PERTURBATIONS OF THE WIGNER-VON NEUMANN POTENTIAL LEAVING THE EMBEDDED EIGENVALUE FIXED

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Abstract. We investigate the Schrödinger operator $H = -d^2/dx^2 + (\gamma/x)\sin\alpha x + V$, acting in $L^p(\mathbf{R})$, $1 \leq p < \infty$, where $\gamma \in \mathbf{R}\setminus\{0\}$, $\alpha > 0$, and $V \in L^1(\mathbf{R})$. For $|\gamma| \leq 2\alpha/p$ we show that H does not have positive eigenvalues. For $|\gamma| > 2\alpha/p$ we show that the set of functions $V \in L^1(\mathbf{R})$, such that H has a positive eigenvalue embedded in the essential spectrum $\sigma_{\mathrm{ess}}(H) = [0,\infty)$, is a smooth unbounded sub-manifold of $L^1(\mathbf{R})$ of codimension one.

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Perturbations du Potentiel Wigner-Von Neumann

Qui Fixent la Valeur Caractéristique Immergée

Résumé. On examine l'opérateur de Schrödinger $H = -d^2/dx^2 + (\gamma/x)\sin\alpha x + V$ défini dans $L^p(\mathbf{R})$, $1 \leq p < \infty$, où $\gamma \in \mathbf{R} \setminus \{0\}$, $\alpha > 0$, et $V \in L^1(\mathbf{R})$. Si $|\gamma| \leq 2\alpha/p$, on montre que H n'a aucune valeur caractéristique positive. Si $|\gamma| > 2\alpha/p$, on montre que l'ensemble des fonctions $V \in L^1(\mathbf{R})$, telles que H a une valeur caractéristique positive immergée dans le spectre essentiel $\sigma_{\mathrm{ess}}(H) = [0, \infty)$, est une sous-variété lisse non-bornée de $L^1(\mathbf{R})$ de codimension égale à un.

1. Introduction

In this paper we consider Schrödinger operators of the form

$$H_{Q,p} = -\frac{d^2}{dx^2} + Q, \tag{1.1}$$

acting in $L^p(\mathbf{R})$, $1 \le p < \infty$, where $Q = W + V$, $W(x) = (\gamma/x)\sin\alpha x$, $\alpha > 0$ and $\gamma \in \mathbf{R}\setminus\{0\}$ are constants, and V is a real-valued function in $L^1(\mathbf{R})$. To give a precise definition of the operator $H_{Q,p}$ we use the *Feynman-Kac formula.* For $f \in \cup_{p\ge 1} L^p(\mathbf{R})$ and $t \ge 0$ we define

$$U_Q(t)f(x) = E_x\left(\exp\left\{-\int_0^t Q(b(s))ds\right\} f(b(t))\right), \tag{1.2}$$

where E_x denotes the expectation with respect to Brownian motion starting at x with Brownian transition function given by

$$p_t(x,y) = \frac{\exp\left(-(x-y)^2/4t\right)}{\sqrt{4\pi t}}, \qquad x,y \in \mathbf{R}, \quad t \ge 0. \tag{1.3}$$

We define $H_{Q,p}$ to be the negative of the infinitesimal generator of the C_0-semigroup $(U_{Q,p}(t); t \ge 0)$, $1 \le p < \infty$, defined for $f \in L^p(\mathbf{R})$ and $t \ge 0$ by $U_{Q,p}(t)f = U_Q(t)f$.

Various classes of operators which contain the ones defined above have been investigated in [6, 12, 16, 17, 18, 26, 28, 30], and it is well known that the spectrum $\sigma(H_{Q,p})$ is p-independent and that $\sigma_{\text{ess}}(H_{Q,p}) = [0,\infty)$ for all $p \ge 1$.

Schrödinger operators of the form (1.1) were introduced by *Wigner* and *Von-Neumann* [29] in order to construct an example of a Schrödinger operator, acting in $L^2(\mathbf{R}^3)$, with a spherically symmetric potential which vanishes at infinity and possesses a positive eigenvalue embedded in the continuum. The significance of the Wigner-Von Neumann example lies in the fact that at the time it contradicted physical intuition, which predicted that bound states of positive energy could not occur if the potential tended to zero at infinity.

In this paper we study the structure of the set of functions $V \in L^1(\mathbf{R})$ for which the operator $H_{Q,p}$ has a positive eigenvalue.

