Observable effects and parametrized scaling limits of a model in nonrelativistic quantum electrodynamics

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Abstract

Scaling limits of the Hamiltonian H of a system of N charged particles coupled to a quantized radiation field are considered. Ultraviolet cutoffs, $\hat{\lambda}_1, ..., \hat{\lambda}_N$, are imposed on the radiation field and the Coulomb gauge is taken. It is so called the Pauli-Fierz model in nonrelativistic quantum electrodynamics. We mainly consider two cases: (i) all the ultraviolet cutoffs are identical, $\hat{\lambda}_1 = \cdots = \hat{\lambda}_N$, (ii) supports of ultraviolet cutoffs have no intersection, $\mathrm{supp}\hat{\lambda}_i \cap \mathrm{supp}\hat{\lambda}_j = \emptyset$, $i \neq j$. The Hamiltonian acts on $L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}$, where $\mathcal{F}$ is a symmetric Fock space and has the form $H = H_{\mathrm{el}} \otimes 1 + B + 1 \otimes H_{\mathrm{quad}}$. Here H_{el} denotes a particle Hamiltonian, H_{quad} a quadratic field operator, and B an interaction term. The scaling is introduced as $H(\kappa) = H_{\mathrm{el}} \otimes 1 + \kappa^l B + \kappa^2 1 \otimes H_{\mathrm{quad}}$, where κ is a scaling parameter and $l \leq 2$ a parameter of the scaling. Performing a mass renormalization we consider the scaling limit of $H(\kappa)$ as $\kappa \to \infty$ in the strong resolvent sense. Then effective Hamiltonians H_{eff} in $L^2(\mathbb{R}^{dN})$ infected with reaction of effect of the radiation field is derived. In particular (1) effective Hamiltonians with an effective potential for $l = 2$, and (2) effective Hamiltonians with an observed mass for $l = 1$, are obtained.

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Contents

1 Preliminaries

1.1 Observable effects and scaling limits

In this paper we consider the Hamiltonian H of a system of N nonrelativistic charged particles interacting with a quantized radiation field. Ultraviolet cutoffs are imposed on the radiation field and recoil of the particles and the radiation field (photons) are excluded, i.e., the dipole approximation is made. Taking scaling limits of the Hamiltonian, we derive effective Hamiltonians. The main idea is to diagonalize H by a canonical transform derived from a symplectic structure.

The Schrödinger Hamiltonian of one charged particle with external potential V is of the form

$$\frac{1}{2m}p^2 + V, \tag{1.1}$$

where p denotes the momentum operator and m the bare mass of the particle. To interpret the Lamb shift [1] intuitively, T. A. Welton [2] formally derived an effective Schrödinger Hamiltonian taking into account of effects of a radiation field. He expected that the position fluctuation of the particle by the radiation field will effectively modify the external potential V. The fluctuation was thought of as a Gaussian random variable Δx, then the effective potential is formally given by the mean value of $V(x+\Delta x)$;

$$\begin{aligned} V_{\mathrm{eff}}(x) := \langle V(x+\Delta x)\rangle_{\mathrm{AVE}} &= (2\pi)^{-3/2}\int_{\mathbb{R}^3} \hat{V}(k) e^{ikx} \langle e^{ik\Delta x}\rangle_{\mathrm{AVE}} \\ &= (2\pi C)^{-3/2}\int_{\mathbb{R}^3} V(y) e^{-|x-y|^2/2C} dy, \end{aligned}$$

with a certain positive constant C. Then (1.1) effectively turns out to be the effective Hamiltonian, Welton's Hamiltonian, given by

$$\frac{1}{2m}p^2 + V_{\mathrm{eff}}. \tag{1.2}$$

He gave an interpretation of the Lamb shift as the difference of the spectrum between the original Hamiltonian and the effective one.

On the other hand it was found in [3] that photons enhances the bare mass m. Then m amounts to the observed mass $m+\delta m$, $\delta m > 0$, through the coupling to a radiation filed. Then (1.1) formally turns out to be

$$\frac{1}{2(m+\delta m)}p^2 + V. \tag{1.3}$$

In this paper performing a mass renormalization to eliminate a divergence contribution, we present a rigorous derivation of (1.2) and (1.3) by a parametrized scaling limits of Hamiltonians describing an interaction between particles and radiation fields. Thus we interpolate (1.2) and (1.3). We suppose that N particles move in the d-dimensional space and a radiation field has the helicity $d-1$. Throughout this paper we assume

$$d \geq 3.$$

$\mathcal{F}$ denotes the Boson Fock space over $W := \oplus^{d-1} L^2(\mathbb{R}^d)$, which is defined by

$$\mathcal{F} := \oplus_{n=0}^{\infty} [\otimes_s^n W],$$

where $\otimes_s^n W$ denotes the n-fold symmetric tensor product of W, and $\otimes_s^0 W := \mathbb{C}$. The bare vacuum Ω is defined by

$$\Omega := \{1, 0, 0, \cdots\}.$$

Hamiltonians under consideration are of the form

$$H := H_{\rm el} \otimes 1 + B + 1 \otimes H_{\rm quad}$$

acting on the Hilbert space

$$\mathcal{H} := L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}.$$

Here $H_{\rm el}$ describes the particle Hamiltonian of the form

$$H_{\rm el} := \sum_{j=1}^N \frac{1}{2m_j} {p_j}^2 + V.$$

Here $p_j := -i\hbar \vec{\nabla}_j$ denotes the momentum operator of the jth particle canonically conjugate to the position operator x_j, $V : \mathbb{R}^{dN} \to \mathbb{R}$ an external potential, $\hbar$ the Planck constant divided by 2π, m_j the mass of the jth particle. $H_{\rm quad}$ a quadratic field operator in $\mathcal{F}$, and B an interaction term. We scale H as follows:

$$H(\kappa) := H_{\rm el} \otimes 1 + \kappa^l B + \kappa^2 1 \otimes H_{\rm quad}.$$

Davies [4, 5] considered the scaling limit $\kappa \to \infty$ of the case of

$$l = 1$$

for certain simple models to derive N-body Schrödinger Hamiltonians. He called this scaling limit *the weak coupling limit*. See also [6, 7]. On the other hand Arai [8] considered the case of

$$l = 2$$

for the one particle Pauli-Fierz model in the dipole approximation without the self-interacting term A^2. Then he derived effective Hamiltonians with an effective potential. In this paper we consider scaling limits of

$$l \leq 2$$

for the N-particle Pauli-Fierz model in the dipole approximation. In particular, performing a mass renormalization, subtracting from $H(\kappa)$ some energy reset, and taking $\kappa \to \infty$ in the resolvent sense, we derive, for $l = 2$, Welton's Hamiltonians, and for $l = 1$, Hamiltonians with an observed mass.

1.2 Nonrelativistic models

Let $a^\dagger(f)$ and $a(f)$, $f \in W$, denote the smeared creation and annihilation operators acting on $\mathcal{F}$, respectively. $a^\sharp$ stands for a or $a^\dagger$. We set

$$a^{\sharp r}(f) := a^\dagger(\underbrace{0 \oplus 0 \cdots \oplus \overset{r_{\rm th}}{f} \oplus \cdots 0 \oplus 0}_{d-1 \text{ times}}).$$

They satisfy the canonical commutation relations,

$$[a^s(g), a^{\dagger r}(f)] = \delta_{rs} \int_{\mathbb{R}^d} f(k)g(k)dk,$$

$$[a^{\dagger r}(f), a^{\dagger s}(g)] = [a^r(f), a^s(g)] = 0$$

on the finite particle subspace of $\mathcal{F}$. We take the Coulomb gauge. The radiation field with the ultraviolet cutoff $\hat{\lambda}_j \in L^2(\mathbb{R}^d)$ is defined by

$$A_{j\mu}(x_j) := A_\mu(x_j, \hat{\lambda}_j) := \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \left\{ a^{\dagger r}(e^r_\mu e^{-ikx_j} \tilde{\hat{\lambda}}_j) + a^r(e^r_\mu e^{ikx_j} \hat{\lambda}_j) \right\},$$

where $e^r(k) = (e^r_1(k), ..., e^r_d(k))$ denotes polarization vectors satisying $e^r(k) \cdot e^s(k) = \delta_{rs}$ and $e^r(k) \cdot k = 0$,

$$\tilde{\hat{\lambda}}_j(k) := \hat{\lambda}_j(-k),$$

and $\hat{f}$ denotes the Fourier transform of f. We assume

$$\hat{\lambda}_j(k) = \hat{\lambda}_j(-k)$$

to ensure that $A_{j\mu}(x_j)$ is symmetric for each $x \in \mathbb{R}^d$. The physically reasonable choice of the ultraviolet cutoff is

$$\hat{\lambda}_j(k) = \frac{\sqrt{c}\hbar\hat{\rho}_j(k)}{\sqrt{(2\pi)^d\omega(k)}},$$

where $\hat{\rho}_j$ denotes the Fourier transform of the charge distribution ρ_j of the jth particle and c denotes the speed of the light. The dispersion relation $\omega(k)$ is given by

$$\omega(k) := |k|.$$

The free Hamiltonian H_{f} in $\mathcal{F}$ is defined by

$$H_{\mathrm{f}} := \hbar c \sum_{r=1}^{d-1} \int \omega(k) a^{\dagger r}(k) a^r(k) dk,$$

where $a^{\dagger r}(k)$ and $a^r(k)$ denote the kernels of $a^{\dagger r}(f)$ and $a^r(f)$, respectively. The total Hamiltonian is given by

$$H_{\mathrm{total}} := \sum_{j=1}^{N} \frac{1}{2m_j} \left(p_j \otimes 1 - \frac{\alpha_j}{c} A_j(x_j) \right)^2 + 1 \otimes H_{\mathrm{f}} + V \otimes 1,$$

where α_j's are non zero coupling constants. We take the dipole approximation, i.e., we replace $A_j(x_j)$ with

$$A_j := A_j(0).$$

Then the Hamiltonian in the dipole approximation is given by

$$H := \sum_{j=1}^{N} \frac{1}{2m_j} \left(p_j \otimes 1 - \frac{\alpha_j}{c} 1 \otimes A_j \right)^2 + 1 \otimes H_{\mathrm{f}} + V \otimes 1.$$

We work with the unit $c = \hbar = 1$ in what follows. Since

$$[p_{j_\mu} \otimes 1, 1 \otimes A_{j_\nu}] = 0,$$

we have

$$H = H_{\mathrm{el}} \otimes 1 - \sum_{j=1}^{N} \frac{\alpha_j}{m_j} p_j \otimes A_j + 1 \otimes \left(\sum_{j=1}^{N} \frac{\alpha_j^2}{2m_j} {A_j}^2 + H_{\mathrm{f}} \right).$$

It is of mathematical interest to see A_j^2-corrections in scaling limits. Then we define, for $\epsilon \geq 0$,

$$H_\epsilon := H_{\rm el} \otimes 1 - \sum_{j=1}^N \frac{\alpha_j}{m_j} p_j \otimes A_j + 1 \otimes \left(\sum_{j=1}^N \frac{\epsilon \alpha_j^2}{2m_j} A_j^2 + H_{\rm f} \right)$$

with the domain

$$D(H_\epsilon) = D(H_{\rm el} \otimes 1) \cap D(1 \otimes H_{\rm f}) \cap D(1 \otimes \sum_{j=1}^N \frac{\epsilon \alpha_j^2}{2m_j} A_j^2).$$

Actually, for $\epsilon < 1$, H_ϵ turns out to be an unphysical Hamiltonian for sufficiently large coupling constants, since in this case H_ϵ is unbounded from below. See Remark 3.10 for details. Let

$$\kappa > 1$$

be a scaling parameter. We introduce the scaled Hamiltonian $H_\epsilon(\kappa)$ as follows:

$$H_\epsilon(\kappa) := H_{\rm el} \otimes 1 - \kappa^l \sum_{j=1}^N \frac{\alpha_j}{m_j} p_j \otimes A_j + \kappa^2 1 \otimes \left(\sum_{j=1}^N \frac{\epsilon \alpha_j^2}{2m_j} A_j^2 + H_{\rm f} \right),$$

where

$$l \leq 2.$$

In particular we set

$$H(\kappa) := H_1(\kappa).$$

Unless confusions arise we omit $\otimes 1$ and $1 \otimes$ for simplicity in what follows. Throughout this paper we suppose that V satisfies Hypothesis **V**:

Hypothesis V: V is infinitesimally small with respect to $\sum_{j=1}^N \frac{p_j^2}{2m_j}$.

