ORNSTEIN-ZERNIKE THEORY FOR THE FINITE RANGE ISING MODELS ABOVE T_c

MASSIMO CAMPANINO, DMITRY IOFFE, AND YVAN VELENIK

Abstract. We derive precise Ornstein-Zernike asymptotic formula for the decay of the two-point function $\langle \sigma_0 \sigma_x \rangle_\beta$ in the general context of finite range Ising type models on $\mathbb{Z}^d$. The proof relies in an essential way on the a-priori knowledge of the strict exponential decay of the two-point function and, by the sharp characterization of phase transition due to Aizenman, Barsky and Fernández, goes through in the whole of the high temperature region $\beta < \beta_c$. As a byproduct we obtain that for every $\beta < \beta_c$, the inverse correlation length ξ_β is an analytic and strictly convex function of direction.

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1. Introduction

The classical Ornstein-Zernike (OZ) formula [OZ],[Th] gives a sharp asymptotic description of the density pair correlation functions away from the critical point. The original OZ argument is, essentially, a local limit type computation based on an ad hoc assumptions on the validity of a certain renewal structure of the correlations.

In this work we give a proof of what happens to be the rigorous counterpart of the OZ structural assumption and, subsequently, derive the corresponding asymptotic formula in the context of the high temperature finite range Ising models on $\mathbb{Z}^d$. After describing the model and formulating the results we give a brief heuristic explanation of the OZ formula in terms of the probabilistic local limit theory. In the last part of the Introduction we outline the content of the subsequent technical sections.

We would like to mention that although we discuss here only high temperature Ising models, many of the ideas we develop could be applied in a broader context of various random line type models which possess appropriate uniform exponential mixing properties, such as, for example, low temperature Pirogov-Sinai interfaces in two dimensions. The corresponding study will appear elsewhere.

1.1. **The model.** In this work we are considering the class of Ising models with finite-range ferromagnetic two-body interactions. To each site $x \in \mathbb{Z}^d$ we associate a non-negative real number $J(x) = J(-x) \geqslant 0$; we suppose that there exists $R > 0$ such that $J(x) = 0$ if $|x| > R$. The collection of these coupling constants is denoted by $\boldsymbol{J}$. We consider $\mathbb{Z}^d$ as a graph $(\mathbb{Z}^d, \mathcal{E}_{\boldsymbol{J}})$, with set of vertices $\mathbb{Z}^d$ and set of unoriented edges $\mathcal{E}_{\boldsymbol{J}} \stackrel{\Delta}{=} \big\{(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d \,:\, J(x-y) > 0\big\}$. Let $B \Subset \mathcal{E}_{\boldsymbol{J}}$ and $\beta > 0$. We denote by V_B the set of sites associated to the edges of B: given an edge $e \in \mathcal{E}_{\boldsymbol{J}}$ and a site $x \in \mathbb{Z}^d$ we say that $x \in e$ if x is an endpoint of e. Then the vertex set V_B is defined as, $V_B \stackrel{\Delta}{=} \big\{x \in \mathbb{Z}^d \,:\, \exists\, e \in B \text{ with } x \in e\big\}$. The Gibbs measure on the graph (V_B, B) at inverse temperature β is the probability measure on $\{-1,1\}^{V_B}$ defined by

$$\mu_{B,\beta}(\sigma) = \frac{1}{Z_\beta(B)} \exp\big\{-\beta \sum_{(x,y)\in B} J(x-y)\sigma_x\sigma_y\big\}, \qquad \sigma \in \{-1,1\}^{V_B}\,.$$

A standard argument using Griffiths' second inequality shows that the corresponding infinite-volume measure exists; we denote it by μ_β. Expectation values with respect to the measures $\mu_{B,\beta}$ and μ_β, are denoted respectively by $\langle\,\cdot\,\rangle_{B,\beta}$ and $\langle\,\cdot\,\rangle_\beta$.

The central quantity of our study is the 2-point correlation function

$$g_\beta(x) \stackrel{\Delta}{=} \langle \sigma_0\sigma_x\rangle_\beta .$$

It plays in the models under consideration precisely the role of the density-density correlation function of classical fluids, as can be seen going to the lattice gas interpretation of the model, $n_x = \frac{1}{2}(\sigma_x + 1)$, where the site x is occupied by a particle iff $n_x = 1$.

We also introduce the corresponding inverse correlation length: For any $x \in \mathbb{R}^d$ let

$$\xi_\beta(x) \stackrel{\Delta}{=} -\lim_{k\to\infty} \frac{1}{k} \log g_\beta([kx]), \tag{1.1}$$

where for any $y \in \mathbb{R}^d$, $[y] \in \mathbb{Z}^d$ is the componentwise integer part of y. A standard sub-additivity argument based on Griffiths' second inequality implies that this limit is well-defined and, moreover, letting $\vec{\mathfrak{n}}(x) \stackrel{\Delta}{=} x/|x|$,

$$g_\beta(x) \leqslant e^{-\xi_\beta(x)} = e^{-\xi_\beta(\vec{\mathfrak{n}}(x))\,|x|} \tag{1.2}$$

for all $x \in \mathbb{Z}^d$. It also follows from Griffiths' second inequality that the function ξ_β is convex.

It is important to know for which values of β the 2-point function decays exponentially, i.e. $\xi_\beta > 0$ on $\mathbb{R}^d \setminus \{0\}$. Let $\beta_c = \beta_c(\boldsymbol{J})$ be the inverse critical temperature of the model, i.e.

$$\beta_c \stackrel{\Delta}{=} \sup\{\beta \,:\, \text{there is a unique Gibbs state at inverse temperature } \beta\}.$$

It is well-known and easy to check that $\infty > \beta_c > 0$ when $d \geqslant 2$. An important and highly non-trivial fact is the following theorem due to Aizenman, Barsky and Fernández [ABF], which asserts that ξ_β is in fact an equivalent norm on $\mathbb{R}^d$,

Theorem 1.1. *$\xi_\beta > 0$ if and only if $\beta < \beta_c$.*

This theorem shows that exponential decay of the 2-point function characterizes the high-temperature regime and it provides the basic input for the techniques we develop here.

1.2. **The Results.** Our main result describes sharp Ornstein-Zernike-type asymptotics for the 2-point function of the models introduced in the previous subsection.

Theorem A. *Let $\beta < \beta_c$. Uniformly in $|x| \to \infty$*

$$\langle \sigma_0 \sigma_x \rangle_\beta \;=\; \frac{\Phi_\beta\left(\vec{\mathfrak{n}}(x)\right)}{\sqrt{|x|^{d-1}}}\, \mathrm{e}^{-|x|\xi_\beta(\vec{\mathfrak{n}}(x))}\left(1 + o(1)\right), \tag{1.3}$$

where $\vec{\mathfrak{n}}(x)$ is the unit vector in the direction of x; $\vec{\mathfrak{n}}(x) = x/|x|$ and Φ_β is a strictly positive locally analytic function on $\mathbb{S}^{d-1}$.

As a byproduct of the techniques employed for the proof of Theorem A we deduce that the inverse correlation length ξ_β is an analytic and strictly convex function of the direction. In order to formulate this in a precise way recall that ξ_β is a convex, homogeneous of order one strictly positive (on $\mathbb{R}^d \setminus \{0\}$) function. As such it is an equivalent norm on $\mathbb{R}^d$ and it is the support function of the compact convex set

$$\mathbf{K}_\beta \;=\; \bigcap_{n\in\mathbb{S}^{d-1}} \left\{ t \in \mathbb{R}^d \,:\, (t,n)_d \leqslant \xi_\beta(n) \right\} \tag{1.4}$$

with a non-empty interiour $0 \in \mathrm{int}\mathbf{K}_\beta$.

Theorem B. *Let $\beta < \beta_c$. Then $\mathbf{K}_\beta$ has a locally analytic strictly convex boundary $\partial\mathbf{K}_\beta$. Furthermore, the Gaussian curvature κ_β of $\partial\mathbf{K}_\beta$ is uniformly positive,*

$$\bar{\kappa}_\beta \stackrel{\Delta}{=} \min_{t\in\partial\mathbf{K}_\beta} \kappa_\beta(t) \ > \ 0. \tag{1.5}$$

In two dimensions $\mathbf{K}_\beta$ is reminiscent of the Wulff shape (by duality it is precisely the low temperature Wulff shape in the case of the nearest neighbour interactions). The inequality (1.5) is called then the positive stiffness condition, and one of the consequences of Theorem B is the validity of the following strict triangle inequality [Io1],[PV2]: Uniformly in $x, y \in \mathbb{R}^2$,

$$\xi_\beta(u) + \xi_\beta(v) - \xi_\beta(u+v) \ \geqslant \bar{\kappa}_\beta\left(|u| + |v| - |u+v|\right).$$

In two dimensions $\bar{\kappa}_\beta$ is the minimal radius of curvature of the Wulff shape $\partial\mathbf{K}_\beta$.

In the 2D nearest neighbour case one, using the duality transformation, can apply our results to study fluctuations of the $\pm$ interface up to the critical temperature. Similarly, an adjustment of our approach to the general Pirogov-Sinai context in two dimensions should, in principle, lead to a comprehensive description of the fluctuation structure of one-dimensional low temperature interfaces. The corresponding results will appear elsewhere.

Along with $\mathbf{K}_\beta$ we shall consider the set

$$\mathbf{U}_\beta \ \stackrel{\Delta}{=} \ \left\{x \in \mathbb{R}^d \ : \ \xi_\beta(x) \leqslant 1\right\} \ = \ \left\{x \in \mathbb{R}^d \ : \ \max_{t\in\mathbf{K}_\beta}(t, x)_d \leqslant 1\right\}. \tag{1.6}$$

Of course, $\mathbf{U}_\beta$ is just the unit ball in the ξ_β-norm. It is bounded, convex , and has non-empty interiour for every $\beta < \beta_c$. Furthermore, the polar restatement of Theorem B implies that the boundary $\partial\mathbf{U}_\beta$ is also locally analytic and strictly convex (c.f. [CIo]).

Results similar or even stronger than Theorem A have been obtained in the perturbative regime $\beta << 1$ [AK],[MZ] and [P-L]. Most recently, the OZ asymptotic has been recovered for the interfaces of the (very) low temperature 2D Blume-Capel model in [HK].

Of course, for the particular case of nearest-neighbour interactions in dimension 2, the Ornstein-Zernike behaviour of the 2-point function is well-known, through explicit computations, see e.g. [MW]. However, non-perturbative, dimension independent results of this type have previously been restricted to simpler models: Self-avoiding walks have been studied in [CC] (for directions x close to the axis) and [Io2] (general directions); Bernoulli percolation model has been analyzed in [CCC] (for directions x close to the axis) and [CIo] (general directions).

¿From a different perspective, Alexander [Al] proved non-perturbative lower bounds on two point functions with almost the correct order on the prefactor near the decay exponent. Though being weaker than the sharp asymptotics presented in Theorem A and failing to capture the fluctuation picture behind the phenomenon, these results have the advantage that they can be applied to a large variety of models. The core renormalization procedure which we develop in Section 2 is inspired by the ideas of [Al] (see also the references therein to his previous works).

1.3. **Probabilistic picture behind the OZ formula.** Let us first explain the order $|x|^{-(d-1)/2}$ of the prefactor in (1.3): Consider a random walk $S_n = V_1 + \dots V_n$ on $\mathbb{Z}^d$ with i.i.d increments V_i. Let us assume that the moment generating function $\mathbb{E}\mathrm{e}^{(t,V_i)_d}$ is finite in a neighbourhood of zero in $\mathbb{R}^d$, that the distribution of V_i is non-lattice and that the walk S_n is forward in the following sense: any point x from the support of the distribution of V_i has a positive projection on $\mu \triangleq \mathbb{E}V_i$. Given a point x on the direction of the principle advance of S_n; $\min_{t>0}|x - t\mu| < 1$, the probability that S_n "steps" on x is given by

$$\sum_{n=1}^{\infty} \mathbb{P}(S_n = x). \tag{1.7}$$

By the usual local limit theorems the term $-\log \mathbb{P}(S_n = x)$ is of the order $|x - n\mu|^2/n$. Hence, the main contribution to the above sum comes from roughly $\sqrt{|x|}$ terms n around $n_0 = |x|/|\mu|$. In other words, up to asymptotically (with $|x| \to \infty$) negligible terms, the sum in (1.7) is given by the Gaussian summation formula,

$$\frac{c_1}{\sqrt{|x|^d}} \sum_{n=1}^{\infty} \exp\left\{-c_2 \frac{(n-n_0)^2}{|x|}\right\} = \frac{c_3}{\sqrt{|x|^{d-1}}}.$$

Of course, it is not difficult to give the exact formula for c_3 in terms of μ and the covariance matrix of V_i.

The above sketch almost literally corresponds to the last step of the proof of the OZ asymptotic formula in the case of the Bernoulli bond percolation in [CIo]. The main effort in the latter paper was to show that the percolation cluster from the origin to a (distant) point $x \in \mathbb{Z}^d$ could be typically split into a density of irreducible pieces with the displacements along the endpoints of these pieces playing the role of the i.i.d steps $V_1, V_2 \dots$ of the random walk S_n.

In the case of Ising models the two-point function $g_\beta(x)$ also admits a geometric random line type representation. Unlike the Bernoulli percolation case, however, different portions of this random line interact, whatever splitting rules we employ. In other words in the induced random walk picture the increments $V_1, V_2, \dots$ are dependent. Local limit description of dependent variables is, in general, a rather delicate matter. Fortunately, random lines which show up in the representation of the Ising two-point function possess a certain exponential decoupling property. The renormalization procedure which we develop in Section 2 gives rise to an irreducible splitting of the random path in such a way, that the dependence between various sub-paths of the splitting has already a uniform exponential decay. The resulting system fits the framework of the Ruelle shift operator on a countable alphabet (of irreducible sub-paths), and, as we shall see in the sequel, the associated local limit results are precisely of the same analytic nature as in the independent case.

1.4. **Organization of the paper.** In Section 2 we develop a renormalization procedure leading to an irreducible decomposition (2.13) of the two point function $g_\beta(x) = \langle \sigma_0 \sigma_x \rangle_\beta$. This decomposition is translated to the Ruelle context in Section 3, where we prove both Theorem A and Theorem B. The proofs rely on general spectral properties of the Ruelle operator on countable alphabets described in Section 4 and on the local limit analysis of the associated observables which is developed in the concluding Section 5

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2. Renormalization

We start by setting up the notation and recalling the well known random line representation of the two-point function $g_\beta(x)$. On the microscopic level these random paths wiggle in a messy way. Our main renormalization result Theorem 2.1 asserts, however, that on sufficiently large scales the random path from 0 to x exhibits, with an overwhelming probability, a regular behaviour, in a sense that it could be split into a density of irreducible pieces. The space of irreducible paths is defined in Subsection 2.6 and the corresponding irreducible representation of $\langle\sigma_0\sigma_x\rangle_\beta$ is given by the formula (2.13) there. The role of the cone confinement condition in **(P2)-(P4)** will become apparent in Section 5: by the bound (3.1) it is precisely what one needs in order to represent the system in terms of the action of Ruelle operator with a uniformly Hölder continuous potential.

2.1. **The random-line representation.** Recall that given a set of edges $B \Subset \mathcal{E}_{\boldsymbol{J}}$ we have defined the associated set of vertices as $V_B \triangleq \{x \in \mathbb{Z}^d : \exists\, e \in B \text{ with } x \in e\}$. For any vertex $x \in V_B$, we define the *index* of x in B by $\mathrm{ind}(x,B) \triangleq \sum_{e\in B} 1_{e\ni x}$ (as before, $x \in e$ means that x is an endpoint of e). The *boundary* of B is defined by $\partial B \triangleq \{x \in V_B : \mathrm{ind}(x,B) \text{ is odd}\}$.

At each $x \in \mathbb{Z}^d$, we fix (in an arbitrary way) an ordering of the x-incident edges of the graph:

$$B_x \triangleq \{e \in B : \mathrm{ind}(x,\{e\}) > 0\} = \{e_1^x \dots, e^x_{\mathrm{ind}(x,B)}\},$$

and for two incident edges $e = e_i \in B_x$, $e' = e_j \in B_x$ we say that $e \leqslant e'$ if the corresponding inequality holds for their sub-indices ; $i \leqslant j$.

Using the identity $e^{\beta J(e)\sigma_t\sigma_{t'}} = \cosh(\beta J(e))\bigl(1 + \sigma_t\sigma_{t'}\tanh(\beta J(e))\bigr)$, we obtain the following expression for the 2-point function of the model in B,

$$\langle\sigma_x\sigma_y\rangle_{B,\beta} = Z_\beta(B)^{-1} \sum_{\substack{D\subset B\\ \partial D=\{x,y\}}} \prod_{e\in D} \tanh\beta J(e)\,.$$

¿From $D \subset B$ with $\partial D = \{x,y\}$, we would like to extract a "self-avoiding path". We use the following procedure:

STEP 1 Set $t'_0 = y$, $j = 0$ and $\Delta_0 = \emptyset$.

STEP 2 Let $e'_j = (t'_j, t'_{j+1})$ be the first edge in $B_{t'_j} \setminus \Delta_j$ (in the ordering of $B_{t'_j}$ fixed above) such that $e_j \in D$. This defines t'_{j+1}.

STEP 3 Set $\Delta_{j+1} = \Delta_j \cup \Bigl\{e \in B_{t'_j} : e \leqslant e'_j\Bigr\}$. If $t'_{j+1} = x$, then set $n = j+1$ and stop. Otherwise update $j \triangleq j+1$ and return to STEP 2.

