting dimension of the set $S \subset X$. The situation with $D^\pm(0)$ is trickier.

Below we list our results concerning the point $q = 0$. The results follows from Theorem 4.3, point (ii) and Theorem 4.2, point (ii).

Theorem 1.5 *Let (X, ϱ) and μ as in Theorem 1.1.*
(i) Suppose that

$$\limsup_{\varepsilon \downarrow 0} \frac{\log\log 1/g(\varepsilon)}{\log 1/\varepsilon} = 0 \ . \tag{1.15}$$

(In particular, this is true if $g^+ < +\infty$, i.e. if $\mu \in \mathcal{P}_g(X)$). Then $D^\pm(0) = P_c D^\pm(0) = C_c D^\pm(0) = \dim_B^\pm(\operatorname{supp}\mu)$.
(ii) Suppose that $\limsup_{\varepsilon\downarrow 0}[\log\log 1/g(\varepsilon)]/[\log 1/\varepsilon] < +\infty$ and $\dim_B^+(\operatorname{supp}\mu) = +\infty$ (resp. $\dim_B^-(\operatorname{supp}\mu) = +\infty$), then $D^+(0) = +\infty$ (resp. $D^-(0) = +\infty$) as well.

The paper is organized as follows. In Section 2 we define the generalized fractal dimensions and supply some relations between them. In Section 3 we introduce the function $g(\varepsilon)$, we illustrate its links to the doubling condition and derive some basic links with the dimensions $D^\pm(q)$ and $P_c D^\pm(q)$, in particular their asymptotic behaviour at $-\infty$. In Section 4 we prove our main results that concern about the finiteness of the dimensions $D^\pm(q)$ and their equality to $P_c D^\pm(q)$. In Section 5 we propose some counter-examples. In particular a measure that lies in $\mathcal{P}_g(X)$ but which is not doubling, and a family of measures where equality of the lower dimensions does not hold.

2 Definitions of generalized fractal dimensions

2.1 General setup

Let (X, ϱ) be a complete separable metric space, ϱ being the distance on X. We denote by $B(x, \varepsilon)$ the *closed* ball centered at x and of radius ε, *i.e.* $B(x, \varepsilon) = \{y \in X, \varrho(x, y) \leq \varepsilon\}$. Let $\mathcal{M}(X)$ be the set a positive regular Borel measures on X. The results of this paper hold under the additional assumption that the total mass of μ is finite: $\mu(X) < +\infty$. However, it will be clear from our proofs that the results remain unchanged if one rescales the mass of the measure μ. Thus, with no loss of generality we shall suppose that μ is a probability measure:

$$\mu(X) = 1 \ . \tag{2.1}$$

We denote by $\mathcal{P}(X)$ the set of probability measures on X:

$$\mathcal{P}(X) = \{\mu \in \mathcal{M}(X),\ \mu(X) = 1\} \ . \tag{2.2}$$

We denote by $\operatorname{supp}\mu$ the support of the measure μ, that is the smallest closed set F such that $\mu(X\backslash F) = 0$ [Fe] [M]. It is well-known (e.g. [M]) that for Borel measures on a separable metric space, $\operatorname{supp}\mu$ is a well defined (unique) set and one has

$$\operatorname{supp}\mu = \{x \in X,\ \mu(B(x,\varepsilon)) > 0 \text{ for any } \varepsilon > 0\}. \tag{2.3}$$

Note that with our assumptions on X and μ, one has

$$\mu(X\backslash \operatorname{supp}\mu) = 0\,. \tag{2.4}$$

Recall that the functions $x \to \mu(B(x,\varepsilon))$ are μ-measurable.

We recall the following well known inequalities.

$$\liminf_{\varepsilon\downarrow 0}(u(\varepsilon) - v(\varepsilon)) \geq \liminf_{\varepsilon\downarrow 0} u(\varepsilon) - \limsup_{\varepsilon\downarrow 0} v(\varepsilon), \tag{2.5}$$

$$\limsup_{\varepsilon\downarrow 0}(u(\varepsilon) - v(\varepsilon)) \geq \limsup_{\varepsilon\downarrow 0} u(\varepsilon) - \limsup_{\varepsilon\downarrow 0} v(\varepsilon). \tag{2.6}$$

2.2 Hentschel-Procaccia dimensions $D^{\pm}(q)$

Let $\mu \in \mathcal{P}(X)$ be a probability measure on X. For $q \in \mathbb{R}$ and $\varepsilon \in (0,1)$, we consider the following functions with values in $\mathbb{R} \cup \{+\infty\}$:

$$I(q,\varepsilon) = \int_{\operatorname{supp}\mu} \mu(B(x,\varepsilon))^{q-1} \mathrm{d}\mu(x)\ . \tag{2.7}$$

We make the following important remark: thanks to (2.4), the integration over X in (2.7) actually leads to the same value when computing $I(q,\varepsilon)$. This fact will be used implicitely many times in the paper when minorating $I(q,\varepsilon)$.

Definition 2.1 *The Hentschel-Procaccia dimensions.*
We define the following functions on $\mathbb{R}$ with values in $\mathbb{R} \cup \{+\infty\}$:

$$\tau^-(q) = \liminf_{\varepsilon\downarrow 0} \frac{\log I(q,\varepsilon)}{\log(1/\varepsilon)}\ , \qquad \tau^+(q) = \limsup_{\varepsilon\downarrow 0} \frac{\log I(q,\varepsilon)}{\log(1/\varepsilon)}\ , \tag{2.8}$$

with the understanding that $\tau^{\pm}(q) = +\infty$ if for some $\varepsilon > 0$, $I(q,\varepsilon) = +\infty$, and the Hentschel-Procaccia dimensions:

$$D^-(q) = \liminf_{\varepsilon\downarrow 0} \frac{\log I(q,\varepsilon)}{(1-q)\log(1/\varepsilon)}\ , \quad D^+(q) = \limsup_{\varepsilon\downarrow 0} \frac{\log I(q,\varepsilon)}{(1-q)\log(1/\varepsilon)}\ , \quad q \neq 1, \tag{2.9}$$

with values in $[0,+\infty]$.

In [BGT2] basic regularity properties of these functions $\tau^{\pm}(q)$ and $D^{\pm}(q)$ are proven in the case $X = \mathbb{R}$. As a matter of fact, many of results of [BGT2] can be easily generalized to any metric space X. In particular, define

$$q^* = \inf\{q \in \mathbb{R}, \tau^+(q) < +\infty\}, \quad q^* \leq 1. \tag{2.10}$$

It is shown in [BGT2] that $\tau^{\pm}(q)$ and $D^{\pm}(q)$ are decreasing functions continuous on $(q^*,+\infty)$ and on $(q^*,1) \cup (1,+\infty)$ respectively. Moreover, for any $A > q^*$, $\tau^{\pm}(q)$ and $D^{\pm}(q)$ are Lipschitz continuous on $[A,+\infty)$ and on $[A,1) \cup (1,+\infty)$ respectively. (It follows directly for $\tau^+(q), D^+(q)$ from the convexity of $\tau^+(q)$, but for the lower dimensions it is not trivial).

2.3 Packings, coverings and generalized Rényi dimensions

To define the generalized Rényi dimensions we shall need to approximate the support of the measure with balls of arbitrary small radius. One can do that using packings or coverings. We first describe packings or coverings of a subset S of X (not necessarily compact), and then turn to the definition of the dimensions.

A finite or countable family $u = (B(x_i, \varepsilon))_{i \in I}$ is a collection of closed balls of radius ε, where $x_i \in X$ and I is a set of index. For a sake of simplicity, if $u = (B(x_i, \varepsilon))_{i \in I}$, we shall denote by u either the set of balls $B(x_i, \varepsilon)$, $i \in I$ (i.e. the family itsef), or the union of these balls $\cup_{i \in I} B(x_i, \varepsilon)$. With obvious notations, we write $u \subset u'$ if the family u' contains the family u.

Coverings. We shall say that $u = (B(x_i, \varepsilon))_{i \in I}$ is an ε-covering of S (finite or countable) if $S \subset \cup_{i \in I} B(x_i, \varepsilon)$. We denote by $\mathcal{C}^{(\varepsilon)}(S)$ the set of such coverings, and by $\mathcal{C}_c^{(\varepsilon)}(S) \subset \mathcal{C}^{(\varepsilon)}(S)$ the set of centered ε-coverings of S, *i.e.* ε-coverings for which in addition $x_i \in S$ for all i.

Packings. We shall say that $u = (B(x_i, \varepsilon))_{i \in I}$ is an ε-packing of S (finite or countable) if $B(x_i, \varepsilon) \cap S \neq \emptyset$, $i \in I$, and $B(x_i, \varepsilon) \cap B(x_j, \varepsilon) = \emptyset$, $i \neq j$. It is a centered ε-packing if in addition $x_i \in S$ for all i. We denote by $\mathcal{P}^{(\varepsilon)}(S)$ the set of ε-packings, and by $\mathcal{P}_c^{(\varepsilon)}(S) \subset \mathcal{P}^{(\varepsilon)}(S)$ the set of centered ε-packings.

We shall say that a centered ε-packing $u \in \mathcal{P}_c^{(\varepsilon)}(S)$ is *maximal* if one cannot add to u another centered ball of radius ε without intersecting the family u; in other terms u is a *maximal centered ε-packing* if for any $x \in S$, $u \cup B(x, \varepsilon)$ does not belong to $\mathcal{P}_c^{(\varepsilon)}(S)$ anymore. The set of maximal centered ε-packings will be denoted by $\mathcal{P}_{c,+}^{(\varepsilon)}(S)$.

In the sequel and throughout the paper, we shall drop the reference to the set S and write $\mathcal{C}^{(\varepsilon)}$, $\mathcal{C}_c^{(\varepsilon)}$, $\mathcal{P}^{(\varepsilon)}$, $\mathcal{P}_c^{(\varepsilon)}$, $\mathcal{P}_{c,+}^{(\varepsilon)}$. Note that in practice S will be the support of the measure μ.

We make the following basic observations, which will be quite useful in the sequel (see [BSa] for related observations).

Observation 2.1 *Given a set S, $\varepsilon > 0$ and $u \in \mathcal{P}_c^{(\varepsilon)}$, one can complete u either to get a centered ε-packing u' with infinite cardinality, or to obtain a maximal centered ε-packing with finite cardinality. In other terms, there exists $u' \in \mathcal{P}_c^{(\varepsilon)}$, $u \subset u'$, such that either* $\operatorname{card} u' = \infty$, *or* $u' \in \mathcal{P}_{c,+}^{(\varepsilon)}$ *and* $\operatorname{card} u' < \infty$.

Indeed, take $\varepsilon > 0$ and pick u a centered ε-packing: $u = (B(x_i, \varepsilon))_{i \in J}$. Suppose its cardinal is finite and equal to N. Consider all the balls $B(y, \varepsilon)$, $y \in \operatorname{supp}\mu$. If one can find such a ball so that $B(y, \varepsilon) \cap B(x_i, \varepsilon) = \emptyset$ for any $i \in J$, then one adds it to the family u and one obtains a new centered ε-packing with cardinality $N + 1$. If one cannot find such a ball $B(y, \varepsilon)$, then that means that the packing is maximal. Iterating this procedure leads to Observation 2.1.

It is of interest to get maximal packings because of the following link with coverings:

Observation 2.2 *Let $u = (B(x_i, \varepsilon))_{i \in I}$ be in $\mathcal{P}_{c,+}^{(\varepsilon)}$. Then $v = (B(x_i, 2\varepsilon))_{i \in I}$ belongs to $\mathcal{C}_c^{(\varepsilon)}$.*

As an immediate consequence of Observation 2.1 and Observation 2.2 we get:

Lemma 2.1 *A closed subset $S \subset X$ is not compact if and only if for any $\varepsilon > 0$ small enough, one can find a centered ε-packing of S with infinite cardinality.*

Proof of Lemma 2.1:
First, recall that since X is complete, S compact is equivalent to S precompact, that is: for any $\varepsilon > 0$, there exists an ε-covering of S with finite cardinality (e.g. [D]).

Suppose that S is not compact, but there exists a sequence (ε_k) going to zero such that for any k there does not exist a centered ε_k-packing with infinite cardinality. Combining Observations 1 and 2 above implies that for each ε_k there exists a $2\varepsilon_k$-covering of S with finite cardinality, and thereby for any $\varepsilon > 0$. It implies that S is compact, which is impossible. Thus, the first (direct) statement of the Lemma is proved.