First of all we have to establish self-adjointness of $H_\epsilon(\kappa)$. Note that if we assume that $\hat{\lambda}_j/\sqrt{\omega}, \hat{\lambda}_j, \omega\hat{\lambda}_j \in L^2(\mathbb{R}^d)$, then

$$\| \left(- \sum_{j=1}^N \frac{\alpha_j}{m_j} p_j A_j + \sum_{j=1}^N \frac{\epsilon \alpha_j^2}{2m_j} A_j^2 \right) \Psi \| \leq a' \| \left(\sum_{j=1}^N \frac{p_j^2}{2m_j} + H_{\rm f} \right) \Psi \| + b' \|\Psi\|$$

for $\Psi \in D(H_{\rm f})$ with some a' and b'. Let

$$\mathcal{D} := D(\sum_{j=1}^N \frac{p_j^2}{2m_j} \otimes 1) \cap D(1 \otimes H_{\rm f}).$$

Proposition 1.1 *Let $\epsilon = 1$. We suppose that $\hat{\lambda}_j/\sqrt{\omega}, \hat{\lambda}_j, \omega\hat{\lambda}_j \in L^2(\mathbb{R}^d)$. (1) $H_{\rm total}$ and H are self-adjoint on $\mathcal{D}$ and bounded from below. (2) Let $l = 1$. Then $H(\kappa)$ is self-adjoint on $\mathcal{D}$ and bounded from below for all κ.*

Proof: See [9, 10] for (1). In the case of $l = 1$, $H(\kappa)$ is derived from the scaling $\alpha_j \to \kappa\alpha_j$ and $\omega \to \kappa\omega$ in H. Thus the self-adjointness follows from (1) □

Proposition 1.2 *Let $\epsilon \neq 1$ and $V = 0$. Suppose $\hat{\lambda}_j/\sqrt{\omega}, \hat{\lambda}_j, \omega\hat{\lambda}_j \in L^2(\mathbb{R}^d)$. Then $H_\epsilon(\kappa)$ is essentially self-adjoint on any core of $\sum_{j=1}^N p_j^2/(2m_j) + H_{\rm f}$.*

Proof: The proof is due to Arai [11, Theorem 2.2]. Let $L := \sum_{j=1}^N p_j^2/(2m_j) + H_{\rm f} + 1$. Then we see that

$$\|H_\epsilon(\kappa)\Psi\| \leq c\|L\Psi\|,$$

and

$$|(H_\epsilon(\kappa)\Psi, L\Psi) - (L\Psi, H_\epsilon(\kappa)\Psi)| \leq c'\|L^{1/2}\Psi\|\|L^{1/2}\Psi\|$$

with some c and c'. Thus, by the Nelson commutator theorem [14, Theorem X.36], $H_\epsilon(\kappa)$ is essentially self-adjoint on any core of L. Thus the proposition follows. □

Self-adjointness of $H_\epsilon(\kappa)$ for $\epsilon < 1$ or $l \neq 1$ is discussed later (Theorem 3.11). We denote the self-adjoint extensions of H_ϵ and $H_\epsilon(\kappa)$ by the same symbols.

1.3 Problems

For the case of $N \geq 2$ we have to take care of the choice of ultraviolet cutoffs. For simplicity we mainly consider two cases: identical case, $\hat{\lambda}_1 = \cdots = \hat{\lambda}_N := \hat{\lambda}$, and independent case, supp $\hat{\lambda}_j \cap$ supp $\hat{\lambda}_i = \emptyset$ for $i \neq j$.

Arai [8] considered the scaling limit of the case of $l = 2$, $\epsilon = 0$ and $N = 1$. Moreover in [12, 13] distinct scaling limits were studied, but unfortunately A^2-corrections did not appear in scaling limits. In this paper we want to improve [12, 13], and furthemore to see the relationship between Davies' scaling limit and Arai's one. As was seen in subsection 1.1, Davies' scaling limits correspond to $l = 1$.

We set

$$H_\epsilon^0 := H_\epsilon\lceil_{V=0}.$$

Since H_ϵ^0 commutes with $p_{j\mu}$, H_ϵ is decomposable in the spectrum of the particle momentum operators;

$$H_\epsilon^0 = \int_{\mathbb{R}^{dN}}^{\oplus} H_\epsilon^0(q)dq.$$

$H_\epsilon^0(q)$ acts on $\mathcal{F}$. Its ground state energy is denoted by

$$E(q) := \inf \sigma(H_\epsilon^0(q)).$$

Then we call

$$m^*(\mu,\nu,j) := \left(\frac{\partial^2}{\partial q_{j\mu}\partial q_{j\nu}} E(q) \Big\lceil_{p=0} \right)^{-1}$$

an *observed mass*. In this paper we explicitly give ground state energy of H_ϵ^0 and its observed mass.

Thus the list of problems considered in this paper is epitomized as follows.

(1) Explicit ground state energy, $\inf \sigma(H_\epsilon^0)$.

(2) Observed mass, $m^*(\mu,\nu,j)$.

(3) Deriving effective Hamiltonians $H_{\rm eff}$ through scaling limits of $H_\epsilon(\kappa)$ subtracting a divergent contribution $E_{\rm ren}(\kappa)$,

$$s-\lim_{\kappa\to\infty}(H_\epsilon(\kappa) - E_{\rm ren}(\kappa) - z)^{-1} = (H_{\rm eff} - z)^{-1} \otimes P_\epsilon$$

for $z \in \mathbb{C}\backslash\mathbb{R}$, where P_ϵ denotes a projection to the subspace spanned by ground states of $\sum_{j=1}^N \epsilon\alpha_j^2 A_j^2/(2m_j) + H_{\rm f}$.

(4) What is a reasonable choice of $E_{\rm ren}(\kappa)$?

(5) l-dependence of $H_{\rm eff}$.

(6) A_j^2-dependence (i.e., ϵ-dependence) of $H_{\rm eff}$.

(7) Ultraviolet cutoff dependence of $H_{\rm eff}$.

We give answers for (1) in Lemma 3.3, (2) in Corollary 3.7, (3) in Theorems 4.9-4.12, (4) in (4.5), (5), (6) and (7) in (4.7).

This paper is organized as follows. In Section 2 we discuss symplectic group structures of H_ϵ^0. In Section 3 we construct the canonical transformation and diagonalize H_ϵ^0. In Section 4 we derive effective Hamiltonians by scaling limits. In Section 5 we give a remark on g-factor shift. Sections 6 and 7 are appendixes.

2 Symplectic group structures

2.1 Unitary representations

Let $B(W)$ denote the set of bounded operators on W. We define for $K \in B(W)$

$$\overline{K} f := \overline{K \overline{f}}.$$

We introduce a notation. For $A, B, C, D \in B(W)$ we define

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a(f) \\ a^\dagger(g) \end{pmatrix} := \begin{pmatrix} a(Af) + a^\dagger(Bg) \\ a(Cf) + a^\dagger(Dg) \end{pmatrix}.$$

Set

$${}^T\!\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} A & C \\ B & D \end{pmatrix}.$$

Definition 2.1 *Let $S, T \in B(W)$ and set*

$$\xi := \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} : W \oplus W \to W \oplus W.$$

(1) We say that ξ is of symplectic group Σ if

$$\xi^* J \xi = \xi J \xi^* = J, \tag{2.1}$$

where

$$\xi^* = \begin{pmatrix} S^* & T^* \\ \overline{T}^* & \overline{S}^* \end{pmatrix}, \quad J := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

(2) We say that ξ is of Σ_{B} if (i) $\xi \in \Sigma$ and (ii) T is a Hilbert-Schmidt operator.

Σ_{B} is a subgroup of Σ. We define

$$\mathcal{H}_m := L^2(\mathbb{R}^m) \otimes \mathcal{F} \cong \int_{\mathbb{R}^m}^{\oplus} \mathcal{F} dq.$$

Suppose that

$$\xi = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma_{\mathrm{B}},$$

and

$$L = (L_1, ..., L_m) \in \oplus^m W.$$

For $f \in W$ we let

$$\begin{pmatrix} b(f) \\ b^\dagger(f) \end{pmatrix} :=^T \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \begin{pmatrix} a(f) \\ a^\dagger(f) \end{pmatrix}$$

and

$$B(f) := 1 \otimes b(f) + \sum_{l=1}^m (L_l, f) p_l \otimes 1,$$

$$B^\dagger(f) := 1 \otimes b^\dagger(f) + \sum_{l=1}^m (\overline{L_l}, f) p_l \otimes 1,$$

where p_l, $l = 1, ..., m$, denote the momentum operators in $L^2(\mathbb{R}^m)$. It follows that

$$[b(f), b^\dagger(g)] = (\overline{f}, g),$$

$$[b(f), b(g)] = [b^\dagger(f), b^\dagger(g)] = 0.$$

Moreover since

$$\xi^{-1} = J\xi^* J = \begin{pmatrix} S^* & -T^* \\ -\overline{T}^* & \overline{S}^* \end{pmatrix},$$

we have

$$\begin{pmatrix} a(f) \\ a^\dagger(f) \end{pmatrix} =^T \begin{pmatrix} S^* & -T^* \\ -\overline{T}^* & \overline{S}^* \end{pmatrix} \begin{pmatrix} b(f) \\ b^\dagger(f) \end{pmatrix} \tag{2.2}$$

and

$$\begin{pmatrix} 1 \otimes a(f) \\ 1 \otimes a^\dagger(f) \end{pmatrix} =^T \begin{pmatrix} S^* & -T^* \\ -\overline{T}^* & \overline{S}^* \end{pmatrix} \begin{pmatrix} B(f) \\ B^\dagger(f) \end{pmatrix} - \sum_{l=1}^m \begin{pmatrix} (SL_l - \overline{TL_l}, f) p_l \otimes 1 \\ (\overline{SL_l} - TL_l, f) p_l \otimes 1 \end{pmatrix}. \tag{2.3}$$

Define

$$R_l := TL_l - \overline{SL_l},$$

and

$$\theta(q, \xi, L) := \exp\left\{\sum_{l=1}^m q_l \left\{a^\dagger(R_l) - a(\overline{R}_l)\right\}\right\}, \quad q = (q_1, \cdots, q_m) \in \mathbb{R}^m.$$

Furthermore set

$$\theta(\xi, L) := \int_{\mathbb{R}^m}^{\oplus} \theta(q, \xi, L) dq = \exp\left\{\sum_{l=1}^m p_l \otimes \left\{a^\dagger(R_l) - a(\overline{R}_l)\right\}\right\}.$$

Proposition 2.2 *Suppose* $\xi = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma_{\mathrm{B}}$. *Then there exists a unitary operator* $U(\xi)$ *of* $\mathcal{F}$ *such that for all* $f \in W$

(1) $U(\xi)^{-1}b^\sharp(f)U(\xi) = a^\sharp(f)$,

(2) $U(\xi)^{-1}\theta(\xi, L)^{-1}B^\sharp(f)\theta(\xi, L)U(\xi) = a^\sharp(f)$.