This procedure produces a sequence $(t'_0 \equiv y, \dots, t'_n \equiv x)$. Let $t_k \triangleq t'_{n-k}$ and $e_k \triangleq e'_{n-k}$. We, thus, constructed a path $\lambda \triangleq \lambda(D) \triangleq (t_0 \equiv x, \dots, t_n \equiv y)$ such that

- $(t_i, t_{i+1}) \in B$, $i = 0, \dots, n-1$,
- $(t_i, t_{i+1}) \neq (t_j, t_{j+1})$ for $i \neq j$,

(but $t_i = t_j$ for $i \neq j$ is allowed); such a sequence will be called a backward edge-self-avoiding line from x to y [1]. The construction also yields a set of edges

$$\Delta(\lambda) \equiv \Delta_n = \bigcup_{i=1}^{n} \{e \in B_{t_i} : e \leqslant e_i\}\,.$$

Notice that $\Delta(\lambda)$ depends only on λ (and the order chosen for the edges). We use the convenient notation $\sum_{\lambda:\, x\mapsto y}$ to represent the summation over all (backward) self-avoiding

[1] We prefer the backward construction of the line λ because it happens to be more convenient when reducing to Ruelle's formalism in Subsection 3.2.

lines from x to y. Observe that for any $D \subset B$ with $\partial D = \{x, y\}$, $\lambda(D) = \lambda$ if and only if (considering λ as a set of edges) $\lambda \subset D$ and $(\Delta(\lambda) \setminus \lambda) \cap D = \emptyset$. We can therefore write

$$\langle \sigma_x \sigma_y \rangle_{B,\beta} = \sum_{\lambda : x \to y} q_{B,\beta}(\lambda) , \tag{2.1}$$

where, writing $w(\lambda) = \prod_{e \in \lambda} \tanh(\beta J(e))$,

$$q_{B,\beta}(\lambda) = w(\lambda) \, \frac{Z_\beta(B \setminus \Delta(\lambda))}{Z_\beta(B)} . \tag{2.2}$$

Equations (2.1) and (2.2) define the *random-line representation* for the 2-point function of the Ising model on the graph $\mathcal{G}$. It has been studied in detail in [PV1, PV2] and is essentially equivalent (though the derivations are quite different) to the random-walk representation of [Az]. We'll need a version of this representation on the infinite graph $(\mathbb{Z}^d, \mathcal{E}_{\boldsymbol{J}})$. To this end, we use the following result ([PV2], Lemmas 6.3 and 6.9): For all $\beta < \beta_c$,

$$\langle \sigma_x \sigma_y \rangle_\beta = \sum_{\lambda : x \mapsto y} q_\beta(\lambda) , \tag{2.3}$$

where $q_\beta(\lambda) \stackrel{\Delta}{=} \lim_{B_n \nearrow \mathcal{E}_{\boldsymbol{J}}} q_{B_n}(\lambda)$ is well defined.

We finally need some rules on how to cut a random-line into pieces. Let $\lambda = (t_0, t_1, \ldots, t_n)$, $z \in \lambda$ and let $t_{k(z)}$ be the last hitting of z by λ. We write $\lambda_<(z) \stackrel{\Delta}{=} (t_0 \ldots, t_{k(z)})$ and $\lambda_>(z) \stackrel{\Delta}{=} (t_{k(z)}, \ldots, t_n)$; notice that (as a set of edges) $\lambda_<(z) \cap \Delta(\lambda_>(z)) = \emptyset$. By the notation $\lambda = \lambda_1 \amalg \lambda_2$, we mean that there exists $z \in \lambda$ such that $\lambda_1 = \lambda_<(z)$ and $\lambda_2 = \lambda_>(z)$. Concatenation of more than two paths is defined by iterating this procedure, e.g. $\lambda_1 \amalg \lambda_2 \amalg \lambda_3 = (\lambda_1 \amalg \lambda_2) \amalg \lambda_3$.

We then have the following BK-type inequality:

$$\sum_{\substack{\lambda : x \to y \\ \lambda \ni t}} q_\beta(\lambda) \leqslant \sum_{\lambda_1 : x \to t} q_\beta(\lambda_1) \sum_{\lambda_2 : t \to y} q_\beta(\lambda_2) . \tag{2.4}$$

Indeed, by Griffiths' inequality,

$$\begin{aligned} \sum_{\substack{\lambda : x \to y \\ \lambda \ni t}} q_{\mathcal{E},\beta}(\lambda) &= \sum_{\lambda_2 : t \to y} q_{\mathcal{E},\beta}(\lambda_2) \sum_{\substack{\lambda : x \to y \\ \lambda = \lambda_1 \amalg \lambda_2}} q_{\mathcal{E} \setminus \Delta(\lambda_2),\beta}(\lambda_1) \; = \sum_{\lambda_2 : t \to y} q_{\mathcal{E},\beta}(\lambda_2) \, \langle \sigma_x \sigma_t \rangle_{\mathcal{E} \setminus \Delta(\lambda_2),\beta} \\ &\leqslant \sum_{\lambda_2 : t \to y} q_{\mathcal{E},\beta}(\lambda_2) \, \langle \sigma_x \sigma_t \rangle_{\mathcal{E},\beta} \; = \sum_{\lambda_1 : x \to t} q_{\mathcal{E},\beta}(\lambda_1) \sum_{\lambda_2 : t \to y} q_{\mathcal{E},\beta}(\lambda_2) , \end{aligned}$$

2.2. **K-skeletons.** We coarse-grain microscopic self-avoiding lines via an appropriate covering by inflated $\mathbf{U}_\beta$ shapes (see (1.6)): Given a self-avoiding line $\lambda = (t_0, ..., t_n)$ and a positive number $K > 0$ construct the K-skeleton $\lambda_K = (x_0, ..., x_N)$ of λ as follows (Figure 1):

STEP 1 Set $x_0 = t_0$, $j = 0$ and $k = 0$.

STEP 2 If the rest of the line $\{t_{j+1}, ..., t_n\} \subseteq K\mathbf{U}_\beta(x_k)$, then set $N = k + 1$ and $x_N = t_n$ and stop. Otherwise proceed to STEP 3.

STEP 3 Find $j^* = \min\{i > j \, : \, t_i \notin K\mathbf{U}_\beta(x_k)\}$. Set $x_{k+1} = t_{j^*}$. Update $j \stackrel{\Delta}{=} j^*$, $k \stackrel{\Delta}{=} k + 1$ and return to STEP 2.

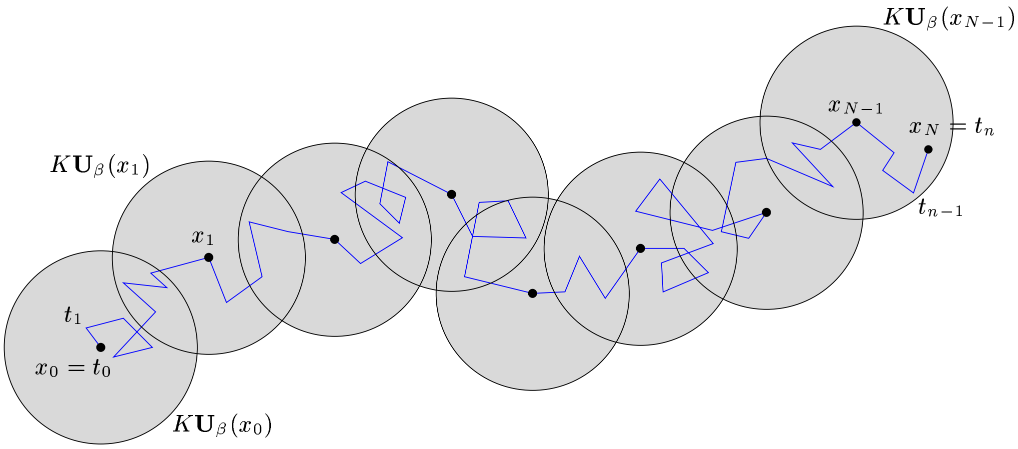

FIGURE 1. A contour $\lambda = (t_0, \ldots, t_n)$ and its K-skeleton $\lambda_K = (x_0, \ldots, x_N)$.

Let us use the notation $\lambda \overset{K}{\sim} \lambda_K$ to stress the fact that λ_K is the K-skeleton of λ. As in the case of paths we say that a skeleton $\lambda_K = (x_0, ..., x_N)$ connects its endpoints, $\lambda_K : x_0 \mapsto x_N$.

Of course a particular skeleton $\lambda_K = (x_0, ..., x_N)$ can be compatible with many different self-avoiding paths, and we introduce the weight

$$q_\beta(\lambda_K) \;=\; \sum_{\lambda \overset{K}{\sim} \lambda_K} q_\beta(\lambda)\,.$$

On any renormalization scale K the BK-inequality (2.4) implies:

$$q_\beta(\lambda_K) \;\leqslant\; \prod_{l=1}^{N} g_\beta(x_l - x_{l-1}) \;\leqslant\; \mathrm{e}^{-(N-1)K}. \tag{2.5}$$

2.3. **The surcharge inequality.** For $t \in \partial\mathbf{K}_\beta$ let us define the surcharge function $\mathfrak{s}_t : \mathbb{Z}^d \mapsto \mathbb{R}_+$ as $\mathfrak{s}_t(x) = \xi_\beta(x) - (t, x)_d$. Then, given a skeleton $\lambda_K = (x_0, ..., x_N)$ we define its surcharge as $\mathfrak{s}_t(\lambda_K) = \sum \mathfrak{s}_t(x_{k+1} - x_k)$. By the first of the inequalities in (2.5),

$$q_\beta(\lambda_K) \;\leqslant \mathrm{e}^{-(t, x_N)_d - \mathfrak{s}_t(\lambda_K)}, \tag{2.6}$$

uniformly in $t \in \partial\mathbf{K}_\beta$, scales K and in K-skeletons λ_K. Furthermore, the following crucial surcharge inequality holds:

Lemma 2.1. *For any $\nu > 0$ there exists a finite renormalization scale $K_0 = K_0(\nu)$ such that*

$$\sum_{\substack{\lambda_K : 0 \mapsto x \\ \mathfrak{s}_t(\lambda_K) \,\geqslant\, 2\nu|x|}} q_\beta(\lambda_K) \;\leqslant c_1(\beta)\mathrm{e}^{-(t,x)_d - \nu|x|}, \tag{2.7}$$

uniformly in $t \in \partial\mathbf{K}_\beta$, $K \geqslant K_0$ and $x \in \mathbb{Z}^d$

Proof. There are at most $c_2(d)K^{d-1}$ choices for each incoming skeleton step. Thus, there are at most $\exp\{c_3(d)N \log K\}$ different K-skeletons of N steps emerging from zero. By (2.5) we can restrict attention only to those skeletons $\lambda_K : 0 \mapsto x$ which comprise at most

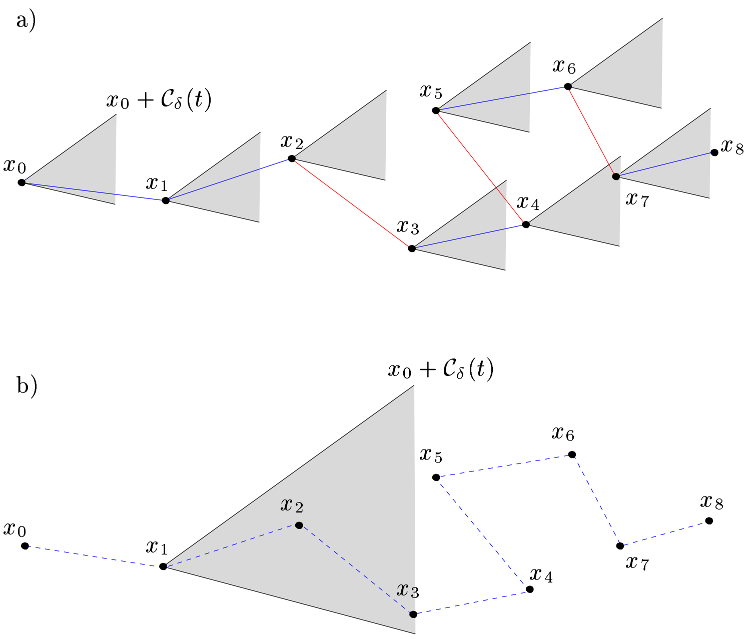

FIGURE 2. a) A skeleton $\lambda_K = (x_0, \dots, x_8)$. The increments $[x_2, x_3]$, $[x_4, x_5]$, $[x_6, x_7]$ are backtracks. Thus, $\#^{\text{back}}_{t,\delta}(\lambda_K) = 3$. b) The same skeleton $\lambda_K = (x_0, \dots, x_8)$. The vertex x_1 is a cone point of λ_K.

$N \leqslant c_4|x|/K$ steps. Choosing K_0 so large that $c_3 c_4 \log K_0/K_0 < \nu$ we, in view of the surcharge bound (2.6), arrive to the conclusion of the lemma. □

There are two types of skeletons with large surcharges which we need to control in order to implement the path decomposition procedure:

2.4. **Forward cones and backtracks.** Let us fix $\delta \in (0, 1/2)$. For any $t \in \partial \mathbf{K}_\beta$ define the forward cone

$$\mathcal{C}_\delta(t) \;=\; \left\{ x \in \mathbb{Z}^d \,:\, \mathfrak{s}_t(x) < \delta \xi_\beta(x) \right\}.$$

Given a K-skeleton $\lambda_K = (x_0, ..., x_N)$ let us define the number $\#^{\text{back}}_{t,\delta}(\lambda_K)$ of (t,δ)-backtracks (Figure 2 a)) of λ_K,

$$\#^{\text{back}}_{t,\delta}(\lambda_K) \;=\; \#\left\{ l \,:\, x_{l+1} - x_l \not\in \mathcal{C}_\delta(t) \right\}.$$

If $x_{l+1} - x_l \in \mathcal{C}_\delta(t)$, we shall say that x_l is a forward point of λ_K.

Notice that the surcharge price of λ_K satisfies

$$\mathfrak{s}_t(\lambda_K) \;\geqslant \delta K \left(\#^{\text{back}}_{t,\delta}(\lambda_K) - 1 \right). \tag{2.8}$$

2.5. **Cone points of skeletons.** Given a skeleton $\lambda_K = (x_0, ..., x_N)$ let us say that x_l is a (t,δ)-cone point of λ_K if (Figure 2 b))

$$\{x_{l+1}, ..., x_N\} \;\subset\; x_l + \mathcal{C}_\delta(t).$$

Of course, each cone point of λ_K is, in particular, a forward point. If a skeleton λ_K contains points which do not satisfy the above condition, define

$l_1 \;=\; \min\{j \,:\, x_j \text{ is not a } (t,\delta) - \text{cone point of } \lambda_K\}$
$r_1 \;=\; \min\{j > l_1 \,:\, x_j - x_{l_1} \not\in \mathcal{C}_\delta(t)\}$
$l_2 \;=\; \min\{j \geqslant r_1 \,:\, x_j \text{ is not a } (t,\delta) - \text{cone point of } \lambda_K\}$

$r_2 \;=\; \min\left\{ j > l_2 \,:\, x_j - x_{l_1} \notin \mathcal{C}_\delta\left(t\right)\right\}$
$\cdots$

Let us say that j is a (t,δ)-marked point of λ_K if it belongs to the (disjoint) union; $j \in \bigvee_k [l_k, ..., r_k)$. Notice that each point of λ_K which is not marked is, automatically, a (t,δ)-cone (or simply a cone point, if no ambiguity with respect to t and δ arises) point of λ_K. We use $\#^{\rm mark}_{t,\delta}\left(\lambda_K\right)$ to denote the number of all the marked points of λ_K.

Lemma 2.2. *Uniformly in K, λ_K and $t \in \partial\mathbf{K}_\beta$, the surcharge cost $\mathfrak{s}_t(\lambda_K)$ is controlled in terms of the number of marked points as*

$$\mathfrak{s}_t(\lambda_K) \;\geqslant\; \tfrac{1}{7}\,\delta K \#^{\rm mark}_{t,\delta}\left(\lambda_K\right). \tag{2.9}$$

Proof. Of course, $\#^{\rm mark}_{t,\delta}\left(\lambda_K\right) = \sum_k (r_k - l_k)$. We claim that for every marked interval $[l_k, ..., r_k)$,

$$\sum_{j=l_k+1}^{r_k} \mathfrak{s}_t(x_j - x_{j-1}) \;\geqslant\; \tfrac{1}{7}\,\delta K(r_k - l_k). \tag{2.10}$$

Indeed, consider two cases:
CASE 1 $(t, x_{r_k} - x_{l_k}) \geqslant \frac{2}{7}K(1-\delta)(r_k - l_k)$. Then, since $x_{r_k} - x_{l_k}$ is a back-track, and since $\mathfrak{s}_t$ evidently inherits from ξ_β convexity and homogeneity of order one,

$$\sum_{j=l_k+1}^{r_k} \mathfrak{s}_t(x_j - x_{j-1}) \geqslant \mathfrak{s}_t(x_{r_k} - x_{l_k}) \;\geqslant \delta\,\xi_\beta(x_{r_k} - x_{l_k}) \geqslant \delta\,(t, x_{r_k} - x_{l_k})_d \geqslant \delta\,\tfrac{1}{7}\,K\,(r_k - l_k).$$

CASE 2 $(t, x_{r_k} - x_{l_k}) < \frac{2}{7}K(1-\delta)(r_k - l_k)$. Notice first that $\mathbf{K}_\beta$ is symmetric, so that $t \in \partial\mathbf{K}_\beta \implies -t \in \partial\mathbf{K}_\beta$. Therefore, the worst possible displacement of the t-projection satisfies (recall that we are assuming $K \gg R$)

$$\min_{i\in\{l_k+1,\ldots,r_k\}} (t, x_i - x_{i-1}) > -2K\,.$$

This allows us to bound below the number N_k of increments $x_j - x_{j-1}$ from the marked interval $j = l_k + 1, ..., r_k$ that are (t,δ)-backtracking. Indeed,

$$(t, x_{r_k} - x_{l_k}) \geqslant (r_k - l_k - N_k)\,(1-\delta)\,K - N_k\,2K\,,$$

which gives, since we have fixed the value of $\delta \in (0, 1/2)$,

$$N_k \geqslant \tfrac{1}{7}\,(r_k - l_k).$$

□

2.6. **Space of irreducible paths.** Given $t \in \partial\mathbf{K}_\beta$ and a path $\lambda = (i_0, \ldots, i_n)$ let us say that i_l; $0 < l < n$, is a t-break point of λ if $i_l \neq i_k$ for all $k \neq l$ and

$$\tilde\lambda \cap \{i_l + \mathcal{H}_t\} \;=\; \{i_k\}\,,$$

where $\mathcal{H}_t$ is the t-orthogonal hyper-plane passing through zero; $\mathcal{H}_t = \left\{x \in \mathbb{R}^d \,:\, (x,t)_d = 0\right\}$, and $\tilde\lambda$ is the embedding of λ with all its edges into $\mathbb{R}^d$. Alternatively, i_l is a t-break point of λ if

$$\max_{k<l} (i_k, t)_d \;<\; (i_l, t)_d \;<\; \min_{k<l} (i_k, t)_d.$$

In addition, given a renormalization skeleton scale K and a forward cone parameter $\delta > 0$, let us say that a break point i_l of $\lambda = (i_0, \ldots, i_n)$ is (t, K, δ)-correct if (Figure 3)

$$\{i_{l+1}, \ldots, i_n\} \;\subseteq\; 2K\mathbf{U}_\beta\left(i_l\right) + \mathcal{C}_\delta\left(t\right).$$

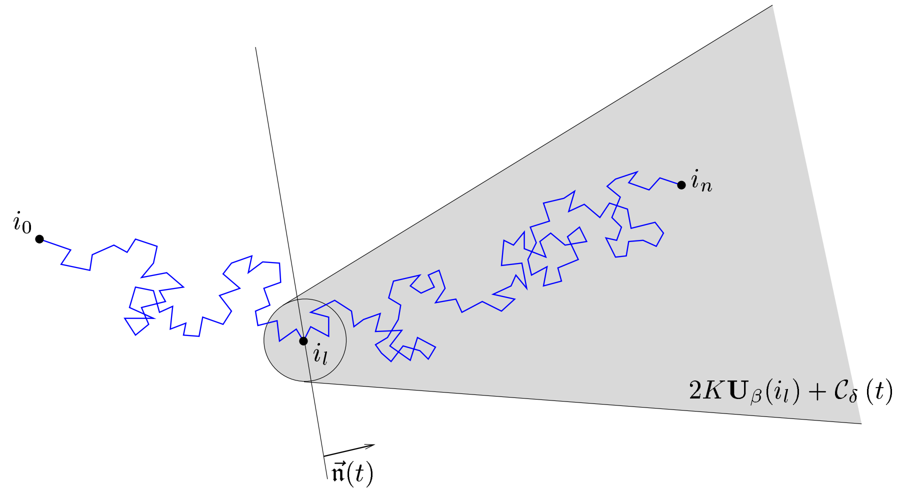

FIGURE 3. i_l is a (t, K, δ)-correct break point of the contour $\lambda = (i_0, \dots, i_n)$.