Assume now that S is compact. Then for any $\varepsilon > 0$ there exists an ε-covering $w = (B(y_j, \varepsilon))_{j\in J}$ of S with finite cardinality. Let $u = (B(x_i, \varepsilon))_{i\in I} \in \mathcal{P}_c^{(\varepsilon)}$ be any centered ε-packing of S. One observes that each x_i belongs to some $B(y_j, \varepsilon)$ and any ball of w contains at most one point x_i. Therefore, $\mathrm{card} I \le \mathrm{card} J < +\infty$ and there is no centered ε-packings with infinite cardinality. $\square$

For a measure $\mu \in \mathcal{P}(X)$ and a family $u = (B(x_i, \varepsilon))_{i\in I}$, define, for $q \in \mathbb{R}$, the following Rényi sums:

$$S(u, q, \varepsilon) = \sum_{i\in I} \mu(B(x_i, \varepsilon)^q, \tag{2.11}$$

where the summation is over i's such that $\mu(B(x_i, \varepsilon)) > 0$. We further define, for $q \in \mathbb{R}$,

$$C(q, \varepsilon) = \inf_{u\in\mathcal{C}^{(\varepsilon)}} S(u, q, \varepsilon), \quad C_c(q, \varepsilon) = \inf_{u\in\mathcal{C}_c^{(\varepsilon)}} S(u, q, \varepsilon), \tag{2.12}$$

and, for $q \in \mathbb{R}$,

$$P(q, \varepsilon) = \sup_{u\in\mathcal{P}^{(\varepsilon)}} S(u, q, \varepsilon), \quad P_c(q, \varepsilon) = \sup_{u\in\mathcal{P}_c^{(\varepsilon)}} S(u, q, \varepsilon). \tag{2.13}$$

Definition 2.2 *The generalized Rényi dimensions*
Let $q \neq 1$. If $V(q, \varepsilon)$ is one the quantities $C(q, \varepsilon)$, $C_c(q, \varepsilon)$, $P(q, \varepsilon)$, $P_c(q, \varepsilon)$ in (2.12) and (2.13), we define its growth exponents by

$$V^-(q) = \liminf_{\varepsilon\downarrow 0} \frac{\log V(q, \varepsilon)}{\log(1/\varepsilon)} \ , \quad V^+(q) = \limsup_{\varepsilon\downarrow 0} \frac{\log V(q, \varepsilon)}{\log(1/\varepsilon)} \ , \tag{2.14}$$

with values in $[0, +\infty]$. We further define the associated lower and upper generalized fractal dimension by

$$VD^-(q) = \liminf_{\varepsilon\downarrow 0} \frac{\log V(q, \varepsilon)}{(1-q)\log(1/\varepsilon)} \ , \quad VD^+(q) = \limsup_{\varepsilon\downarrow 0} \frac{\log V(q, \varepsilon)}{(1-q)\log(1/\varepsilon)} \ , \tag{2.15}$$

with values in $[0, +\infty]$.
In (2.14) and (2.15) the understanding is that $VD^\pm(q) = +\infty$ if for some $\varepsilon > 0$, $V(q, \varepsilon) = +\infty$.
The limits in (2.14) define the numbers $C^\pm(q)$, $C_c^\pm(q)$, $P^\pm(q)$, $P_c^\pm(q)$ and the ones in (2.15) define the covering, and centered covering, packing, centered packing dimensions $CD^\pm(q)$, $C_cD^\pm(q)$, $PD^\pm(q)$, $P_cD^\pm(q)$ for $q \in \mathbb{R}\backslash\{1\}$.

For $q = 0$ one recovers the lower and upper box counting dimensions of the support of μ [Fa][M],

$$P_cD^{\pm}(0) = C_cD^{\pm}(0) = \dim_B^{\pm}(\operatorname{supp}\mu) \ .$$

For $q = 1$ the generalized Rényi dimensions are defined in a different way and are usually called Rényi dimensions or entropy dimensions. Note that the functions $VD^{\pm}(q)$ may be discontinuous at the point $q = 1$. We refer the reader to e.g. [He][O1][BGT2] for the study of this particular point.

2.4 Basic relations between the different dimensions for negative q's

Following immediately from the definitions, one has, for all $q \in \mathbb{R}\backslash\{1\}$, $C^{\pm}(q) \leq C_c^{\pm}(q)$ and $P_c^{\pm}(q) \leq P^{\pm}(q)$, so that if $q < 1$ (with reverse inequalities if $q > 1$):

$$CD^{\pm}(q) \leq C_cD^{\pm}(q) \quad \text{and} \quad P_cD^{\pm}(q) \leq PD^{\pm}(q).$$

As mentioned in the introduction, as far as the regime $q < 0$ is concerned, the dimensions $PD^{\pm}(q)$ are not good objects to look at, for they are often infinite as long as the support of the measure is a strict subset of X. We defined them for a sake of completeness, but we shall not discuss these dimensions anymore.

Proposition 2.1 *Let $\mu \in \mathcal{P}(X)$ be a probability measure (we do not assume here that* $\operatorname{supp}\mu$ *is compact). For any $q \leq 0$,*
(i) $C^{\pm}(q) = C_c^{\pm}(q)$, and thus $CD^{\pm}(q) = C_cD^{\pm}(q)$,
(ii) $C_c^{\pm}(q) = P_c^{\pm}(q)$, and thus $C_cD^{\pm}(q) = P_cD^{\pm}(q)$,
(iii) $\tau^{\pm}(q) \leq C^{\pm}(q)$, and thus $D^{\pm}(q) \leq CD^{\pm}(q)$.

Point (ii) was known in the case $X = \mathbb{R}^d$ [O1]. If the inequality $C_cD^{\pm}(q) \geq P_cD^{\pm}(q)$ was proved for $q \leq 0$ in [O1] in great generality, the proof of the converse was resorting to the Besicovitch covering theorem. We propose a general and elementary proof of this fact that $C_cD^{\pm}(q) \leq P_cD^{\pm}(q)$, $q \leq 0$, using Observations 2.1 and 2.2.

Proof of Proposition 2.1:
(i) Since $C_cD^{\pm}(q) \geq CD^{\pm}(q)$ is trivial, we show the converse inequality. First observe that to calculate $C(q,\varepsilon)$, it is sufficient to consider coverings $u = (B(x_i,\varepsilon))_{i\in I}$ such that $B(x_i,\varepsilon) \cap \operatorname{supp}\mu \neq \emptyset$ for all $i \in I$. Indeed one can always eliminate the balls of u which do not intersect $\operatorname{supp}\mu$, and thus get a smaller covering $v(u)$ of $\operatorname{supp}\mu$. As $S(v(u),q,\varepsilon) \leq S(u,q,\varepsilon)$, the infimum over all coverings u is equal to the infimum over coverings $v(u)$. Let us now pick $v = (B(x_i,\varepsilon))_{i\in I} \in \mathcal{C}^{(\varepsilon)}$ an ε-covering of $\operatorname{supp}\mu$ such that $B(x_i,\varepsilon) \cap \operatorname{supp}\mu \neq \emptyset$ for all $i \in I$. Thus for any $i \in I$, there exists $y_i \in B(x_i,\varepsilon) \cap \operatorname{supp}\mu$ such that $B(x_i,\varepsilon) \subset B(y_i,2\varepsilon)$. It implies that $w = (B(y_i,2\varepsilon))_{i\in I}$ is a centered 2ε-covering of $\operatorname{supp}\mu$. Note that for any $q \leq 0$,

$$S(w,q,2\varepsilon) \leq S(v,q,\varepsilon).$$

As a consequence, for any $q \leq 0$, $C_c(q,2\varepsilon) \leq C(q,\varepsilon)$ and thus $C_cD^{\pm}(q) \leq CD^{\pm}(q)$ which concludes the proof of the first equality.
(ii) We turn to the second one. In [O1] it is shown in full generality that for any $q \leq 0$, $P_cD^{\pm}(q) \leq C_cD^{\pm}(q)$. Since the proof is short, we provide it for the reader's convenience. Let $u = (B(x_i,\varepsilon))_{i\in I}$ be any centered ε-packing of $\operatorname{supp}\mu$ and $v =$

$(B(y_k, \varepsilon/2))_{k\in K}$ any centered $\varepsilon/2$-covering of suppμ. For each $i \in I$ choose an integer $k(i)$ such that $x_i \in B(y_{k(i)}, \varepsilon/2)$ and observe that if $i \neq j$, then $k(i) \neq k(j)$ since $\varrho(x_i, x_j) > \varepsilon$. It is also clear that $B(y_{k(i)}, \varepsilon/2) \subset B(x_i, \varepsilon)$. Since $q \le 0$, we obtain

$$\begin{aligned} S(u,q,\varepsilon) &= \sum_{i\in I} \mu^q(B(x_i,\varepsilon)) \le \sum_{k(i),i\in I} \mu^q(B(y_{k(i)}, \varepsilon/2) \\ &\le \sum_{k\in K} \mu^q(B(y_k, \varepsilon/2) = S(v,q,\varepsilon/2). \end{aligned}$$

Since it is true for any $u \in \mathcal{P}_c^{(\varepsilon)}$, $v \in \mathcal{C}_c^{(\varepsilon/2)}$, we obtain $P_c(q,\varepsilon) \le C_c(q,\varepsilon/2)$ and the desired result follows.

Let us show the converse inequality. Take any $\varepsilon > 0$ and pick $w = (B(x_i,\varepsilon))_{i\in J} \in \mathcal{P}_c^{(\varepsilon)}$ a centered ε-packing. If it has infinite cardinality, then, since $q \le 0$, one has $S(w,q,\varepsilon) \ge \sum_{i\in J} \mu(X)^q = +\infty$. Therefore, $P_c(q,\varepsilon) = +\infty \ge C_c(q, 2\varepsilon)$ whatever $C_c(q,2\varepsilon)$ is (finite or not). Assume now that w has a finite cardinality. Then, due to Observation 2.1, one can complete w to get either a centered ε-packing with infinite cardinality, in which case, by the same reasoning as above $P_c(q,\varepsilon) = +\infty \ge C_c(q,2\varepsilon)$, or a maximal centered ε-packing $w' = (B(x_i', \varepsilon))_{i\in J'}$. In the latter case, due to Observation 2.2, $v = (B(x_i', 2\varepsilon))_{i\in J'}$ belongs to $\mathcal{C}_c^{(2\varepsilon)}$, and one has,

$$S(v,q,2\varepsilon) \le S(w', q, \varepsilon)$$

It follows by (2.12) and (2.13) that $C_c(q,2\varepsilon) \le P_c(q,\varepsilon)$. As a consequence, in any case, we have the inequality $C_c(q,2\varepsilon) \le P_c(q,\varepsilon)$ and thus $C_cD^{\pm}(q) \le P_cD^{\pm}(q)$.
(iii) Let $u = (B(x_i,\varepsilon))_{i\in I}$ be any ε-covering of suppμ. It is clear that

$$I(q,2\varepsilon) \le \sum_{i\in I} \int_{B(x_i,\varepsilon)} (\mu(B(y,2\varepsilon))^{q-1} \mathrm{d}\mu(y). \tag{2.16}$$

As $\mu(B(y,2\varepsilon)) \ge \mu(B(x_i,\varepsilon))$ for any $y \in B(x_i,\varepsilon)$, (2.16) implies for any $q < 1, \varepsilon > 0$ that $I(q,2\varepsilon) \le S(u,q,\varepsilon)$. Thus, $I(q,2\varepsilon) \le C(q,\varepsilon)$ and we get (iii). □

3 The function $g(\varepsilon)$ and related general results

3.1 Definition and link with the compacity of the support

Definition 3.1 *Let $\mu \in \mathcal{P}(X)$ be a probability measure. Define for $\varepsilon > 0$ the increasing function*

$$g(\varepsilon) = \inf_{x\in \operatorname{supp}\mu} \mu(B(x,\varepsilon)), \quad g(\varepsilon) \in [0, \mu(X)] = [0,1], \tag{3.1}$$

and its growth exponents, with values in $[0,+\infty]$, namely,

$$g^- = \liminf_{\varepsilon\downarrow 0} \frac{g(\varepsilon)}{\log \varepsilon} = \liminf_{\varepsilon\downarrow 0} \frac{1/g(\varepsilon)}{\log(1/\varepsilon)}, \quad g^+ = \limsup_{\varepsilon\downarrow 0} \frac{g(\varepsilon)}{\log\varepsilon} = \limsup_{\varepsilon\downarrow 0} \frac{1/g(\varepsilon)}{\log(1/\varepsilon)}, \tag{3.2}$$

with the understanding that $g^+ = g^- = +\infty$ if for some $\varepsilon > 0$, $g(\varepsilon) = 0$.
Among the class of compactly supported measures, a subclass of particular interest will be

$$\mathcal{P}_g(X) = \{\mu \in \mathcal{P}(X) \text{ with compact support, } g^+ < \infty\}. \tag{3.3}$$

Obviously the definition of $\mathcal{P}_g(X)$ above extends to measures $\mu \in \mathcal{M}(X)$ of finite mass (but not necessarily one), as it is written in (1.9).