Proof: See [15, 16, 17] for details [1]. □

We set

$$\mathcal{U}(q, \xi, L) := \theta(q, \xi, L)U(\xi) : \mathcal{F} \to \mathcal{F},$$

and

$$\mathcal{U}(\xi, L) := \theta(\xi, L)U(\xi) : \mathcal{H}_m \to \mathcal{H}_m.$$

We call $\mathcal{U}(\xi, L)$ the canonical transformation associated with $\{\xi, L\} \in \Sigma_{\mathrm{B}} \times \oplus^m W$.

2.2 Average of fluctuations

Define

$$\pi(f) := i\left\{a^\dagger(\bar{f}) - a(f)\right\}, \quad f \in W.$$

We want to obtain the average of fluctuation $F_{\xi L}$ of a function $F : \mathbb{R}^m \to \mathbb{C}$;

$$F_{\xi L} := \mathcal{U}(\xi, L)^{-1} F \mathcal{U}(\xi, L)$$

$$= U(\xi)^{-1} F(x_1 + \pi(R_1), \cdots, x_m + \pi(R_m))U(\xi),$$

with respect to the bare vacuum. Let Γ be the $m \times m$ symmetric matrix defined by

$$\Gamma := ((L_\mu, L_\nu)_W)_{1 \le \mu,\nu \le m},$$

and $\langle a, b\rangle$ denote the scalar product on $\mathbb{C}^m$. Suppose

$$\mathrm{Rank}\,\Gamma = n \le m$$

[1] We easily see that for $\xi_1 := \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma_{\mathrm{B}}$ and $\xi_2 := \begin{pmatrix} U & \overline{V} \\ V & \overline{U} \end{pmatrix} \in \Sigma_{\mathrm{B}}$,

$$U(\xi_2)U(\xi_1)a(f)U(\xi_1)^{-1}U(\xi_2)^{-1} = a\left((US + \overline{V}T)f\right) + a^\dagger\left((VS + \overline{U}T)f\right),$$

$$U(\xi_2)U(\xi_1)a^\dagger(f)U(\xi_1)^{-1}U(\xi_2)^{-1} = a\left((U\overline{T} + \overline{VS})f\right) + a^\dagger\left((V\overline{T} + \overline{US})f\right),$$

and

$$\xi_2 \cdot \xi_1 = \begin{pmatrix} US + \overline{V}T & U\overline{T} + \overline{VS} \\ VS + \overline{U}T & V\overline{T} + \overline{US} \end{pmatrix}.$$

Then $U(\xi_2)U(\xi_1) = U(\xi_2\xi_1)$. Hence the map $\Sigma_{\mathrm{B}} \ni \xi \mapsto U(\xi)$ defines a unitary representation of Σ_{B}.

and T is a unitary matrix such that

$$T\Gamma T^{-1} = \mathrm{diag}\{\mu_1, \cdots, \mu_n, 0, \cdots, 0\}, \quad \mu := \mu_1 \cdots \mu_n \neq 0.$$

Let

$$F_{T^{-1}}(x) := F(T^{-1}x), \quad x \in \mathbb{R}^m.$$

We define

$$F_{\mathrm{eff}}(x) := (2\pi\mu)^{-m/2} \int_{\mathbb{R}^n} F_{T^{-1}}(y_1, \cdots, y_n, (Tx)_{n+1}, \cdots, (Tx)_m) \times$$
$$\times e^{-\sum_{j=1}^n |(Tx)_j - y_j|^2/(2\mu)} dy.$$

In particular if Rank $\Gamma = n$, then

$$F_{\mathrm{eff}}(x) = (2\pi \mathrm{det}\Gamma)^{-m/2} \int_{\mathbb{R}^m} F(y) e^{-|x-y|^2/(2\mathrm{det}\Gamma)} dy.$$

Lemma 2.3 *Let* $\xi = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma_{\mathrm{B}}$ *and* $(L_1, \cdots, L_m) \in \oplus^m W$. *Suppose that* F *is such that*

$$\int_{\mathbb{R}^n} |F_{T^{-1}}|(y_1, \cdots, y_n, (Tx)_{n+1}, \cdots, (Tx)_m) e^{-\sum_{j=1}^n |(Tx)_j - y_j|^2/(2\mu)} dy \in L^1_{\mathrm{loc}}(\mathbb{R}^m, dx).$$

Then for $f \in L^2(\mathbb{R}^m)$, *and* $g \in L^2(\mathbb{R}^m)$ *such that* $\mathcal{U}(\xi, L) g \otimes \Omega \in D(F)$, *we see that* $g \in D(F_{\mathrm{eff}})$ *and*

$$(f \otimes \Omega, F_{\xi L} g \otimes \Omega)_{\mathcal{H}_m} = (f, F_{\mathrm{eff}} g)_{L^2(\mathbb{R}^m)}. \tag{2.4}$$

Proof: We see that by (2.2)

$$a(\overline{R_l}) = b(S^* \overline{R_l}) - b^\dagger(\overline{T}^* \overline{R_l}),$$

$$a^\dagger(R_l) = -b(T^* R_l) + b^\dagger(\overline{S}^* R_l).$$

Thus we have

$$a^\dagger(R_l) - a(\overline{R_l}) = b^\dagger(\overline{S}^* R_l + \overline{T}^* \overline{R_l}) - b(T^* R_l + S^* \overline{R_l})$$

$$:= b^\dagger(R_l') - b(\overline{R_l'}).$$

It follows from $\xi^* J \xi = \xi J \xi^*$ that

$$R_l' = \overline{S}^*(TL_l - \overline{SL_l}) + \overline{T}^*(\overline{TL_l} - SL_l)$$

$$= (\overline{S}^* T - \overline{T}^* S) L_l + (\overline{T}^* \overline{T} - \overline{S}^* \overline{S}) \overline{L_l} = -\overline{L_l}.$$

Hence

$$\pi(R_l) = i \left\{ a^\dagger(R_l) - a(\overline{R_l}) \right\} = i \left\{ b(L_l) - b^\dagger(\overline{L_l}) \right\}.$$

Then we have

$$U(\xi)^{-1} \pi(R_l) U(\xi) = i \left\{ a(L_l) - a^\dagger(\overline{L}_l) \right\} = -\pi(L_l). \tag{2.5}$$

Let

$$F_N(x) := \begin{cases} F(x), & |x| < N, \\ 0, & |x| \geq N. \end{cases}$$

Moreover

$$F_{N\epsilon} := \rho_\epsilon * F_N, \quad \epsilon > 0,$$

where $\rho_\epsilon(x) := \rho(x/\epsilon)/\epsilon^m$, $\rho(x) \geq 0$, $\mathrm{supp}\rho \subset \{x \in \mathbb{R}^m \| x| \leq 1\}$, and $\int_{\mathbb{R}^m} \rho(x) dx = 1$. (2.5) leads to

$$\begin{aligned} & (f \otimes \Omega, (F_{N\epsilon})_{\xi L} g \otimes \Omega) \\ &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} dk e^{ikx} \check{F}_{N\epsilon}(k) (f \otimes \Omega, U(\xi)^{-1} e^{i\langle k, x + \pi(R) \rangle} U(\xi) g \otimes \Omega) \\ &= (2\pi)^{-m/2} \int_{\mathbb{R}^d} dk e^{ikx} \check{F}_{N\epsilon}(k) (f \otimes \Omega, e^{i\langle k, x - \pi(\overline{L}) \rangle} g \otimes \Omega) \\ &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} dk e^{ikx} \check{F}_{N\epsilon}(k) \overline{f(x)} g(x) e^{-\langle k, \Gamma k \rangle/2} \\ &= (f, (F_{N\epsilon})_{\mathrm{eff}} g). \end{aligned}$$

Taking $\epsilon \to 0$, we have $(F_{N\epsilon})_{\mathrm{eff}}(x) \to (F_N)_{\mathrm{eff}}(x)$, and $(F_{N\epsilon})_{\xi L} \to (F_N)_{\xi L}$ strongly. Then the lemma follows for F_N. Let $f \in C_0^\infty(\mathbb{R}^m)$. Then taking $N \to \infty$, we also have $(f, (F_N)_{\xi L} g) \to (f, F_{\xi L} g)$ and $(f, (F_N)_{\mathrm{eff}} g) \to (f, F_{\mathrm{eff}} g)$ by the dominated convergence theorem. Hence the lemma follows. □

Remark 2.4 *F_{eff} is independent of ξ.*

2.3 Physical vacuum

Let $\xi = \begin{pmatrix} S & \overline{T} \\ T & \overline{S} \end{pmatrix} \in \Sigma_{\mathrm{B}}$ and $L \in \oplus^{dN} W$. We define

$$\Omega(q, \xi, L) := \mathcal{U}(q, \xi, L) \Omega.$$

Let K be a Hilbert-Schmidt operator. Then there exist complete orthonormal systems (CONS) $\{\psi_i\}$, $\{\phi_i\}$, and a positive sequence $\{\mu_i\}$ such that

$$Kf = \sum_{i=1}^{\infty} \mu_i(\psi_i, f)\phi_i, \quad f \in \mathcal{H}_m,$$

with

$$\sum_{i=1}^{\infty} \mu_i^2 = \|K\|_{\mathrm{HS}}^2,$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. We define for a finite particle vector Ψ

$$\langle a^\dagger|K|a^\dagger\rangle\Psi := s - \lim_{n\to\infty}\sum_{i=1}^{n} \mu_i a^\dagger(\overline{\psi}_i)a^\dagger(\phi_i)\Psi.$$

The above strong limit actually exists. It is well known that if

$$(1) K \text{ is a Hilbert-Schmidt operator}, \quad (2) \overline{K}^* = K, \quad (3) \|K\| < 1,$$

then

$$U_0(K) := s - \lim_{N\to\infty}\sum_{n=1}^{N} \frac{1}{n!}\left(-\frac{1}{2}\langle a^\dagger|K|a^\dagger\rangle\right)^n \Psi$$

exists for a finite particle vector Ψ. See [18].