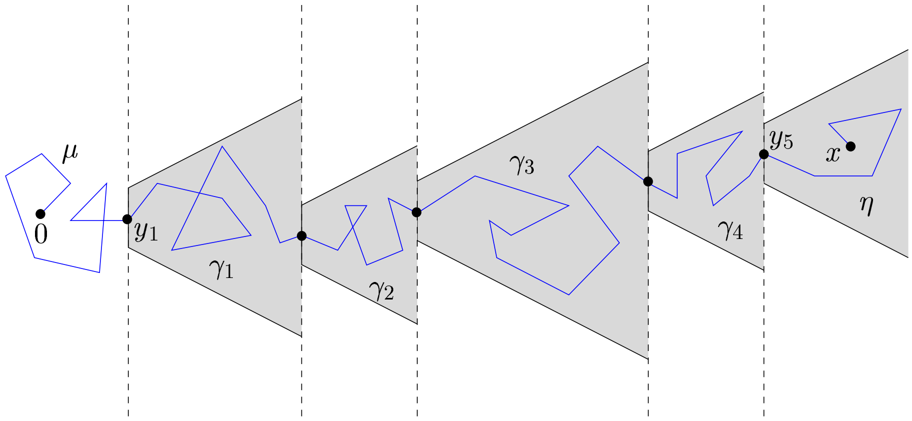

FIGURE 4. The splitting $\lambda = \mu \amalg \gamma_1 \amalg \gamma_2 \amalg \gamma_3 \amalg \gamma_4 \amalg \eta$ into irreducible components.

In particular, if one can find some (t, δ)-cone point x_j of the skeleton λ_K of λ such that the break point i_l is on the piece of λ between x_j and i_n, and $i_l \in K\mathbf{U}_\beta(x_j)$, then i_l is automatically (t, K, δ)-correct.

Theorem 2.1. *Fix a forward cone parameter $\delta \in (0, 1/2)$. There exist a renormalization scale K_0 and positive numbers $\epsilon = \epsilon(\delta, \beta) > 0$, $\nu = \nu(\delta, \beta) > 0$ and $M = M(\beta) < \infty$, such that for all $K \geqslant K_0$, the upper bound*

$$\sum_{\lambda : 0 \mapsto x} q_\beta(\lambda) \, \mathbb{I}_{\left\{ \substack{\lambda \text{ has less than } \epsilon|x|/K \\ (t, K, \delta)\text{-correct break points}} \right\}} \leqslant M \mathrm{e}^{-(t,x)_d - \nu|x|}, \tag{2.11}$$

holds uniformly in the dual directions $t \in \partial\mathbf{K}_\beta$ and in the end-points $x \in \mathbb{Z}^d$.

We relegate the proof of the theorem to the next subsection. Notice, however, that by (1.2) and (2.1) the bound (2.11) is trivial whenever t and x are such that x lies outside the cone $\mathcal{C}_{\nu'}(t)$; $\nu' = \nu \max_{y \neq 0} |y| / \xi_\beta(y)$.

For the forward directions $x \in \mathcal{C}_{\nu'}(t)$ Theorem 2.1 suggests the splitting of a path $\lambda : 0 \mapsto x$ (Figure 4):

$$\lambda \;=\; \mu \amalg \gamma_1 \amalg, \dots, \amalg \gamma_m \amalg \eta, \tag{2.12}$$

which possesses the following set of properties **P1** – **P4**:

(P1) All the points $y_1, \dots, y_{m+1}$ are break points of λ.

(P2) $\eta \subseteq 2K\mathbf{U}_\beta(y_{m+1}) + \mathcal{C}_\delta(t)$ and η does not contain any (t, K, δ)-correct break point.

(P3) For any $l = 1, \dots, m$, the path γ_l does not contain (t, K, δ)-correct break points, but

$$\gamma_l \subset 2K\mathbf{U}_\beta(y_l) + \mathcal{C}_\delta(t).$$

(P4) μ does not contain (t, K, δ)-correct break points.

Notice that the successive application of **P1**-**P4** gives an unambiguous construction of the decomposition (2.12).

Notice, furthermore, that the paths γ_l (or, more precisely, the shifted paths $\gamma_l - y_l$) belong to the following basic countable set $S = S(t, K, \delta)$ of irreducible paths:

Definition (The basic set of irreducible paths S). Let us say that a path $\gamma = (i_0, ..., i_k) \in S$ if

1. $i_0 = 0$ and $(i_0, t) < (i_l, t) < (i_k, t)$ for all $l = 1, \dots, k-1$.
2. $\gamma \subset 2K\mathbf{U}_\beta(i_0) + \mathcal{C}_\delta(t)$.
3. γ does not contain (t, K, δ)-correct break points.

Given a path $\lambda = (t_0, \dots, t_n)$ let us define the displacement $V(\lambda) \in \mathbb{Z}^d$ along λ as the difference between the endpoints $V(\lambda) = t_n - t_0$. By Theorem 2.1, the splitting (2.12) gives rise to the following irreducible representation of the two point function $g_\beta(x) = \langle \sigma_0 \sigma_x \rangle_\beta$: Let ϵ be small enough and $t \in \partial\mathbf{K}_\beta$ be such that $x \in \mathcal{C}_\epsilon(t)$. Then

$$g_\beta(x)\left(1 + \mathrm{o}\left(\mathrm{e}^{-\nu|x|}\right)\right) \;=\; \sum_{\mu,\eta} \sum_{m=0}^{\infty} \sum_{\substack{\gamma_1,\dots,\gamma_m \in \mathcal{S}:\\ V(\mu)+V(\gamma_1)+\dots+V(\eta)=x}} q_\beta\left(\mu \amalg \gamma_1 \amalg \cdots \amalg \gamma_m \amalg \eta\right). \tag{2.13}$$

2.7. **Proof of Theorem 2.1.** The proof is, actually, a modification of the argument developed in [CIo] in the context of the Bernoulli bond percolation. It is based on the skeleton calculus of the preceding subsections and on the following simple finite energy type property: There exists a positive constant $c_5 > 0$, such that for any set $B \subset \mathcal{E}_\mathbf{J}$ and any path $\lambda \subset B$,

$$q_{B,\beta}(\lambda) \;\geqslant \mathrm{e}^{-c_5|\gamma|}. \tag{2.14}$$

Notice that all the estimates we employ in the course of the proof hold uniformly in $t \in \partial\mathbf{K}_\beta$, and so will the result.

Fix a number $\eta \in (0, 1)$.

Definition Given a point $x \in \mathcal{C}_\delta(t)$, a skeleton scale K and a K-skeleton $\lambda_K : 0 \mapsto x$, let us say that λ_K is η-admissible if the number of (t, δ)-marked points

$$\#_{t,\delta}^{\mathrm{mark}}(\lambda_K) < \eta|x|/K.$$

By the surcharge inequality (2.7) and the surcharge function lower bound (2.9), there exists a finite scale $K_0 = K_0(\eta, \delta)$, such that

$$\sum_{\substack{\lambda_K : 0 \mapsto x \\ \lambda \text{ is not } \eta\text{-admissible}}} q_\beta(\lambda_K) \leqslant c_1 \exp\left\{ -\frac{\delta\eta}{14}|x| - (t, x)_d \right\}, \tag{2.15}$$

uniformly in the scales $K \geqslant K_0$ and in $x \in \mathcal{C}_\delta(t)$.

Eventually, we are going to pick up η sufficiently small, which, as the arguments below show, will ensure that up to an exponentially small correction the paths λ compatible with η-admissible skeletons contain a density of (t, K, δ)-correct break points, as has been asserted in (2.11) of Theorem 2.1.

In the sequel we shall tacitly assume that the running skeleton scale K is much larger than the range of the interaction R; $K \gg R$. On every such skeleton scale K let us slice $\mathbb{R}^d$ into the disjoint union of t-oriented slabs: Let $\vec{\mathfrak{n}}(t) = t/|t|$ be the unit vector in the direction of t

$$\mathbb{R}^d = \bigvee_{l=-\infty}^{\infty} \left(l \cdot 8K\vec{\mathfrak{n}}(t) + \mathcal{S}_K(t) \right), \tag{2.16}$$

where the slab $\mathcal{S}_K(t)$ is defined via:

$$\mathcal{S}_K(t) = \left\{ u \in \mathbb{R}^d : 0 \leqslant (\mathfrak{n}(t), u)_d < 8K \right\}.$$

For every $x \in \mathcal{C}_\delta(t)$,

$$(t, x)_d \geqslant (1-\delta)\xi_\beta(x) \geqslant (1-\delta)|x| \min_{\vec{\mathfrak{n}} \in \mathbb{S}^1} \xi_\beta(\vec{\mathfrak{n}}) \triangleq c_6(\beta)(1-\delta)|x|. \tag{2.17}$$

Furthermore, by (1.6) (and in view of the assumption $R \ll K$) $(t, x_{k+1} - x_k)_d < 2K$, whenever $x_{k+1} - x_k$ is a skeleton increment on the K-th skeleton scale. As a result, each skeleton $\lambda_K : 0 \mapsto x$ intersects at least $\left[\frac{c_6(\beta)(1-\delta)|x|}{8K}\right]$ subsequent slabs in the partition (2.16). On the other hand, if λ_K is, in addition, η-admissible, then at most $\eta|x|/K$ of these slabs can possibly contain marked points of λ_K . The two latter remarks prescribe the choice of the number η:

$$0 < \eta < \frac{(1-\delta)c_6(\beta)}{16}. \tag{2.18}$$

Let us summarize: Given a number η as in (2.18) and a skeleton parameter $K > K_0(\eta, \delta)$, then for any $x \in \mathcal{C}_\delta(t)$ and for any η-admissible skeleton $\lambda_K : 0 \mapsto x$ at least $\left[\frac{c_6(\beta)(1-\delta)|x|}{16K}\right]$ of the slabs

$$\mathcal{S}_{K,l}(t) \triangleq l \cdot 8K\vec{\mathfrak{n}}(t) + \mathcal{S}_K(t) ; \qquad l = 1, \cdots, \left[\frac{c_6(\beta)(1-\delta)|x|}{8K}\right] - 1$$

contain only cone points of λ_K. We shall call such slabs λ_K-clean.

From now on let us fix η and K as above. For any $x \in \mathcal{C}_\delta(t)$ and any η-admissible skeleton $\lambda_K : 0 \mapsto x$ let us number the λ_K-clean slabs in the decomposition (2.16) as $l_1, \ldots, l_n$. As we have just seen,

$$n = n(\lambda_K) \geqslant \left[\frac{c_6(\beta)(1-\delta)|x|}{16K}\right], \tag{2.19}$$

uniformly in all the situations of interest.

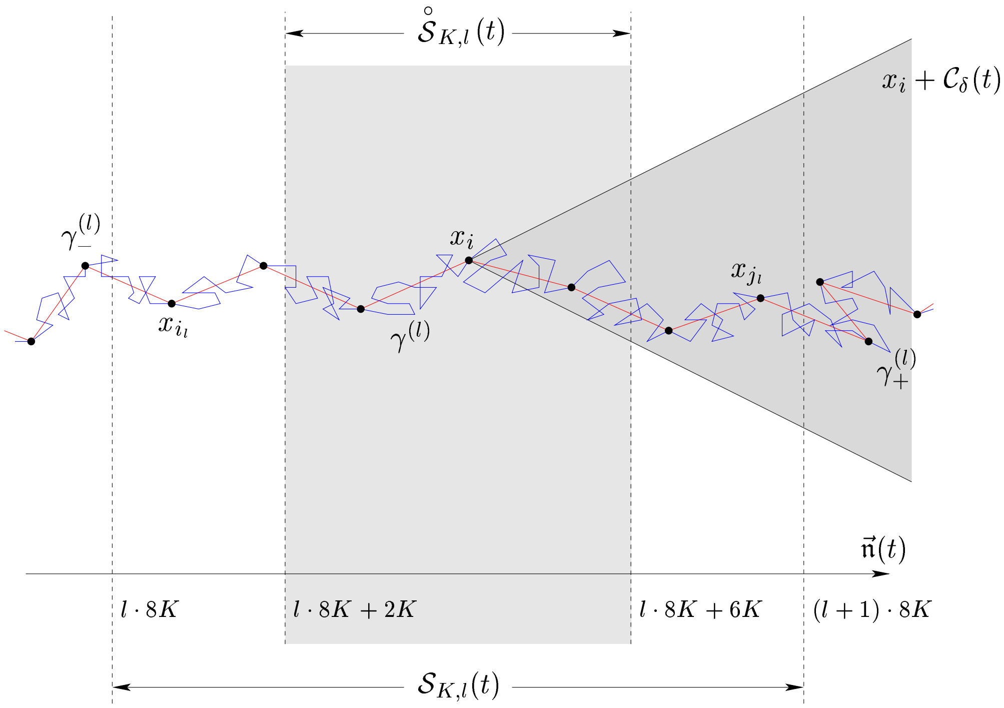

FIGURE 5. Clean slab of a skeleton $(x_0, \dots, x_N)$ (only a piece of which is drawn): Each point x_i; $i = i_l, \dots, j_l$, is a cone point of λ_K. In the decomposition $\gamma = \gamma_-^{(l)} \amalg \gamma^{(l)} \amalg \gamma_+^{(l)}$ of a path $\gamma \overset{K}{\sim} \lambda_K$, the left and right sub-paths $\gamma_-^{(l)}$ and $\gamma_+^{(l)}$ do not intersect $\overset{\circ}{\mathcal{S}}_{K,l}(t)$.

For any λ_K-clean slab $\mathcal{S}_{K,l}(t)$ of the skeleton $\lambda_K = (x_0, \dots, x_N)$ let us introduce the indices i_l and j_l via:

$$i_l = \min\{i : x_i \in \mathcal{S}_{K,l}(t)\} \quad \text{and} \quad j_l = \max\{j \geqslant i_l : x_j \in \mathcal{S}_{K,l}(t)\}.$$

Thus, we can associate with $\mathcal{S}_{K,l}(t)$ the embedded sub-skeleton $\lambda_K^{(l)} = (x_{i_l}, \dots, x_{j_l})$. Similarly, let $\gamma = \gamma_0 \amalg \cdots \amalg \gamma_{N-1}$ be a path compatible with the skeleton λ_K, where $\gamma_i : x_i \mapsto x_{i+1}$ is the corresponding portion of γ between the skeleton vertices x_i and x_{i+1}. Then we defined the embedded paths $\gamma_-^{(l)} = \gamma_0 \amalg \cdots \amalg \gamma_{i_l - 1}$, $\gamma^{(l)} = \gamma_{i_l} \amalg \cdots \amalg \gamma_{j_l}$ and $\gamma_+^{(l)} = \gamma_{j_l+1} \amalg \cdots \amalg \gamma_{N-1}$. In this notation,

$$\gamma = \gamma_-^{(l)} \amalg \gamma^{(l)} \amalg \gamma_+^{(l)}.$$

Let us take a closer look at $\lambda_K^{(l)}$ and $\gamma^{(l)}$ (Figure 5): Introducing the inner half-slab

$$\overset{\circ}{\mathcal{S}}_{K,l}(t) \triangleq \left\{u \in \mathbb{R}^d : l \cdot 8K + 2K \leqslant (t,u)_d \leqslant l \cdot 8K + 6K\right\},$$

notice that by the very construction $x_{i_l}, x_{j_l} \in \mathcal{S}_{K,l}(t) \setminus \overset{\circ}{\mathcal{S}}_{K,l}(t)$. In addition, since all the increments of λ_K on the interval $[i_l, \dots, j_l)$ are forward;

$$2K > (x_{i+1} - x_i, t)_d \geqslant (1-\delta)K \qquad \forall\, i = i_l, \dots, j_l - 1,$$

the number $j_l - i_l$ of vertices in the sub-skeleton $\lambda_K^{(l)}$ is bounded as

$$3 \leqslant j_l - i_l \leqslant 8/(1-\delta). \tag{2.20}$$

Finally, the left and right sub-paths $\gamma_-^{(l)}$ and $\gamma_+^{(l)}$ are disjoint from $\overset{\circ}{\mathcal{S}}_{K,l}(t)$:

$$\gamma_-^{(l)} \cap \overset{\circ}{\mathcal{S}}_{K,l}(t) \;=\; \emptyset \qquad \text{and} \qquad \gamma_+^{(l)} \cap \overset{\circ}{\mathcal{S}}_{K,l}(t) \;=\; \emptyset. \tag{2.21}$$