One can easily see that

$$g^{\pm} \geq \sup_{x \in \mathrm{supp}\mu} \gamma^{\pm}(x),$$

where $\gamma^{\pm}(x)$ are local exponents of the measure μ. Strict inequalities may occur.

Proposition 3.1 *Let $\mu \in \mathcal{P}(X)$ be a probability measure.*
(i) If $\mathrm{supp}\mu$ *is compact, then $g(\varepsilon) > 0$ for any $\varepsilon > 0$.*
(ii) If $\mathrm{supp}\mu$ *is not compact, then $g(\varepsilon) = 0$ for all ε small enough. And thus $g^- = g^+ = +\infty$.*

In particular if μ is compactly supported then the expressions in (3.2) make sense.

Proof of Proposition 3.1:
(i) Suppose $g(\varepsilon) = 0$ for some $\varepsilon > 0$. Then one can construct a sequence $x_n \in \mathrm{supp}\mu$ such that

$$\lim_{n\to\infty} \mu(B(x_n, \varepsilon)) = 0.$$

Since $\mathrm{supp}\,\mu$ is compact and X is complete, one can extract a convergent sub-sequence out of it : $x_{n_k} \to y$, $y \in \mathrm{supp}\mu$. For k_0 large enough, $x_{n_k} \in B(y, \varepsilon/2)$, for all $k \geq k_0$, and then

$$B(y, \varepsilon/2) \subset B(x_{n_k}, \varepsilon), \quad k \geq k_0.$$

Hence $\mu(B(y, \varepsilon/2)) \leq \mu(B(x_{n_k}, \varepsilon))$, $k \geq k_0$, and the latter goes to zero as $k \to \infty$ by construction. On the other hand, according to (2.3) $\mu(B(y, \varepsilon/2)) > 0$ since $y \in \mathrm{supp}\mu$. Contradiction. Hence $g(\varepsilon) > 0$ for all $\varepsilon > 0$.
(ii) Let $v = (B(x_i, \varepsilon))_{i\in I} \in \mathcal{P}_c^{(\varepsilon)}$ be a centered ε-packing of $\mathrm{supp}\mu$. Since $\mathrm{supp}\mu$ is not compact, if ε is small enough, v can be chosen such that its cardinal is infinite by Lemma 2.1. But

$$\sum_{i\in I} \mu(B(x_i, \varepsilon)) \leq \mu(X) < +\infty,$$

and thus we get that $\mu(B(x_i, \varepsilon)) \to 0$, as $i \to \infty$. Hence for $\varepsilon > 0$ small enough, $g(\varepsilon) = 0$. $\square$

3.2 Relation with the doubling condition

Definition 3.2 *A measure $\mu \in \mathcal{M}(X)$ is said to satisfy to a doubling condition (or "μ is doubling") if there exist two constants $K > 1, \nu > 0$ such that uniformly in $x \in \mathrm{supp}\mu$,*

$$K_\nu(x) \equiv \sup_{0<\varepsilon\leq\nu} \frac{\mu(B(x, 2\varepsilon))}{\mu(B(x, \varepsilon))} \leq K, \tag{3.4}$$

or equivalently, there exists a constant $K > 1$,

$$\limsup_{\varepsilon\downarrow 0} \left[\sup_{x\in\mathrm{supp}\,\mu} \frac{\mu(B(x, 2\varepsilon))}{\mu(B(x, \varepsilon))} \right] \leq K. \tag{3.5}$$

One also says that μ is "diametrically regular".

We first make a couple of remarks:

a) We adopt here the definition used by Olsen [O1] rather than the one used by Pesin [P] where (3.4) is required to hold for *all* $x \in X$, and not only in the support of μ. Indeed such a definition dramatically limitates the range of candidates to the doubling condition (for instance most of compactly supported measures would not satisfy to it). Definition (3.4) (like in [O1]) sounds more natural to us.

b) In (3.4) above we used balls of radius 2ε and ε. One could equivalently consider any ratio of the form $\mu(B(x,\gamma\varepsilon))/\mu(B(x,\varepsilon))$, with $\gamma > 1$. Indeed, once μ is shown to be doubling for one particular $\gamma > 1$, then the same property holds for any $\gamma > 1$ [O1].

c) One can find in the litterature an alternative definition, where the bound $K_\nu(x) \leq K$ above is only required to hold for μ-almost all $x \in \operatorname{supp}\mu$, rather than for *all* $x \in \operatorname{supp}\mu$ (e.g. [EJJ]). Using Remark b) above, it is not hard to see that these two points of view are actually equivalent in the case of separable metric space. So that

$$(3.4) \iff (3.5) \iff \limsup_{\varepsilon\downarrow 0}\left[\mu - \text{ess.sup}\ \frac{\mu(B(x,2\varepsilon))}{\mu(B(x,\varepsilon))}\right] \leq K \tag{3.6}$$

d) We stress that not all measures with compact support are doubling: see [EJJ] and Subsection 5.1.

e) One can define a local version of the above doubling condition by inverting the order of the "limsup" and the "ess.sup" in the r.h.s. of (3.6). A weak form of such a local doubling condition is introduced in [BSa], where the ratio $\mu(B(x,2\varepsilon))/\mu(B(x,\varepsilon))$ is allowed to grow, say logarithmically in ε. It is then proved that *any* regular Borel measure on $X = \mathbb{R}^d$ satisfy to such a weak local doubling condition. It is however not anymore the case if one consider the same weak condition but non local (restore the position of the "limsup" and the "ess.sup"). Following Proposition 3.2, a weak, but uniform as in (3.6), doubling condition would actually imply (H2) in (1.15). Note that doubling conditions of local type are not relevant for the study of generalized dimensions, for the latter are objects defined globally.

Proposition 3.2

Let $\mu \in \mathcal{P}(X)$ be with compact support. If μ is doubling, then $\mu \in \mathcal{P}_g(X)$: that is $g^- \leq g^+ < \infty$.

Proof of Proposition 3.2:

Fix $\nu > 0$. Applying (3.4) n times leads to $\mu(B(x,\nu/2^n)) \geq K^{-n}\mu(B(x,\nu))$ for any $x \in \operatorname{supp}\mu$. On the other hand, for all $\varepsilon \in (0,\nu)$ one can find n such that $\nu 2^{-n-1} \leq \varepsilon \leq \nu 2^{-n}$. As a consequence

$$\mu(B(x,\varepsilon)) \geq \mu(B(x,\nu 2^{-n-1})) \geq K^{-n-1}\mu(B(x,\nu)) \geq 1/K\varepsilon^A\mu(B(x,\nu)),$$

where $A = \log K/\log 2$. Taking the infimum over all $x \in \operatorname{supp}\mu$ yields

$$g(\varepsilon) \geq 1/K\varepsilon^A g(\nu), \quad \text{for any } \varepsilon \in (0,\nu),$$

with $g(\nu) > 0$ by Proposition 3.1. The result follows with $g^+ \leq A$. $\square$

The converse to Proposition 3.2 is not true as shown by the example presented in Subsection 5.1: one can have $g^+ < +\infty$ but still the measure is not doubling. Condition

$g^+ < \infty$ can actually be considered as a weak doubling condition. Indeed note that for any $x \in \operatorname{supp} \mu$, $\varepsilon > 0$ and $N \in \mathbb{N}^*$ one has

$$\frac{\mu(B(x,2\varepsilon))}{\mu(B(x,\varepsilon))} \frac{\mu(B(x,3\varepsilon))}{\mu(B(x,2\varepsilon))} \cdots \frac{\mu(B(x,(N+1)\varepsilon))}{\mu(B(x,N\varepsilon))} = \frac{\mu(B(x,(N+1)\varepsilon))}{\mu(B(x,\varepsilon))} \leq \frac{\mu(X)}{g(\varepsilon)}$$

So that for any ε, there exists $k = k(x,\varepsilon,N)$, $1 \leq k \leq N$, such that $\mu(B(x,(k+1)\varepsilon))/\mu(B(x,k\varepsilon)) \leq (1/g(\varepsilon))^{1/N} = \exp(1/N \log(\mu(X)/g(\varepsilon)))$. Now if for instance $N \sim \log(1/\varepsilon)$ then the latter bound leads to $\mu(B(x,(k+1)\varepsilon))/\mu(B(x,k\varepsilon)) \leq K_0 < \infty$ if $g^+ < \infty$. In other terms,

Observation 3.1 *If $\mu \in \mathcal{P}_g(X)$, then there exists $K_0 < \infty$ such that for any $\varepsilon > 0$ small enough and for any $x \in \operatorname{supp} \mu$, there exists $k_{x,\varepsilon} \in \mathbb{N}$, $1 \leq k_{x,\varepsilon} \leq \log(1/\varepsilon)$ such that*

$$\frac{\mu(B(x,(k_{x,\varepsilon}+1)\varepsilon))}{\mu(B(x,k_{x,\varepsilon}\varepsilon))} \leq K_0. \tag{3.7}$$

As one can see the property $g^+ < \infty$ for compactly supported measures is a very natural substitute to the rather strong condition (3.4) that defines doubling measures. If the support of μ is not compact, than the key quantity is not $g(\varepsilon)$ anymore (since it is zero); we refer to [GT] where this situation is handled.

Property (3.7) can be seen as a non uniform doubling condition in the sence that for each $x \in supp\mu$ the radius of the balls for which $\mu(x,(k+1)\varepsilon)$ and $\mu(x,k\varepsilon)$ have comparable sizes (that is the spirit of the doubling condition) depends on x and ε. However one recovers some (crucial) uniformity by the fact that $k_{x,\varepsilon}$ is uniformly bounded in $x \in supp\mu$ by some (increasing) function of ε.

We note that Observation 3.1 constitutes, sort of say, the foundation of our proof of the equivalence of the dimensions. It does not appear clearly in the proof of Proposition 4.1 and Theorem 4.3, because we derive stronger results than just equivalence under the condition $g^+ < \infty$: we derive the equivalence for the larger classes of measures that satisfy (1.12) or (1.15).

As for the equivalence between Hentschel-Procaccia dimensions and generalized Rényi dimensions, the following proposition is rather immediate.

Proposition 3.3 *Suppose that the measure is doubling, then $D^{\pm}(q) = P_cD^{\pm}(q) = C_cD^{\pm}(q)$ for all $q \leq 0$.*

Indeed, one already has $D^{\pm}(q) \leq P_cD^{\pm}(q) = C_cD^{\pm}(q)$, and if $u = (B(x_i,\varepsilon))_{i\in I} \in \mathcal{P}_c^{(\varepsilon)}$, then

$$\begin{aligned} I(q,\varepsilon) &\geq \sum_{i\in I} \int_{B(x_i,\varepsilon)} \mu(B(x,\varepsilon))^{q-1} \mathrm{d}\mu(x) \geq \sum_{i\in I} \mu(B(x_i,\varepsilon))\mu(B(x_i,2\varepsilon))^{q-1} \\ &\geq K^{q-1} \sum_{i\in I} \mu(B(x_i,\varepsilon))^q = K^{q-1} S(u,q,\varepsilon) \ . \end{aligned} \tag{3.8}$$

Since this is true for all $u \in \mathcal{P}_c^{(\varepsilon)}$, one has $I(q,\varepsilon) \geq P_c(q,\varepsilon)$ and Proposition 3.3 follows.