Our main result is:

Theorem 1.1. *Let $H_{Q,p}$ be as in (1.1). If $|\gamma| \leq 2\alpha/p$, then $H_{Q,p}$ does not have positive eigenvalues. If $|\gamma| > 2\alpha/p$, then the set of functions $V \in L^1(\mathbf{R})$ such that $H_{Q,p}$ has a positive eigenvalue embedded in the essential spectrum $\sigma_{\text{ess}}(H_{Q,p}) = [0, \infty)$ is a smooth unbounded sub-manifold of $L^1(\mathbf{R})$ of codimension one. In addition, if V belongs to this sub-manifold then $\alpha^2/4$ is the unique positive eigenvalue of $H_{Q,p}$.*

It is well known that the eigenvalues in the discrete spectrum are, in an appropriate setting, stable under perturbations. On the other hand, it is also known that embedded eigenvalues in the continuum are rather unstable [1, 2, 9, 10, 20]. In [2] *Agmon*, *Herbst*, and *Skibsted* prove that generically, in a Baire category sense, arbitrarily small perturbations of a generalized N-body Hamiltonian remove all non-threshold eigenvalues embedded in the continuum, and conjecture that the set of perturbations that preserve a non-threshold embedded eigenvalue is something like a differentiable manifold. The result presented in this paper shows that the above conjecture is true for the simplest Schrödinger operators which possess an eigenvalue embedded in the continuum. A similar result for $p = 2$ was announced in [11] without proof.

For $\alpha > 0$ and $\gamma \in \mathbf{R} \setminus \{0\}$, let $M(\alpha, \gamma)$ be the set of functions $V \in L^1(\mathbf{R})$ such that, for some $k > 0$, the differential equation

$$-\psi'' + \gamma \frac{\sin \alpha r}{r} \psi + V\psi = k^2 \psi, \qquad r \in \mathbf{R}, \tag{1.4}$$

has a nonzero solution that goes to zero as $|r|$ goes to infinity. We say that a function ψ is a solution of this differential equation if it is continuously differentiable, ψ' is absolutely continuous, and (1.4) holds almost everywhere. Local existence of solutions to (1.4) is well known. We also prove

Theorem 1.2. *Let $M(\alpha, \gamma)$ be as defined above. Then $M(\alpha, \gamma)$ is a smooth unbounded sub-manifold of $L^1(\mathbf{R})$ of codimension one. In addition, if $V \in M(\alpha, \gamma)$ then $k = \alpha/2$.*

Using the terminology of [19], $M(\alpha, \gamma)$ is the set of functions V in $L^1(\mathbf{R})$ such that $H_{Q,2}$ has a half-bound state of positive energy.

To prove the results stated above we determine, following *Cassell* [7], the exact asymptotic behavior at infinity of the solutions to (1.4) and then identify the set of functions V in $L^1(\mathbf{R})$ that produce positive eigenvalues of $H_{Q,p}$ with the zero set of a smooth function on $L^1(\mathbf{R})$ for which zero is a regular value.

Schrödinger operators with eigenvalues in the continuous spectrum have also been investigated in [3, 14, 23], and the asymptotic behavior of the solutions of (1.4) for various classes of potentials has also been studied in [4, 7, 13, 15, 22]. For perturbations of embedded eigenvalues in situations which are relevant to the automorphic Laplacian and N-body Schrödinger operators see [2, 5, 9, 10, 20, 25]. In a different context, results of the type presented here have been obtained in [21].

This paper is organized as follows. In Section 2 we investigate the asymptotic behavior at infinity of solutions to (1.4) and establish the existence of solutions that vanish at infinity. In Section 3 we prove the main results. In the Appendix we establish the connection between the eigenfuctions of $H_{Q,p}$ and the solutions of (1.4) that belong to $L^p(\mathbf{R})$.

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2. Existence of Solutions that Vanish at Infinity

In this section we follow *Cassell* [7] to determine the asymptotic behavior as r goes to infinity of the solutions to (1.4). We will prove

Theorem 2.1 *For $\alpha > 0$, $\gamma \in \mathbf{R} \setminus \{0\}$, $k > 0$, and $V \in L^1(\mathbf{R})$ we have:*

i) If $k \neq \alpha/2$, then (1.4) has solutions ϕ and ψ such that, as r goes to $+\infty$,

$$\phi(r) = \cos kr + o(1) \qquad \textit{and} \qquad \psi(r) = \sin kr + o(1),$$

with

$$\phi'(r) = -k \sin kr + o(1) \qquad \textit{and} \qquad \psi'(r) = k \cos kr + o(1).$$

ii) If $k = \alpha/2$, then (1.4) has solutions ϕ and ψ such that, as r goes to $+\infty$,

$$\phi(r) = r^{-\gamma/2\alpha}(\cos kr + o(1)) \qquad \textit{and} \qquad \psi(r) = r^{\gamma/2\alpha}(\sin kr + o(1)),$$

with

$$\phi'(r) = -kr^{-\gamma/2\alpha}(\sin kr + o(1)) \qquad \textit{and} \qquad \psi'(r) = kr^{\gamma/2\alpha}(\cos kr + o(1)).$$