Consequently, any t-break point of $\gamma^{(l)}$ in the strip $\overset{\circ}{\mathcal{S}}_{K,l}(t)$ is automatically a t-break point of the whole path γ. Furthermore, for any λ_K-compatible path $\gamma = \gamma_-^{(l)} \amalg \gamma^{(l)} \amalg \gamma_+^{(l)}$ one can find $\bar{\gamma}^{(l)} : x_{i_l} \mapsto x_{j_l}$, such that $\gamma_-^{(l)} \amalg \bar{\gamma}^{(l)} \amalg \gamma_+^{(l)}$ is still compatible and λ_K-compatible, but $\bar{\gamma}^{(l)}$ has a t-break point in $\overset{\circ}{\mathcal{S}}_{K,l}(t)$ and $|\bar{\gamma}^{(l)}| \leqslant c_7 K$. By (2.20) the total number of all compatible paths is, uniformly in $\gamma_-^{(l)}, \gamma_+^{(l)}$ and $\lambda_K^{(l)}$, bounded above by $c_8 K^d$. Thus, in view of the finite energy condition (2.14) applied on the set $B = \mathcal{E}_{\mathbf{J}} \setminus \left(\Delta(\gamma_-^{(l)}) \cup \Delta(\gamma_+^{(l)}) \right)$, we infer:

$$\begin{aligned} \sum_{\substack{\gamma^{(l)}:\ \lambda_K \overset{K}{\sim} \gamma_-^{(l)} \amalg \gamma^{(l)} \amalg \gamma_+^{(l)} \\ \gamma^{(l)} \text{ has no } t\text{-break points}}} & q_\beta\left(\gamma_-^{(l)} \amalg \gamma^{(l)} \amalg \gamma_+^{(l)}\right) \\ & \leqslant \left(1 - \mathrm{e}^{-c_8 K^d}\right) \sum_{\gamma^{(l)}:\ \lambda_K \overset{K}{\sim} \gamma_-^{(l)} \amalg \gamma^{(l)} \amalg \gamma_+^{(l)}} q_\beta\left(\gamma_-^{(l)} \amalg \gamma^{(l)} \amalg \gamma_+^{(l)}\right) \end{aligned} \tag{2.22}$$

The estimate (2.22) is uniform in the points $x \in \mathcal{C}_\delta(t)$, skeletons $\lambda_K : 0 \mapsto x$, λ_K-clean slabs $\mathcal{S}_{K,l}(t)$ and in the corresponding embedded sub-paths $\gamma_-^{(l)}$ and $\gamma_+^{(l)}$. Since, by the choice of η in (2.18) we control the number of different clean slabs of η-admissible skeletons, (2.22) implies: Let $K \geqslant K_0(\eta, \delta)$. Then there exist $\epsilon = \epsilon(K) > 0$ and $\nu = \nu(K) > 0$ such that, uniformly in $x \in \mathcal{C}_\delta(t)$ and in the η-admissible skeletons $\lambda_K : 0 \mapsto x$,

$$\sum_{\gamma \overset{K}{\sim} \lambda_K} q_\beta(\gamma)\, \mathbb{I}_{\left\{ \substack{\gamma \text{ has less than } \epsilon|x| \text{ break points} \\ \text{inside } \lambda_K\text{-clean slabs}} \right\}} \;\leqslant\; \mathrm{e}^{-\nu|x|} q_\beta(\lambda_K). \tag{2.23}$$

On the other hand any break-point inside a λ_K-clean slab lies inside $K\mathbf{U}_\beta(x_i)$ for some cone point x_i of λ_K. Thus, such break points are automatically (t, K, δ)-correct, and, thereby, the claim of Theorem 2.1 follows from (2.23) and (2.15). ☐

3. Ornstein-Zernike formula

The basic decoupling estimate (3.1) which we derive in the first subsection, the cone confinement of the irreducible pieces in the decomposition (2.12) and the exponential estimate of Theorem 2.1 enable a reinterpretation of the representation formula (2.13) in terms of the Ruelle operator (3.13) with a uniformly Hölder continuous (3.9) summable (3.10) potential, which paves the way for an application of general spectral and local limit results of Sections 4 and 5. In view of this reinterpretation Theorem B more or less directly follows from the analytic perturbation theory of non-degenerate eigenvalues as it is proved in Subsection 3.3. The Ornstein-Zernike formula (1.3) is derived, along the lines of the general local limit approach of Section 5, in Subsection 3.4.

3.1. **Basic decoupling estimate.** We prove here an important estimate on the dependence between pieces of a path, similar to point 4 of Lemma 5.3 in [PV1]. For two compatible paths γ and λ define the conditional weight

$$q_{\beta,\lambda}(\gamma) \;=\; \frac{q_\beta(\gamma \amalg \lambda)}{q_\beta(\lambda)}.$$

Lemma 3.1. *For every $\beta < \beta_c$ there exists $\theta < 1$ and $c_1 < \infty$, such that for any path γ and for any pair of compatible and γ-compatible paths $\lambda = \eta \amalg \lambda_1$ and $\lambda' = \eta \amalg \lambda_2$ with $\Delta(\gamma) \cap (\Delta(\lambda_1) \cup \Delta(\lambda_2)) = \emptyset$, the following estimate on the ratio of the conditional weights holds:*

$$\frac{q_{\beta,\lambda}(\gamma)}{q_{\beta,\lambda'}(\gamma)} \geqslant \exp\left\{ -c_1 \sum_{\substack{t\in\gamma \\ s\in\lambda_1\cup\lambda_2}} \theta^{|t-s|} \right\}. \tag{3.1}$$

Remark 3.1. *Lemma 3.1 is a principal tool for rewriting the random line weights q_β in terms of the action of Ruelle operator with Hölder continuous potential. In particular, cone confinement conditions* **(P2)** *and* **(P3)** *have been designed in order to ensure appropriate exponential summability properties based on* (3.1)*; see the bound* (3.9) *below.*

Proof. Let us consider a finite graph $(\mathcal{V},\mathcal{E})$ such that $\gamma \amalg \lambda \subset \mathcal{E}$ and $\gamma \amalg \lambda' \subset \mathcal{E}$.

Using $q_{\mathcal{E},\beta,\lambda}(\gamma) = q_{\mathcal{E}\setminus\Delta(\lambda),\beta}(\gamma)$, we see that the ratio (on the finite graph) equals to

$$\frac{q_{\mathcal{E}\setminus\Delta(\lambda),\beta}(\gamma;J)}{q_{\mathcal{E}\setminus\Delta(\lambda'),\beta}(\gamma;J)}\,. \tag{3.2}$$

However, using the following expression for the weight of a contour (see (6.39) in [PV2]),

$$q_{\mathcal{E},\beta}(\lambda;J) = \Big(\prod_{e\in\lambda} \tanh(\beta J(e))\Big) \prod_{\substack{e\in\Delta(\lambda)\\ e=\langle t,t'\rangle}} \cosh(\beta J(e)) \exp[-\beta J(e) \int_0^1 \langle \sigma_t\sigma_{t'}\rangle_{\mathcal{E},\beta}^{J_s(e)}\;\mathrm{d}s]\,, \tag{3.3}$$

where $J_s(e) = \begin{cases} J(e) & \text{if } e \notin \Delta(\lambda) \\ sJ(e) & \text{if } e \in \Delta(\lambda) \end{cases}$, we see that (3.2) is also equal to

$$\prod_{e=\langle t,t'\rangle\in\Delta(\gamma)} \exp\left[-\beta J(e) \int_0^1 \left(\langle\sigma_t\sigma_{t'}\rangle^{J_s}_{\mathcal{E}\setminus\Delta(\lambda),\beta} - \langle\sigma_t\sigma_{t'}\rangle^{J_s}_{\mathcal{E}\setminus\Delta(\lambda'),\beta} \right) \mathrm{d}s \right]\,, \tag{3.4}$$

In view of the strict exponential decay of connectivities in (1.2), it is then sufficient to show that

$$\langle\sigma_t\sigma_{t'}\rangle^{J_s}_{\mathcal{E}\setminus\Delta(\lambda),\beta} - \langle\sigma_t\sigma_{t'}\rangle^{J_s}_{\mathcal{E}\setminus\Delta(\lambda'),\beta} \leqslant \sum_{s\in\lambda_1\cup\lambda_2} \langle\sigma_t\sigma_s\rangle_\beta \langle\sigma_s\sigma_{t'}\rangle_\beta\,.$$

Let us prove the latter bound. Let $\mathcal{E}_1 = \mathcal{E}\setminus\Delta(\lambda)$ and $\mathcal{E}_2 = \mathcal{E}\setminus\Delta(\lambda')$. We have using the random-line representation:

$$\begin{aligned}
\langle\sigma_t\sigma_{t'}\rangle^{J_s}_{\mathcal{E}_1,\beta} &= \sum_{\substack{\lambda:\, t\mapsto t' \\ \lambda\subset\mathcal{E}_1\cap\mathcal{E}_2}} q_{\mathcal{E}_1,\beta}(\lambda;J_s) + \sum_{\substack{\lambda:\, t\mapsto t' \\ \lambda\cap(\mathcal{E}_1\triangle\,\mathcal{E}_2)\neq\emptyset}} q_{\mathcal{E}_1,\beta}(\lambda;J_s) \\
&\leqslant \sum_{\substack{\lambda:\, t\mapsto t' \\ \lambda\subset\mathcal{E}_1\cap\mathcal{E}_2}} q_{\mathcal{E}_1\cap\mathcal{E}_2,\beta}(\lambda;J_s) + \sum_{u\in\lambda_1\cup\lambda_2} \sum_{\substack{\lambda:\, t\mapsto t' \\ \lambda\ni u}} q_{\mathcal{E}_1,\beta}(\lambda;J_s) \\
&\leqslant \langle\sigma_t\sigma_{t'}\rangle^{J_s}_{\mathcal{E}_1\cap\mathcal{E}_2,\beta} + \sum_{u\in\lambda_1\cup\lambda_2} \langle\sigma_t\sigma_u\rangle^J_{\mathcal{E}_1,\beta} \langle\sigma_{t'}\sigma_u\rangle^J_{\mathcal{E}_1,\beta} \\
&\leqslant \langle\sigma_t\sigma_{t'}\rangle^{J_s}_{\mathcal{E}_2,\beta} + \sum_{u\in\lambda_1\cup\lambda_2} \langle\sigma_t\sigma_u\rangle_\beta \langle\sigma_{t'}\sigma_u\rangle_\beta\,.
\end{aligned}$$

The first inequality follows from (3.3), Griffiths inequality and the fact that all paths containing an edge of $\mathcal{E}_1\triangle\,\mathcal{E}_2$ must also contain a site from $\lambda_1\cup\lambda_2$; the second one from the

BK-type inequality (2.4); finally the last one results from another application of Griffiths' inequality. ☐

3.2. **Reduction to Ruelle's setting.** Given $t \in \partial \mathbf{K}_\beta$, $\delta \in (0,1)$, a lattice point $x \in \mathcal{C}_\delta(t)$ and a path $\lambda : 0 \to x$ which admits the irreducible decomposition (2.12), let us rewrite the statistical weight $q_\beta(\lambda)$ as

$$\begin{aligned} q_\beta(\lambda)\, \mathrm{e}^{(t,x)_d} &= q_\beta(\mu \amalg \gamma_1 \amalg \cdots \amalg \gamma_m \amalg \eta)\, \mathrm{e}^{(t,x)_d} \\ &= q_\beta(\eta)\, q_\beta(\mu)\, \mathrm{e}^{(t, V(\eta)+V(\mu))_d} \exp\left\{ \sum_{k=1}^m \psi_\eta^t(\gamma_k, \dots, \gamma_m) \right\} g_{\mu,\eta}^t(\gamma_1, \dots, \gamma_m), \end{aligned}$$

where, as in (2.13), we use $V(\gamma)$ to denote the $\mathbb{Z}^d$-displacement between the endpoints of γ and define the potential ψ_η^t via:

$$\begin{aligned} \mathrm{e}^{\psi_\eta^t(\gamma_k, \dots, \gamma_m)} &= \frac{q_\beta(\gamma_k \amalg \gamma_{k+1} \amalg \cdots \amalg \gamma_m \amalg \eta)}{q_\beta(\gamma_{k+1} \amalg \cdots \amalg \gamma_m \amalg \eta)}\, \mathrm{e}^{(t, V(\gamma_k))_d} \\ &= q_{\beta, \gamma_{k+1} \amalg \cdots \amalg \gamma_m \amalg \eta}(\gamma_k)\, \mathrm{e}^{(t, V(\gamma_k))_d} \end{aligned} \tag{3.5}$$

for $k = 1, \dots, m-1$ and, accordingly, $\mathrm{e}^{\psi_\eta^t(\gamma_m)} = q_{\beta,\eta}(\gamma_m)\, \mathrm{e}^{(t, V(\gamma_m))_d}$.

Similarly, the function $g_{\mu,\eta}^t$ is defined as

$$g_{\mu,\eta}^t(\gamma_1, \dots, \gamma_m) = \frac{q_\beta(\mu \amalg \gamma_1 \amalg \cdots \amalg \gamma_m \amalg \eta)}{q_\beta(\gamma_1 \amalg \cdots \amalg \gamma_m \amalg \eta)\, q_\beta(\mu)} = \frac{q_{\beta, \gamma_1 \amalg \cdots \amalg \gamma_m \amalg \eta}(\mu)}{q_\beta(\mu)} \tag{3.6}$$

Notice that since the irreducible paths $\gamma_1, \dots, \gamma_m$ and the boundary condition η in the decomposition (2.12) always satisfy the K-cone conditions **(P2)** and **(P3)** of Subsection 2.6, the decoupling Lemma 3.1 implies that the conditional weights above are sandwiched between the corresponding unconditional ones: There exists $c_2 = c_2(\theta, K, \delta) < \infty$ such that

$$\frac{1}{c_2} \leqslant \frac{q_{\beta, \gamma_{k+1} \amalg \cdots \amalg \gamma_m \amalg \eta}(\gamma_k)}{q_\beta(\gamma_k)} \leqslant c_2 \quad \text{and} \quad \frac{1}{c_2} \leqslant g_{\mu,\eta}^t(\gamma_1, \dots, \gamma_m) \leqslant c_2 \tag{3.7}$$

uniformly in $t \in \partial \mathbf{K}_\beta$, $x \in \mathcal{C}_\delta(t)$, paths $\lambda : 0 \mapsto x$ and $k = 1, \dots, m$ in the decomposition (2.12) of λ.

In order to enable a uniform local limit study of $g_\beta(x)$ along the lines of the formalism which will be developed in Sections 4 and 5 let us, first of all, extend any finite sequence of paths $(\gamma_1, \dots, \gamma_m)$ to an infinite one by adding dummy empty paths $\emptyset$. In this way any finite sequence of paths $(\gamma_1, \dots, \gamma_m)$ corresponds to the infinite sequence $\underline{\gamma} = (\gamma_1, \dots, \gamma_m, \emptyset, \emptyset, \dots)$. Thus, given $t \in \partial \mathbf{K}_\beta$ and the forward cone parameter $\delta \in (0,1)$ the basic space $\mathcal{S}_\emptyset$ of infinite sequences of irreducible paths can be described as follows:

$$\mathcal{S}_\emptyset = \left\{ \underline{\gamma} = (\gamma_1, \gamma_2, \dots) \in \{S \cup \emptyset\}^{\mathbb{N}} : \gamma_k = \emptyset \Rightarrow \gamma_j = \emptyset \;\forall j > k \right\}, \tag{3.8}$$

where $S = S(t, \delta)$ is the corresponding space of irreducible paths.

The potential ψ_η^t in (3.5) has been defined only for sequences $\underline{\gamma}$ of the type $\underline{\gamma} = (\gamma_1, \dots, \gamma_m, \emptyset, \dots)$. However, the basic decoupling estimate (3.1) implies that for every $t \in \partial \mathbf{K}_\beta$ any two such sequences $\underline{\gamma}$ and $\underline{\lambda}$ with the proximity index $\mathbf{i}(\underline{\gamma}, \underline{\lambda}) \stackrel{\Delta}{=} \min\{k : \gamma_k \neq \lambda_k\} > 1$ satisfy the uniform estimate:

$$\left| \psi_\eta^t(\underline{\gamma}) - \psi_\eta^t(\underline{\lambda}) \right| \leqslant c_3 \theta^{\mathbf{i}(\underline{\gamma}, \underline{\lambda})}, \tag{3.9}$$

where the constant c_3 depends only on the renormalization scale K and on the forward cone parameter δ which specify the set of irreducible paths S. Consequently, ψ_η^t admits a unique Hölder continuous extension to the whole of $\mathcal{S}_\emptyset$.

Finally, in view of (3.7), Theorem 2.1 implies:

$$\sum_{\gamma_1 \in S} \mathrm{e}^{\psi_\eta^t(\gamma_1, \underline{\gamma})} \leqslant c_4 \sum_{x \in \mathcal{C}_\delta(t)} \mathrm{e}^{-\nu|x|} < \infty, \tag{3.10}$$

uniformly in $\underline{\gamma} \in \mathcal{S}_\emptyset$.

By (2.13) we have derived the following representation of the two point function: For every $x \in \mathcal{C}_\delta(t)$;

$$\mathrm{e}^{(t,x)_d} g_\beta(x) = o\left(\mathrm{e}^{-\nu|x|}\right) + \sum_{\mu,\eta} q_\beta(\mu)\, q_\beta(\eta) \mathrm{e}^{(t, V(\mu)+V(\eta))_d} \sum_{n=1}^{\infty} \mathbb{Q}_{n,\mu,\eta}^t \left(x - V(\mu) - V(\eta)\right), \tag{3.11}$$

where the weights $\mathbb{Q}_{n,\mu,\eta}^t(r)$ are given by

$$\mathbb{Q}_{n,\mu,\eta}^t(r) = \sum_{\substack{\underline{\gamma} \in \mathcal{S}_n \\ \sum V(\gamma_i) = r}} \mathrm{e}^{\Psi_{\eta,n}^t(\underline{\gamma} \mid \underline{\emptyset})} g_{\mu,\eta}^t(\underline{\gamma}, \underline{\emptyset}), \tag{3.12}$$

with $\mathcal{S}_n$ being the set of all n-strings of irreducible paths from $S(t,\delta)$, and, for every $\underline{\gamma} \in \mathcal{S}_n$,

$$\Psi_{\eta,n}^t(\underline{\gamma} \mid \underline{\emptyset}) = \psi_\eta^t(\gamma_n, \underline{\emptyset}) + \psi_\eta^t(\gamma_{n-1}, \gamma_n, \underline{\emptyset}) + \cdots + \psi_\eta^t(\gamma_1, \gamma_2, \ldots, \gamma_n, \underline{\emptyset}).$$

Thereby, the weights (3.12) fall into the general framework of the Ruelle operator induced weights (5.1). The local asymptotics of the latter are studied in general in Section 5. In our case, the associated Ruelle operator L_η^t is given by

$$L_\eta^t f(\underline{\gamma}) = \sum_{\gamma_1 \in S} \mathrm{e}^{\psi_\eta^t(\gamma_1, \underline{\gamma})} f(\gamma_1, \underline{\gamma}) \tag{3.13}$$

By (3.9) and (3.10) L_η^t is a bounded linear operator on $\mathcal{G}_\theta(\mathcal{S}_\emptyset)$ (see Subsection 4.1 for the definition of the space $\mathcal{G}_\theta(\mathcal{S}_\emptyset)$ of Hölder continuous functions on $\mathcal{S}_\emptyset$).