3.3 Basic general relations with the dimensions $D^{\pm}(q)$ and $P_cD^{\pm}(q)$

Theorem 3.1 *Let $\mu \in \mathcal{P}(X)$ be a probability measure with compact support.*
i) If $\mu \in \mathcal{P}_g(X)$, i.e. $g^+ < \infty$, then for all $q \leq 0$ (actually $q < 1$),

$$(1-q)g^{\pm} - g^+ \leq \tau^{\pm}(q) \leq (1-q)g^{\pm}, \tag{3.9}$$

and thus

$$g^{\pm} - \frac{g^+}{(1-q)} \leq D^{\pm}(q) \leq g^{\pm}. \tag{3.10}$$

In particular, if $g^+ < \infty$ then $D^{\pm}(-\infty) = g^{\pm}$.
ii) If $g^+ = \infty$ but $g^- < \infty$, then the right inequalities in (3.9) and (3.10) still hold for $\tau^-(q)$ and $D^-(q)$.

Proof of Theorem 3.1:
The measure μ has compact support, so that by Proposition 3.1, $g(\varepsilon) > 0$. Since $q - 1 < 0$, it follows that

$$I(q,\varepsilon) = \int_{\mathrm{supp}\mu} \mu(B(x,\varepsilon))^{q-1} \mathrm{d}\mu(x) \leq g(\varepsilon)^{q-1}\mu(X) = g(\varepsilon)^{q-1}.$$

And the right inequalities in (3.9) and (3.10) follow.

We turn to the left inequalities. Let $\varepsilon > 0, \eta > 0$. From the definition of $g(\eta+\varepsilon)$, for any $\nu > 0$ one can find a point $y \in \mathrm{supp}\mu$ such that $\mu(B(y,\eta+\varepsilon)) \leq g(\eta+\varepsilon)+\nu$. One can then estimate (using the fact that $\mu(X \setminus \mathrm{supp}\mu) = 0$):

$$\begin{aligned} I(q,\varepsilon) &\geq \int_{B(y,\eta)} \mu(B(x,\varepsilon))^{q-1}\mathrm{d}\mu(x) \geq \int_{B(y,\eta)} \mu(B(y,\eta+\varepsilon))^{q-1}\mathrm{d}\mu(x) \\ &\geq (g(\eta+\varepsilon)+\nu)^{q-1}\mu(B(y,\eta)). \end{aligned}$$

Since $y \in \mathrm{supp}\mu$, $\mu(B(y,\eta)) \geq g(\eta)$. Letting then ν going to 0, one gets

$$I(q,\varepsilon) \geq (g(\eta+\varepsilon))^{q-1}g(\eta), \quad \varepsilon,\eta > 0. \tag{3.11}$$

We shall use many times this bound later in the paper. In particular, (3.11) is true for $\eta = \varepsilon$. Taking the log and dividing by $\log(1/\varepsilon)$, yields

$$\frac{\log I(q,\varepsilon)}{\log(1/\varepsilon)} \geq (1-q)\frac{\log(g(2\varepsilon))}{\log\varepsilon} - \frac{\log g(\varepsilon)}{\log\varepsilon}.$$

Estimates (2.5)-(2.6) finish the proof. □

Theorem 3.2 *Let $\mu \in \mathcal{P}(X)$ be a probability measure.*
(i) Suppose μ has a compact support. Then for all $q \leq 0$,

$$(-q)g^{\pm} \leq P_c^{\pm}(q) \leq (1-q)g^{\pm}, \tag{3.12}$$

and thus

$$\frac{-q}{(1-q)}g^{\pm} \leq P_cD^{\pm}(q) \leq g^{\pm}, \tag{3.13}$$

with the understanding that if $g^+ = +\infty$ (resp. $g^- = +\infty$) then for all $q < 0$ $P_c^+(q) = P_cD^+(q) = +\infty$ (resp. $P_c^-(q) = P_cD^-(q) = +\infty$).
In particular, if $g^+ < \infty$ (resp. $g^- < \infty$) then $P_cD^+(-\infty) = g^+$ (resp. $P_cD^-(-\infty) = g^-$).
(ii) If suppμ *is not compact, then $P_cD^{\pm}(q) = +\infty$ for any $q \leq 0$.*

Proof of Theorem 3.2: (i) Since μ has a compact support, $g(\varepsilon) > 0$ by Proposition 3.1. From the definition of $g(\varepsilon)$, there exists $x \in \text{supp}\mu$ so that $\mu(B(x,\varepsilon)) \leq 2g(\varepsilon)$. Consider the ball $B(x,\varepsilon)$. It is a particular packing $u = (B(x,\varepsilon)) \in \mathcal{P}_c^{(\varepsilon)}$ of $\text{supp}\mu$, so that for $q \leq 0$

$$P_c(q,\varepsilon) \geq S(u,q,\varepsilon) = \mu(B(x,\varepsilon))^q \geq (2g(\varepsilon))^q. \tag{3.14}$$

On the other hand, note that, for any $q < 1$ and for any centered ε-packing $u = (B(x_i,\varepsilon))_{i\in I} \in \mathcal{P}_c^{(\varepsilon)}$,

$$\begin{aligned} S(u,q,\varepsilon) &= \sum_{i\in I} \mu(B(x_i,\varepsilon))^q = \sum_{i\in I} \mu(B(x_i,\varepsilon))\mu(B(x_i,\varepsilon))^{q-1} \\ &\leq g(\varepsilon)^{q-1}\mu(X) = g(\varepsilon)^{q-1}. \end{aligned} \tag{3.15}$$

Estimates (3.14) and (3.15) lead to the desired result.
(ii) Assume that μ has a non compact support. By Lemma 2.1, for any $\varepsilon > 0$ small enough there exists a centered ε-packing with infinite cardinality. This yields immediately $P_c(q,\varepsilon) = +\infty$ for any $q \leq 0$ and thus $P_cD^{\pm}(q) = +\infty$. □

4 General results for $D^{\pm}(q)$

4.1 General result for $D^+(q)$

Theorem 4.1 *Let $\mu \in \mathcal{P}(X)$ be a probability measure on X with compact support. Suppose that $g^+ = +\infty$. Then for any $q < 0$, $D^+(q) = +\infty$.*

Combining this result with that of Theorem 3.1 shows that the class of measures $\mathcal{P}_g(X)$ introduced in (3.3) characterizes the measures which give rise to finite upper Hentschel-Procaccia dimensions. We have:

Corollary 4.1 *Suppose that $\mu \in \mathcal{P}(X)$ has a compact support. Then $D^+(q)$ is finite for all $q < 0$ if and only if $D^+(q)$ is finite for some $q < 0$ if and only if $\mu \in \mathcal{P}_g(X)$.*

Proof of Theorem 4.1:
By Proposition 3.1, one has $g(\eta) > 0$ for any $\eta > 0$. We define, for $\eta > 0$, the increasing finite function

$$f(\eta) = \log(1/g(\eta)).$$

Suppose the theorem does not hold: there exists $q < 0$ such that $D^+(q) < +\infty$. Thus there exists $A > 0$ finite such that

$$I(q,\varepsilon) \leq \left(\frac{1}{\varepsilon}\right)^A, \quad \text{for all } \varepsilon > 0 \text{ small enough.} \tag{4.1}$$

Set

$$B = \max(2, 2/|q|), \quad \text{and} \quad K = 2AB/|q| \ .$$

Since $g^+ = +\infty$, we can pick $0 < \eta_0 < 1/3$ small enough so that

$$f(\eta_0) > K\,|\log \eta_0|\,. \tag{4.2}$$

The bound (3.11) yields for any $\varepsilon \in (0, \eta_0)$:

$$I(q, \varepsilon) \geq g(\eta_0 - \varepsilon)(g(\eta_0))^{-(1+|q|)}. \tag{4.3}$$

By taking the log, (4.3) together with (4.1) leads to

$$f(\nu) \geq (1 + |q|)f(\nu + \varepsilon) - A \log(1/\varepsilon) \tag{4.4}$$

for all $\nu \in (0, \eta_0)$, $\varepsilon > 0$. We define now a decreasing sequence $(\eta_k)_{k\geq 0}$ as follows: η_0 is defined as in (4.2), and for all $k \geq 0$,

$$\eta_{k+1} = \eta_k - \varepsilon_k, \quad \varepsilon_k = \exp\left(-\frac{|q|}{2A} f(\eta_k)\right).$$

Of course, this definition is correct only if $\eta_k - \varepsilon_k > 0$ for all k. We will show this is indeed the case. Define for $k \geq 1$ the numbers

$$C_k = \eta_0 - \sum_{j=0}^{k-1} \eta_0^{B(1+j|q|/2)}.$$

Since $B = \max(2, 2/|q|)$ and $\eta_0 \leq 1/3$, it is easy to check that $C_k \geq C_\infty = \eta_0 - \sum_{j=0}^{\infty} \eta_0^{B(1+j|q|/2)} > 0$. We shall show now that for all $k \geq 1$ the following bounds hold:

$$\eta_k \geq C_k, \quad f(\eta_k) \geq (1 + |q|/2)^k f(\eta_0). \tag{4.5}$$

First, one observes that (4.2) implies

$$\varepsilon_0 = \exp\left(-\frac{|q|}{2A} f(\eta_0)\right) \leq \exp\left(\frac{|q|}{2A} K \log \eta_0\right) = \eta_0^B.$$

Thus, $\eta_1 = \eta_0 - \varepsilon_0 \geq \eta_0 - \eta_0^B = C_1 > 0$ since $B = \max(2, 2/|q|)$. Next, (4.4) with $\nu = \eta_1$, $\varepsilon = \varepsilon_0$ yields

$$f(\eta_1) \geq (1 + |q|)f(\eta_0) - A \log 1/\varepsilon_0 = (1 + |q|/2)f(\eta_0).$$

Therefore, (4.5) is true for $k = 0$. Assume now that (4.5) holds for all $k \leq p$ and show that it holds for $k = p+1$. Since $\eta_p > 0$, the number ε_p is well defined and due to (4.2) and (4.5) we obtain

$$\varepsilon_p \leq \exp\left(-\frac{|q|}{2A}(1 + |q|/2)^p f(\eta_0)\right) \leq \eta_0^{B(1+|q|/2)^p} \leq \eta_0^{B(1+p|q|/2)}.$$

Therefore,

$$\eta_{p+1} = \eta_p - \varepsilon_p \geq C_p - \eta_0^{B(1+p|q|/2)} = C_{p+1} > 0.$$

Next, (4.4) and (4.5) yield

$$f(\eta_{p+1}) \geq (1 + |q|)f(\eta_p) - A \log 1/\varepsilon_p = (1 + |q|/2)f(\eta_p) \geq (1 + |q|/2)^{p+1} f(\eta_0).$$

We see that (4.5) hold for $k = p + 1$ and thus for all k.

Now we can finish the proof of the Theorem. Since $\eta_k \geq C_k \geq C_\infty > 0$ for all k, (4.5) implies

$$f(C_\infty) \geq f(\eta_k) \geq (1 + |q|/2)^k f(\eta_0).$$

Letting k go to ∞, we obtain $f(C_\infty) = +\infty$. This is impossible since $C_\infty > 0$ and thus $g(C_\infty) > 0$. The theorem is proved. $\square$

4.2 A technical lower bound

In this subsection we derive an abstract lower bound for the integral $I(q,\delta)$. It is the basic result we shall use in the next two subsections.