Proof. Setting $\xi(r) = \psi(r/k)$, $\sigma = \alpha/k > 0$, and $\eta = \gamma/k \in \mathbf{R} \setminus \{0\}$ we find that ξ satisfies

$$-\xi'' + \eta\frac{\sin \sigma r}{r}\xi + W\xi = \xi, \tag{2.1}$$

where $W(r) = V(r/k)/k^2 \in L^1(\mathbf{R})$. Using the transformation

$$x = \begin{pmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{pmatrix} \begin{pmatrix} \xi \\ \xi' \end{pmatrix},$$

we see that (2.1) is equivalent to

$$x' = A(r)x, \tag{2.2}$$

where

$$A(r) = a(r)\begin{pmatrix} \sin r \cos r & \sin^2 r \\ -\cos^2 r & -\sin r \cos r \end{pmatrix},$$

and $a(r) = -\eta\dfrac{\sin \sigma r}{r} - W(r)$.

Next we write $A(r) = (\eta/r)G(r) + R(r)$, where $R(r)$ is the L^1-matrix given by

$$R(r) = -W(r)\begin{pmatrix} \sin r \cos r & \sin^2 r \\ -\cos^2 r & -\sin r \cos r \end{pmatrix}, \tag{2.3}$$

and

$$G(r) \equiv \begin{pmatrix} g_1(r) & g_2(r) \\ g_3(r) & -g_1(r) \end{pmatrix}, \tag{2.4}$$

with

$$g_1(r) = -\frac{1}{4}(\cos(\sigma - 2)r - \cos(\sigma + 2)r),$$

$$g_2(r) = -\frac{1}{4}(2\sin\sigma r - \sin(\sigma+2)r - \sin(\sigma-2)r),$$

and

$$g_3(r) = \frac{1}{4}(2\sin\sigma r + \sin(\sigma+2)r + \sin(\sigma-2)r).$$

Now we decompose G as $G = G_1 + G_2$, where $G_1 = 0$ and $G_2 = G$ for $\sigma \neq 2$, and

$$G_1 = \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

and

$$G_2(r) = \frac{1}{4}\begin{pmatrix} \cos 4r & -2\sin 2r + \sin 4r \\ 2\sin 2r + \sin 4r & -\cos 4r \end{pmatrix}$$

for $\sigma = 2$.

Setting $S(r) = I + (\eta/r)G_2^*$, a crude approximation to a solution of $S' = (\eta/r)G_2 S$, where

$$G_2^*(r) \equiv \int_0^r G_2(u)du,$$

we find that if a is large then $S(r)^{-1}$ exists for $r \geq a$ and $\sup\{\|S(r)^{-1}\| : r > a\} < \infty$. Hence setting

$$\tilde{R} = S^{-1}((\eta/r)^2(GG_2^* - G_2^*G_1) + RS + (\eta/r^2)G_2^*)$$

we have that $\tilde{R} \in L^1(a,\infty)$ and defining $B = (\eta/r)G_1 + \tilde{R}$ we have that

$$SB = AS - S'. \tag{2.5}$$

Therefore setting $x = S(r)y$ and using (2.5) we find that (2.2) is equivalent to

$$y' = B(r)y. \tag{2.6}$$

To finish the proof we proceed as follows:

i) If $\sigma \neq 2$ then $B = \tilde{R} \in L^1(a,\infty)$. Hence, proceeding as in the proof of Theorem XI.65 of [22] we find that, as r goes to $+\infty$, (2.6) has a fundamental matrix $X = I + o(1)$, where I denotes the 2×2 identity matrix. Thus (2.1) has solutions ψ_1, ψ_2 such that

$$\psi_1(r) = \cos r + o(1), \qquad \psi_2(r) = \sin r + o(1),$$

$$\psi_1'(r) = -\sin r + o(1), \qquad \text{and} \qquad \psi_2'(r) = \cos r + o(1),$$

from which i) of Theorem 2.1 follows.