3.3. **The geometry of $\partial\mathbf{K}_\beta$ and the spectral radius $\rho_{\mathbf{S}}^t(s)$.** Since $g_\beta(x)$ is logarithmically asymptotic (see (1.1)) to $\mathrm{e}^{-\xi_\beta(x)}$,

$$\lim_{|x| \to \infty} \frac{1}{|x|} \left(\xi_\beta(x) + \log g_\beta(x)\right) = 0$$

the shape $\mathbf{K}_\beta$ could be alternatively described as the closure of the domain of convergence of the series

$$s \to \sum_{x \in \mathbb{Z}^d} \mathrm{e}^{(s,x)_d} g_\beta(x).$$

Let us fix $t \in \partial\mathbf{K}_\beta$ and $\nu > 0$ small. For every $|s| < \nu/2$ the convergence of the series

$$\sum_x g_\beta(x) \mathrm{e}^{(t+s,x)_d} \tag{3.14}$$

depends, by the very definition of the surcharge costs, only of the behaviour of $g_\beta(x)\mathrm{e}^{(t+s,x)_d}$ along the directions x satisfying $\mathfrak{s}_t(x) \leqslant \nu|x|$. For such x-s, however, the paths $\lambda : 0 \mapsto x$ admit the irreducible decomposition (2.12) with respect to the dual direction $t \in \partial\mathbf{K}_\beta$, and we are entitled to employ the representation (3.11). Therefore, for $|s| < \nu/2$ the convergence in (3.14) is equivalent to the convergence of the following series:

$$\sum_n \sum_{\mu,\eta} q_\beta(\mu) q_\beta(\eta) \mathrm{e}^{(t+s, V(\mu)+V(\eta))_d} \left[L^t_{\eta,s}\right]^n g^t_{\mu,\eta}(\underline{\emptyset}), \tag{3.15}$$

where we have introduced the "tilted" operator

$$L^t_{\eta,s} f(\underline{\gamma}) \;=\; \sum_{\gamma_1 \in S} \mathrm{e}^{\psi^t_\eta(\gamma_1, \underline{\gamma}) + (s, V(\gamma_1))_d} f(\gamma_1, \underline{\gamma}) \;=\; L^t_\eta \left(\mathrm{e}^{(s,V(\cdot))_d} f\right)(\underline{\gamma}).$$

(3.10) insures that the operator $L^t_{\eta,s}$ is well defined for all $s < \nu/2$.

By Theorem 2.1 the series

$$\sum_{\mu,\eta} q_\beta(\mu) q_\beta(\eta) \mathrm{e}^{(t+s, V(\mu)+V(\eta))_d} \tag{3.16}$$

converges. On the other hand, (3.7) suggests the substitution of the $\left[L^t_{\eta,s}\right]^n g^t_{\mu,\eta}(\underline{\emptyset})$ terms in (3.15) by

$$\left[L^t_{\eta,s}\right]^n \mathbb{I}(\emptyset) \;=\; \sum_{\underline{\gamma} \in \mathcal{S}_n} \mathrm{e}^{\Psi^t_{\eta,n}(\underline{\gamma} \mid \underline{\emptyset}) + \sum (s, V(\gamma_i))_d},$$

where $\mathbb{I}$ denotes the constant function on $\mathcal{S}_\emptyset$. As the in the cases of (3.7) and (3.9), the cone confinement properties **(P2)** and **(P3)** of the irreducible paths and the basic decoupling estimate (3.1) imply:

$$\sup_n \sup_{\eta,\eta'} \sup_{\underline{\gamma}} \left\{ \Psi^t_{\eta,n}(\underline{\gamma} \mid \underline{\emptyset}) - \Psi^t_{\eta',n}(\underline{\gamma} \mid \underline{\emptyset}) \right\} \;\leqslant\; c_5 < \infty. \tag{3.17}$$

In view of (3.16) this means that the convergence in (3.14) is equivalent to the convergence of

$$\sum_n \left[L^t_{\eta,s}\right]^n \mathbb{I}(\emptyset)$$

for some (and hence for all η). Since (3.17) evidently implies that the spectral radius $\rho_{\mathbf{S}}\left(L^t_{\eta,s}\right) \stackrel{\Delta}{=} \rho^t_{\mathbf{S}}(s)$ does not depend on η, we arrive to the following characterization of $\partial \mathbf{K}_\beta$ around t: For $|s| < \nu/2$,

$$t + s \in \partial \mathbf{K}_\beta \iff \rho_{\mathbf{S}}\left(L^t_{\eta,s}\right) \stackrel{\Delta}{=} \rho^t_{\mathbf{S}}(s) = 1\,. \tag{3.18}$$

Moreover, by Theorem 2.1, the conditions **A1** and **A2** of Section 5 are satisfied for the path displacement observable $V : S \mapsto \mathbb{Z}^d$. Consequently, by the analytic perturbation theory and the non-degeneracy of $\mathrm{Hess}(\log \rho^t_{\mathbf{S}})(0)$ established in Subsection 5.4 below, the equation (3.18) implies that the compact surface $\partial \mathbf{K}_\beta$ is locally analytic and has a uniformly positive Gaussian curvature. In particular, the map

$$t \;\mapsto\; \frac{\nabla \rho^t_{\mathbf{S}}(0)}{|\nabla \rho^t_{\mathbf{S}}(0)|}$$

is a diffeomorphism from $\partial \mathbf{K}_\beta$ to $\mathbb{S}^{d-1}$. Since by the general dual description of support functions $\xi_\beta(x) = (t, x)_d$ if and only if x is orthogonal to a supporting hyperplane to $\mathbf{K}_\beta$ at t, we conclude: For any $x \in \mathbb{R}^d \setminus 0$ and $t \in \partial \mathbf{K}_\beta$,

$$\xi_\beta(x) = (t, x)_d \iff \exists\, \alpha \in \mathbb{R}_+ \text{ such that } x = \alpha \nabla \log \rho^t_{\mathbf{S}}(0). \tag{3.19}$$

3.4. **Proof of the OZ formula.** We shall recover the asymptotic behaviour of the two point function $g_\beta(x) = \langle\sigma_0\sigma_x\rangle_\beta$ from the representation (3.11). The crucial fact is that the local limit analysis which will be developed in Section 5 applies for the operators L^t_η (defined in (3.13)) and the functions $g^t_{\mu,\eta}$ (defined in (3.6)) *uniformly* in $t \in \partial\mathbf{K}_\beta$ and in boundary conditions μ,η satisfying properties **(P2)** and **(P4)** of Subsection 2.6. Indeed, in the language of Section 4 the inequalities (3.7) and (3.9) imply that

$$\sup_{t,\mu,\eta}\left\{\|\psi^t_\eta\|_\theta + \|g^t_{\mu,\eta}\|_\theta + \|L^t_\eta\|_\theta\right\} \;<\; \infty.$$

In particular (see Theorem 4.1 below), there exists $\epsilon > 0$ such that the spectrum $\Sigma_{\mathbf{S}}\left(L^t_\eta\right)$ of L^t_η satisfies

$$\Sigma_{\mathbf{S}}\left(L^t_\eta\right)\bigcap\left\{\mu \,:\, |\mu| > (1-2\epsilon)\rho^t_{\mathbf{S}}(0)\right\} \;=\; \left\{\rho^t_{\mathbf{S}}(0)\right\}$$

uniformly in $t \in \partial\mathbf{K}_\beta$ and in the boundary conditions η. Consequently one can find an open neighbourhood $\mathcal{U}$ of the origin in $\mathbb{C}^d$, such that the family of analytic functions (see Subsection 4.2 for the definition of the spectral projector P_L),

$$\left\{\xi \;\mapsto\; P_{L^t_{\eta,\xi}}\mathbb{I}(\emptyset)\right\}_{t,\eta}$$

is uniformly continuous on $\overline{\mathcal{U}}$. By the second of the inequalities in (3.7) it follows that the family of the analytic functions

$$\left\{\xi \;\mapsto\; \chi^t_{\mu,\eta}(\xi) \;\triangleq\; P_{L^t_{\eta,\xi}}g^t_{\mu,\eta}(\emptyset)\right\}_{t,\mu,\eta}$$

is uniformly bounded away from zero and infinity on $\mathcal{U}$. By the Cauchy formula the sequence $\left\{\nabla\chi^t_{\mu,\eta}(0)\right\}$ is also uniformly bounded in $t \in \partial\mathbf{K}_\beta$ and boundary conditions μ and η satisfying properties **(P2)** and **(P4)** of Subsection 2.6.

By the preceding discussion the asymptotic results of Subsection 5.1 below hold uniformly in $t \in \partial\mathbf{K}_\beta$ and in the boundary conditions μ,η. For each particular choice of the data we shall distinguish between three different cases:

Let us fix (see (5.6) below) $\nu \in (0, 1/2)$ and define

$$R^t_{n,\nu} \;=\; \left\{r \in \mathbb{Z}^d \,:\, |r - n\nabla\log\rho^t_{\mathbf{S}}(0)| < n^{1-\nu}\right\}.$$

CASE 1 $r \in R^t_{n,\nu}$. Then, by Theorem 5.1

$$\mathbb{Q}^t_{n,\mu,\eta}(r) \;=\; \frac{\chi^t_{\mu,\eta}(0)}{\sqrt{(2\pi n)^d \det A^t_{\mathbf{S}}}}\exp\left\{-\frac{1}{2n}\mathcal{A}^t_{\mathbf{S}}\left(r - n\nabla\log\rho^t_{\mathbf{S}}(0)\right)\right\}(1+o(1)), \tag{3.20}$$

where the quadratic form $\mathcal{A}^t_{\mathbf{S}}$ is given by $\mathcal{A}^t_{\mathbf{S}}(v) = \left(\left[A^t_{\mathbf{S}}\right]^{-1}v, v\right)_d$ and $A^t_{\mathbf{S}} \triangleq \mathrm{Hess}\left(\log\rho^t_{\mathbf{S}}\right)(0)$.

Pick now a large enough number M.

CASE 2 $r \notin R^t_{n,\nu}$, but $|r| \leqslant Mn$. Then, as it follows from Lemma 5.1,

$$\mathbb{Q}^t_{n,\mu,\eta}(r) \leqslant \mathrm{e}^{-c_6 n^{1-2\nu}} \;\leqslant\; \mathrm{e}^{-c_7|r|^{1-2\nu}}. \tag{3.21}$$

CASE 3 Finally, let $r \notin R^t_{n,\nu}$, and $|r| > Mn$. In view of Theorem 2.1,

$$\mathbb{Q}^t_{n,\mu,\eta}(r) \;\leqslant\; \mathrm{e}^{-c_8|r|}, \tag{3.22}$$

once M has been chosen large enough. This is just an exponential form of Markov's inequality.

Turning back to the expansion (3.11), for each $x \in \mathbb{Z}^d$ define the dual direction $t = t(x) \in \partial \mathbf{K}_\beta$ and the coefficient $\alpha = \alpha(x)$ as in (3.19). Set also $n_0 = n_0(x) = [\alpha(x)]$. Of course,

$$n_0(x) \;=\; \frac{|x|}{|\nabla \log \rho^t_{\mathbf{S}}(0)|}\,(1 + o(1)) \tag{3.23}$$

uniformly in $|x| \to \infty$.

For every pair of boundary conditions (μ, η) with

$$|V(\mu)| + |V(\eta)| \;\leqslant\; n_0^{1/2-\nu} \tag{3.24}$$

we, using the asymptotic estimates (3.20), (3.21) and (3.22), infer that the second sum in (3.11) admits the following uniform (in $|x| \to \infty$ and in (μ, η) satisfying (3.24)) asymptotic expression:

$$\frac{\chi^t_{\mu,\eta}(0)}{\sqrt{(2\pi n_0)^{d-1}\mathcal{A}^t_{\mathbf{S}}\left(\nabla \log \rho^t_{\mathbf{S}}(0)\right)\det A^t_{\mathbf{S}}}}\,(1 + o(1)) \;\triangleq\; \frac{\phi_{\mu,\eta}(t)}{\sqrt{|x|^{d-1}}}\,(1 + o(1))\,, \tag{3.25}$$

with

$$\phi_{\mu,\eta}(t) \;=\; \frac{\chi^t_{\mu,\eta}(0)\sqrt{|\nabla \log \rho^t_{\mathbf{S}}(0)|^{d-1}}}{\sqrt{(2\pi)^{d-1}\mathcal{A}^t_{\mathbf{S}}\left(\nabla \log \rho^t_{\mathbf{S}}(0)\right)\det A^t_{\mathbf{S}}}}.$$

On the other hand, in view of the irreducibility of the boundary conditions (μ, η) in the decomposition (2.12), the mass-gap estimate (2.11) of Theorem 2.1 implies that

$$q_\beta\,(\mu)\,q_\beta\,(\eta)\,\mathrm{e}^{(t, V(\mu)+V(\eta))_d} \;\leqslant\; c_9\mathrm{e}^{-c_{10}(|V(\mu)|+|V(\eta)|)},$$

uniformly in $t \in \partial \mathbf{K}_\beta$ and in (μ, η). Consequently, the total contribution to the right-hand side of (3.11) from the terms corresponding to those boundary conditions (μ, η) which do not comply with (3.24) is at most $\exp\left\{-c_{11}|x|^{1/2-\nu}\right\}$ for some $c_{11} > 0$. This is negligible as compared to (3.25), and the Ornstein-Zernike formula (1.3) follows with the pre-factor $\Phi_\beta\,(\vec{\mathfrak{n}}(x))$ being identified as

$$\Phi_\beta\,(\vec{\mathfrak{n}}(x)) \;=\; \sum_{\mu,\eta} \phi_{\mu,\eta}(t(x))\,q_\beta(\mu) q_\beta(\eta)\mathrm{e}^{(t, V(\mu)+V(\eta))_d}.$$

□

4. Ruelle's Perron-Frobenius Theorem for Countable Alphabets

The results and the methods of this section are not particularly new. A general treatment of the subshifts on countable alphabets could be found in [Br] and in [Sa]. Full shifts are studied in the recent preprint [Is] based on the earlier work [CIs]. Unfortunately, the setup in the abovementioned papers is different from ours and we cannot rely directly on the corresponding techniques therein. In particular, in all these works the authors assumed one or another form of irreducibility of the shift, whereas in our context it happens to be natural to permit an additional transient class. Thus, for the reader's convenience we prefer to formulate the theory in a closed form as we need it here, giving exact references whenever possible and providing brief proofs otherwise.

4.1. **The Setup.** Let S be a countable set. We use $\mathcal{S}_n$ to denote the set of n-strings $\underline{x} = (x_1, ..., x_n)$ of elements of S and $\mathcal{S}$ to denote the set of countable $\underline{x} = (x_1, x_2, \dots)$ string of elements of S. Eventually, we shall study functions defined on set of all finite and infinite strings,

$$\mathcal{S} \bigcup \left(\bigcup_{n=1}^{\infty} \mathcal{S}_n \right) .$$

It happens to be convenient to introduce a dummy element $\emptyset$ and define

$$\mathcal{S}_{\emptyset} = \left\{ \underline{x} \in \{S \cup \emptyset\}^{\mathbb{N}} \, : \, x_i = \emptyset \Rightarrow x_j = \emptyset \; \forall j > i \right\} . \tag{4.1}$$

In other words, the infinite strings $\mathcal{S} \subset \mathcal{S}_{\emptyset}$, and for every $n \in \mathbb{N}$ we extend finite strings from $\mathcal{S}_n$ by attaching to it the infinite sequence $\underline{\emptyset}$ of empty elements.

For every $\theta \in (0,1)$ one can define the distance d_θ on $\mathcal{S}_{\emptyset}$ via

$$\mathrm{d}_\theta(\underline{x}, \underline{y}) = \theta^{\mathbf{i}(\underline{x}, \underline{y})},$$

where the proximity index between the strings $\underline{x} \neq \underline{y}$ is given by

$$\mathbf{i}(\underline{x}, \underline{y}) \ \stackrel{\Delta}{=} \ \min \{ k \, : \, x_k \neq y_k \} .$$

Notice that $\mathcal{S}$ is a closed subset of $\mathcal{S}_{\emptyset}$ in the d_θ metrics.

Given a function $f : \mathcal{S}_{\emptyset} \mapsto \mathbb{C}$ and a number $k \in \mathbb{N}$ define the k-th variation of f,

$$\mathbf{var}_k (f) \ = \ \sup_{\{\underline{x}, \underline{y} : \mathbf{i}(\underline{x}, \underline{y}) \geqslant k\}} \left| f(\underline{x}) - f(\underline{y}) \right| .$$

We say that f is continuous (or more exactly locally uniformly continuous) if

$$\lim_{k \to \infty} \mathbf{var}_k (f) \ = \ 0.$$

The space $\mathcal{C} = \mathcal{C}(\mathcal{S}_{\emptyset})$ of bounded continuous functions equipped with the usual sup-norm $\| \cdot \|_\infty$ is Banach.

Also, given a number $\theta \in (0,1)$, we say that f is uniformly Hölder continuous (or, equivalently, uniformly Lipschitz continuous in the d_θ metrics of $\mathcal{S}_{\emptyset}$) if

$$|f|_\theta \ \stackrel{\Delta}{=} \ \sup_{k > 1} \frac{\mathbf{var}_k (f)}{\theta^k} \ < \ \infty.$$

Of course, $|f|_\theta < \infty$ does not imply that $\mathbf{var}_1 (f) < \infty$, and hence uniformly Hölder continuous functions can be unbounded. However, the functional space

$$\mathcal{G}_\theta = \mathcal{G}_\theta(\mathcal{S}_{\emptyset}) \ = \ \{ f \, : \, f \in \mathcal{C} \text{ and } |f|_\theta < \infty \}$$

is Banach with respect to the norm $\| \cdot \|_\theta = \| \cdot \|_\infty + | \cdot |_\theta$.

Let a real uniformly Hölder continuous function ψ; $|\psi|_\theta < \infty$, be such that

$$\sup_{\underline{x} \in \mathcal{S}_{\emptyset}} \sum_{z \in S} \mathrm{e}^{\psi(z, \underline{x})} \ < \ \infty \tag{4.2}$$

Then the linear operator

$$L f(\underline{x}) \ = \ \sum_{z \in S} \mathrm{e}^{\psi(z, \underline{x})} f(z, \underline{x}) \tag{4.3}$$

is well-defined and bounded on both $\mathcal{C}$ and $\mathcal{G}_\theta$.