Proposition 4.1 *Let $q \le 0$, and $\Delta, \delta, \varepsilon > 0$. Then, for Δ/δ large enough (depending only on q),*

$$I(q,\delta) \ge K(q,\varepsilon,[\Delta/\delta])\, P_c(q,\varepsilon+\Delta), \tag{4.6}$$

($[\Delta/\delta]$ denotes the integer part of Δ/δ), where

$$\begin{cases} K(q,\varepsilon,N) = \exp\left(-2|q|\xi^N \log 1/g(\varepsilon)\right), \quad \xi = \frac{1}{1+|q|}, & \textit{for } q<0, \\ K(0,\varepsilon,N) = \exp\left(-1/N \log 1/g(\varepsilon)\right). \end{cases} \tag{4.7}$$

Proof of Proposition 4.1:
Let $q \le 0$, $\delta, \varepsilon > 0, \Delta \ge \delta$ and set $N = [\Delta/\delta] \in \mathbb{N}^*$. We define $\eta = \varepsilon + \Delta$. Let $(B(x_i,\eta))_{i\in I} \in \mathcal{P}_c^{(\eta)}$ be a centered η-packing of suppμ. Recalling (2.4), one has

$$I(q,\delta) \ge \sum_{i\in I} \int_{B(x_i,\eta)} \mu(B(x,\delta))^{q-1} \mathrm{d}\mu(x). \tag{4.8}$$

We shall prove that for any $w \in$ suppμ,

$$A(w,q,\varepsilon,\delta,N) \equiv \int_{B(w,\eta)} \mu(B(x,\delta))^{q-1}\mathrm{d}\mu(x) \ge K(q,\varepsilon,N)\mu(B(w,\eta))^q\ , \tag{4.9}$$

where $K(q,\varepsilon,N)$ is the finite positive constant defined in (4.7). Note that $K(q,\varepsilon,N)$ is uniform in w, which is crucial. Let k be any integer between 0 and N. Obviously, since $q \le 0$ and $B(x,\delta) \subset B(w,\varepsilon+(k+1)\delta)$ for any $x \in B(w,\varepsilon+k\delta)$, one has the following inequalities

$$\begin{aligned} A(w,q,\varepsilon,\delta,N) &\ge \int_{B(w,\,\varepsilon+N\delta)} \mu(B(x,\delta))^{q-1}\mathrm{d}\mu(x) \ge \int_{B(w,\,\varepsilon+k\delta)} \mu(B(x,\delta))^{q-1}\mathrm{d}\mu(x) \\ &\ge \mu(B(w,\varepsilon+k\delta))\mu(B(w,\varepsilon+(k+1)\delta))^{q-1}. \end{aligned}$$

Therefore,

$$\frac{A(w,q,\varepsilon,\delta,N)}{\mu(B(w,\eta))^q} \ge \frac{\mu(B(w,\varepsilon+k\delta))}{\mu(B(w,\eta))}\left(\frac{\mu(B(w,\varepsilon+(k+1)\delta))}{\mu(B(w,\eta))}\right)^{q-1} \tag{4.10}$$

for any $k = 0,1,2,\ldots,N-1$. We define positive numbers

$$t_k = \frac{\mu(B(w,\varepsilon+k\delta))}{\mu(B(w,\eta))} = \frac{\mu(B(w,\varepsilon+k\delta))}{\mu(B(w,\varepsilon+\Delta))}, \quad k = 0,1,2,\ldots,N.$$

Note that, since $w \in$ suppμ and $\mu(B(w,\eta)) \le 1$, one has $t_0 \ge \mu(B(w,\varepsilon)) \ge g(\varepsilon)$, and thus,

$$g(\varepsilon) \le t_0 \le t_1 \le \ldots \le t_N = \frac{\mu(B(w,\varepsilon+[\Delta/\delta]\delta))}{\mu(B(w,\varepsilon+\Delta))} \le 1. \tag{4.11}$$

Since (4.10) is true for any $k = 0,1,2,\ldots,N-1$, one gets

$$\frac{A(w,q,\varepsilon,\delta,N)}{\mu(B(w,\eta))^q} \ge \max_{k\in[0,N-1]} t_k t_{k+1}^{q-1} \equiv L \tag{4.12}$$

We would like to prevent L from becoming too small. We shall get a control from below for L using the function $g(\varepsilon)$. What we shall do here is actually similar in its spirit to the simple arguments that led to Observation 3.1. For $q = 0$ it is basically the same, as for $q < 0$ we derive a better (but slightly trickier) bound. Consider first the case $q < 0$. Then for any $k = 0, 1, \cdots, N-1$ one has $t_k t_{k+1}^{-(1+|q|)} \le L$, which yields

$$\log t_{k+1} \ge \frac{1}{1+|q|}(\log t_k - \log L). \tag{4.13}$$

We proceed to the following natural change of variables $z_k = \log t_k + (\log L)/|q|$. Then (4.13) leads to

$$z_{k+1} \ge \xi z_k, \quad \text{where} \quad \xi = \frac{1}{1+|q|} < 1 .$$

By a repeated use of that inequality one gets

$$z_N \ge \xi^N z_0. \tag{4.14}$$

Moreover from (4.11) one derives $z_0 = \log t_0 + (\log L)/|q| \ge \log g(\varepsilon) + (\log L)/|q|$ and $z_N \le (\log L)/|q|$. Then (4.14) gives

$$\frac{\log L}{|q|} \ge \xi^N \left(\log g(\varepsilon) + \frac{\log L}{|q|} \right)$$

and finally

$$\log L \ge |q| \frac{\xi^N}{1-\xi^N} \log g(\varepsilon).$$

Since $\xi = 1/(1+|q|) < 1$, one can assume that N is large enough so that $\xi^N < 1/2$. Hence, since $\log g(\varepsilon) \le 0$, we have $\log L \ge 2|q|\xi^N \log g(\varepsilon)$.

As for the case $q = 0$, (4.13) leads to $\log t_{k+1} \ge \log t_k - \log L$ and thus

$$0 \ge \log t_N \ge \log t_0 - N \log L \ge \log g(\varepsilon) - N \log L ,$$

which implies $\log L \ge 1/N \log g(\varepsilon)$.

So depending on $q < 0$ or $q = 0$ we have

$$\begin{cases} L \ge K_1(q, \varepsilon, N) \equiv \exp\left(-2|q|\xi^N \log 1/g(\varepsilon)\right) & \text{if } q < 0, \\ L \ge K_2(\varepsilon, N) \equiv \exp\left(-1/N \log 1/g(\varepsilon)\right) & \text{if } q = 0. \end{cases} \tag{4.15}$$

Clearly, putting together Inequalities (4.12) and (4.15) leads to (4.9). So for any centered η-packing $u = (B(x_i, \eta))_{i \in I}$, combining (4.8) and (4.9) one gets, for any $q \le 0$,

$$I(q, \delta) \ge K(q, \varepsilon, N) \sum_{i \in I} \mu(B(x_i, \eta))^q,$$

where $\eta = \varepsilon + \Delta$. Taking the supremum other all such packings yields (4.6). $\square$

4.3 Criteria for non finiteness of $D^-(q)$

In this subsection we study the finiteness, or more precisely the non finiteness, of the lower dimensions $D^-(q)$. Of course, if $g^- < +\infty$ then by Theorem 3.1, $D^-(q)$ is finite for any $q < 0$. So, as far as the finiteness of $D^-(q)$ is concerned, the remaining open question concerns the case $g^- = +\infty$. Since the latter implies that $g^+ = +\infty$, we already know by Theorem 4.1 that $D^+(q) = +\infty$ for any $q < 0$, and by Theorem 3.2 we also know that $P_cD^-(q) = P_cD^+(q) = +\infty$ for any $q < 0$. What happens to $D^-(q)$? Is it always infinite as well? Or could it be finite?

One may think that typically the dimensions $D^-(q)$ should be infinite if $g^- = +\infty$, juste like $P_cD^-(q)$. However in Subsection 5.2 Remark 5.1, we give an example where $g^- = +\infty$ but the dimensions $D^-(q)$ are all finite for $q < 0$. So the question is: are there any conditions which will ensure that the dimensions are indeed infinite. Below we provide a series of such criteria, in terms of the behaviour of the function $g(\varepsilon)$.

For any $p = 1, 2, ...$ we define

$$\log_1 u = \log u, \quad \log_{p+1} u = \log(\log_p u), \tag{4.16}$$

assuming that u is big enough so that the arguments of each logarithm are positive, and also for $p = 0, 1, ...$

$$\exp_0 u = u, \quad \exp_1 u = \exp u, \quad \exp_{p+1} u = \exp(\exp_p u), \tag{4.17}$$

Theorem 4.2 *(i) Assume that $g^- = +\infty$ and that*

$$\limsup_{\varepsilon \downarrow 0} \frac{\log_3 1/g(\varepsilon)}{\log 1/\varepsilon} < +\infty. \tag{4.18}$$

Then $D^-(q) = +\infty$ for all $q < 0$.
(ii) Assume that

$$\limsup_{\varepsilon \downarrow 0} \frac{\log_2 1/g(\varepsilon)}{\log 1/\varepsilon} < +\infty$$

and $P_cD^+(0) = \dim_B^+(\mathrm{supp}\mu) = +\infty$ (resp. $P_cD^-(0) = \dim_B^-(\mathrm{supp}\mu) = +\infty$). Then $D^+(0) = +\infty$ (resp. $D^-(0) = +\infty$) as well.
(iii) Assume that for some $p = 2, 3...$ and for some $\alpha \in]0, +\infty]$, the two conditions are fulfilled:

$$\liminf_{\varepsilon \downarrow 0} \frac{\log_p 1/g(\varepsilon)}{\log 1/\varepsilon} = \alpha, \tag{4.19}$$

$$\limsup_{\varepsilon \downarrow 0} \frac{\log_{p+2} 1/g(\varepsilon)}{\log 1/\varepsilon} < \alpha. \tag{4.20}$$

Then $D^-(q) = +\infty$ for all $q < 0$.

Remarks
a) Note that (i) is similar to (iii) with $p = 1$ and $\alpha = +\infty$.
b) The example described in Subsection 5.2 Remark 5.1 shows that the condition (4.18) is actually optimal.
c) Points (i) and (ii) extend the results of Theorem 4.3 below if $P_cD^-(q)$ turns out

to be infinite. Indeed, in this case, Points (i) and (ii) say that the limit in (H1) and (H2) needs not to be zero but only finite, to get the equivalence (in the sense that both dimensions $D^-(q)$ and $P_cD^-(q)$ are infinite).
d) One can give an alternative proof of points (i), (iii) of this Theorem in spirit of the proof of Theorem 4.1. Namely, assuming that $D^-(q) < +\infty$ for some $q < 0$, one obtains (4.4) on some sequence $\varepsilon_n \to 0$. Iterating this inequality, one can show that it is incompatible with the two conditions of the Theorem for $p = 1$ ($g^- = +\infty$ and (4.18)) or for $p \geq 2$ ((4.19) and (4.20)).

Proof of Theorem 4.2:
Point (i). Set

$$\nu = 1 + \limsup_{\varepsilon \downarrow 0} \frac{\log_3 1/g(\varepsilon)}{\log 1/\varepsilon}.$$

Let $q < 0, \varepsilon > 0$. Apply Proposition 4.1 with $\Delta = \varepsilon$, $\delta = \varepsilon^{1+\nu}$, and thus $N = [\varepsilon^{-\nu}]$. Since $\log_3 1/g(\varepsilon) \leq (\nu - 1/2) \log 1/\varepsilon$ for ε small enough, one gets, with $\gamma = |\log \xi|$,

$$\begin{aligned} I(q,\delta) &\geq \exp\left(-2|q| \exp(-\gamma((1/\varepsilon)^\nu - 1) + (1/\varepsilon)^{\nu - 1/2})\right) P_c(q, 2\varepsilon) \\ &\geq \exp(-|q|) P_c(q, 2\varepsilon), \end{aligned}$$

for ε small enough. Then, using that $\log(1/\delta) = (1+\nu)\log(1/\varepsilon)$, we obtain that,

$$\liminf_{\delta \downarrow 0} \frac{\log I(q,\delta)}{\log(1/\delta)} \geq \frac{1}{1+\nu} P_c^-(q).$$

Since $g^- = +\infty$, one knows that $P_c^-(q) = +\infty$ and thus the latter inequality yields the result.
Point (ii). The proof is similar and based on the bound of Proposition 4.1 for $q = 0$.
Point (iii). We suppose $\alpha \neq +\infty$. The proof for $\alpha = +\infty$ is similar.
First recall (3.14): in full generality $P_c(q, \varepsilon) \geq (2g(\varepsilon))^q$. It follows from Proposition 4.1, with $\Delta = \varepsilon$, $N = [\varepsilon/\delta]$, that, for $q < 0$ and all $\delta, \varepsilon > 0$

$$\frac{1}{|q|} \log I(q,\delta) \geq \log(1/2g(2\varepsilon))) - 2\xi^{\varepsilon/\delta - 1} \log(1/g(\varepsilon)), \tag{4.21}$$

with $\xi = (1 + |q|)^{-1}$. Let $\delta > 0$, we define ε, via the relation

$$\log(1/\delta) = \exp_{p-2}((1/2\varepsilon)^{\alpha - 2\eta}),$$

where the choice of $\eta > 0$ is fixed so that

$$0 \leq \limsup_{\varepsilon \downarrow 0} \frac{\log_{p+2} 1/g(\varepsilon)}{\log 1/\varepsilon} < \alpha - 4\eta < \alpha = \liminf_{\varepsilon \downarrow 0} \frac{\log_p 1/g(\varepsilon)}{\log 1/\varepsilon}.$$