Lemma 2.5 *We have*

$$\Omega(q,\xi,L) = \det(1-(TS^{-1})^*TS^{-1})^{1/4}\theta(q,\xi,L)U_0(TS^{-1})\Omega.$$

Proof: See [12] for details. □

3 Canonical transformations

3.1 Fixed particle-momentum Hamiltonians

We define the fixed particle-momentum Hamiltonian by

$$H_\epsilon^0(q) := \sum_{j=1}^{N} \frac{1}{2m_j} q_j^2 - \sum_{j=1}^{N} \frac{\alpha_j}{m_j} q_j \cdot A_j + \sum_{j=1}^{N} \frac{\epsilon\alpha_j^2}{2m_j} A_j^2 + H_{\mathrm{f}}.$$

Here q is a fixed particle-momentum:

$$q = (q_1, \cdots, q_N) \in \mathbb{R}^{dN}.$$

Proposition 3.1 *Suppose that $\hat{\lambda}_j/\sqrt{\omega}, \hat{\lambda}_j, \omega\hat{\lambda}_j \in L^2(\mathbb{R}^d)$ for $j = 1, ..., N$. Then for all $\{\alpha_1, \cdots, \alpha_N\}$ and all $q \in \mathbb{R}^{dN}$, $H_\epsilon^0(q)$ is self-adjoint on $D(H_{\rm f})$ and bounded from below.*

Proof: The proof is similar to Proposition 1.2. Let $L := H_{\rm f} + 1$. Then we easily have

$$\|H_\epsilon^0(q)\Psi\| \le c\|L\Psi\|,$$

and

$$|(H_\epsilon^0(q)\Psi, L\Phi) - (L\Psi, H_\epsilon^0(q)\Phi)| \le c'\|L^{1/2}\Psi\|\|L^{1/2}\Phi\|$$

with some constants c and c' for $\Psi, \Phi \in D(H_{\rm f})$. Hence $H_\epsilon^0(q)$ is essentially self-adjoint on $D(H_{\rm f})$ by the Nelson commutator theorem. Moreover it is seen that

$$c''\left(\|H_{\rm f}\Psi\|^2 + \|\sum_{j=1}^N\left(\frac{q_j^2}{2m_j} - \frac{\alpha_j}{m_j}q_j\cdot A_j + \frac{\epsilon\alpha_j}{2m_j}A_j^2\right)\Psi\|^2\right) \le \|H_\epsilon^0(q)\Psi\|^2 + c'''\|\Psi\|^2$$

with some c'' and c'''. Then $H_\epsilon^0(q)$ is closed on $D(H_{\rm f})$. Thus $H_\epsilon^0(q)$ is self-adjoint on $D(H_{\rm f})$. □

By a canonical transformation $\mathcal{U}(q)$, we will diagonalize $H_\epsilon^0(q)$ as

$$\mathcal{U}(q)^{-1}H_\epsilon^0(q)\mathcal{U}(q) = E(q) + H_{\rm f} \tag{3.1}$$

with some constant $E(q)$. In [12], (3.1) was established for $N = 1$, in [13] for $N \ge 1$ but sufficiently small $|\alpha_j|$'s. In this section we improve it as follows:

(1) we extend (3.1) to $N \ge 1$ and arbitrary α_j's,

(2) we derive an explicit form of $E(q)$,

(3) we show analytic properties of $E(q)$ in each α_j's.

3.2 Assumptions on ultraviolet cutoffs and coupling constants

In this subsection we introduce assumptions on $\hat{\lambda}_j$ and α_j. We suppose the following assumptions on $\hat{\lambda}_j$'s.

(A.1) $\hat{\lambda}_j$ is rotation invariant, i.e., there exits a function $\varphi_j : [0, \infty) \to \mathbb{R}$ such that $\hat{\lambda}_j(k) = \varphi_j(|k|)$, (A.2) $\hat{\lambda}_j/\omega, \omega\hat{\lambda}_j \in L^2(\mathbb{R}^d)$,

(A.3) $\varphi_{ij}(s) := \varphi_i(\sqrt{s})\varphi_j(\sqrt{s})s^{(d-1)/2} \in L^\delta([0,\infty), ds)$ for some $0 < \delta < 1$, and there exists $0 < C_{ij} < 1$ such that, for arbitrary $s, t \geq 0$,

$$|\varphi_{ij}(s) - \varphi_{ij}(t)| \leq C_{ij}|s-t|.$$

(A.4) $\|\hat{\lambda}_j\omega^{(d-2)/2}\|_\infty < \infty$ and $\|\hat{\lambda}_j\omega^{d/2}\|_\infty < \infty$, where $\|g\|_\infty := \sup_{x\in\mathbb{R}^d}|g(x)|$.

Above assumptions are sufficient conditions to construct a canonical transformation. Let

$$m_{ab} := \begin{cases} m_j, & a = b = j, \\ 0, & a \neq b. \end{cases}$$

Define

$$\begin{aligned} D_{ij}(k) &:= m_{ij} - \epsilon\alpha_i\alpha_j K \lim_{\eta\downarrow 0}\int_{\mathbb{R}^d} \frac{\omega(k')\hat{\lambda}_i(k')\hat{\lambda}_j(k')}{\omega(k)^2 - \omega(k')^2 + i\eta} dk' \\ &= m_{ij} - \epsilon\alpha_i\alpha_j \frac{K}{2} S_{d-1}\left(\lim_{\eta\downarrow 0}\int_{|\omega(k)^2 - x| > \eta, x > 0} \frac{\varphi_{ij}(x)}{\omega(k)^2 - x}dx - 2\pi i \varphi_{ij}(\omega(k)^2)\right), \end{aligned} \tag{3.2}$$

where S_{d-1} denotes the volume of the $(d-1)$-dimensional unit sphere and

$$K := \frac{d-1}{d}.$$

We define

$$D := D(k) = (D_{ij}(k))_{1\leq i,j\leq N} : \oplus^N L^2(\mathbb{R}^d) \to \oplus^N L^2(\mathbb{R}^d).$$

We introduce two assumptions of ultraviolet cutoffs, I_λ and II_λ, as follows:

Hypothesis I_λ**:** $\hat{\lambda}_1 = \cdots = \hat{\lambda}_N := \hat{\lambda}$ and $\Im\hat{\lambda}(k) \neq 0$ for all $k \in \mathbb{R}^d$.

Hypothesis II_λ**:** Each $\hat{\lambda}_j$ has a compact support such that supp $\hat{\lambda}_j \cap$ supp $\hat{\lambda}_i = \emptyset$ for $i \neq j$, and $\left|\frac{\hat{\lambda}_j(k)}{D_{jj}(k)}\right| \to 0$ as k converges to the boundary of supp$\hat{\lambda}_j$.

In what follows we mainly study above two cases. The next assumptions are needed to see the boundedness of the Hamiltonians from below.

Hypothesis $\mathrm{I}_\lambda^\alpha(\epsilon)$**:** Ultraviolet cutoffs satisfies I_λ, and $\{\alpha_1, \cdots, \alpha_N\}$ satisfies that

$$\begin{cases} \sum_{j=1}^N \frac{\alpha_j^2}{m_j} < \frac{1}{\|\hat{\lambda}/\sqrt{\omega}\|^2}\frac{1}{1-\epsilon}, & 0 \leq \epsilon < 1, \\ \\ \{\alpha_1, \cdots, \alpha_N\} \in \mathbb{R}^N, & \epsilon \geq 1. \end{cases}$$

Hypothesis $\mathrm{II}^\alpha_\lambda(\epsilon)$**:** Ultraviolet cutoffs satisfies II_λ, and $\{\alpha_1, \cdots, \alpha_N\}$ satisfies that

$$\begin{cases} \frac{\alpha_j^2}{m_j} \le \frac{1}{\|\hat{\lambda}_j/\sqrt{\omega}\|^2}\frac{1}{1-\epsilon}, & j=1,...,N, \quad 0 \le \epsilon < 1, \\ \{\alpha_1, \cdots, \alpha_N\} \in \mathbb{R}^N, & \epsilon \ge 1. \end{cases}$$

3.3 Diagonalization of $H^0_\epsilon(q)$

Since

$$D_{ij}(0) = m_{ij} + \epsilon\alpha_i\alpha_j \frac{K}{2} S_{d-1} \int_0^\infty \frac{\varphi_{ij}(x)}{x} dx,$$

we easily see that $D(0)^{-1}$ exists as an $N \times N$ matrix for $\mathrm{I}_\lambda$and II_λ. Set

$$\begin{pmatrix} R_1 \\ \vdots \\ R_N \end{pmatrix} := \sqrt{\omega} D(0)^{-1} \begin{pmatrix} \hat{\lambda}_1 \\ \vdots \\ \hat{\lambda}_N \end{pmatrix}.$$

We define

$$\theta(q) := \exp\left(\sum_{j=1}^N \sum_{r=1}^{d-1} \sum_{\mu=1}^d \frac{\alpha_j q_{j\mu}}{\sqrt{2}} \left\{ a^r\left(\overline{\frac{e^r_\mu R_j}{\omega^{3/2}}}\right) - a^{\dagger r}\left(\frac{e^r_\mu R_j}{\omega^{3/2}}\right)\right\}\right).$$

Proposition 3.2 *Suppose* I_λ*or* II_λ.

(1) *There exist* $\mathbb{W} \in \Sigma_{\mathrm{B}}$ *and* Q_j, $j=1,...,N$, *such that*

$$\theta(q, \mathbb{W}, L) = \theta(q),$$

where $L = (L^1_1, ..., L^d_1, ..., L^1_N, ..., L^d_N) \in \oplus^{dN} W$ *with*

$$L^\mu_j := \frac{\alpha_j}{\sqrt{2}} \oplus^{d-1}_{r=1} \frac{e^r_\mu Q_j}{\omega^{3/2}}.$$

(2) *Let*

$$\mathcal{U}(q) := \theta(q) U(\mathbb{W}).$$

Then $\mathcal{U}(q)$ *maps* $D(H_{\mathrm{f}})$ *onto itself, and*

$$\mathcal{U}(q)^{-1} H^0_\epsilon(q) \mathcal{U}(q) = E(q) + H_{\mathrm{f}}$$

with some $E(q)$.

(3) $E(q)$ *can be analytically continued to a neighborhood* $\mathcal{O}$ *of* $\mathbb{R}$ *in each* α_j*'s.*

Proof: See Appendix. $\square$

3.4 Ground state energy

In this subsection we show an explicit form of $E(q)$ for both cases I_λ and II_λ. In the case of $N=1$ it is shown in [19]. For the case of $N \geq 2$ one needs a slight modification of [19]. We do not, however, need to go into the minor details for the moment. So a proof of the following lemma is taken up in Appendix.

Lemma 3.3 *(1) Suppose* I_λ. *Then we have*

$$E(q) = \frac{1}{2}\sum_{j=1}^N \frac{q_j^2}{m_j} - \frac{1}{2}E_{\mathrm{I}}\left|\sum_{j=1}^N \frac{\alpha_j}{m_j}q_j\right|^2 + g_{\mathrm{I}}(m)$$

where

$$E_{\mathrm{I}} := \frac{K\|\hat{\lambda}/\sqrt{\omega}\|^2}{1+\epsilon\rho\|\hat{\lambda}/\sqrt{\omega}\|^2},$$

$$g_{\mathrm{I}}(m) := \frac{d}{2\pi}\int_{-\infty}^{\infty} \frac{\epsilon\rho s^2\|\sqrt{\omega}\hat{\lambda}/(s^2+\omega^2)\|^2}{1+\epsilon\rho\|\sqrt{\omega}\hat{\lambda}/\sqrt{s^2+\omega^2}\|^2}ds,$$

$$\rho := K\sum_{j=1}^N \frac{\alpha_j^2}{m_j}.$$

(2) Suppose II_λ. *Then we have*

$$E(q) = \frac{1}{2}\sum_{j=1}^N \frac{q_j^2}{m_j} - \frac{1}{2}\sum_{j=1}^N \frac{q_j^2}{m_j}E_{\mathrm{II}}^j + g_{\mathrm{II}}(m),$$

where

$$E_{\mathrm{II}}^j := \frac{\rho_j\|\hat{\lambda}_j/\sqrt{\omega}\|^2}{1+\epsilon\rho_j\|\hat{\lambda}_j/\sqrt{\omega}\|^2},$$

$$g_{\mathrm{II}}(m) := \sum_{j=1}^N \frac{d}{2\pi}\int_{-\infty}^{\infty} \frac{\epsilon\rho_j s^2\|\sqrt{\omega}\hat{\lambda}_j/(s^2+\omega^2)\|^2}{1+\epsilon\rho_j\|\sqrt{\omega}\hat{\lambda}_j/\sqrt{s^2+\omega^2}\|^2}ds,$$

$$\rho_j := K\frac{\alpha_j^2}{m_j}.$$

Proof: See Appendix. □

It is an interesting case where $\epsilon = 1$.