Furthermore, given a potential ψ as above and an observable $V : S \mapsto \mathbb{Z}^d$, the complex operator

$$L_{i\tau} f(\underline{x}) \;\triangleq\; \sum_{z\in S} \mathrm{e}^{\psi(z,\underline{x})+i(\tau,V(z))_d}\, f(z,\underline{x}), \tag{4.4}$$

where $(\cdot,\cdot)_d$ denotes the scalar product in $\mathbb{C}^d$, is also defined and bounded on $\mathcal{G}_\theta$ and $\mathcal{C}$ for every $\tau \in [-\pi,\pi)^d$. The original operator L corresponds in the latter notation to $\tau = 0$.

4.2. **Spectral properties of L and $L_{i\tau}$.** Given a bounded linear operator T on $\mathcal{G}_\theta$ let $\Sigma_{\mathbf{S}}(T)$ and $\Sigma_{\mathbf{F}}(T)$ denote the spectrum and, respectively, the Fredholm spectrum [AKPRS] of T. We use $\rho_{\mathbf{S}}(T)$ and $\rho_{\mathbf{F}}(T)$ to denote the corresponding spectral radiuses. Any point $\lambda \in \Sigma_{\mathbf{S}}(T) \cap \{\lambda : |\lambda| > \rho_{\mathbf{F}}(L)\}$ is an isolated eigenvalue of T ([AKPRS], Subsection 2.6.12), and there exists $\epsilon > 0$, such that

$$\{\mu : |\mu - \lambda| < 2\epsilon\} \cap \Sigma_{\mathbf{S}} \;=\; \{\lambda\}\,.$$

Furthermore, for such points λ the associated spectral projector

$$P_\lambda \;=\; \frac{1}{2\pi i}\oint_{|\mu-\lambda|=\epsilon} (\mu I - T)^{-1}\,\mathrm{d}\mu \tag{4.5}$$

is finite dimensional. The dimension of Range (P_λ) is called the algebraic multiplicity of λ. An isolated point $\lambda_0 \in \Sigma_{\mathbf{S}}$ of algebraic multiplicity 1 called a **non-degenerate eigenvalue** of L. There is a well-developed analytic perturbation theory of non-degenerate eigenvalues, which, in our context, leads to crucial local limit type results. We shall describe this in detail in Section 5.

With the above notions in mind let us turn to the spectral properties of the operators L and $L_{i\tau}$ which were defined in (4.3) and in (4.4) respectively.

Theorem 4.1. *Assume that a uniformly Hölder continuous real interaction potential ψ; $\|\psi|_\theta < \infty$, satisfies the summability condition* (4.2)*. Then for every $V : S \mapsto \mathbb{Z}^d$ and for each $\tau \in [-\pi,\pi)^d$ (in particular for $\tau = 0$)*

$$\rho_{\mathbf{F}}(L_{i\tau}) \;<\; \rho_{\mathbf{S}}(L). \tag{4.6}$$

Furthermore, $\rho_{\mathbf{S}} = \rho_{\mathbf{S}}(L)$ is a non-degenerate eigenvalue of L on $\mathcal{G}_\theta$ and the corresponding eigenfunction h is strictly positive;

$$\inf_{\underline{x}\in\mathcal{S}_\emptyset} h(\underline{x}) \;>\; 0. \tag{4.7}$$

Finally, the rest of the spectrum of L on $\mathcal{G}_\theta$ satisfies

$$\sup_{\lambda\in\Sigma_{\mathbf{S}}\setminus\rho_{\mathbf{S}}} |\lambda| \;<\; \rho_{\mathbf{S}}. \tag{4.8}$$

In particular, there exist $C < \infty$ and $\delta > 0$, such that for any $f \in \mathcal{G}_\theta$ one can find a coefficient $c = c(f)$ satisfying:

$$\|\frac{1}{\rho_{\mathbf{S}}^n} L^n f \;-\; c(f)h\|_\theta \;\leqslant\; C\|f\|_\theta (1-\delta)^n. \tag{4.9}$$

The above coefficient $c(f)$ satisfies $c(f)h = P_L f$, where we use P_L to denote the spectral projector (4.5) *associated with $\lambda = \rho_{\mathbf{S}}$.*

The rest of the section is devoted to the proof of the theorem.

4.3. **Fredholm spectrum.** In this subsection we establish the spectral gap assertion (4.6) of Theorem 4.1. Without loss of the generality we may assume that $\rho_{\mathbf{S}}(L) = 1$. By a version of the Nussbaum's formula [Nus1], [AKPRS] it suffices to show that there exists a compact subset $K = K(\tau)$ of $\mathcal{G}_\theta$ and a number $n = n(\tau) \in \mathbb{N}$, such that

$$\sup_{\|f\|_\theta \leqslant 1} \inf_{g \in K} \|L^n_{i\tau} f - g\|_\theta \,<\, 1. \tag{4.10}$$

The n-th power of L is given by

$$L^n f(\underline{x}) \;=\; \sum_{\underline{z} \in \mathcal{S}_n} \mathrm{e}^{\Psi_n(\underline{z} \mid \underline{x})} f(\underline{z}, \underline{x}),$$

where

$$\Psi_n(\underline{z} \mid \underline{x}) \;=\; \psi(z_n, \underline{x}) + \psi(z_{n-1}, z_n, \underline{x}) + ... + \psi(z_1, z_2, ...z_n, \underline{x}). \tag{4.11}$$

It is easy to check that for every $n \in \mathbb{N}$ and for every $\underline{x}, \underline{y} \in \mathcal{S}_\emptyset$

$$\Psi_n(\underline{z} \mid \underline{y}) - \Psi_n(\underline{z} \mid \underline{x}) \leqslant \beta \theta^{\mathbf{i}(\underline{x}, \underline{y})}, \tag{4.12}$$

with

$$\beta = \beta(\psi) \;=\; \frac{|\psi|_\theta}{1 - \theta}. \tag{4.13}$$

Lemma 4.1. *Assume that $\rho_{\mathbf{S}}(L) = 1$. Then,*

$$\sup_n \sum_{\underline{z} \in \mathcal{S}_n} \|\mathrm{e}^{\Psi_n(\underline{z} \mid \cdot)}\|_\theta \;\stackrel{\Delta}{=}\; M = M(\psi) < \infty \tag{4.14}$$

Proof. By the assumption on $\rho_{\mathbf{S}}(L)$, $\inf_{\underline{x}} L^n \mathbb{I}(\underline{x}) \leqslant 1$, where $\mathbb{I}$ is the constant function identically equal to 1. Let us pick $\underline{x}_0$ such that $L^n \mathbb{I}(\underline{x}_0) \leqslant 2$. Then, for every $\underline{y} \in \mathcal{S}_\emptyset$, we,using (4.12), estimate:

$$\mathrm{e}^{\Psi_n(\underline{z} \mid \underline{y})} \;=\; \mathrm{e}^{\Psi_n(\underline{z} \mid \underline{x}_0)} \mathrm{e}^{\Psi_n(\underline{z} \mid \underline{y}) - \Psi_n(\underline{z} \mid \underline{x}_0)} \;\leqslant\; \mathrm{e}^{\Psi_n(\underline{z} \mid \underline{x}_0)} \mathrm{e}^{\beta\theta}.$$

Therefore, by the choice of $\underline{x}_0$,

$$\sup_n \sum_{\underline{z} \in \mathcal{S}_n} \|\mathrm{e}^{\Psi_n(\underline{z} \mid \cdot)}\|_\infty \;\leqslant\; 2\mathrm{e}^{\beta\theta}. \tag{4.15}$$

Moreover, since for every n, by (4.12),

$$\mathbf{var}_k \left(\mathrm{e}^{\Psi_n(\underline{z} \mid \cdot)}\right) \;\leqslant\; \|\mathrm{e}^{\Psi_n(\underline{z} \mid \cdot)}\|_\infty \left(\mathrm{e}^{\beta\theta^k} - 1\right), \tag{4.16}$$

the sup-norm estimate (4.15) readily implies the conclusion of the Lemma with

$$M(\psi) = 2\mathrm{e}^{\beta\theta} \left(1 + \sup_{t \in (0,1]} \frac{\mathrm{e}^{\beta t} - 1}{t}\right)$$

$\square$

Given $n \in \mathbb{N}$, $\tau \in [-\pi, \pi)^d$ and $\underline{z} \in \mathcal{S}_n$ set

$$g^n_{\tau, \underline{z}}(\underline{x}) \;=\; \exp\left\{\Psi_n(\underline{z} \mid \underline{x}) + i \sum_1^n (\tau, V(z_k))_d\right\}.$$

By Lemma 4.1 we (assuming that $\rho_{\mathbf{S}}(L)=1$) obtain the following estimate:

$$\sup_n \sum_{\underline{z}\in\mathcal{S}_n} \|g^n_{\tau,\underline{z}}\|_\theta \;\leqslant M<\infty. \tag{4.17}$$

We shall construct the compact sets $K(\tau)$ in (4.10) from finite linear combinations of functions from the family $\{g^n_{\tau,\underline{z}}\}_{\underline{z}\in\mathcal{S}_n}$: Given $f\in\mathcal{G}_\theta$ with $\|f\|_\theta\leqslant 1$ and $n\in\mathbb{N}$ let us represent $L^{n+1}_{i\tau}f$ as

$$L^{n+1}_{i\tau}f(\underline{x}) \;=\; \sum_{\underline{z}\in\mathcal{S}_n} g^n_{\tau,\underline{z}}(\underline{x})L_{\underline{z},i\tau}f(\underline{x}), \tag{4.18}$$

where the operator $L_{\underline{z},i\tau}$ is defined by

$$L_{\underline{z},i\tau}f(\underline{x}) \;=\; \sum_{u\in S} \mathrm{e}^{\psi(u,\underline{z},\underline{x})+i(\tau,V(u))_d} f(u,\underline{z},\underline{x}).$$

Using the obvious inequalities: For every $\phi_1,\phi_2\in\mathcal{G}_\theta$, $\|\phi_1\phi_2\|_\infty\leqslant\|\phi_1\|_\infty\|\phi_2\|_\infty$ and, for each $k\in\mathbb{N}$,

$$\mathbf{var}_k\left(\phi_1\phi_2\right) \;\leqslant \|\phi_1\|_\infty \mathbf{var}_k\left(\phi_2\right)+\|\phi_2\|_\infty\mathbf{var}_k\left(\phi_1\right),$$

we infer from (4.17):

$$\|L_{\underline{z},i\tau}f(\cdot)\|_\infty\leqslant M \quad\text{and}\quad |L_{\underline{z},i\tau}f(\cdot)|_\theta \;\leqslant 2M\theta^n, \tag{4.19}$$

uniformly in n, $\underline{z}\in\mathcal{S}_n$ and in $\|f\|_\theta\leqslant 1$. Fix now a large enough power n satisfying $4M^2\theta^n<1/2$ and a reference point $\underline{x}_0\in\mathcal{S}_\emptyset$. Defining the coefficients

$$\mathbf{a}_{\underline{z}}[f] \;=\; L_{\underline{z},i\tau}f(\underline{x}_0),$$

we can rewrite (4.18) as

$$L^{n+1}_{i\tau}f(\underline{x}) \;=\; \sum_{\underline{z}\in\mathcal{S}_n}\mathbf{a}_{\underline{z}}[f]g^n_{\tau,\underline{z}}(\underline{x}) \;+\; \sum_{\underline{z}\in\mathcal{S}_n} g^n_{\tau,\underline{z}}(\underline{x})\left(L_{\underline{z},i\tau}f(\underline{x})-L_{\underline{z},i\tau}f(\underline{x}_0)\right). \tag{4.20}$$

Since we have adjusted the choice of the power n to the estimates in (4.17) and in (4.19) (notice that the latter also implies $\|L_{\underline{z},i\tau}f(\cdot)-L_{\underline{z},i\tau}f(\underline{x}_0)\|_\infty\leqslant 2M\theta^n$), we obtain

$$\|\sum_{\underline{z}\in\mathcal{S}_n} g^n_{\tau,\underline{z}}(\cdot)\left(L_{\underline{z},i\tau}f(\cdot)-L_{\underline{z},i\tau}f(\underline{x}_0)\right)\|_\theta \;<\; 1/2. \tag{4.21}$$

On the other hand, by the first of the inequalities in (4.19), the sequence of the coefficients $\{\mathbf{a}_{\underline{z}}[f]\}$ is a bounded one; $|\mathbf{a}_{\underline{z}}[f]|\leqslant M$. Since by (4.17) for every $\epsilon>0$ one can choose a finite subset $\mathcal{S}_{n,\epsilon}\Subset\mathcal{S}_n$ such that

$$\sum_{\underline{z}\notin\mathcal{S}_{n,\epsilon}}\|g^n_{\tau,\underline{z}}\|_\theta \;<\; \frac{\epsilon}{M},$$

we are able to derive the following estimate which holds uniformly in $\|f\|_\theta\leqslant 1$:

$$\|L^{n+1}_{i\tau}f(\cdot)-\sum_{\underline{z}\in\mathcal{S}_{n,\epsilon}}\mathbf{a}_{\underline{z}}[f]g^n_{\tau,\underline{z}}(\cdot)\|_\theta \;<\; \frac{1}{2}+\epsilon.$$

It remains to define the compact $K(\tau)\Subset\mathcal{G}_\theta$ as the set of all M-bounded linear combinations of the finite family $\{g^n_{\tau,\underline{z}}\}_{\underline{z}\in\mathcal{S}_{n,\epsilon}}$;

$$K(\tau) \;\stackrel{\Delta}{=}\; \left\{\sum_{\underline{z}\in\mathcal{S}_{n,\epsilon}} a_{\underline{z}}\, g^n_{\tau,\underline{z}}(\cdot)\;:\;\max_{\underline{z}}|a_{\underline{z}}|\leqslant M\right\},$$

and the target assertion (4.10) follows.

4.4. **The principal eigenfunction of L.** Two main complications we encounter here, as compared to the classical setup of subshifts over finite alphabets [Bow],[PP], are the non-compactness of the space $\mathcal{S}_\emptyset$ and the reducibility of the shift $(x_1, x_2, \dots) \mapsto (x_2, \dots)$ on $\mathcal{S}_\emptyset$. The latter is merely a nuisance. Nevertheless, it precludes an immediate reference to [Sa], where a non-compact version of Ruelle's Perron-Frobenius theorem has been established in a rather general irreducible context.

The results on the existence and strict positivity of the principal eigenfunction in the form we need them here, that is as asserted in Theorem 4.1, can be deduced from a generalized version of Krein-Rutman theorem [Nus2] on the set-condensing linear maps on cones. However, possibly the most transparent way to prove (4.24) is to use an approximation procedure similar to the one suggested in [CIs]: Let us enumerate the elements of S as $x_1, x_2, x_3, \dots$ For every $N \in \mathbb{N}$ define the truncated state space $S^{(N)} = \{x_1, \dots, x_N\}$, and, accordingly, define the space $\mathcal{S}_\emptyset^{(N)}$ of countable strings of elements from $S^{(N)} \cup \{\emptyset\}$ as in (4.1).

For every $\theta \in (0,1)$ $\mathcal{S}_\emptyset^{(N)}$ is a compact shift-invariant subset of $\mathcal{S}_\emptyset$ in the d_θ-distance (the topology does not depend on θ, of course). Let us use $\mathcal{G}_\theta^{(N)}$ to denote the restriction of $\mathcal{G}_\theta$ to $\mathcal{S}_\emptyset^{(N)}$. Proceeding along these lines, given an interaction potential ψ which satisfies the assumptions of Theorem 4.1 define the truncated operator $L^{(N)}$ on $\mathcal{G}_\theta^{(N)}$,

$$L^{(N)} f(\underline{x}) \;=\; \sum_{z \in S^{(N)}} \mathrm{e}^{\psi(z, \underline{x})} f(z, \underline{x}).$$

By Lemma 4.1,

$$\lim_{N \to \infty} \rho_{\mathbf{S}} \left(L^{(N)} \right) \;=\; \rho_{\mathbf{S}} \left(L \right). \tag{4.22}$$

On the other hand, despite the reducibility, the arguments of [PP](pp. 22-24) directly apply in the $(\mathcal{G}_\theta^{(N)}, L^{(N)})$-setup above. Consequently, there exists a positive eigenfunction $h^{(N)} \in \mathcal{G}_\theta^{(N)}$;

$$L^{(N)} h^{(N)} \;=\; \rho_{\mathbf{S}} \left(L^{(N)} \right) h^{(N)},$$

which, moreover, satisfies the following bound:

$$\forall\, \underline{x}, \underline{y} \in \mathcal{S}_\emptyset^{(N)} \quad h^{(N)}(\underline{x}) \geqslant \mathrm{e}^{-\beta \theta^{\mathbf{i}(\underline{x}, \underline{y})}} h^{(N)}(\underline{y}), \tag{4.23}$$

where the constant β has been defined in (4.13). Notice that the estimate (4.23) holds uniformly in the cutoffs N.