As a consequence, for ε small enough (depending on η), $[\log_p 1/g(2\varepsilon)]/[\log(1/2\varepsilon)] \geq \alpha - \eta$, and thus

$$\frac{\log 1/(g(2\varepsilon))}{\log(1/\delta)} \geq \frac{\exp_{p-2}((1/2\varepsilon)^{\alpha-\eta})}{\log(1/\delta)} = \exp_{p-2}((1/2\varepsilon)^{\alpha-\eta}) / \exp_{p-2}((1/2\varepsilon)^{\alpha-2\eta}), \tag{4.22}$$

which goes to infinity as $\delta \to 0$. On the other hand, for any ε small enough,

$$\frac{\log_{p+2} 1/(g(\varepsilon))}{\log(1/\varepsilon)} < \alpha - 3\eta.$$

It follows that for $\gamma = |\log \xi| > 0$ given, and for any ε small enough,

$$\begin{aligned} \xi^{\varepsilon/\delta} \log 1/g(\varepsilon) &\leq \xi^{\varepsilon/\delta} \exp_p((1/\varepsilon)^{\alpha-3\eta}) \\ &= \exp(-\gamma\varepsilon \exp_{p-1}((1/2\varepsilon)^{\alpha-2\eta})) \exp_p((1/\varepsilon)^{\alpha-3\eta}), \end{aligned} \tag{4.23}$$

the latter going to zero as $\delta \to 0$. It follows from (4.21)-(4.23) that

$$\lim_{\delta \downarrow 0} \frac{\log I(q,\delta)}{\log(1/\delta)} = +\infty,$$

or in other terms $D^-(q) = +\infty$. □

4.4 Criteria for the equivalence of $D^\pm(q)$ and $P_cD^\pm(q)$

So far we investigated when are the dimensions $D^\pm(q)$ finite or infinite for a given compactly supported measure. We showed that the finiteness of $D^+(q)$ and $P_cD^+(q)$ are totally characterized by the one of g^+. As for the lower dimensions, if the finiteness of $P_cD^-(q)$ is equivalent to the one of g^-, this is not the case with $D^-(q)$. The next question, which is as natural as important, is: if, for $q < 0$ (or $q = 0$), the dimensions are finite, then do the Hentschel-Procaccia and the generalized Rényi dimensions coincide? We shall answer positively to this question, for both $D^+(q)$ and $D^-(q)$, provided $g^+ < \infty$. If now $g^+ = \infty$, but $g^- < \infty$, then we give a criterium that implies $D^-(q) = P_cD^-(q)$. In Subsection 5.2 we exhibit a family of measures such that $D^-(q) < P_cD^-(q) < +\infty$, on some interval $(-\infty, q_0)$.

Consider the following two conditions on the rate of decay of the function $g(\varepsilon)$.

$$\text{(H1)} \qquad \limsup_{\varepsilon \downarrow 0} \frac{\log\log\log(1/g(\varepsilon))}{\log 1/\varepsilon} = 0,$$

$$\text{(H2)} \qquad \limsup_{\varepsilon \downarrow 0} \frac{\log\log(1/g(\varepsilon))}{\log 1/\varepsilon} = 0.$$

Note that

$$(\mu \text{ is doubling with compact support}) \implies (g^+ < +\infty) \implies \text{(H2)} \implies \text{(H1)}\ .$$

Theorem 4.3 *Suppose μ has a compact support.*
(i) Assume Hypothesis (H1) holds. Then for any $q < 0$,

$$D^\pm(q) = P_cD^\pm(q). \tag{4.24}$$

(ii) Assume Hypothesis (H2) holds. Then for any $q \leq 0$

$$D^\pm(q) = P_cD^\pm(q). \tag{4.25}$$

Corollary 4.2 *One has*
(i) If $g^+ < +\infty$, then for any $q \le 0$, $D^\pm(q) = P_cD^\pm(q) < +\infty$.
(ii) If $g^+ = +\infty$, $g^- < +\infty$ and (H1) holds, then $D^-(q) = P_cD^-(q) < +\infty$ for any $q < 0$.
(iii) If $g^+ = +\infty$ but (H2) holds, then $D^\pm(0) = P_cD^\pm(0)$ (both being finite or infinite).

Remarks
a) If $g^+ = +\infty$, $g^- < +\infty$ but (H1) does not hold, it is possible that $D^-(q) < P_cD^-(q)$, both dimensions being finite according to Theorems 3.1 and 3.2. We give such examples in Section 5.2, which shows that the condition (H1) above is optimal: for any $\delta > 0$ one can construct examples for which $g^- < +\infty$, $\limsup_{\varepsilon\downarrow 0} \log_3(1/g(\varepsilon))/\log 1/\varepsilon = \delta$ but with $D^-(q) < P_cD^-(q)$ on some interval $(-\infty, q_0)$.
b) If $q = 0$ and $g^+ = +\infty$, in general one cannot say whether $D^+(0), P_cD^+(0)$ are finite or not (the same for g^- and $D^-(0), P_cD^-(0)$). The second point of Theorem 4.3, provides a criterium for their equality.
c) If $g^- = +\infty$, then Point (i) of Theorem 4.3 says that $D^-(q) = P_cD^-(q) = +\infty$ if $\limsup \log_3(1/g(\varepsilon))/\log 1/\varepsilon = 0$. This a particular case of Point (i) of Theorem 4.2 which says that this is still the case if $\limsup \log_3(1/g(\varepsilon))/\log 1/\varepsilon < +\infty$.

Proof of Theorem 4.3:
The heart of Theorem 4.3 is Proposition 4.1.

The fact that $D^\pm(q) \le C_cD^\pm(q) = P_cD^\pm(q)$ for all $q \le 0$ follows from Proposition 2.1. We show the converse inequalities $P_cD^\pm(q) \le D^\pm(q)$.

Let $\nu > 0$. Apply Proposition 4.1 with $\Delta = \varepsilon$, and $\delta = \varepsilon^{1+\nu}$ (hence $N = [\varepsilon^{-\nu}]$). Hypotheses (H1) if $q < 0$ and (H2) if $q = 0$ are so that for $\nu > 0$ being given, and for *any* $\varepsilon > 0$ small enough, $K(q, \varepsilon, [\varepsilon^{-\nu}])$ is non smaller than some constants $K^*(q, \nu) > 0$, uniformly in ε (this is similar to the proof of Point (i) of Theorem 4.2). So that (4.6) yields, for *any* $\varepsilon > 0$ small enough,

$$I(q, \varepsilon^{1+\nu}) \ge K^*(q, \nu) P_c(q, 2\varepsilon).$$

Thus taking the log, dividing by $(1-q)\log 1/\varepsilon$ and taking the lim inf or lim sup, one gets, for any $\nu > 0$,

$$D^\pm(q) \ge \frac{1}{1+\nu} P_cD^\pm(q). \tag{4.26}$$

And the theorem follows. □

5 Examples and counter examples

Throughout this section we mean by $f(n) \sim g(n)$, as $n \to \infty$, that there exists two constants C_1 and C_2 such that for n large enough

$$C_1 g(n) \le f(n) \le C_2 g(n).$$

5.1 Example where $g^- < g^+ < +\infty$

Let γ and a two positive reals. We define a measure μ on $\mathbb{R}$ by $\mu = \sum_{n\ge 1} a_n \delta_{x_n}$, where

$$x_n = \exp(-\exp \gamma n) \quad \text{and} \quad a_n = \mu(\{x_n\}) = x_n^a. \tag{5.1}$$

Note that $\operatorname{supp}\mu = (\bigcup_{n\ge 1}\{x_n\}) \cup \{0\} \subset [0,1]$. We shall show that

Proposition 5.1
(i) μ is not doubling.
(ii) One has $a = g^- < g^+ = ae^\gamma < +\infty$. In particular $\mu \in \mathcal{P}_g(\mathbb{R})$.

This compactly supported measure μ is then a very simple exemple both of a measure with $g^- < g^+$ and of a measure which is not doubling, but that still belongs to the class $\mathcal{P}_g(\mathbb{R})$, since $g^+ < \infty$. The latter then implies that for this measure μ one has $D^\pm(q) = P_c D^\pm(q)$ for any $q \le 0$, by Corollary 1.1.

Proof of Proposition 5.1:
To verify Point (i), note that for n large enough so that $x_n/2 > x_{n+1}$, $\mu(0, x_n/2) = \sum_{k=n+1}^\infty a_k \sim a_{n+1}$, and $\mu(0, x_n) = \sum_{k=n}^\infty a_k \sim a_n$. As a consequence

$$\frac{\mu(0, x_n)}{\mu(0, x_n/2)} \sim \left(\frac{x_n}{x_{n+1}}\right)^a,$$

which goes to infinity as n goes to infinity. Hence $\limsup_{\varepsilon\downarrow 0}[\mu(0, 2\varepsilon)/\mu(0,\varepsilon)] = +\infty$, and μ is not doubling.
We turn to Point (ii). Let $\varepsilon > 0$. There exists a unique n such that $x_{n+1} \le \varepsilon < x_n$. Note that at the point $x = 0$ one has

$$\mu(-\varepsilon, \varepsilon) = \sum_{k=n+1}^\infty a_k \sim a_{n+1} = x_{n+1}^a \tag{5.2}$$

One easily sees that $g(\varepsilon) = \inf_{x\in \operatorname{supp}\mu} \mu(x-\varepsilon, x+\varepsilon) = \mu(-\varepsilon,\varepsilon)$. Indeed if $x = x_k$ with $k \ge n+1$ then clearly $\mu(x_k - \varepsilon, x_k + \varepsilon) \ge \mu(0,\varepsilon) = \mu(-\varepsilon,\varepsilon)$, and if $k \le n$, then $\mu(x_k - \varepsilon, x_k+\varepsilon) \ge \mu(\{x_k\}) = a_k \ge a_n >> a_{n+1}$, and thus by (5.2), $\mu(x_k-\varepsilon, x_k+\varepsilon) \ge \mu(-\varepsilon,\varepsilon)$. As a consequence for $\varepsilon \in [x_{n+1}, x_n)$ small enough, $x_{n+1}^a \le g(\varepsilon) \le 2x_{n+1}^a$. Thus,

$$\frac{a\log 1/x_{n+1} - \log 2}{\log 1/x_{n+1}} \le \frac{\log 1/g(\varepsilon)}{\log 1/\varepsilon} \le \frac{a \log 1/x_{n+1}}{\log 1/x_n} = ae^\gamma,$$

and $a \le g^- \le g^+ \le ae^\gamma$. Since the left and right bound are reached for particular sequences $\varepsilon_n = x_{n+1}$ and $\varepsilon_n = x_n/2$ respectively, it implies that $g^- = a$ and $g^+ = ae^\gamma$.
□

5.2 Examples where $D^-(q) < P_c D^-(q)$, with $P_c D^-(q)$ finite or not

We pick $\alpha_1 > \alpha_2 > 0$ two reals. Let a_n be sequence of real in $(0,1]$, a_0 a small real to be Line (5.7) below, and

$$a_{n+1} = \exp\left(-\exp a_n^{-\alpha_1}\right). \tag{5.3}$$

The sequence a_n is monotone and fast decaying. We define intervals $I_n = [a_n, a_n^\nu]$, for $\nu \in](1+\alpha_2)^{-1}, 1[$ (this condition comes from Lemma 5.2). Note that due to the fast decay of the a_n's, the intervals I_n are disjoints. We further define the measure μ on each interval I_n as follows:

$$\mu(\mathrm{d}x) = C_n \rho(x)\mathrm{d}x, \quad \rho(x) = \exp(-\exp x^{-\alpha_2}), \tag{5.4}$$

and the constant C_n is such that

$$\mu(I_n) = a_n^{K\nu}, \tag{5.5}$$

with $K\nu > 1$. Note that

$$\mathrm{supp}\mu = \{0\} \cup \bigcup_{p=0}^{\infty} I_p \subset [0,1].$$

We first make useful observations. Straightforward computations show that

$$a_n^{(K-1)\nu} \rho^{-1}(a_n^{-\nu}) \leq C_n \leq 2a_n^{(K-1)\nu} \rho^{-1}(a_n^{-\nu}/2). \tag{5.6}$$

We choose a_0 small enough such that

$$C_n \geq 1 \ , \quad \text{forall } n \geq 0. \tag{5.7}$$

In the following we shall need the following bound: for any $\beta > 0$ for n large enough (depending on $\beta, \alpha_1, \alpha_2$),

$$\rho(x) \geq a_n^{\beta} \text{ for all } x \in I_p, \quad p \leq n-1. \tag{5.8}$$

Indeed observe that since $\rho(x)$ is increasing and a_p decreasing, we have for $x \in I_p$, $p \leq n-1$: $\rho(x) \geq \rho(a_p) \geq \rho(a_{n-1})$. Due to the definition of $\rho(x)$ and a_n, $\rho(a_{n-1}) \geq a_n^{\beta}$ for any $\beta > 0$ for n large enough, so we obtain (5.8).