ii) If $\sigma = 2$ and $\gamma > 0$, then the change of variables $\tau = \eta \log r$ transforms (2.6) into

$$\frac{d\varphi}{d\tau} = (G_1 + L)\,\varphi, \tag{2.7}$$

where L is in $L^1(\tau_0, \infty)$ for some τ_0 independent of V. It is easily verified that this last system of O.D.E.s satisfies the conditions of a theorem due to Levinson. See Theorem 8.1 in Ch. 3 of [8]. Thus, as τ goes to $+\infty$, (2.7) has solutions φ_1 and φ_2 such that

$$\lim_{\tau\to\infty} \exp(\tau/4)\varphi_1(\tau) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \lim_{\tau\to\infty} \exp(-\tau/4)\varphi_2(\tau) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence (2.6) has solutions of the form

$$y_1 = r^{-\eta/4}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1)\right), \qquad y_2 = r^{\eta/4}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)\right), \tag{2.8}$$

and therefore (2.1) has solutions ψ_1, ψ_2 such that

$$\psi_1(r) = r^{-\eta/4}(\cos r + o(1)), \qquad \psi_2(r) = r^{\eta/4}(\sin r + o(1)),$$

$$\psi_1'(r) = -r^{-\eta/4}(\sin r + o(1)), \qquad \text{and} \qquad \psi_2'(r) = r^{\eta/4}(\cos r + o(1)).$$

Changing the signs that need to be changed, we see that the same result is true when $\gamma < 0$, and thus ii) of Theorem 2.1 follows. //

Remark. Clearly an analogous result holds in a neighborhood of $-\infty$.

3. Proof of the main result

First we prove Theorem 1.2 and then Theorem 1.1.

Proof of Theorem 1.2. Clearly we may assume $\gamma > 0$. Let $M(\alpha, \gamma)$ be as in the statement of Theorem 1.2. First we show that $M(\alpha, \gamma)$ is nonempty. By Theorem 2.1, for any given $V \in L^1(\mathbf{R})$ we may choose nonzero solutions ψ_- and ψ_+ of (1.4) with $k = \alpha/2$, such that $\psi_-(r)$ goes to zero as r goes to $-\infty$, and $\psi_+(r)$ goes to zero as r goes to $+\infty$. By the same theorem we can also choose $a > 0$ such that $\psi_-(-a)\psi_+(a) > 0$. Now, if we define $\psi(r) = \psi_-(r)$ for $r \leq -a$, $\psi(r) = \psi_+(r)$ for $r \geq a$, and $\psi(r) = \varphi(r)$ for $|r| \leq a$, where φ is any C^2 function of constant sign that smoothly joins ψ_- and ψ_+ on $[-a, a]$, and set $\tilde{V}(r) = V(r)$ for $|r| \geq a$ and $\tilde{V}(r) = (\alpha^2/4) - (\gamma/r)\sin\alpha r + \varphi''/\varphi$ for $|r| \leq a$, then $\tilde{V} \in L^1(\mathbf{R})$ and ψ is a nonzero continuously differentiable function which goes to zero as $|r|$ goes to infinity and satisfies

$$-\psi'' + \gamma\frac{\sin\alpha r}{r}\psi + \tilde{V}\psi = (\alpha^2/4)\psi, \qquad \text{a.e in } \mathbf{R}.$$

Hence $\tilde{V} \in M(\alpha, \gamma)$. In addition, it follows from the construction of $\tilde{V}$ that $M(\alpha, \gamma)$ is unbounded in $L^1(\mathbf{R})$. Furthermore, if V belongs to $M(\alpha, \gamma)$, then in view of Theorem 2.1 we must have $k = \alpha/2$.

To complete the proof of Theorem 1.2 we only need to show that $M(\alpha, \gamma)$ is a smooth sub-manifold of $L^1(\mathbf{R})$ of codimension one. This is proved in the following lemma.

Lemma 3.1. *Let $M(\alpha, \gamma)$ be as in Theorem 1.2. Then there exists a C^∞ function $F : L^1(\mathbf{R}) \longrightarrow \mathbf{R}$ such that zero is a regular value of F and $M(\alpha, \gamma) = F^{-1}(\{0\})$.*

Proof. For every $V \in L^1(\mathbf{R})$ let ψ_+ be the solution of

$$-\psi'' + \gamma\frac{\sin\alpha r}{r}\psi + V\psi = (\alpha^2/4)\psi, \qquad r \in \mathbf{R}, \tag{3.1}$$

which coincides for large positive r with the function ϕ given in (ii) of Theorem 2.1. First we will show that ψ_+ and ψ_+' depend smoothly on V. For $r_0 \in \mathbf{R}$, let X_{r_0} be the Banach space of continuous functions φ from $[r_0, \infty)$ into $\mathbf{R}^2$ with the norm

$$\|\varphi\|_{r_0} \equiv \sup_{\tau \geq r_0} \|\varphi(\tau)\exp(\tau/4)\| < \infty.$$