Corollary 3.4 *(1) Suppose* II_λ*and* $\epsilon = 1$. *Then*

$$E(q) = \sum_{j=1}^N \frac{1}{2m_j^{\mathrm{eff}}} q_j^2 + \sum_{j=1}^N \frac{d}{2\pi} \int_{-\infty}^{\infty} \frac{\rho_j s^2 \|\sqrt{\omega}\hat{\lambda}_j/(s^2+\omega^2)\|^2}{1+\rho_j\|\sqrt{\omega}\hat{\lambda}_j/\sqrt{s^2+\omega^2}\|^2} ds,$$

where

$$m_j^{\mathrm{eff}} := m_j + \alpha_j^2 K \|\hat{\lambda}_j/\sqrt{\omega}\|^2.$$

(2) Suppose $N=1$ *and* $\epsilon = 1$. *Then*

$$E(q) = \frac{1}{2m^{\mathrm{eff}}} q^2 + \frac{d}{2\pi} \int_{-\infty}^{\infty} \frac{K\alpha_1^2 s^2 \|\sqrt{\omega}\hat{\lambda}_1/(s^2+\omega^2)\|^2}{m + K\alpha_1^2\|\sqrt{\omega}\hat{\lambda}_1/(s^2+\omega^2)\|^2} ds,$$

where

$$m^{\mathrm{eff}} := m_1 + \alpha_1^2 K \|\hat{\lambda}_1/\sqrt{\omega}\|^2.$$

Lemma 3.5 *(1) Suppose* $\mathrm{I}_\lambda^\alpha(\epsilon)$. *Then for all* $q \in \mathbb{R}^{dN}$

$$\sum_{j=1}^N \frac{q_j^2}{2m_j} - \frac{1}{2} E_{\mathrm{I}} \left| \sum_{j=1}^N \frac{\alpha_j}{m_j} q_j \right|^2 > 0.$$

(2) Suppose $\mathrm{II}_\lambda^\alpha(\epsilon)$. *Then for all* $q \in \mathbb{R}^{dN}$

$$\sum_{j=1}^N \frac{q_j^2}{2m_j} - \frac{1}{2} \sum_{j=1}^N \frac{q_j^2}{m_j} E_{\mathrm{II}}^j > 0.$$

Proof: By the Schwartz inequality we have

$$\frac{1}{2} E_{\mathrm{I}} \left| \sum_{j=1}^N \frac{\alpha_j}{m_j} q_j \right|^2 \leq \frac{\rho \|\hat{\lambda}/\sqrt{\omega}\|^2}{1+\epsilon\rho\|\hat{\lambda}/\sqrt{\omega}\|^2} \sum_{j=1}^N \frac{q_j^2}{2m_j}.$$

Hence

$$\sum_{j=1}^N \frac{q_j^2}{2m_j} - \frac{1}{2} E_{\mathrm{I}} \left| \sum_{j=1}^N \frac{\alpha_j}{m_j} q_j \right|^2 \geq \frac{1}{2} \left(1 - \frac{\rho \|\hat{\lambda}/\sqrt{\omega}\|^2}{1+\epsilon\rho\|\hat{\lambda}/\sqrt{\omega}\|^2} \right) \sum_{j=1}^N \frac{q_j^2}{2m_j}.$$

Since Hypothesis $\mathrm{I}_\lambda^\alpha(\epsilon)$ implies that

$$1 - \frac{\rho \|\hat{\lambda}/\sqrt{\omega}\|^2}{1+\epsilon\rho\|\hat{\lambda}/\sqrt{\omega}\|^2} > 0,$$

(1) follows. (2) is similarly proven. $\square$

3.5 Observed mass

We define the observed mass by

$$m^*(\mu,\nu,j) := \left(\frac{\partial^2}{\partial q_{j\mu}\partial q_{j\nu}} E(q) \Big\lceil_{q=0} \right)^{-1}.$$

Put

$$m^*(\mu,\mu,j) := m^*(\mu,j).$$

By Lemma 3.3 we have the following corollaries.

Corollary 3.6 *(1) Suppose* I_λ. *Then*

$$m^*(\mu,\nu,j) = \delta_{\mu\nu} m_j \left(\frac{1+K(\epsilon\sum_{i=1}^N \frac{\alpha_i^2}{m_i} - \frac{\alpha_j^2}{m_j})\|\hat{\lambda}/\sqrt{\omega}\|^2}{1+\epsilon K \sum_{j=1}^N \frac{\alpha_j^2}{m_j}\|\hat{\lambda}/\sqrt{\omega}\|^2} \right)^{-1}.$$

In particular for the case of $\epsilon = 1$,

$$m^*(\mu,\nu,j) = \delta_{\mu\nu} m_j \left(\frac{1+K\sum_{i\neq j} \frac{\alpha_i^2}{m_i}\|\hat{\lambda}/\sqrt{\omega}\|^2}{1+K\sum_{j=1}^N \frac{\alpha_j^2}{m_j}\|\hat{\lambda}/\sqrt{\omega}\|^2} \right)^{-1}.$$

(2) Suppose II_λ. *Then*

$$m^*(\mu,\nu,j) = \delta_{\mu\nu} \frac{m_j + \epsilon K \alpha_j^2 \|\hat{\lambda}_j/\sqrt{\omega}\|^2}{1+(\epsilon-1)K\frac{\alpha_j^2}{m_j}\|\hat{\lambda}_j/\sqrt{\omega}\|^2}.$$

In particular for the case of $\epsilon = 1$,

$$m^*(\mu,\nu,j) = \delta_{\mu\nu} \left(m_j + K\alpha_j^2 \|\hat{\lambda}_j/\sqrt{\omega}\|^2 \right).$$

Proof: It is a direct calculation. □

It is interesting to compare the sizes of observed masses for $\mathrm{I}_\lambda$and II_λ.

Corollary 3.7 *Suppose* $\epsilon = 1$. *Let* $m_{\mathrm{I}}^*(\mu,j)$ *be the observed mass for* I_λ *and* $m_{\mathrm{II}}^*(\mu,j)$ *for* II_λ. *Suppose that* $\|\hat{\lambda}/\sqrt{\omega}\| \le \|\hat{\lambda}_j/\sqrt{\omega}\|$ *for* j. *Then*

$$m_{\mathrm{I}}^*(\mu,j) < m_{\mathrm{II}}^*(\mu,j).$$

Proof: Note that

$$f(x) = \frac{x}{a+x}, \quad a \geq 0,$$

is a monotonously increasing function of x. Directly we have

$$\frac{1}{m_{\mathrm{I}}^*(\mu, j)} = \frac{1}{m_j}\left(\frac{1 + K\sum_{i\neq j}\frac{\alpha_i^2}{m_i}\|\hat{\lambda}/\sqrt{\omega}\|^2}{1 + K\sum_{j=1}^N \frac{\alpha_j^2}{m_j}\|\hat{\lambda}/\sqrt{\omega}\|^2}\right)$$

$$> \frac{1}{m_j}\left(\frac{1}{1 + K\frac{\alpha_j^2}{m_j}\|\hat{\lambda}/\sqrt{\omega}\|^2}\right) \geq \frac{1}{m_j}\left(\frac{1}{1 + K\frac{\alpha_j^2}{m_j}\|\hat{\lambda}_j/\sqrt{\omega}\|^2}\right)$$

$$= \frac{1}{m_{\mathrm{II}}^*(\mu, j)}.$$

Hence the corollary follows. □

3.6 Diagonalization of H_ϵ^0 and its self-adjointness

We define

$$\theta := \exp\left(\sum_{j=1}^N \sum_{r=1}^{d-1}\sum_{\mu=1}^d \frac{\alpha_j}{\sqrt{2}} p_{j\mu} \otimes \left\{ a^r\left(\overline{\frac{e_\mu^r R_j}{\omega^{3/2}}}\right) - a^{\dagger r}\left(\frac{e_\mu^r R_j}{\omega^{3/2}}\right)\right\}\right)$$

and

$$\mathcal{U} := \theta U(\mathbb{W}).$$

Note that $\mathcal{U}$ maps $\mathcal{D}$ onto itself:

$$\mathcal{U} : \mathcal{D} \to \mathcal{D}.$$

Moreover we define $E(p)$ by the pseudo differential operator with the kernel $E(q)$ in $L^2(\mathbb{R}^{dN})$.

Lemma 3.8 *Suppose* I_λ *or* II_λ. *Then for* $\Phi \in C_0^\infty(\mathbb{R}^{dN})\hat{\otimes} D(H_{\mathrm{f}})$,

$$\mathcal{U}^{-1} H_\epsilon^0 \mathcal{U}\Phi = (E(p) \otimes 1 + 1 \otimes H_{\mathrm{f}})\Phi. \tag{3.3}$$

Proof: Let $\Phi = g \otimes \psi$ and $\Psi = f \otimes \phi$, $g, f \in C_0^\infty(\mathbb{R}^{dN})$, $\psi, \phi \in D(H_{\mathrm{f}})$. We note that $\mathcal{U}\Phi \in \mathcal{D}$. Then the left hand side of (3.3) is well defined. We have by Proposition 3.2

$$(\Psi, \mathcal{U}^{-1} H_\epsilon^0 \mathcal{U}\Phi)_{\mathcal{H}}$$

$$= \int_{\mathbb{R}^{dN}} dq(\widehat{\mathcal{U}\Psi}(q), H_\epsilon^0 \widehat{\mathcal{U}\Phi}(q))_{\mathcal{F}} = \int_{\mathbb{R}^{dN}} dq(\mathcal{U}(q)\hat{\Psi}(q), H_\epsilon^0(q)\mathcal{U}(q)\hat{\Phi}(q))_{\mathcal{F}}$$

$$\int_{\mathbb{R}^{dN}} dq\overline{\hat{f}}(q)\hat{g}(q)(\mathcal{U}(q)\psi, H_\epsilon^0(q)\mathcal{U}(q)\phi)_{\mathcal{F}} = \int_{\mathbb{R}^{dN}} dq\overline{\hat{f}}(q)\hat{g}(q)(\psi, (E(q) + H_{\mathrm{f}})\phi)_{\mathcal{F}}$$

$$= (\Psi, (E(q) \otimes 1 + 1 \otimes H_{\mathrm{f}})\Phi)_{\mathcal{H}}.$$

Thus the lemma follows. □

Lemma 3.9 *Suppose* $\mathrm{I}_\lambda^\alpha(\epsilon)$ *or* $\mathrm{II}_\lambda^\alpha(\epsilon)$. *Then*

$$\mathcal{U}^{-1} H_\epsilon^0 \mathcal{U} = E(p) \otimes 1 + 1 \otimes H_{\mathrm{f}} \tag{3.4}$$

holds on $\mathcal{D}$. *In particular* H_ϵ *is self-adjoint on* $\mathcal{D}$ *and bounded from below.*