Proposition 5.2 *For the measure μ described above one has*
(i)

$$g^- = K, \quad \text{and} \quad \limsup_{\varepsilon \downarrow 0} \frac{\log\log\log 1/g(\varepsilon)}{\log 1/\varepsilon} = \alpha_1. \tag{5.9}$$

As a consequence $g^+ = +\infty$ and thus $D^+(q) = P_c D^+(q) = +\infty$.
(ii) Moreover, one has

$$\tau^-(q) = \max(K\nu|q|, 1+|q|) \quad \textit{and} \quad P_c^-(q) = \max(K|q|, 1+|q|).$$

Or in other terms

$$\begin{cases} \tau^-(q) = P_c^-(q) = 1+|q| & \textit{if } -(K-1)^{-1} \leq q \leq 0, \\ \tau^-(q) = 1+|q| < K|q| = P_c^-(q) & \textit{if } -(K\nu-1)^{-1} \leq q < -(K-1)^{-1}, \\ \tau^-(q) = K\nu|q| < K|q| = P_c^-(q) & \textit{if } q \leq -(K\nu-1)^{-1}. \end{cases}$$

In particular $D^-(q) < P_c D^-(q) < +\infty$ for $q < -(K-1)^{-1}$, and $D^-(-\infty) = K\nu < K = P_c D^-(-\infty) = g^-$.

Remark 5.1 *One can also construct an example, based on the one proposed above, where $g^- = +\infty$ and $\alpha_1 = +\infty$ in (5.9), but still $D^-(q) < \infty$. To get such an example, it is enough to let the parameters α_1, K, ν vary with the interval one considers: $\alpha_{1,n}, \alpha_{2,n}, K_n, \nu_n$ on I_n, with $\alpha_{1,n} > \alpha_{2,n}$ going to infinity as well as K_n, and ν_n is going to zero. We keep the following relations : $K_n \nu_n = \gamma > 1$ fixed, and $\nu_n \in](1+\alpha_{2,n})^{-1}, 1[$ (so that Lemma 5.2 is satisfied).*

As a result, $g^- = \lim_{n\to\infty} K_n = +\infty$ and

$$\limsup_{\varepsilon \downarrow 0} \frac{\log\log\log(1/g(\varepsilon))}{\log(1/\varepsilon)} = \lim_{n\to\infty} \alpha_{1,n} = +\infty.$$

So for any $q < 0$, $D^+(q) = P_c D^{\pm}(q) = +\infty$, but $D^-(q) = \max(\gamma|q|/(1+|q|, 1) < D^-(-\infty) = \gamma < +\infty$.

To prove Point (i) of Proposition 5.2, we shall first show that the infimum, when computing $g(\varepsilon) = \inf_{x\in\operatorname{supp}\mu} \mu(x-\varepsilon, x+\varepsilon)$, is obtained for $x = 0$.

Lemma 5.1 *For any $\varepsilon > 0$ small enough, one has $g(\varepsilon) = \mu([0,\varepsilon])$. Moreover $g(\varepsilon) \leq C\varepsilon^K$ for some constant $C > 0$.*

Proof of Lemma 5.1:

For any $\varepsilon > 0$ one can find a unique n such that $a_{n+1}^\nu < \varepsilon \leq a_n^\nu$. We shall assume that n is large enough so that (5.8) holds.

Since $\varepsilon > a_{n+1}^\nu$, for any $x \in I_p,\ p \geq n+1$ we have $\mu([x-\varepsilon, x+\varepsilon]) \geq \mu([0,\varepsilon]) = \mu([0-\varepsilon, 0+\varepsilon])$. Therefore, when calculating $g(\varepsilon)$, one does not need to consider x from intervals $I_p,\ p \geq n+1$. It is sufficient to take $x = 0$ and $x \in I_p,\ p \leq n$.

Consider $x \in I_p,\ p \leq n-1$. Since $\varepsilon \leq a_n^\nu$ and $C_p \geq 1$, using (5.8), we obtain:

$$\mu([x-\varepsilon, x+\varepsilon]) = \mu([x-\varepsilon, x+\varepsilon] \cap I_p) \geq a_n^\beta \varepsilon \geq \varepsilon^{1+\beta/\nu}. \tag{5.10}$$

We turn to the case where $x \in I_n$. We shall study separately the cases $\varepsilon \in]a_{n+1}^\nu, a_n]$ and $\varepsilon \in]a_n, a_n^\nu]$.

1) Assume $a_{n+1}^\nu < \varepsilon \leq a_n$. Recall $x \in I_n$. Using (5.8), we have

$$\mu([x-\varepsilon, x+\varepsilon]) \geq a_{n+1}^\beta \varepsilon \geq a_{n+1}^{\nu+\beta}. \tag{5.11}$$

Further, since $\varepsilon \leq a_n$ one has $\mu([0,\varepsilon]) = \mu([0, a_{n+1}^\nu])$, and the fast decay in p of $\mu(I_p) = a_p^{K\nu}$ implies, as n goes to infinity,

$$\mu([0,\varepsilon]) = \sum_{p=n+1}^{\infty} \mu(I_p) \sim \mu(I_{n+1}) = a_{n+1}^{K\nu}. \tag{5.12}$$

Since $K > 1$ one can take β small so that $\nu + \beta < K\nu$ and (5.11)-(5.12) yield

$$\mu([x-\varepsilon, x+\varepsilon]) \geq \mu([0,\varepsilon]), \quad x \in I_n, \tag{5.13}$$

for n large enough, so we do not need to take into account the points x coming from I_n. Next, the bound (5.12) implies $\mu([0,\varepsilon]) \leq \varepsilon^K$. Since $K > 1$, taking β small enough, we see from (5.10) that $\mu([x-\varepsilon, x+\varepsilon]) \geq \mu([0,\varepsilon])$ for $x \in I_p,\ p \leq n-1$. Finally, as n goes to infinity,

$$g(\varepsilon) = \mu([0,\varepsilon]) \sim a_{n+1}^{K\nu} \leq \varepsilon^K, \quad \varepsilon \in [a_{n+1}^\nu, a_n]. \tag{5.14}$$

2) Assume now $\varepsilon \in]a_n, a_n^\nu]$ and recall $x \in I_n$. Observe that

$$\mu([0,\varepsilon]) = \mu([0, a_{n+1}^\nu]) + \mu([a_n, \varepsilon]). \tag{5.15}$$

Since ρ is increasing $\mu([x, x+\varepsilon]) \geq \mu([a_n, a_n+\varepsilon])$, and thereby

$$\mu([x-\varepsilon, x+\varepsilon]) \geq \mu([a_n, a_n+\varepsilon]) = \mu([a_n, \varepsilon]) + \mu([\varepsilon, \varepsilon + a_n]). \tag{5.16}$$

Since $C_n \geq 1$, due to (5.8)

$$\mu([\varepsilon, \varepsilon + a_n]) \geq \rho(a_n) a_n \geq a_{n+1}^\beta a_n \geq a_{n+1}^{2\beta} \geq a_{n+1}^{K\nu/2} \geq \mu([0, a_{n+1}^\nu]), \tag{5.17}$$

provided β is small enough and n large enough. The bounds (5.15)-(5.17) yield $\mu([x-\varepsilon, x+\varepsilon]) \geq \mu([0,\varepsilon])$ for $x \in I_n$, and to calculate $g(\varepsilon)$, it is thus sufficient to consider $\mu([0,\varepsilon])$ and to compare it with $\mu([x-\varepsilon, x+\varepsilon])$, $x \in I_p$, $p \leq n-1$ (bounded from below by (5.10)). Assume first that $a_n < \varepsilon \leq a_n^\nu/3$. Then

$$\mu([a_n,\varepsilon]) \leq C_n \rho(a_n^\nu/3)\varepsilon \leq 2a_n^{(K-1)\nu}\rho^{-1}(a_n^\nu/2)\rho(a_n^\nu/3)\varepsilon \leq \varepsilon^M \tag{5.18}$$

for any $M > 0$ if n is large enough. On the other hand, if $\varepsilon \in [a_n^\nu/3, a_n^\nu]$, then

$$\mu([a_n,\varepsilon]) \leq \mu(I_n) = a_n^{K\nu} \leq (3\varepsilon)^K. \tag{5.19}$$

Finally, (5.15), (5.18) and (5.19) imply

$$\mu([0,\varepsilon]) \leq 2a_{n+1}^{K\nu} + \varepsilon^M + \varepsilon^K \leq 2\varepsilon^K, \quad \varepsilon \in [a_n, a_n^\nu]. \tag{5.20}$$

Since $K > 1$, for β small enough $K > 1 + \beta/\nu$, and (5.10) implies $\mu([x-\varepsilon, x+\varepsilon]) > \mu([0,\varepsilon])$ and thus $g(\varepsilon) = \mu([0,\varepsilon])$. □

Proof of Proposition 5.2:

Concerning g^-, note that Lemma 5.1 implies that $g^- \geq K$. Consider then the subsequence $\varepsilon_n = a_n^\nu$. One has $g(\varepsilon_n) = \mu([0,a_n^\nu]) \sim \mu(I_n) = a_n^{K\nu} = \varepsilon_n^K$. Therefore, $g^- = K$.

To show the second claim of (5.9), we need to bound $g(\varepsilon)$ from below. First, it is clear that

$$g(\varepsilon) = \mu([0,\varepsilon]) \geq \mu(I_{n+1}) = a_{n+1}^{K\nu}. \tag{5.21}$$

for any $\varepsilon \in [a_{n+1}^\nu, a_n^\nu]$.