It is easy to verify that the smoothness of ψ_+ and ψ'_+ with respect to V follows from the fact that for all $r_0 \in \mathbf{R}$, the solution φ_1 of (2.7) that satisfies

$$\lim_{\tau \to +\infty} \exp(\tau/4)\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

is a smooth a function of $L \in L^1(\mathbf{R})$ with values in X_{r_0}. To prove this last define $\Phi : L^1(\mathbf{R}) \times X_{\tau_0} \to X_{\tau_0}$ as

$$\Phi(L,\varphi)(\tau) = \varphi(\tau) - \psi_1(\tau) + \int_\tau^\infty \Psi(\tau)\Psi^{-1}(s)L(s)\varphi(s)ds,$$

where, τ_0 is as in (2.7), $\psi_1(\tau) = \exp(-\tau/4)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and

$$\Psi(\tau) = \begin{pmatrix} \exp(-\tau/4) & 0 \\ 0 & \exp(\tau/4) \end{pmatrix}.$$

Note that $\Phi(L,\varphi) = 0$ if and only if $\varphi = \varphi_1$. Next we fix $L_0 \in L^1(\mathbf{R})$ and let $\varphi_0 \in X_{\tau_0}$ be so that $\Phi(L_0, \varphi_0) = 0$. We prove that if τ_0 is sufficiently large, then $\Phi(L,\varphi) = 0$ implicitly defines φ_1 as a smooth function of L, with values in X_{τ_0}, on a neighborhood of L_0. Since $\Phi(\cdot,\cdot)$ is jointly smooth, by the implicit function theorem it suffices to show that for τ_0 sufficiently large the operator $d_2\Phi(L_0,\varphi_0)$ is invertible from X_{τ_0} onto X_{τ_0}. Clearly

$$(d_2\Phi(L_0,\varphi_0))(H) = H + P(H), \qquad \text{for all } H \in X_{\tau_0},$$

where

$$P(H)(\tau) \equiv \int_\tau^\infty \Psi(\tau)\Psi^{-1}(s)L_0(s)H(s)ds.$$

Using the definition of Ψ it is easily verified that

$$\|P(H)\|_{\tau_0} \leq \|H\|_{\tau_0} \int_{\tau_0}^\infty \|L_0(s)\|ds, \qquad \text{for all } H \in X_{\tau_0},$$

from which the invertibility of $d_2\Phi(L_0,\varphi_0)$ for large τ_0 follows. Thus φ_1 is smooth in L in a neighborhood $\mathcal{O}$ of L_0 in the Banach space X_{τ_0}. Since $\varphi_1(\tau_0)$ is smooth in L, the smoothness of φ_1 as a function from $\mathcal{O}$ to X_{r_0} follows from the fact that the solutions to the initial value problem

$$\frac{d\varphi}{d\tau} = (G_1 + L)\varphi, \qquad\qquad \varphi(\tau_0) = \varphi_1(\tau_0),$$

are smooth in L and the initial value $\varphi_1(\tau_0)$. This last can be proved in the usual way using the integral equation.

Analogous arguments show that the solution ψ_- of (3.1) that satisfies

$$\psi_-(r) = |r|^{-\gamma/2\alpha}(\cos kr + o(1)) \qquad \text{and} \qquad \psi'_-(r) = -k|r|^{-\gamma/2\alpha}(\sin kr + o(1)),$$

as $r \to -\infty$, is a smooth function of V and so is ψ'_-.

Now we define $F : L^1(\mathbf{R}) \to \mathbf{R}$ as

$$F(V) = \begin{vmatrix} \psi_+(r,V) & \psi_-(r,V) \\ \psi'_+(r,V) & \psi'_-(r,V) \end{vmatrix}.$$

This function is well defined since the Wronskian of any pair of solutions of (3.1) is constant as a function of r. Moreover $V \in M(\alpha,\gamma)$ if and only if $F(V) = 0$; or equivalently, if and only if $\psi_- = \lambda\psi_+$, where $\lambda \neq 0$ is a function of V, constant as a function of r. Thus $M(\alpha,\gamma) = F^{-1}(\{0\})$.

Since F is a smooth function of V, to finish the proof it remains to show that zero is a regular value of F, that is to say that for every $V \in M(\alpha,\gamma)$ we have $dF(V) \neq 0$. Differentiating F with respect to V we find that for every V and H in $L^1(\mathbf{R})$ we have

$$dF(V)(H) = \begin{vmatrix} d\psi_+(r,V)(H) & \psi_-(r,V) \\ d\psi'_+(r,V)(H) & \psi'_-(r,V) \end{vmatrix} + \begin{vmatrix} \psi_+(r,V) & d\psi_-(r,V)(H) \\ \psi'_+(r,V) & d\psi'_-(r,V)(H) \end{vmatrix}, \tag{3.2}$$

where d indicates differentiation with respect to V and the prime differentiation with respect to r, with V fixed.