It is natural to normalize $h^{(N)}(\underline{\emptyset}) = 1$, so that for all N; $\mathrm{e}^{-\beta} \leqslant h^{(N)}(\cdot) \leqslant \mathrm{e}^{\beta}$. Then for every $N \in \mathbb{N}$ the restriction to $\mathcal{S}_\emptyset^{(N)}$ of the family $\{h^{(M)}\}_{M \geqslant N}$ is bounded in $\mathcal{G}_\theta^{(N)}$. Using the diagonal procedure, one can extract a subsequence, $\left\{h^{(M_k)}\right\}$ which converges in the $\|\cdot\|_\infty$-norm on each of the $\mathcal{S}_\emptyset^{(N)}$ sets. The limiting function, let us call it h, is defined on $\cup_N \mathcal{S}_\emptyset^{(N)}$ and inherits the following properties:

$$h(\underline{\emptyset}) = 1 \qquad \text{and} \qquad \forall\, \underline{x}, \underline{y} \in \bigcup_N \mathcal{S}_\emptyset^{(N)} \quad h(\underline{x}) \geqslant \mathrm{e}^{-\beta \theta^{\mathbf{i}(\underline{x}, \underline{y})}} h(\underline{y}).$$

Therefore, it can be extended by continuity to the whole of $\mathcal{S}_\emptyset$, and it is straightforward to check from (4.22) and (4.14) that the extension, which we continue to call h, is a strictly positive principal eigenvalue of L;

$$Lh = \rho_{\mathbf{S}}(L)h \qquad \text{and} \qquad \mathrm{e}^{-\beta} \leqslant h(\cdot) \leqslant \mathrm{e}^{\beta}. \tag{4.24}$$

This establishes (4.7) of Theorem 4.1

4.5. **Properties of $\rho_{\mathbf{S}}(L)$.** In principle it is possible to complete the proof of Theorem 4.1 along the lines of [Ru] (Proposition 5.4 on p.90) with necessary adjustments due to the fact that the unit ball of $\mathcal{G}_\theta$ is no longer compact in the space of continuous functions $\mathcal{C}$. Such an approach might also lead to additional complications related to the existence of transient states. In fact, the invariant measure in our case will be concentrated on the infinite strings of elements from S proper and put zero weight on the extended (by $\underline{\emptyset}$) finite strings from $\mathcal{S}_n$. Since the latter is the main object to be studied in the application to the sharp decay asymptotics of the two-point functions in the high temperature Ising models, we shall follow a different route:

Using (4.24) of the previous subsection we can normalize L, and, apart from the conditions imposed on the interaction ψ in the statement of Theorem 4.1, there is no loss of generality to assume that

$$\rho_{\mathbf{S}}(L) = 1 \ \text{ and, moreover, } \ L\mathbb{I}(\cdot) \equiv 1, \tag{4.25}$$

We need to show:

(A) $\lambda = 1$ is the only spectral point of $\Sigma_{\mathbf{S}}(L)$ on the spectral circle $\{z \in \mathbb{C} : |z| = 1\}$.

(B) The algebraic multiplicity of $\lambda = 1$ equals to one, or, equivalently, Range (P_L) is a one-dimensional sub-space spanned by the eigenfunction $\mathbb{I}(\cdot)$, where, as in the statement of Theorem 4.1, P_L is the spectral projector (4.5) at the principal eigenvalue $\lambda = 1$.

Once **(A)** and **(B)** above are verified, we readily recover the remaining exponential convergence result (4.9). Indeed, by **(B)**, for every $f \in \mathcal{G}_\theta$ there exists a number $c(f)$, such that $P_L f = c(f)\mathbb{I}$. On the other hand, the spectral radius of $L(I - P_L)$ is, by **(A)** above, strictly less than 1.

Let $\lambda \in \{z : |z| = 1\} \cap \Sigma_{\mathbf{S}}$. Since we have already established that $\rho_{\mathbf{F}} < 1$, it follows ([AKPRS], Subsection 2.6.12) that the eigenspace $\mathcal{N}(\lambda I - L)$ is not empty and finite-dimensional. Let h_λ be an eigenfunction; $Lh_\lambda = \lambda h_\lambda$. By the positivity of L,

$$L\,|h_\lambda|\,(\underline{x}) \geqslant |h_\lambda|\,(\underline{x}) \qquad \forall\, \underline{x} \in \mathcal{S}_\emptyset\,. \tag{4.26}$$

Since, for every $\underline{x} \in \mathcal{S}_\emptyset$ and each $n \in \mathbb{N}$, the probability distribution $\mathrm{e}^{\Psi_n(\cdot|\underline{x})}$ is strictly positive on $\mathcal{S}_n$, we infer from (4.26) that

$$\sup_{\underline{x} \in \mathcal{S}_\emptyset} |h_\lambda|\,(\underline{z}, \underline{x}) = \sup_{\underline{x} \in \mathcal{S}_\emptyset} |h_\lambda|\,(\underline{x}) \qquad \forall\, n \text{ and } \forall\, \underline{z} \in \mathcal{S}_n\,. \tag{4.27}$$

Indeed, taking $\sup_{\underline{x}}$ in both sides of (4.26) certainly does not change the " $\geqslant$ " sign of the latter inequality. On the other hand, $\sup_{\underline{x} \in \mathcal{S}_\emptyset} |h_\lambda|\,(\underline{z}, \underline{x}) \leqslant \sup_{\underline{x} \in \mathcal{S}_\emptyset} |h_\lambda|\,(\underline{x})$ for any n and $\underline{z}$.

The relation (4.27) suggests to consider the restriction of h_λ to the closed shift invariant subset $\mathcal{S} = \{\underline{x} : x_i \neq \emptyset\ \forall i\}$ of $\mathcal{S}_\emptyset$: The Hölder continuity of h_λ and the $n \to \infty$ limit in (4.27) readily imply:

$$|h_\lambda|\,(\cdot) \equiv \sup_{\underline{x} \in \mathcal{S}_\emptyset} |h_\lambda|\,(\underline{x}).$$

on $\mathcal{S}$. It is natural to normalize h_λ as $|h_\lambda| \equiv 1$ on $\mathcal{S}$. But then, given any $\underline{x} \in \mathcal{S}$, the function

$$\underline{z} \rightarrow \frac{h_\lambda(\underline{z}, \underline{x})}{\lambda^n h_\lambda(\underline{x})}$$

is also unimodal on every $\mathcal{S}_n$; $n = 1, 2, \dots$. Since

$$\sum_{\underline{z} \in \mathcal{S}_n} \mathrm{e}^{\Psi_n(\underline{z}|\underline{x})} \frac{h_\lambda(\underline{z}, \underline{x})}{\lambda^n h_\lambda(\underline{x})} = 1,$$

the normalization assumption (4.25), strict positivity of the weights $\mathrm{e}^{\Psi_n(\underline{z}|\underline{x})}$ and elementary convexity considerations imply that

$$h_\lambda(\underline{z}, \underline{x}) \;=\; \lambda^n h_\lambda(\underline{x}) \tag{4.28}$$

for every $\underline{x} \in \mathcal{S}$, $n \in \mathbb{N}$ and $\underline{z} \in \mathcal{S}_n$. Consequently, for every $n \in \mathbb{N}$,

$$\sup_{\underline{x}, \underline{y} \in \mathcal{S}} \left| h_\lambda(\underline{x}) - h_\lambda(\underline{y}) \right| \;\leqslant \theta^n \|h_\lambda\|_\theta,$$

or, in other words, h_λ is a multiple of $\mathbb{I}$ on $\mathcal{S}$. In particular, (4.28) already implies that $\lambda = 1$, and **(A)** follows. Furthermore, since $L^n h_\lambda(\underline{x}) = h_\lambda(\underline{x})$ for every $\underline{x} \in \mathcal{S}_\emptyset$ and h_λ is Hölder continuous, we, taking the limit $n \to \infty$, readily infer that, actually, $h_\lambda \equiv \mathbb{I}$ on the whole of $\mathcal{S}_\emptyset$.

In order to prove **(B)** notice, first of all, that the argument above implies that the eigenspace $\mathcal{N}(I - L)$ is, actually, spanned by $\mathbb{I}$, and, hence, $\lambda = 1$ is a simple eigenvalue. Now, since $\lambda = 1$ is a Fredholm point; $1 > \rho_{\mathbf{F}}(L)$, there exists a power $n_0 < \infty$, such that $(I - L)^{n_0} f = 0$ for every function f from the range of the projector $f \in \mathrm{Range}(P_L)$ ([Ka], Section III.6.5). If $n_0 = 1$, then, by the preceding remark, we are done. Otherwise, if $n_0 > 1$, then for every $f \in \mathrm{Range}(P_L)$ there exists a number $d(f) \in \mathbb{C}$, such that

$$(I - L)^{n_0 - 1} f \;=\; d(f)\mathbb{I}.$$

However, the equation $(I - L)g = d\mathbb{I}$ does not have solutions unless $d = 0$. Indeed, we may assume that both d and g are real and, in addition, that d is non-negative. But, by (4.25),

$$d \;=\; \inf_{\underline{x}} (g - Lg)(\underline{x}) \;\leqslant 0.$$

As a result $(I - L)^{n_0 - 1} f = 0$ for every $f \in \mathrm{Range}(P_L)$. This reduces the discussion back to the case of $n_0 = 1$, and **(B)** follows.

5. Local Limit Theorem

We continue to work in the framework and the notation of Section 4 and derive strong local limit type results associated with the Ruelle's operator L. The basic tool is to use the spectral theory in order to control the analytic expansions of the corresponding log-moment generating functions. We refer to [DS] for a thorough exposition of the local limit analysis of dependent $\mathbb{Z}^d$-valued random variables. See also [AD] where similar results in the CLT region (and, more generally, in the appropriate scaling regions for various stable laws) have been established for Gibbs-Markov maps.

5.1. **The setup and the result.** Let $V : S \mapsto \mathbb{Z}^d$ be an observable and $g \in \mathcal{G}_\theta$ is a positive function; $\inf_{\underline{x}} g(\underline{x}) > 0$. Assuming that the potential ψ; $|\psi|_\theta < \infty$, satisfies the finiteness assumption (4.2), we associate with each $\underline{x} \in \mathcal{S}_\emptyset$ and every $n \in \mathbb{N}$ the weight function $\mathbb{Q}_{n,\underline{x}}$ on $\mathbb{Z}^d$ via

$$\mathbb{Q}_{n,\underline{x}}(r) \ = \sum_{\underline{z}\in\mathcal{S}_n:\sum_1^n V(z_i)=r} \mathrm{e}^{\Psi_n(\underline{z}\ |\ \underline{x})} g(\underline{z},\underline{x}). \tag{5.1}$$

Our prime task here is to develop a sharp (as $n\to\infty$) asymptotic formula for the weights $\mathbb{Q}_{n,\underline{x}}$. The term "sharp" will always mean "up to zero order terms". An example of such a sharp asymptotic expression is provided by (4.9): There exists $c_1 > 0$, such that

$$\sum_r \mathbb{Q}_{n,\underline{x}}(r) \ = \ L^n g(\underline{x}) \ = \ d_g(\underline{x})\rho_{\mathbf{S}}^n(L)\left(1 \ + \ o(\mathrm{e}^{-c_1 n})\right),$$

where $d_g(\underline{x}) = P_L g(\underline{x})$. Since d_g is strictly positive and bounded away from zero, there is no loss of generality to assume that $\mathbb{Q}_{n,\underline{x}}$ is a probability measure on $\mathbb{Z}^d$:

$$\sum_r \mathbb{Q}_{n,\underline{x}}(r) \ = \ L^n g(\underline{x}) \ \equiv \ \rho_{\mathbf{S}}(L) \ = \ 1. \tag{5.2}$$

The essential assumptions are, of course, those imposed on the observable V:

A1. Range(V) generates $\mathbb{Z}^d$, in particular V is truly d-dimensional, in the sense that $\forall \xi \in \mathbb{R}^d \setminus 0$, the scalar product $(V(\cdot),\xi)_d \neq const.$

A2. There exists $\delta > 0$, such that for every $\xi \in \mathbb{R}^d$ with $|\xi| < \delta$

$$\sum_{\underline{z}\in\mathcal{S}_n} \mathrm{e}^{\Psi_n(\underline{z}\ |\ \underline{x})+\sum_1^n(\xi,V(z_k))_d} \ < \ \infty. \tag{5.3}$$

Notice that by the rigidity bound (4.12), the assumption **A2** is not sensitive to the choices of boundary condition $\underline{x}$ and powers $n = 1, 2, \ldots$. In particular,

$$L_\xi f(\underline{x}) \ = \ \sum_{z\in S} \mathrm{e}^{\psi(z,\underline{x})+(\xi,V(z))_d} f(z,\underline{x}), \tag{5.4}$$

is a well defined bounded linear operator on $\mathcal{G}_\theta$ for every $\xi \in \mathbb{C}^d$ with $|\mathrm{Re}(\xi)| < \delta$.

Our first result is a rough Gaussian large deviation upper bound which enables to focus the attention on the values of r near the running average

$$n v_{n,\underline{x}} \ \stackrel{\Delta}{=} \ \mathbb{E}_{n,\underline{x}} \sum_{k=1}^n V(z_k).$$

Lemma 5.1. *For every $\nu > 0$ there exist $c_2, c_3 > 0$, such that*

$$\sum_{r:|r-nv_{n,\underline{x}}|\ \geqslant\ n^{1-\nu}} \mathbb{Q}_{n,\underline{x}}(r) \ < c_2 \mathrm{e}^{-c_3 n^{1-2\nu}}. \tag{5.5}$$

Lemma 5.1 is a standard consequence of the exponential Markov inequality and the non-degeneracy condition (5.9) which is formulated below (and, subsequently, is proved in Subsection 5.4).

¿From now on we fix $\nu \in (0, 1/2)$ and concentrate on deriving uniform sharp asymptotics of $\mathbb{Q}_{n,\underline{x}}(r)$ over the set

$$R_{n,\nu} \ = \ \left\{ r \in \mathbb{Z}^d \ : \ |r - nv_{n,\underline{x}}| < n^{1-\nu} \right\}. \tag{5.6}$$

It is exactly on this stage that we shall extensively rely on the spectral analysis of Section 4. In order to structure our main result here in an optimal way let us formulate it in the form of several separate propositions:

We claim that there exists an open neighbourhood $\mathcal{U}$ of the origin in $\mathbb{C}^d$, such that, uniformly in boundary conditions $\underline{x} \in \mathcal{S}_\emptyset$, all the properties listed below hold:

Lemma 5.2. *The functions*

$$\rho_{\mathbf{S}}(\xi) \;\stackrel{\Delta}{=}\; \rho_{\mathbf{S}}(L_\xi) \qquad \text{and} \qquad \chi_{\underline{x}}(\xi) \;\stackrel{\Delta}{=}\; P_{L_\xi} g(\underline{x})$$

are analytic and bounded away from zero on $\mathcal{U}$. Furthermore, for every $\xi \in \mathcal{U}$, $\rho_{\mathbf{S}}(\xi)$ is (c.f. Subsection 4.1) a non-degenerate eigenvalue of L_ξ and, independently of a particular choice of $\xi \in \mathcal{U}$, there exists $\epsilon > 0$ such that the rest of the spectrum of L_ξ lies inside the circle of the radius $(1-\epsilon)|\rho_{\mathbf{S}}(\xi)|$.

This is a rather standard assertion of the analytic perturbation theory based on Theorem 4.1 and assumption **A2**. We shall explain it in more detail (and with the appropriate references to [Ka]) in Subsection 5.3.

As it follows from Lemma 5.2, the log-Laplace transforms

$$\mathbb{H}_{n,\underline{x}}(\xi) \;\stackrel{\Delta}{=}\; \frac{1}{n} \log L_\xi^n g(\underline{x}) \tag{5.7}$$

are defined and analytic on $\mathcal{U}$. Moreover,

Lemma 5.3. *There exist $c_4 > 0$ such that, uniformly in $\underline{x} \in \mathcal{S}_\emptyset$ and $\xi \in \mathcal{U}$,*

$$\mathbb{H}_{n,\underline{x}}(\xi) \;=\; \log \rho_{\mathbf{S}}(\xi) + \frac{1}{n} \log \chi_{\underline{x}}(\xi) + o\left(\mathrm{e}^{-c_4 n}\right). \tag{5.8}$$

In addition the Hessians $\mathrm{Hess}\left(\mathbb{H}_{n,\underline{x}}\right)$ *are uniformly non-degenerate at $\xi = 0$;*

$$\inf_{\underline{x} \in \mathcal{S}_\emptyset} \left|\det\left(\mathrm{Hess}(\mathbb{H}_{n,\underline{x}})(0)\right)\right| \;>\; 0. \tag{5.9}$$

The non-degeneracy condition (5.9) is responsible for the Gaussian form of our main uniform local limit result: Define

$$A_{\mathbf{S}} \;=\; \mathrm{Hess}\left(\log \rho_{\mathbf{S}}\right)(0).$$

Theorem 5.1. *Uniformly in $r \in R_{n,\nu}$ (see (5.6)) and $\underline{x} \in \mathcal{S}_\emptyset$*

$$\mathbb{Q}_{n,\underline{x}}(r) \;=\; \frac{d_g(\underline{x}) \rho_{\mathbf{S}}^n(0)}{\sqrt{(2\pi n)^d \det(A_{\mathbf{S}})}} \exp\left\{ -\frac{1}{2n} \left(A_{\mathbf{S}}^{-1}(r - n v_{n,\underline{x}}), (r - n v_{n,\underline{x}}) \right)_d \right\} (1 \;+\; o(1)). \tag{5.10}$$

Notice that since the running average $v_{n,\underline{x}} = \nabla \mathbb{H}_{n,\underline{x}}(0)$, the uniform analytic expansion (5.8) implies,

$$v_{n,\underline{x}} \;=\; \nabla \log \rho_{\mathbf{S}}(0) + \frac{1}{n} \nabla \log \chi_{\underline{x}}(0) + o\left(\mathrm{e}^{-c_5 n}\right), \tag{5.11}$$

and, consequently, we could have written the $\underline{x}$-independent term $\nabla \log \rho_{\mathbf{S}}(0)$ instead of $v_{n,\underline{x}}$ in the target asymptotic formula (5.10).

Proof of Theorem 5.1. The proof is a blend of the conventional local CLT techniques and the (equally conventional) change of measure by exponential tilts argument reinforced with an analytic control over log-Laplace transforms through the expansion (5.8). We shall merely sketch it here with an emphasis on how the spectral analysis of the Ruelle's operator enters the picture. We refer to [DS] for a comprehensive general exposition of the local limit theory and also to [PP], where similar results are obtained for the Ruelle's operators over finite alphabets.

As before, there is no loss of generality to assume that $\mathbb{Q}_{n,\underline{x}}$ is a probability measure on $\mathbb{Z}^d$, in particular, that $d_g(\underline{x}) \equiv 1$ and that $\rho_{\mathbf{S}}(L) = \rho_{\mathbf{S}}(0) = 1$.