Proof: By the assumption on $\{\alpha_1, \cdots, \alpha_N\}$, $E(p)$ is nonnegative. Then $E(p) + H_{\mathrm{f}}$ is self-adjoint on $\mathcal{D}$ and bounded from below. For $\Phi \in \mathcal{D}$, we can take a sequence $\Phi_n \in C_0^\infty(\mathbb{R}^{dN})\hat{\otimes} D(H_{\mathrm{f}})$ such that $\Phi_n \to \Phi$ and $(E(p)\otimes 1 + 1 \otimes H_{\mathrm{f}})\Phi_n \to (E(p)\otimes 1 + 1\otimes H_{\mathrm{f}})\Phi$ strongly as $n \to \infty$. Since H_ϵ is closed, $H_\epsilon \mathcal{U}\Phi_n \to H_\epsilon\mathcal{U}\Phi$ as $n \to \infty$. Thus (3.4) holds. Since $\mathcal{U}$ maps $\mathcal{D}$ onto itself, H_ϵ is self-adjoint on $\mathcal{D}$ and bounded from below.
□

Remark 3.10 *Let* $\epsilon < 1$. *Then* $E(p)$ *is unbounded from below for sufficiently large coupling constants. Actually in the case of* $N = 1$, *for*

$$\alpha_1^2 \geq m_1 \left(\|\hat{\lambda}_1/\sqrt{\omega}\|^2 (1-\epsilon) \right)^{-1},$$

$E(p)$ *is unbounded from below. Hence* H_ϵ^0 *is unbounded from below for sufficiently large coupling constants. We should not overlook that the approximation of neglecting the self-interaction* A^2 *in* H *is not reasonable for sufficiently large coupling constants.*

Theorem 3.11 *Suppose* $\mathrm{I}_\lambda^\alpha(\epsilon)$ *or* $\mathrm{II}_\lambda^\alpha(\epsilon)$. *Then* H_ϵ *is self-adjoint on* $\mathcal{D}$ *and bounded from below.*

Proof: By Lemma 3.9 and the closed graph theorem there exists a constant c such that

$$\|(\sum_{j=1}^N \frac{p_j^2}{2m_j} + H_{\mathrm{f}})\Psi\| \leq c(\|H_\epsilon\Psi\| + \|\Psi\|), \quad \Psi \in \mathcal{D}.$$

Thus

$$\|V\Psi\| \le ac(\|H_\epsilon\Psi\| + \|\Psi\|) + b\|\Psi\|$$

with some sufficiently small a, and some b, the Kato-Rellich theorem leads to the lemma. $\square$

Corollary 3.12 *Suppose* I_λ. *Then*

$$\inf\sigma(H_\epsilon^0) = g_{\mathrm{I}}(m) = \frac{d}{2\pi}\int_{-\infty}^{\infty} \frac{\epsilon K \sum_{j=1}^N \frac{\alpha_j^2}{m_j} s^2 \|\sqrt{\omega}\hat{\lambda}/(s^2+\omega^2)\|^2}{1+\epsilon K \sum_{j=1}^N \frac{\alpha_j^2}{m_j}\|\sqrt{\omega}\hat{\lambda}/\sqrt{s^2+\omega^2}\|^2} ds.$$

Suppose II_λ. *Then*

$$\inf\sigma(H_\epsilon^0) = g_{\mathrm{II}}(m) = \sum_{j=1}^N \frac{d}{2\pi}\int_{-\infty}^{\infty} \frac{\epsilon K \frac{\alpha_j^2}{m_j} s^2 \|\sqrt{\omega}\hat{\lambda}_j/(s^2+\omega^2)\|^2}{1+\epsilon K \frac{\alpha_j^2}{m_j}\|\sqrt{\omega}\hat{\lambda}_j/\sqrt{s^2+\omega^2}\|^2} ds.$$

Proof: It follows from Lemmas 3.3 and 3.8. $\square$

4 Effective Hamiltonian

4.1 Scaled Hamiltonians

Here we recall the scaled Hamiltonian $H_\epsilon(\kappa)$;

$$H_\epsilon(\kappa) := H_{\mathrm{el}}\otimes 1 - \kappa^l \sum_{j=1}^N \frac{\alpha_j}{m_j} p_j \otimes A_j + \kappa^2 1 \otimes \left(\sum_{j=1}^N \frac{\epsilon\alpha_j^2}{2m_j} A_j^2 + H_{\mathrm{f}} \right).$$

$H_\epsilon(\kappa)$ is derived from H_ϵ scaled as

$$\omega \to \kappa^2\omega, \tag{4.1}$$

$$\alpha_j \to \kappa^l \alpha_j, \tag{4.2}$$

$$\epsilon \to \kappa^{2-2l}\epsilon. \tag{4.3}$$

Replacing $\omega, \alpha_j, \epsilon$ as in (4.1)–(4.3), we see that

$$\theta \longrightarrow \exp\left(\kappa^{l-2} \sum_{j=1}^N \sum_{\mu=1}^d \frac{\alpha_j}{\sqrt{2}} p_{j\mu} \otimes \left\{ a^r\left(\frac{\overline{e_\mu^r R_j}}{\omega}\right) - a^{\dagger r}\left(\frac{e_\mu^r R_j}{\omega}\right) \right\} \right), \tag{4.4}$$

$$-\frac{1}{2}E_{\rm I}\left|\sum_{j=1}^N \frac{\alpha_j}{m_j}q_j\right|^2 \longrightarrow -\kappa^{2l-2}\frac{1}{2}E_{\rm I}\left|\sum_{j=1}^N \frac{\alpha_j}{m_j}q_j\right|^2,$$

$$-\frac{1}{2}\sum_{j=1}^N \frac{q_j^2}{m_j}E_{\rm II}^j \longrightarrow -\kappa^{2l-2}\frac{1}{2}\sum_{j=1}^N \frac{q_j^2}{m_j}E_{\rm II}^j,$$

$$g_{\rm I}(m)\to g_{\rm I}(m/k^2),$$

$$g_{\rm II}(m)\to g_{\rm II}(m/k^2).$$

Lemma 4.1 *$U(\mathbb{W})$ leaves invariant under the scaling (4.1),(4.2) and (4.3).*

Proof: See Lemma 6.5 in Appendix. □

Let

$$E_{\rm ren}(p) := \begin{cases} -\frac{1}{2}E_{\rm I}\left|\sum_{j=1}^N \frac{\alpha_j}{m_j}p_j\right|^2, & \text{for } {\rm I}_\lambda, \\ \\ -\frac{1}{2}\sum_{j=1}^N \frac{p_j^2}{m_j}E_{\rm II}^j, & \text{for } {\rm II}_\lambda. \end{cases} \tag{4.5}$$

Moreover we set

$$g(m) := \begin{cases} g_{\rm I}(m), & \text{for } {\rm I}_\lambda, \\ \\ g_{\rm II}(m), & \text{for } {\rm II}_\lambda. \end{cases}$$

We define $\mathcal{U}_\kappa$ by $\mathcal{U}$ with ω,α_j and ϵ replaced as in (4.1),(4.2), and (4.3), respectively.

Lemma 4.2 *Suppose ${\rm I}_\lambda^\alpha(\kappa^{2l-2}\epsilon)$ or ${\rm II}_\lambda^\alpha(\kappa^{2l-2}\epsilon)$. Then*

$$\mathcal{U}_\kappa^{-1}H_\epsilon(\kappa)\mathcal{U}_\kappa = \frac{1}{2}\sum_{j=1}^N \frac{p_j^2}{m_j} + \kappa^{2l-2}E_{\rm ren}(p) + \kappa^2 H_{\rm f} + g(m/\kappa^2) + \mathcal{U}_\kappa^{-1}V\mathcal{U}_k.$$

In particular $H_\epsilon(\kappa)$ is self-adjoint on $\mathcal{D}$ and bounded from below.

Proof: It follows from Lemma 3.9. □

Lemma 4.3 *Suppose ${\rm I}_\lambda$or ${\rm II}_\lambda$. Then there exists the unique ground state Ω_ϵ of $\sum_{j=1}^N \epsilon\alpha_j^2 A_j^2/(2m_j)+H_{\rm f}$. Moreover*

$$P_\epsilon := U(\mathbb{W})P_\Omega U(\mathbb{W})^{-1} \tag{4.6}$$

is the projection to the eigenspace spanned by Ω_ϵ.

Proof: Set $q=0$ in Lemma 3.2 (2). Then

$$\mathcal{U}(0)^{-1}\left(\epsilon\sum_{j=1}^N \frac{\alpha_j^2}{2m_j}A_j^2+H_{\rm f}\right)\mathcal{U}(0)=H_{\rm f}+g(m).$$

The right hand side has the unique ground state Ω. Thus $\mathcal{U}(0)\Omega=U(\mathbb{W})\Omega=\Omega_\epsilon$ is the unique ground state of $\sum_{j=1}^N \epsilon\alpha_j^2A_j^2/(2m_j)+H_{\rm f}$. Hence the lemma follows. $\square$

Lemma 4.4 *(1) Suppose $l=2$. Then $\mathcal{U}_\kappa$ is independent of κ. (2) Suppose $l<2$. Then $s-\lim_{\kappa\to\infty}\mathcal{U}_\kappa=U(\mathbb{W})$.*

Proof: It follows from (4.1) and (4.4). $\square$

We write as

$$\mathcal{U}_\infty:=\begin{cases}\mathcal{U}, & l=2,\\ \\ U(\mathbb{W}), & l<2.\end{cases}$$

4.2 Effective potentials

Define $V_{\rm eff}$ acting on $L^2(\mathbb{R}^{dN})$ by

$$(f,V_{\rm eff}g)_{L^2(\mathbb{R}^{dN})}:=(f\otimes\Omega,\left(\mathcal{U}_\infty^{-1}V\mathcal{U}_\infty\right)g\otimes\Omega)_{\mathcal{H}}.$$

Note

$$D(V_{\rm eff})\supset D(\sum_{j=1}^N\frac{p_j^2}{2m_j}).$$

We define the $N\times N$ correlation matrix $\Delta=(\Delta_{ij})_{1\le i,j\le N}$ by

$$\Delta_{ij}:=\frac12\alpha_i\alpha_jK\left(\frac{Q_i}{\omega^{3/2}},\frac{Q_j}{\omega^{3/2}}\right).$$

Note that

$$\left(L_i^\mu,L_j^\nu\right)=\frac{\alpha_i\alpha_j}{2}K\delta_{\mu\nu}\left(\frac{Q_i}{\omega^{3/2}},\frac{Q_j}{\omega^{3/2}}\right)=\delta_{\mu\nu}\Delta_{ij}.$$

Suppose that ${\rm Rank}\Delta=M$ and $T=T_\Delta$ is an $N\times N$ unitary matrix such that

$$T\Delta T^{-1}={\rm diag}\{\mu_1,\cdots,\mu_M,0,\cdots,0\},\ \mu=\mu_1\cdots\mu_M\neq0.$$

Let

$$\widetilde{T}:=\begin{pmatrix}T_{11}1_d & \cdots & T_{1N}1_d\\ \vdots & & \vdots\\ T_{N1}1_d & \cdots & T_{NN}1_d\end{pmatrix}.$$

Here 1_d denotes the $d\times d$ identity matrix.