In order to prove that $dF(V)$ is not zero we note first that, for any fixed $a \in \mathbf{R}$,

$$\psi'_+(r,V) = \psi'_+(a,V) + \int_a^r \Lambda(t,V)\psi_+(t,V)\,dt,$$

where $\Lambda(r,V) \equiv (\gamma/r)\sin\alpha r + V - (\alpha^2/4)$. Using the fact that for any interval $[c,d]$ the map $V \mapsto \psi_+(\cdot,V)$, from $L^1(\mathbf{R})$ to the space $C[c,d]$ is smooth we have

$$d\psi'_+(r,V)(H) = d\psi'_+(a,V)(H) + \int_a^r H(t)\psi_+(t,V)dt + \int_a^r \Lambda(t,V)d\psi_+(t,V)(H)dt.$$

Thus $d\psi'_+(r,V)(H)$ is absolutely continuous with derivative

$$(d\psi'_+(r,V)(H))' = H(r)\psi_+(r,V) + \Lambda(r,V)d\psi_+(r,V)(H) \qquad \text{a.e.}$$

Now for any fixed $b \in \mathbf{R}$ and $\beta \geq b$ we consider

$$\int_b^\beta \psi_+(t,V)(d\psi'_+(t,V)(H))'dt$$
$$= \int_b^\beta (\psi_+(t,V))^2 H(t) + \Lambda(t,V)\psi_+(t,V)d\psi_+(t,V)(H)dt.$$

Integrating by parts we also have

$$\int_b^\beta \psi_+(t,V)(d\psi'_+(t,V)(H))'dt$$
$$= d\psi'_+(t,V)(H)\psi_+(t,V)\Big|_b^\beta - \int_b^\beta \psi'_+(t,V)d\psi'_+(t,V)(H)dt.$$

Since $\psi_+(t,V)$ is not a jointly C^2-function of t and V, in order to perform another integration by parts we show first that for any $t \in \mathbf{R}$ and V and H in $L^1(\mathbf{R})$ we have

$$d\psi'_+(t,V)(H) = (d\psi_+(t,V)(H))'. \tag{3.3}$$

To prove (3.3) just note that $\psi_+(t,V) = \psi_+(c,V) + \int_c^t \psi'_+(\tau,V)d\tau$. Since $\psi'_+(\cdot,V)$ is smooth in V as a function in $C[c,d]$ for any $d > c$, we can differentiate with respect to V under the integral sign and obtain

$$d\psi_+(t,V)(H) = d\psi_+(c,V)(H) + \int_c^t d\psi'_+(\tau,V)(H)d\tau,$$

where for fixed V and H, $d\psi'_+(\tau,V)(H)$ is continuous in τ. Thus (3.3) follows immediately.

So another integration by parts yields

$$\int_b^\beta (\psi_+(t,V))^2 H(t) + \Lambda(t,V)\psi_+(t,V)d\psi_+(t,V)(H))dt$$

$$= \psi_+(t,V)d\psi_+'(t,V)(H)\Big|_b^\beta - \psi_+'(t,V)d\psi_+(t,V)(H)\Big|_b^\beta + \int_b^\beta \Lambda(t,V)\psi_+(t,V)d\psi_+(t,V)(H)dt,$$

which gives

$$\int_b^\beta \psi_+(t,V)^2 H(t)dt = \big(\psi_+(t,V)d\psi_+'(t,V)(H) - \psi_+'(t,V)d\psi_+(t,V)(H)\big)\Big|_b^\beta .$$

Taking β to infinity and utilizing the fact that for fixed V and H, the functions ψ_+, ψ_+', $d\psi_+$, and $d\psi_+'$ all approach zero at infinity we obtain

$$\int_b^\infty \psi_+(t,V)^2 H(t)dt = \psi_+'(b,V)d\psi_+(b,V)(H) - \psi_+(b,V)d\psi_+'(b,V)(H).$$

Analogously,

$$\int_{-\infty}^b \psi_-(t,V)^2 H(t)dt = \psi_-(b,V)d\psi_-'(b,V)(H) - \psi_-'(b,V)d\psi_-(b,V)(H).$$