1) Assume that $\varepsilon \in [a_{n+1}^\nu, 2a_n]$. Then (5.21) and the definition of a_n yield

$$\log\log 1/g(\varepsilon) \leq \log(K\nu) + \frac{1}{a_n^{\alpha_1}} \leq \frac{2}{a_n^{\alpha_1}}$$

for n large enough. Therefore,

$$\log\log\log 1/g(\varepsilon) \leq \log 2 + \alpha_1 \log\frac{1}{a_n} \leq \log 2 + \alpha_1 \log\frac{2}{\varepsilon} \tag{5.22}$$

2) Let now $\varepsilon \in [2a_n, a_n^\nu]$. Since $C_n \geq 1$, we can estimate:

$$g(\varepsilon) = \mu([0,\varepsilon]) \geq \mu([a_n,\varepsilon]) \geq \mu([\varepsilon/2,\varepsilon]) \geq \varepsilon/2\rho(\varepsilon/2) = \varepsilon/2\exp(-\exp(2/\varepsilon)^{\alpha_2}). \tag{5.23}$$

Since $\alpha_2 < \alpha_1$, one can easily see that (5.22) and (5.23) imply

$$\limsup_{\varepsilon\downarrow 0} \frac{\log\log\log 1/g(\varepsilon)}{\log 1/\varepsilon} \leq \alpha_1. \tag{5.24}$$

Then, considering the sequence $\varepsilon_n = a_n$ shows that the equality actually holds in (5.24). Indeed, due to (5.12), one has

$$g(\varepsilon_n) = \mu([0,a_{n+1}]) \sim a_{n+1}^{K\nu} = \exp(-K\nu\exp(a_n^{-\alpha_1})) = \exp(-K\nu\exp(\varepsilon_n^{-\alpha_1})). \tag{5.25}$$

We turn to the second part of the Proposition, i.e. the one concerning the dimensions. It is clear that (5.9) implies $g^+ = +\infty$ and thus (Theorem 4.1) that the upper dimensions are infinite: $D^+(q) = P_c D^+(q) = +\infty$ for all $q < 0$. As to the lower dimensions, they are finite since $g^- = K < +\infty$. We first compute $P_c^-(q)$ and show that

$$P_c^-(q) = \max(K|q|, 1 + |q|).$$

We already know that $P_c^-(q) \geq g^-|q| = K|q|$. Note that $P_c(q,\varepsilon) \geq P_{c,I_0}(q,\varepsilon)$, where $P_{c,I_0}(q,\varepsilon)$ stands for the generalized Rényi sums of μ restricted to the first interval I_0. One thus trivially gets $P_{c,I_0}(q,\varepsilon) \geq C\varepsilon^{q-1}$, and thus $P_c^-(q) \geq 1 + |q|$. It remains to show that $P_c^-(q) \leq \max(K|q|, 1+|q|)$. To that aim, pick a sequence $\varepsilon_n = a_n^\nu$. Then centered packings of $\operatorname{supp}\mu$ consist in the interval $[0, a_n^\nu]$ plus centered packings of the intervals I_k, $k \leq n-1$ (since for any $x \in I_k$, $k \geq n$, the ball $[x - \varepsilon_n, x + \varepsilon_n]$ contains all the other intervals I_k, $k \geq n$). We recall (5.10): for $x \in I_p$, $p \leq n-1$ one has $\mu([x-\varepsilon, x+\varepsilon]) \geq \varepsilon^{1+\beta/\nu}$. In addition, note that $\operatorname{supp}\mu \subset [0,1]$, so that for any centered packing of $\operatorname{supp}\mu$, the number of intervals of radius ε_n and centered in I_k, $k \leq n-1$, is less than ε_n^{-1}. It follows that, for any $\beta > 0$ (provided n is large enough),

$$\begin{aligned} P_c(q,\varepsilon) &\leq \mu([0,a_n^\nu])^q + \varepsilon_n^{-1}\varepsilon_n^{(1+\beta/\nu)q} \\ &\leq C\left(\frac{1}{\varepsilon_n}\right)^{K|q|} + \left(\frac{1}{\varepsilon_n}\right)^{1+(1+\beta/\nu)|q|} \end{aligned}$$

As a consequence, for any $\beta > 0$, $P_c^-(q) \leq \max(K|q|, 1 + (1+\beta/\nu)|q|)$. The result follows.

We turn to $\tau^-(q)$ and show that

$$\tau^-(q) = \max(K\nu|q|, 1 + |q|).$$

The main part of the work is to show that $\tau^-(q) \leq \max(K\nu|q|, 1+|q|)$. That's what we start with, and to that aim, we shall control the integral $I(q,\varepsilon)$ on each interval I_k: we set $J_k(q,\varepsilon) := \int_{I_k} \mu([x-\varepsilon, x+\varepsilon])^{q-1}\mathrm{d}\mu(x)$. We pick a sequence $\varepsilon_n = 2a_n$.

1) We start with the integrals $J_k(q,\varepsilon)$ for $k \geq n+1$, and show they can be bounded by a constant. First note that if $x \in I_k$, $k \geq n+1$, then (5.8) implies $\mu([x-\varepsilon_n, x+\varepsilon_n]) \geq \mu([a_n, 2a_n]) \geq a_n C_n \rho(a_n) \geq a_n a_{n+1}^\beta \geq a_{n+1}^{2\beta}$ for any $\beta > 0$ and n large enough depending on β. Now, for n large enough,

$$\begin{aligned} \sum_{k\geq n+1} J_k(q,\varepsilon_n) &\leq a_{n+1}^{2\beta(q-1)} \sum_{k\geq n+1} \mu(I_k) && (5.26) \\ &\leq 2a_{n+1}^{2\beta(q-1)}\mu(I_{n+1}) = 2a_{n+1}^{2\beta(q-1)+K\nu} && (5.27) \\ &\leq 2, && (5.28) \end{aligned}$$

where we took $\beta \leq 1/2K\nu(1+|q|)^{-1}$.

2) We now evaluate the integral $J_n(q,\varepsilon_n)$ that we split in two parts: a left part $L_n = [a_n, a_n^\nu - 2a_n]$ and a right part $R_n = [a_n^\nu - 2a_n, a_n^\nu]$. Note that for any $x \in L_n$,

$$\mu(x-\varepsilon_n, x+\varepsilon_n) \geq \mu(x+\varepsilon_n/2, x+\varepsilon_n) = C_n \int_{x+\varepsilon_n/2}^{x+\varepsilon_n} \rho(x)\mathrm{d}x \geq C_n \frac{\varepsilon_n}{2}\rho(x+\varepsilon_n/2). \quad (5.29)$$

As a consequence, on $L_n = [a_n, a_n^\nu - 2a_n]$,

$$\int_{L_n} \mu(x-\varepsilon_n, x+\varepsilon_n)^{q-1}\mathrm{d}\mu(x) \leq C_n^q \left(\frac{\varepsilon_n}{2}\right)^{q-1} \int_{L_n} \rho(x+\varepsilon_n/2)^{q-1}\rho(x)\mathrm{d}x. \quad (5.30)$$

One has the following trivial lemma:

Lemma 5.2 *Recall $\nu \in](1+\alpha_2)^{-1}, 1[$. For any $q<0$, and for any n large enough (depending on ν and q) and for all $x \in L_n = [a_n, a_n^\nu - 2a_n]$, one has $\rho(x+\varepsilon_n/2)^{q-1}\rho(x) < 1$.*

Indeed, $\rho(x+\varepsilon_n/2)^{q-1}\rho(x) < 1$ is equivalent to (after taking a double log),

$$\begin{aligned} & \log(1+|q|) + \frac{1}{x+\varepsilon_n/2} < \frac{1}{x} \\ \iff \quad & \varepsilon_n/2 > x\left((1-x\log(1+|q|))^{-1/\alpha_2} - 1\right) \sim \frac{\log(1+|q|)}{\alpha_2} x^{1+\alpha_2}, \end{aligned} \tag{5.31}$$

as x goes to zero (i.e. n goes to $+\infty$). Since $\varepsilon_n = 2a_n$, $(1+\alpha_2)\nu > 1$ and $x \le a_n^\nu$ the last inequality is true for n large enough, which proves Lemma 5.2.

It thus follows from (5.29), using (5.6), that

$$\begin{aligned} \int_{L_n} \mu(x-\varepsilon_n, x+\varepsilon_n)^{q-1} \mathrm{d}\mu(x) &\le C_n^q \left(\frac{\varepsilon_n}{2}\right)^{q-1} a_n^\nu && (5.32) \\ &\le a_n^{q-1} a_n^{\nu + q(K-1)\nu} \rho^{|q|}(a_n^\nu) && (5.33) \\ &\le 1 && (5.34) \end{aligned}$$

for n large enough. It is on $L_n = [a_n, a_n^\nu - 2a_n]$ that the difference between the Hentshel-Proccacia and the Rényi dimensions takes place. Instead of being large, like with Rényi sums, the part of $J_n(q,\varepsilon_n)$ computed on $[a_n, a_n^\nu - 2a_n]$ is very small. The key is the double exponential in the definition of $\rho(x)$.

We turn to the right part of I_n: $R_n := [a_n^\nu - 2a_n, a_n^\nu]$. On this part, one has $\mu(x-\varepsilon_n, x+\varepsilon_n) \ge \mu(R_n)$ for any $x \in R_n$. Let us show that $\mu(R_n) \ge 1/2\mu(I_n)$ for n large enough. In fact,

$$\mu(L_n) = C_n \int_{L_n} \rho(x)dx \le C_n a_n^\nu \rho(a_n^\nu - 2a_n),$$

and

$$\mu(R_n) = C_n \int_{R_n} \rho(x)dx \ge C_n \int_{a_n^\nu - a_n}^{a_n^\nu} \rho(x)dx \ge C_n a_n \rho(a_n^\nu - a_n).$$

Using again Lemma 5.2, with $x = a_n^\nu - 2a_n$ (and thus $x+\varepsilon_n/2 = a_n^\nu - a_n$) and say $q=-1$, one gets $\rho(a_n^\nu - a_n) > \rho(a_n^\nu - 2a_n)^{1/2}$, and thereby, for n large enough,

$$\begin{aligned} \mu(R_n) &\ge C_n a_n \rho(a_n^\nu - 2a_n)^{1/2} \ge C_n a_n^\nu \rho(a_n^\nu - 2a_n) \\ &\ge \mu(L_n). \end{aligned}$$

As a consequence, $\mu(R_n) \ge 1/2\mu(I_n)$ for n large enough.
We can now estimate:

$$\int_{R_n} \mu(x-\varepsilon_n, x+\varepsilon_n)^{q-1} \mathrm{d}\mu(x) \le \mu(R_n)^q \le (\mu(I_n)/2)^q = C_1 a_n^{qK\nu} = C_2 \varepsilon_n^{qK\nu} \tag{5.35}$$

with some finite constants C_1, C_2 uniform in n. Putting together (5.34) and (5.35), one gets

$$J_n(q,\varepsilon_n) = \int_{L_n} + \int_{R_n} \le C(q)\varepsilon_n^{qK\nu}. \tag{5.36}$$

3) We turn to $J_k(q,\varepsilon)$, $k \le n-1$. Recall (5.10): $\mu(x-\varepsilon_n, x+\varepsilon_n) \ge \varepsilon_n^{1+\beta/\nu}$ for any $\beta > 0$ for n large enough. Thus,

$$\sum_{k=1}^{n-1} J_k(q,\varepsilon_n) \le \varepsilon_n^{(1+\beta/\nu)(q-1)} \sum_{k=1}^{n-1} \mu(I_k) \le C\varepsilon_n^{(1+\beta/\nu)(q-1)}. \tag{5.37}$$

Finally, (5.28), (5.36) and (5.37) yield

$$I(q,\varepsilon_n) \le C(\varepsilon_n^{qK\nu} + \varepsilon_n^{(1+\beta/\nu)(q-1)}) \tag{5.38}$$

for n large enough depending on β. As a consequence,

$$\tau^-(q) \le \max\left(K\nu|q|, (1+\beta/\nu)(1+|q|)\right)$$

for any $\beta > 0$. It follows that $\tau^-(q) \le \max(K\nu|q|, 1+|q|)$.

It remains to show the converse inequality. First, $I(q,\varepsilon) \ge J_0(q,\varepsilon) = \int_{I_0} \mu([x-\varepsilon, x+\varepsilon])^{q-1} \mathrm{d}\mu(x) \ge C\varepsilon^{q-1}$. Therefore $\tau^-(q) \ge 1+|q|$.

Now, if $\varepsilon \in [a_n/2, a_n^\nu]$, note that for any $x \in I_n$, $\mu([x-\varepsilon, x+\varepsilon]) \le \mu([0, a_n^\nu]) = \sum_{k=0}^n \mu(I_k) \le 2\mu(I_n)$, for large n. Thus, for large n,

$$I(q,\varepsilon) \ge J_n(q,\varepsilon) \ge C\mu(I_n)^q = Ca_n^{K\nu q} \ge C\left(\frac{2}{\varepsilon}\right)^{K\nu|q|} .$$

And if $\varepsilon \in [a_{n+1}^\nu, a_n/2]$, then

$$\begin{aligned} I(q,\varepsilon) &\ge \int_{I_{n+1}} \mu([x-\varepsilon, x+\varepsilon])^{q-1} \mathrm{d}\mu(x) \ge C\mu(I_{n+1})^q = Ca_{n+1}^{K\nu q} \\ &\ge C\left(\frac{1}{\varepsilon}\right)^{K|q|} \ge C\left(\frac{1}{\varepsilon}\right)^{K\nu|q|} . \end{aligned}$$

As a consequence $\tau^-(q) \ge K\nu|q|$ and the Proposition is proved. □

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