Lemma 4.5 *Let $l = 2$. Suppose*

$$\int_{\mathbb{R}^{dM}} |V_{\widetilde{T}^{-1}}|(y_1, \cdots, y_M, (\widetilde{T}x)_{M+1}, \cdots, (\widetilde{T}x)_N) e^{-\sum_{j=1}^K |(\widetilde{T}x)_j - y_j|^2/(2\mu)} dy \in L^1_{\rm loc}(\mathbb{R}^{dN}).$$

(1) *Suppose that* $\mathrm{Rank}\Delta = N$. *Then we have*

$$V_{\rm eff}(x) = (2\pi)^{-dN/2} (\det\Delta)^{-d/2} \int_{\mathbb{R}^{dN}} e^{-|x-y|^2/(2\det\Delta)} V(y) dy.$$

(2) *Suppose that* $\mathrm{Rank}\Delta = M < N$. *Then we have*

$$V_{\rm eff}(x) = (2\pi)^{-dN/2} \mu^{-d/2} \int_{\mathbb{R}^{dM}} V_{\widetilde{T}^{-1}}(y_1, \cdots, y_M, (\widetilde{T}x)_{M+1}, \cdots, (\widetilde{T}x)_N)$$

$$\times e^{-\sum_{j=1}^K |(\widetilde{T}x)_j - y_j|^2/(2\mu)} dy.$$

Proof: It follows from Lemma 2.3. □

Lemma 4.6 *Let $l < 2$. Then $V_{\rm eff} = V$.*

Proof: Since $\mathcal{U}_\infty$ and V is commutative, the lemma follows. □

4.3 Scaling limits

Let

$$V_\kappa := \mathcal{U}_\kappa^{-1} V \mathcal{U}_\kappa.$$

We define

$$H_\epsilon^{\rm ren}(\kappa) := \begin{cases} H_\epsilon(\kappa) - \kappa^{2l-2} E_{\rm ren}(p) - g(m/\kappa^2), & 1 < l \le 2, \\ H_\epsilon(\kappa) - g(m/\kappa^2), & l \le< 1, \end{cases}$$

and

$$H_{\rm eff} := \begin{cases} \sum_{j=1}^N \frac{p_j^2}{2m_j} + V_{\rm eff}, & l = 2, \\ \sum_{j=1}^N \frac{p_j^2}{2m_j} + V, & 1 < l < 2, \\ \sum_{j=1}^N \frac{p_j^2}{2m_j} + E_{\rm ren}(p) + V, & l = 1, \\ \sum_{j=1}^N \frac{p_j^2}{2m_j} + V, & l < 1, \end{cases} . \tag{4.7}$$

Lemma 4.7 *Let κ be sufficiently large. Define*

$$H_{\rm eff}(\kappa):=\left\{\begin{array}{ll}\sum_{j=1}^N \frac{p_j^2}{2m_j}+\kappa^2 H_{\rm f}+V, & 1<l\leq 2,\\ \\ \sum_{j=1}^N \frac{p_j^2}{2m_j}+\kappa^{2l-2}E_{\rm ren}(p)+\kappa^2 H_{\rm f}+V, & l\leq 1,\end{array}\right. .$$

(1) Suppose I_λ*and* II_λ*, and* $l\neq 1$*. Then*

$$\mathcal{U}_\kappa H_\epsilon^{\rm ren}(\kappa)\mathcal{U}_\kappa=H_{\rm eff}(\kappa). \tag{4.8}$$

In particular, for all $\{\alpha_1,\cdots,\alpha_N\}$*,* $H_\epsilon^{\rm ren}(\kappa)$ *is self-adjoint on* $\mathcal{D}$ *and bounded from below. (2) Suppose* $\mathrm{I}_\lambda^\alpha(\epsilon)$ *or* $\mathrm{II}_\lambda^\alpha(\epsilon)$*, and* $l=1$*. Then (4.8) holds. In particular* $H_\epsilon^{\rm ren}(\kappa)$ *is self-adjoint on* $\mathcal{D}$ *and bounded from below.*

Proof: By Hypothesis **V**we see that for $\Psi\in\mathcal{D}$

$$\|V_\kappa\Psi\|\leq a\|\sum_{j=1}^N\frac{p_j^2}{2m_j}\Psi\|+b\|\Psi\|.$$

Thus by the Kato-Rellich theorem $H_{\rm eff}(\kappa)$ for $1<l\leq 2$ is self-adjoint on $\mathcal{D}$ and bounded from below. For $l<1$, since κ is sufficiently large, we have $\sum_{j=1}^N\frac{p_j^2}{2m_j}+\kappa^{2l-2}E_{\rm ren}(p)$ is positive and

$$\|(\sum_{j=1}^N\frac{p_j^2}{2m_j}+E_{\rm ren}(p))f\|\geq\epsilon(\alpha_1,\cdots,\alpha_N)\|\sum_{j=1}^N\frac{p_j^2}{2m_j}f\|$$

with some $\epsilon(\alpha_1,\cdots,\alpha_N)>0$ independent of large κ. Then

$$\|V_\kappa\Psi\|\leq\frac{a}{\epsilon(\alpha_1,\cdots,\alpha_N)}\|(\sum_{j=1}^N\frac{p_j^2}{2m_j}+E_{\rm ren}(p))\Psi\|+b\|\Psi\|. \tag{4.9}$$

Hence $H_{\rm eff}(\kappa)$ for $l\neq 1$ is self-adjoint on $\mathcal{D}$ and bounded from below. By (3.8), (4.8) holds on $C_0^\infty(\mathbb{R}^{dN})\widehat{\otimes}D(H_{\rm f})$, and $\mathcal{U}_\kappa$ maps $\mathcal{D}$ onto itself. Then (4.8) holds on $\mathcal{D}$ by a limiting argument. Thus (1) follows. For $l=1$, $\sum_{j=1}^N p_j^2/(2m_j)+E_{\rm ren}(p)$ is a nonnegative self-adjoint operator. Thus $H_{\rm eff}(\kappa)$ is self-adjoint on $\mathcal{D}$ and bounded from below. Thus (2) follows similarly. $\square$

Lemma 4.8 *(1) Suppose* $\mathrm{I}_\lambda^\alpha(\epsilon)$ *or* $\mathrm{II}_\lambda^\alpha(\epsilon)$*, and* $l=1$*. Then* $H_{\rm eff}$ *is self-adjoint on* $D(\sum_{j=1}^N p_j^2/(2m_j))$ *and bounded from below. (2) Let* $l\neq 1$*. Then* $H_{\rm eff}$ *is self-adjoint on* $D(\sum_{j=1}^N p_j^2/(2m_j))$ *and bounded from below.*

Proof: It is clear. We omit it. □

Let P_ϵ denotes the projection to the subspace spanned by the unique ground state of $\sum_{j=1}^N \frac{\epsilon\alpha_j^2}{2m_j} A_j^2 + H_{\rm f}$. Note $P_0 := P_\Omega$ is the projection to Ω.

Theorem 4.9 *Let $l=2$. Suppose* I_λ *or* II_λ. *Moreover suppose* $|V_{\rm eff}| \in L^1_{\rm loc}(\mathbb{R}^{dN})$. *Then for all* $\{\alpha_1, \cdots, \alpha_N\}$ *and* $z \in \mathbb{C}\setminus\mathbb{R}$,

$$s - \lim_{\kappa\to\infty} (H_\epsilon^{\rm ren}(\kappa) - z)^{-1} = \theta\left\{(H_{\rm eff} - z)^{-1} \otimes P_\epsilon\right\}\theta^{-1}.$$

Theorem 4.10 *Let $1 < l < 2$. Suppose* I_λ *or* II_λ. *Then for all* $\{\alpha_1, \cdots, \alpha_N\}$ *and* $z \in \mathbb{C}\setminus\mathbb{R}$,

$$s - \lim_{\kappa\to\infty} (H_\epsilon^{\rm ren}(\kappa) - z)^{-1} = (H_{\rm eff} - z)^{-1} \otimes P_\epsilon. \tag{4.10}$$

Theorem 4.11 *Let $l = 1$. Suppose* $\mathrm{I}_\lambda^\alpha(\epsilon)$ *or* $\mathrm{II}_\lambda^\alpha(\epsilon)$. *Then (4.10) holds.*

Theorem 4.12 *Let $l < 1$. Suppose* I_λ *or* II_λ. *Then for all* $\{\alpha_1, \cdots, \alpha_N\}$ *and* $z \in \mathbb{C}\setminus\mathbb{R}$, *(4.10) holds.*

Before proving Theorems 4.9–4.12 we show a lemma.

Lemma 4.13 *Suppose $z \in \mathbb{C}\setminus\mathbb{R}$.*

(1) *Let $l = 2$. Suppose* $|V|_{\rm eff} \in L^1_{\rm loc}(\mathbb{R}^{dN})$, *and* I_λ *or* II_λ. *Then for all* $\{\alpha_1, \cdots, \alpha_N\}$

$$s - \lim_{\kappa\to\infty} (H_{\rm eff}(\kappa) - z)^{-1} = (H_{\rm eff} - z)^{-1} \otimes P_\Omega. \tag{4.11}$$

(2) *Let $1 < l < 2$. Suppose* I_λ *or* II_λ. *Then for all* $\{\alpha_1, \cdots, \alpha_N\}$, *(4.11) holds.*

(3) *Let $l = 1$. Suppose* $\mathrm{I}_\lambda^\alpha(\epsilon)$ *or* $\mathrm{II}_\lambda^\alpha(\epsilon)$. *Then (4.11) holds.*

(4) *Let $l < 1$. Suppose* I_λ *or* II_λ. *Then for all* $\{\alpha_1, \cdots, \alpha_N\}$, *(4.11) holds.*

Proof: By an abstract theorem in [8, p.2654, (i),(ii),(iii)] it is enough to prove that

(1) $D(V_\kappa) \supset D(H_{\rm eff})$ and $V_\kappa(H_{\rm eff} - z)^{-1}$ is a bounded operator for sufficiently large z with $\lim_{z\to\infty} \|V_\kappa(H_{\rm eff} - z)^{-1}\| = 0$ uniformly in κ,

(2) $V_\kappa(H_{\rm eff} - z)^{-1}$ is strongly continuous in κ and $s - \lim_{\kappa\to\infty} V_\kappa(H_{\rm eff} - z)^{-1} = V_{\rm eff}(H_{\rm eff} - z)^{-1}$.