Finally, combining (3.2) with these last two identities and using the fact that for $V \in M(\alpha,\gamma)$ we have $\psi_- = \lambda\psi_+$, $\lambda \neq 0$, we find that

$$\begin{aligned} dF(V)(H) &= \lambda \int_b^\infty \psi_+(t,V)^2 H(t)dt + \frac{1}{\lambda}\int_{-\infty}^b \psi_-(t,V)^2 H(t)dt \\ &= \int_{-\infty}^\infty \psi_-(t,V)\psi_+(t,V)H(t)dt, \end{aligned}$$

for all $H \in L^1(\mathbf{R})$. Therefore, if $V \in M(\alpha,\gamma)$ then $dF(V)$ is the linear functional on $L^1(\mathbf{R})$ defined by the function $\psi_-\psi_+ \in L^\infty(\mathbf{R}) \setminus \{0\}$. //

Proof of Theorem 1.1. For $p \geq 1$, $\alpha > 0$, and $\gamma \in \mathbf{R} \setminus \{0\}$, let $M_p(\alpha,\gamma)$ be the set of functions $V \in L^1(\mathbf{R})$ for which the operator $H_{Q,p}$ has a positive eigenvalue. It follows from Theorem 2.1 and Proposition A.1 that

$$M_p(\alpha,\gamma) = \begin{cases} \emptyset, & \text{if } |\gamma| \leq 2\alpha/p, \\ M(\alpha,\gamma), & \text{if } |\gamma| > 2\alpha/p. \end{cases} \qquad //$$

4. Appendix

Here we establish the connection between the eigenfunctions of the operator $H_{Q,p}$ and the decaying solutions of (1.4).

Below we use *Duhamel's formula* [27] in the form

$$(\varphi, U_Q(t)f) = (\varphi, U_0(t)f) - \int_0^t (U_Q(t-u)\varphi, QU_0(u)f)du, \tag{A.1}$$

for $\varphi \in C_0^\infty(\mathbf{R})$ and $f \in L^\infty(\mathbf{R}) \cap L^p(\mathbf{R})$, where Q is as in (1.1) and $U_Q(t)$ is as introduced in (1.2). Formula (A.1) is readily established by an approximation argument starting with bounded Q. Here

$$(\phi, \psi) = \int_{-\infty}^{\infty} \overline{\phi(x)}\psi(x)dx.$$

The main result of this section is

Proposition A.1. *Let p, Q, and $H_{Q,p}$ be as in (1.1). Then $f \in L^p(\mathbf{R})$ is an eigenfunction of $H_{Q,p}$ corresponding to the eigenvalue $\lambda \in \mathbf{R}$ if and only if f is a differentiable function that vanishes at infinity, such that f' is absolutely continuous on every finite interval of $\mathbf{R}$ and*

$$-f'' + Qf = \lambda f \qquad \text{a.e.} \tag{A.2}$$

Proof. Suppose $f \in L^p(\mathbf{R})$ is a differentiable function that vanishes at infinity, such that f' is absolutely continuous on every finite interval of $\mathbf{R}$ and that (A.2) is satisfied. We will show that

$$U_{Q,p}(t)f = \exp(-\lambda t)f, \qquad t \geq 0. \tag{A.3}$$

For any given $\varphi \in C_0^\infty$ we define

$$g(s) = (\varphi, U_Q(s)f), \qquad s \geq 0.$$

We show first that for any $s \geq 0$ we have

$$\lim_{t \to 0^+} \frac{g(s+t) - g(s)}{t} = -\lambda g(s).$$

Set $\psi = U_Q(s)\varphi$. Using Duhamel's formula and the fact [6] that $(\phi, U_Q(s)\xi) = (U_Q(s)\phi, \xi)$, for $\phi \in L^p(\mathbf{R})$ and $\xi \in L^{p'}(\mathbf{R})$, we find that

$$
\begin{aligned}
\frac{g(s+t) - g(s)}{t} &= \left(\psi, \frac{U_Q(t) - 1}{t} f\right) \\
&= \left(\psi, \frac{U_0(t) - 1}{t} f\right) - \frac{1}{t}\int_0^t (U_Q(t-u)\psi, QU_0(u)f)du.
\end{aligned}
\tag{A.4}
$$