STEP 1 Fix a small $\epsilon > 0$. We shall start by proving (5.10) for the values of r satisfying (see (5.11))

$$|r - n\nabla \log \rho_{\mathbf{S}}(0)| \leqslant n^{1/2-2\epsilon}. \tag{5.12}$$

In this case the target asymptotic expression (5.10) of Theorem 5.1 takes a simpler form:

$$\mathbb{Q}_{n,\underline{x}}(r) = \frac{1}{\sqrt{(2\pi n)^d \det(A_{\mathbf{S}})}} (1 + o(1)). \tag{5.13}$$

Let $\widehat{\mathbb{Q}}_{n,\underline{x}}$ denote the Fourier transform of $\mathbb{Q}_{n,\underline{x}}$,

$$\widehat{\mathbb{Q}}_{n,\underline{x}}(\tau) = \sum_{t\in\mathbb{Z}^d} \mathbb{Q}_{n,\underline{x}}(t) \mathrm{e}^{i(\tau,t)_d} = L^n_{i\tau} g(\underline{x}).$$

By the Fourier inversion formula,

$$\mathbb{Q}_{n,\underline{x}}(r) = \frac{1}{(2\pi)^d} \int \cdots \int_{[-\pi,\pi]^d} \mathrm{e}^{-i(\tau,r)_d} \widehat{\mathbb{Q}}_{n,\underline{x}}(\tau) \mathrm{d}\tau. \tag{5.14}$$

Given $\delta > 0$, we split $[-\pi,\pi]^d$ into three disjoint regions of integration:

$$\begin{aligned} [-\pi,\pi]^d &= A_\epsilon \vee A_{\epsilon,\delta} \vee A_\delta \\ &\triangleq \left\{\tau : |\tau| < n^{-1/2+\epsilon}\right\} \vee \left\{\tau : n^{-1/2+\epsilon} \leqslant |\tau| < \delta\right\} \vee \{\tau : |\tau| \geqslant \delta\}. \end{aligned} \tag{5.15}$$

The integral over A_δ could be ignored by the virtue of the following proposition, which we shall prove in Subsection 5.2:

Proposition 5.1. *For every $\delta > 0$ there exists $\eta = \eta(\delta) > 0$, such that*

$$\sup_{\tau\in[-\pi,\pi]^d\setminus(-\delta,\delta)^d} \rho_{\mathbf{S}}(L_{i\tau}) \leqslant 1 - \eta. \tag{5.16}$$

An immediate consequence is that, uniformly in $\underline{x} \in \mathcal{S}_\emptyset$ and $\tau \in A_\delta$,

$$|\widehat{\mathbb{Q}}_{n,\underline{x}}(\tau)| \leqslant \mathrm{e}^{n\log(1-\eta)}. \tag{5.17}$$

Turning to $A_{\epsilon,\delta}$ notice that if $\delta > 0$ is sufficiently small, then $iA_{\epsilon,\delta} \subset \mathcal{U}$, and, consequently,

$$\widehat{\mathbb{Q}}_{n,\underline{x}}(\tau) = \mathrm{e}^{n\mathbb{H}_{n,\underline{x}}(i\tau)}.$$

Choosing, if necessary, $\delta > 0$ even smaller, we infer from the analytic expansion formula (5.8) and the non-degeneracy of Hess $(\log \rho_{\mathbf{S}})(0)$, that there exists $c_5 > 0$, such that ,

$$|\widehat{\mathbb{Q}}_{n,\underline{x}}(\tau)| \leqslant \mathrm{e}^{-c_5 n|\tau|^2} \leqslant \mathrm{e}^{-c_5 n^{2\epsilon}}, \tag{5.18}$$

uniformly in $\underline{x} \in \mathcal{S}_\emptyset$ and $\tau \in A_{\epsilon,\delta}$.

Finally, uniformly in τ from the remaining region A_ϵ,

$$\begin{aligned} \widehat{\mathbb{Q}}_{n,\underline{x}}(\tau)\mathrm{e}^{-i(\tau,r)_d} &\overset{(5.12)}{=} \exp\left\{n\mathbb{H}_{n,\underline{x}}(i\tau) - in\left(\tau, \nabla \log \rho_{\mathbf{S}}(0)\right)_d + o(1)\right\} \\ &\overset{(5.8)}{=} \exp\left\{-\frac{n}{2}\left(A_{\mathbf{S}}\tau,\tau\right)_d + o(1)\right\}, \end{aligned}$$

and (5.13) follows.

STEP 2 In order to extend the result to the full range of $r \in R_{n,\nu}$ as it has been asserted in Theorem 5.1, consider the family of "tilted" measures $\{\mathbb{Q}^{\xi}_{n,\underline{x}}\}$ (indexed by $\xi \in \mathcal{U} \cap \mathbb{R}^d$):

$$\mathbb{Q}^{\xi}_{n,\underline{x}}(r) \;=\; \frac{\mathrm{e}^{(\xi,r)_d}}{L_{\xi} g(\underline{x})} \mathbb{Q}_{n,\underline{x}}(r) \;=\; \exp\left\{(\xi,r)_d - n\mathbb{H}_{n,\underline{x}}(\xi)\right\} \mathbb{Q}_{n,\underline{x}}(r).$$

The expectation $nv_{n,\underline{x}}(\xi)$ under the measure $\mathbb{Q}^{\xi}_{n,\underline{x}}$ is, according to (5.8), given by the following asymptotic expression:

$$v_{n,\underline{x}}(\xi) \;=\; \nabla \log \rho_{\mathbf{S}}(\xi) + \frac{1}{n}\nabla \log \chi_{\underline{x}}(\xi) + o\left(\mathrm{e}^{-c_6 n}\right).$$

Since the Hessian $\mathrm{Hess}(\log \rho_{\mathbf{S}})$ is non-degenerate at $\xi = 0$, we, actually independently from $\underline{x} \in \mathcal{S}_{\emptyset}$, can pick a small $\delta > 0$, such that the map $\xi \mapsto v_{n,\underline{x}}(\xi)$ has an analytic inverse on $\{\xi \,:\, |\xi| < \delta\} \subset \mathcal{U}$. Since, in this case,

$$R_{n,\nu} \;\subset\; \mathrm{Range}\left(nv_{n,\underline{x}}(\xi)\Big|_{|\xi|<\delta} \right),$$

as soon as n is sufficiently large (also uniformly in $\underline{x} \in \mathcal{S}_{\emptyset}$), we are entitled to introduce the notation

$$\xi_{n,\underline{x}} = \xi_{n,\underline{x}}(r) \;=\; v^{-1}_{n,\underline{x}}(r/n) \qquad \text{or, equivalently,} \qquad \frac{r}{n} \;=\; \nabla \mathbb{H}_{n,\underline{x}}(\xi_{n,\underline{x}}). \tag{5.19}$$

Then the analytic implicit function theorem (c.f. [DS]) implies that, uniformly in $r \in R_{n,\nu}$ and $\underline{x} \in \mathcal{S}_{\emptyset}$,

$$\xi_{n,\underline{x}}(r) \;=\; A^{-1}_{\mathbf{S}}\left(\frac{r}{n} - \nabla \log \rho_{\mathbf{S}}(0)\right) \;+\; \mathrm{O}\left(n^{-2\nu}\right). \tag{5.20}$$

As a result, we conclude that, uniformly in $\underline{x} \in \mathcal{S}_{\emptyset}$ and $r \in R_{n,\nu}$,

$$\begin{aligned}\mathbb{Q}_{n,\underline{x}}(r) \;&=\; \exp\left\{-n\left((\frac{r}{n}, \xi_{n,\underline{x}}(r))_d - \mathbb{H}_{n,\underline{x}}(\xi_{n,\underline{x}})\right)\right\} \mathbb{Q}^{\xi_{n,\underline{x}}}_{n,\underline{x}}(r) \\ &\overset{(5.8),(5.20)}{=} \exp\left\{-\frac{1}{2n}\left(A^{-1}_{\mathbf{S}}(r - n\nabla \log \rho_{\mathbf{S}}(0)), (r - n\nabla \log \rho_{\mathbf{S}}(0))\right)_d\right\} \mathbb{Q}^{\xi_{n,\underline{x}}}_{n,\underline{x}}(r)\,(1 + o(1)).\end{aligned}$$

Finally, by the very choice of the tilt $\xi_{n,\underline{x}}(r)$ in (5.19), the results of **Step 1** apply to yield the desirable prefactor expression for $\mathbb{Q}^{\xi_{n,\underline{x}}}_{n,\underline{x}}(r)$. ◻

5.2. **Decay off the real axis.** In this subsection we establish the claim of Proposition 5.1. The proof involves three steps:

STEP 1 Fredholm spectrum of $L_{i\tau}$

This has been already performed in Subsection 4.3, and by (4.6) of Theorem 4.1, $\rho_{\mathbf{F}}(L_{i\tau}) \;<\; 1$ holds for every $\tau \in [-\pi,\pi]^d$.

STEP 2 Spectrum of $L_{i\tau}$ for $\tau \neq 0$.

Lemma 5.4. *Assume that $\tau \neq 0$. Then,*

$$\rho_{\mathbf{S}}(L_{i\tau}) \;<\; 1. \tag{5.21}$$

Proof. If there exists $\lambda \in \Sigma_{\mathbf{S}}(L_{i\tau})$ with $|\lambda| \geqslant 1$, then, by the preceding step, λ is a Fredholm point and, as such, is, necessarily, an eigenvalue of $L_{i\tau}$. Let $h_{\lambda} \in \mathcal{G}_{\theta}$ be a corresponding eigenfunction;

$$L_{i\tau} h_{\lambda} \;=\; \lambda h_{\lambda}.$$

Taking the absolute values,

$$L|h_\lambda| \ \geqslant |\lambda|\,|h_\lambda|.$$

Since L is normalized we, following the line of reasoning employed in Subsection 4.5, infer that $|\lambda| = 1$ as well as that h_λ is unimodal, $|h_\lambda| \equiv 1$. Consequently, for every $\underline{x} \in \mathcal{S}_\emptyset$, every $n \in \mathbb{N}$ and each $\underline{z} \in \mathcal{S}_n$,

$$\mathrm{e}^{i\sum_1^n (\tau, V(z_k))_d} h_\lambda(\underline{z}, \underline{x}) \ = \ \lambda^n h_\lambda(\underline{x}), \tag{5.22}$$

and then, taking $n \to \infty$, conclude that h_λ is a multiple of $\mathbb{I}$. In view of (5.22) this means that $(\tau, V(z))_d$ is independent of $z \in S$, a contradiction to the Assumption **A1** of Subsection 5.1. □

STEP 3 Uniform estimate on $\rho_{\mathbf{S}}\left(L_{i\tau}\right)$.

It remain to show that, given $\delta > 0$, the inequality (5.21) holds uniformly over $\tau \in [-\pi, \pi]^d \setminus (-\delta, \delta)^d$.This follows from well known facts on the lower-semicontinuity of the spectrum. Assume that this is not the case, and there exists a sequence $\{\tau_k\} \subset [-\pi, \pi]^d \setminus (-\delta, \delta)^d$ and a sequence of numbers $\lambda_k \in \Sigma_{\mathbf{S}}\left(L_{i\tau_k}\right)$, such that

$$\lim_{k\to\infty} |\lambda_k| \ = \ 1.$$

Without loss of the generality we may assume that $\{\tau_k\}$ converges to some $\tau \neq 0$ and $\{\lambda_k\}$ converges to some λ with $|\lambda| = 1$. By Lemma 5.4, however, $\rho_{\mathbf{S}}\left(L_{i\tau}\right) < 1$. Therefore, λ belongs to the resolvent set of $L_{i\tau}$. The latter is open, and one can find an $\epsilon > 0$, such that the operator norm

$$\|\left(\mu I - L_{i\tau}\right)^{-1}\|_\theta \ \leqslant \epsilon^{-1} \tag{5.23}$$

for every $|\mu - \lambda| \leqslant \epsilon$. On the other hand, $L_{i\tau_k}$ converges to $L_{i\tau}$ in the strong operator topology: For every $f \in \mathcal{G}_\theta$,

$$\|\left(L_{i\tau_k} - L_{i\tau}\right) f\|_\infty \ \leqslant \|f\|_\infty \sum_{z\in S} \left|1 - \mathrm{e}^{i(\tau-\tau_k, V(z))_d}\right| \phi(z),$$

where we have introduced the notation

$$\phi(z) \ = \ \sup_{\underline{x}} \mathrm{e}^{\psi(z,\underline{x})} \qquad \left(\text{Notice that by (4.12)}, \quad \sum_z \phi(z) < \infty\right).$$

Similarly, using (4.12),

$$\|\left(L_{i\tau_k} - L_{i\tau}\right) f\|_\theta \ \leqslant \ \left(c_3(\psi)\|f\|_\infty + \|f\|_\theta\right) \sum_{z\in S} \left|1 - \mathrm{e}^{i(\tau-\tau_k, V(z))_d}\right| \phi(z),$$

with

$$c_3(\psi) \ = \ \sup_{t\in(0,1)} \frac{\mathrm{e}^{\beta t} - 1}{t},$$

and β specified in (4.13). Thus, $\lim_{k\to\infty} \|L_{i\tau_k} - L_{i\tau}\|_\theta \ = \ 0$, follows by the bounded convergence theorem. As a result, it follows from (5.23) that $(\mu I - L_{i\tau_k})$ is invertible on $\{\mu \,:\, |\mu - \lambda| < \epsilon\}$, as soon as τ_k is close enough to τ, which is, of course, a contradiction. □

5.3. **Perturbation theory of non-degenerate eigenvalues.** Let $\mathcal{F}$ be a Banach space, $\mathcal{F}^*$ its dual and $B_\delta \subset \mathbb{C}$ is an open ball $B_\delta = \{z \,:\, |z| < \delta\}$.
Definition. A uniformly bounded family of linear operators $\{T(\xi)\}_{\xi\in D}$ is said to be holomorphic on B_δ if

$$\forall f \in \mathcal{F} \text{ and } \forall f^* \in \mathcal{F}^* \qquad \text{the map } \xi \mapsto (T(\xi)f, f^*) \text{ is holomorphic in } B_\delta .$$

We rely on the following statement of the analytic perturbation theory (c.f. [Ka], Section VII.1.3):
Let $\{T(\xi)\}$ be a holomorphic family of operators on B_δ, and assume that $\lambda = \lambda(0)$ is a *non-degenerate* eigenvalue of $T(0)$. Then given a closed contour Γ with $\mathrm{ext}(\Gamma) \cap \Sigma_{\mathbf{S}}(T(0)) = \{\lambda\}$, there exists $\epsilon \in (0, \delta)$, such that:
1) For every $T(\xi)$ with $|\xi| < \epsilon$, there is exactly one spectral point $\lambda(\xi)$, such that $\{\lambda(\xi)\} = \mathrm{ext}(\Gamma) \cap \Sigma_{\mathbf{S}}(T(\xi))$.
2) $\lambda(\xi)$ is a non-degenerate eigenvalue of $T(\xi)$ and the map $\xi \mapsto \lambda(\xi)$ is analytic on B_ϵ.
We use this result in the following way: By (5.3) the family of operators $\{L_\xi\}$ on $\mathcal{G}_\theta$ is holomorphic on B_δ for some $\delta > 0$. According to Theorem 4.1, $\lambda(0) = \rho_{\mathbf{S}}(L)$ is a non-degenerate eigenvalue of $L = L_0$ and, moreover, there exists $\nu > 0$, such that the exteriour of $\Gamma_\nu \triangleq \{z \in \mathbb{C} \,:\, |z| = (1-2\nu)\rho_{\mathbf{S}}(L)\}$ satisfies

$$\mathrm{ext}\,(\Gamma_\nu) \cap \Sigma_{\mathbf{S}}\,(L_0) \;=\; \{\lambda(0)\}.$$

Consequently, there exists $\epsilon > 0$ and an analytic function $\lambda(\xi)$ on B_ϵ, such that for every $|\xi| \leqslant \epsilon$ the number $\lambda(\xi) \triangleq \rho_{\mathbf{S}}(L_\xi)$ is a non-degenerate eigenvalue of L_ξ, $|\lambda(\xi) - \lambda(0)| < \nu$, and

$$\mathrm{ext}\,(\Gamma_\nu) \cap \Sigma_{\mathbf{S}}\,(L_\xi) \;=\; \{\lambda(\xi)\}.$$

It follows that the family of projectors

$$P_{L_\xi} \;\triangleq\; I \;+\; \frac{1}{2\pi i}\oint_{\Gamma_\nu} (\mu I - L_\xi)^{-1}\, \mathrm{d}\mu$$

is analytic on B_ϵ, and so is the family

$$\chi_{\underline{x}}(\xi) \triangleq P_{L_\xi} g(\underline{x}),$$

which shows up in the statement of Lemma 5.2. Since $\inf_{\underline{x}} Lg(\underline{x}) > 0$, we can, if necessary, choose ϵ so small that $\{\chi_{\underline{x}}(\xi)\}$ is, uniformly in $\underline{x} \in \mathcal{S}_\emptyset$ and $\xi \in B_\epsilon$, bounded away from zero. All the conclusions of Lemma 5.2 are, thereby, verified. Furthermore, for every $\xi \in B_\epsilon$ and $\underline{x} \in \mathcal{S}_\emptyset$,

$$L_\xi^n g(\underline{x}) \;=\; \chi_{\underline{x}}(\xi)\rho_{\mathbf{S}}^n(\xi) \;+\; o\left((1-\nu)^n |\rho_{\mathbf{S}}(\xi)|^n\right).$$

The expansion (5.8) follows.

5.4. **Non-degeneracy of** $\mathrm{Hess}\,(\log\rho_{\mathbf{S}})\,(0)$**.** One has to show that there exists a positive $\alpha > 0$, such that the variance

$$\left(\mathrm{Hess}(\mathbb{H}_{n,\underline{x}})l, l\right)_d \;=\; \frac{1}{n}\mathbb{V}\mathrm{ar}_{n,\underline{x}}\left(\sum_{k=1}^{n} (V(z_i), l)_d\right) \;\geqslant \alpha |l|^2,$$

uniformly in n, $\underline{x} \in \mathcal{S}_\emptyset$ and $l \in \mathbb{R}^d$. This follows from the conditional variance argument based on the assumptions **A1**,**A2** and the Hölder upper bound (4.12). Indeed, let $n =$

$km + l$. Then,

$$\mathbb{V}\mathrm{ar}_{n,\underline{x}}\left(\sum_{i=1}^{n}(V(z_i),l)_d\right) \geqslant \min_{\bar{z}_j\in S\, j\neq 0 \bmod (m)} \mathbb{V}\mathrm{ar}_{n,\underline{x}}\left(\sum_{i=1}^{k}(V(z_{im}),l)_d \mid z_j=\bar{z}_j\right)$$

However the conditional variances of the variables $V(z_{im})$ are, uniformly in $\{\bar{z}_j\}$, bounded away both from zero and ∞, whereas the correlation coefficient between different $V(z_{im})$'s decays to zero exponentially fast with m. $\square$

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Dipartimento di Matematica, Università di Bologna, piazza di Porta S. Donato 5, I-40126 Bologna, Italy
E-mail address: campanin@@dm.unibo.it

Faculty of Industrial Engineering, Technion, Haifa 3200, Israel
E-mail address: ieioffe@@ie.technion.ac.il

Laboratoire d'Analyse, Topologie et Probabilités, UMR-CNRS 6632, CMI, Université de Provence, 39 rue Joliot Curie, 13453 Marseille, France
E-mail address: velenik@cmi.univ-mrs.fr