We show next that as $t \to 0^+$ the right side of (A.4) goes to $(\psi, f'') - (\psi, Qf) = -\lambda g(s)$. In fact, using the kernel $p_t(x, y)$ of $U_0(t)$ introduced in (1.3) we have

$$
\begin{aligned}
\left(\psi, \frac{U_0(t) - 1}{t} f\right) &= \left(\psi, \frac{1}{t}\int_{-\infty}^{\infty} p_t(x, y)(f(y) - f(x))dy\right) \\
&= \left(\psi, \frac{1}{t}\int_{-\infty}^{\infty} p_t(0, y)(f(x+y) - f(x))dy\right) \\
&= \left(\psi, \frac{1}{t}\int_{-\infty}^{\infty} p_t(0, y)\int_0^y (y-u)f''(x+u)dudy\right) \\
&= \left(\psi, f'' + \int_{-\infty}^{\infty} p_t(0, y)\int_0^y (y-u)\frac{(f''(x+u) - f''(x))}{t}dudy\right),
\end{aligned}
$$

where in the third equality we have used Taylor's formula

$$
f(x+u) - f(x) = yf'(x) + \int_0^y (y-u)f''(x+u)du.
$$

Setting $z = y/\sqrt{t}$ and then $u = \sqrt{t}w$ we find that

$$
\begin{aligned}
&\int_{-\infty}^{\infty} p_t(0, y)\int_0^y (y-u)\frac{(f''(x+u) - f''(x))}{t}dudy \\
&\qquad = \int_{-\infty}^{\infty} p_1(0, z)\int_0^z (z-w)(f''(x+\tau w) - f''(x))dw\,dz,
\end{aligned}
$$

where $\tau \equiv \sqrt{t}$. Hence

$$
\begin{aligned}
&\left|\left(\psi, \int_{-\infty}^{\infty} p_t(0, y)\int_0^y (y-u)\frac{(f''(x+u) - f''(x))}{t}dudy\right)\right| \\
&\quad \le \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} |\psi(x)|\int_{-\infty}^{\infty} \exp(-z^2/4)\int_{-|z|}^{|z|} 2|z||f''(x+\tau w) - f''(x)|dw\,dz\,dx \\
&\quad \le \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} |\psi(x)|\int_{-\infty}^{\infty}\int_{|w|}^{\infty} 2z\exp(-z^2/4)|f''(x+\tau w) - f''(x)|dz\,dw\,dx \\
&\quad = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} |\psi(x)|\int_{-\infty}^{\infty} \exp(-w^2/4)|f''(x+\tau w) - f''(x)|dw\,dx
\end{aligned}
$$

Thus, using (A.2), the fact that $f \in L^p(\mathbf{R}) \cap L^\infty(\mathbf{R})$, and the dominated convergence theorem we see that the right side of the last inequality goes to zero as $t \to 0^+$ and therefore

$$\lim_{t \to 0^+} \left(\psi, \frac{U_0(t) - 1}{t} f \right) = (\psi, f'').$$

By the continuity of the function $(U_Q(u)\psi, Qf)$ with respect to u, the second term on the right side of (A.4) approaches $-(\psi, Qf)$ since

$$\frac{1}{t} \int_0^t (U_Q(t-u)\psi, Q(U_0(u) - 1)f)\, du$$

goes to zero as $t \to 0^+$. To see this last we use the fact that $U_Q(t)$ maps $L^\infty(\mathbf{R})$ into $L^\infty(\mathbf{R})$ and that $\|U_Q(t)\|_{L^\infty \to L^\infty} \leq C$, for small t, that $Q = W + V$, with $V \in L^1(\mathbf{R})$ and $W \in L^p(\mathbf{R})$, for $p > 1$, and that $(U_0(t) - 1)f$ converges uniformly to zero as $t \to 0^+$ since f vanishes at infinity and hence is uniformly continuous on $\mathbf{R}$.

Thus we have proved that the right derivative $D_+ g$ of the function $g(s)$ satisfies $D_+ g(s) = -\lambda g(s)$, for all $s \geq 0$. It follows that $D_+(\exp(\lambda s) g(s)) = 0$ for $s \geq 0$ and therefore [24] that $g(s) = \exp(-\lambda s) g(0)$, which proves (A.3).

Suppose now that $f \in L^p(\mathbf{R})$ satisfies (A.3). Then [6, 26] $f \in L^\infty(\mathbf{R})$, f is continuous, vanishes at infinity, and f' exists and belongs to $L^2_{\mathrm{loc}}(\mathbf{R})$. By Duhamel's formula, for every $\varphi \in C_0^\infty(\mathbf{R})$ we have

$$(\varphi, U_Q(t)f) = (\varphi, U_0(t)f) - \int_0^t (U_Q(t-u)\varphi, QU_0(u)f)\, du.$$

Differentiating this last at $t = 0$, using (A.3), we obtain $-\lambda(\varphi, f) = (\varphi'', f) - (\varphi, Qf)$ and thus $(\varphi', f') = (\varphi, (\lambda - Q)f)$ for all $\varphi \in C_0^\infty(\mathbf{R})$. Standard approximation arguments show that f' is almost everywhere equal to an absolutely continuous function and that (A.2) is satisfied. //

5. References

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