	$H_\epsilon^{\mathrm{ren}}(\kappa)+g(m/\kappa^2)$	H_{eff}
$l=2$, I_λ	$H_\epsilon(\kappa)+\frac{\kappa^2}{2}E_{\mathrm{I}}\left\lvert\sum_{j=1}^N \frac{\alpha_j}{m_j}p_j\right\rvert^2$	$\frac{1}{2m}p^2+V_{\mathrm{eff}}$
$l=2$, II_λ	$H_\epsilon(\kappa)+\frac{\kappa^2}{2}\sum_{j=1}^N \frac{p_j^2}{m_j}E_{\mathrm{II}}^j$	$\frac{1}{2m}p^2+V_{\mathrm{eff}}$
$1<l<2$, I_λ	$H_\epsilon(\kappa)+\frac{\kappa^{2l-2}}{2}E_{\mathrm{I}}\left\lvert\sum_{j=1}^N \frac{\alpha_j}{m_j}p_j\right\rvert^2$	$\frac{1}{2m}p^2+V$
$1<l<2$, II_λ	$H_\epsilon(\kappa)+\frac{\kappa^{2l-2}}{2}\sum_{j=1}^N \frac{p_j^2}{m_j}E_{\mathrm{II}}^j$	$\frac{1}{2m}p^2+V$
$l=1$, I_λ	$H_\epsilon(\kappa)$	$\sum_{j=1}^N \frac{p_j^2}{2m_j}-\frac{1}{2}E_{\mathrm{I}}\left\lvert\sum_{j=1}^N \frac{\alpha_j}{m_j}p_j\right\rvert^2+V$
$l=1$, II_λ	$H_\epsilon(\kappa)$	$\sum_{j=1}^N \frac{p_j^2}{2m_j}-\frac{1}{2}\sum_{j=1}^N \frac{p_j^2}{m_j}E_{\mathrm{II}}^j+V$
$l<1$	$H_\epsilon(\kappa)$	$\frac{1}{2m}p^2+V$

Table 1: Effective Hamiltonians

By the definition of $V_{\rm eff}$ and Hypothesis **V**, we have

$$\|V_{\rm eff} f\| \leq a' \| \sum_{j=1}^N \frac{p_j^2}{2m_j} f\| + b'\|f\|$$

with some positive a' and b'. Together with (4.9) we get

$$\|V_\kappa f\| \leq a''\|H_{\rm eff} f\| + b''\|f\|$$

with some positive a'' and b''. Moreover

$$\|V_\kappa (H_{\rm eff} - z)^{-1} f\| \leq a''\|H_{\rm eff}(H_{\rm eff} - z)^{-1} f\| + b''\|(H_{\rm eff} - z)^{-1} f\|.$$

Hence $V_\kappa (H_{\rm eff} - z)^{-1} f \to 0$ as $z \to \infty$. Then (1) follows. Since $\mathcal{U}_\kappa$ is continuous in κ and $s - \lim_{\kappa\to\infty} \mathcal{U}_k = \mathcal{U}_\infty$, we have

$$\begin{aligned}
\|(V_\kappa - V_{\kappa'}) f\| &\leq \|\mathcal{U}_\kappa^{-1} V \mathcal{U}_k f - \mathcal{U}_\kappa^{-1} V \mathcal{U}_{\kappa'} f\| + \|\mathcal{U}_\kappa^{-1} V \mathcal{U}_{\kappa'} f - \mathcal{U}_{\kappa'}^{-1} V \mathcal{U}_{\kappa'} f\| \\
&\leq \|V(\mathcal{U}_\kappa - \mathcal{U}_{\kappa'}) f\| + \|(\mathcal{U}_\kappa^{-1} - \mathcal{U}_{\kappa'}^{-1}) V \mathcal{U}_{\kappa'} f\| \\
&\leq a\|(\mathcal{U}_\kappa - \mathcal{U}_{\kappa'}) \sum_{j=1}^N \frac{p_j^2}{2m_j} f\| + b\|(\mathcal{U}_\kappa - \mathcal{U}_{\kappa'}) f\| + \|(\mathcal{U}_\kappa^{-1} - \mathcal{U}_{\kappa'}^{-1}) V \mathcal{U}_{\kappa'} f\|.
\end{aligned}$$

Here in the third inequality we use that $\mathcal{U}_k$ and $p_j, j = 1, ..., N$ are commutative. Then the first statement of (2) follows. The second statement is also proven similarly. □

A proof of Theorems 4.9–4.12

Suppose $l = 2$. Then $\mathcal{U}_\kappa = \mathcal{U}$ is independent of κ. We have by (4.8)

$$(H_\epsilon^{\rm ren}(\kappa) - z)^{-1} = \mathcal{U}(H_{\rm eff}(\kappa) - z)^{-1} \mathcal{U}^{-1}.$$

Taking $\kappa \to \infty$, we have

$$\begin{aligned}
\lim_{\kappa\to\infty} (H_\epsilon^{\rm ren}(\kappa) - z)^{-1} &= \mathcal{U} \left\{ (H_{\rm eff} - z)^{-1} \otimes P_\Omega \right\} \mathcal{U}^{-1} \\
&= \theta \left\{ (H_{\rm eff} - z)^{-1} \otimes U(\mathbb{W}) P_\Omega U(\mathbb{W})^{-1} \right\} \theta^{-1}.
\end{aligned}$$

Since $U(\mathbb{W}) P_\Omega U(\mathbb{W})^{-1} = P_\epsilon$, Theorem 4.9 follows. Next suppose $l = 1$.

$$(H_\epsilon^{\rm ren}(\kappa) - z)^{-1} = \mathcal{U}_\kappa \left(H_{\rm eff}(\kappa) - z \right)^{-1} \mathcal{U}_\kappa^{-1}.$$

Since

$$s - \lim_{\kappa\to\infty} \mathcal{U}_\kappa = U(\mathbb{W}),$$

taking $k \to \infty$, we have

$$\lim_{\kappa\to\infty} (H_\epsilon^{\rm ren}(\kappa) - z)^{-1} = U(\mathbb{W}) \left\{ (H_{\rm eff} - z)^{-1} \otimes P_\Omega \right\} U(\mathbb{W})^{-1}.$$

Hence Theorem 4.11 follows. Theorems 4.10 and 4.12 are proven similarly. □

4.4 Examples

In this subsection we assume that $|V_{\rm eff}| \in L^1_{\rm loc}(\mathbb{R}^{dN})$. A typical example is for $N = 1$ and $\epsilon = 1$. Put $A_1 = A$, $m_1 = m$, $p_1 = p$, $\alpha_1 = \alpha$.

Corollary 4.14 *Let $N = 1$ and $\epsilon = 1$. Suppose* I_λ *or* II_λ. *Then for all $\alpha \in \mathbb{R}$ and $z \in \mathbb{C} \setminus \mathbb{R}$*

($l = 2$)

$$s - \lim_{\kappa\to\infty} \left(H(\kappa) + \kappa^2 \frac{1}{2(m+\delta m)} \frac{\delta m}{m} p^2 - g(m/\kappa^2) - z \right)^{-1}$$
$$= \theta \left\{ \left(\frac{1}{2m} p^2 + V_{\rm eff} - z \right)^{-1} \otimes P_1 \right\} \theta^{-1}.$$

($1 < l < 2$)

$$s - \lim_{\kappa\to\infty} \left(H(\kappa) + \kappa^{2l-2} \frac{1}{2(m+\delta m)} \frac{\delta m}{m} p^2 - g(m/\kappa^2) - z \right)^{-1}$$
$$= \left(\frac{1}{2m} p^2 + V - z \right)^{-1} \otimes P_1.$$

($l = 1$)

$$s - \lim_{\kappa\to\infty} \left(H(\kappa) - g(m/\kappa^2) - z \right)^{-1} = \left(\frac{1}{2(m+\delta m)} p^2 + V - z \right)^{-1} \otimes P_1.$$

($l < 1$)

$$s - \lim_{\kappa\to\infty} \left(H(\kappa) - g(m/\kappa^2) - z \right)^{-1} = \left(\frac{1}{2m} p^2 + V - z \right) \otimes P_1.$$

	$H^{\rm ren}(\kappa)+g(m/\kappa^2)$	$H_{\rm eff}$
$l=2$	$H(\kappa)+\kappa^2\frac{1}{2(m+\delta m)}\frac{\delta m}{m}p^2$	$\frac{1}{2m}p^2+V_{\rm eff}$
$1<l<2$	$H(\kappa)+\kappa^{2l-2}\frac{1}{2(m+\delta m)}\frac{\delta m}{m}p^2$	$\frac{1}{2m}p^2+V$
$l=1$	$H(\kappa)$	$\frac{1}{2(m+\delta m)}p^2+V$
$l<1$	$H(\kappa)$	$\frac{1}{2m}p^2+V$

Table 2: Effective Hamiltonians for $N=1$ and $\epsilon=1$

By virtue of Corollary 4.14 we see that in the case of $l=2$ the Welton Hamiltonian dose appear and in the case of $l=1$ the Hamiltonian with observed mass does.

Corollary 4.15 *Suppose* I_λ*and* $l=2$. *Moreover we assume that* $\alpha_1=\cdots=\alpha_N:=\alpha$ *and* $m_1=\cdots=m_N:=m$. *Then*

$$V_{\rm eff}=(2\pi C)^{-d/2}\int_{\mathbb{R}^d}V_{\widetilde{T}^{-1}}(y,(\widetilde{T}x)_2,\cdots,(\widetilde{T}x)_N))e^{-|(\widetilde{T}x)_1-y|^2/2C}dy,$$

where

$$C:=\frac{N\alpha^2}{2}\frac{d-1}{d}\|\frac{Q}{\omega^{3/2}}\|^2,$$

Proof: Since the correlation matrix is given by

$$\Delta=\frac{\alpha^2}{2}\|\frac{Q}{\omega^{3/2}}\|^2K\begin{pmatrix}1&\cdots&1\\ \vdots&&\vdots\\ 1&\dots&1\end{pmatrix},$$

where

$$Q=\sqrt{\omega}D^{-1}\begin{pmatrix}\hat{\lambda}\\ \vdots\\ \hat{\lambda}\end{pmatrix},$$

we have Rank $\Delta=1$. Thus the corollary follows. □

Corollary 4.16 *Suppose* II_λ *and* $l = 2$. *Then*

$$V_{\rm eff} = (2\pi C)^{-d/2} \int_{\mathbb{R}^{dN}} V(y) e^{-|x-y|^2/2C} dy,$$

where

$$C := \Pi_{j=1}^N \left(\frac{d-1}{2d} \frac{\alpha_j^2}{m_j} \int_{\mathbb{R}^d} \frac{|\hat{\lambda}_j(k)|^2}{\omega(k)^2(m_j - \epsilon\delta C_j)} dk \right),$$

$$\delta C_j := \frac{d-1}{2d} \alpha_j^2 \int_{\mathbb{R}^d} \frac{\omega(k)|\hat{\lambda}_j(k')|^2}{\omega(k)^2 - \omega(k')^2 + i0} dk'.$$

Proof: Since the correlation matrix is given by

$$\Delta = \mathrm{diag}\{\frac{1}{2}\alpha_1^2 K \| \frac{Q_1}{\omega^{3/2}} \|^2, \cdots, \frac{1}{2}\alpha_N^2 K \| \frac{Q_N}{\omega^{3/2}} \|^2\},$$

where

$$Q_j = \frac{\sqrt{\omega}\hat{\lambda}_j}{D_{jj}},$$

the corollary follows. □

5 Concluding Remarks (g-factor shift)

For simplicity we set $\epsilon = 1$, $N = 1$, and $d = 3$. Put $A_1 = A$, $m_1 = m$, $p_1 = p$, $\alpha_1 = \alpha$. We suppose that spin 1/2 is included in the particle and an external vector potential $a(x)$ is imposed. Thus the Hamiltonian is read

$$H_{\rm spin} := \frac{1}{2m} \{\sigma \cdot (p - a(x) - \alpha A)\}^2 + V + H_{\rm f}$$

acting on the Hilbert space

$$\mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \otimes \mathcal{F}.$$

Here $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ denotes the Pauli matrices:

$$\sigma_1 := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 := \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

$$H_{\rm spin} = \frac{1}{2m}(p - \alpha A)^2 + H_{\rm f} + V - \frac{g}{2m} \sigma \cdot {\rm curl}\ a + H_{\rm I},$$

where

$$g = 2$$

and

$$H_{\mathrm{I}} := -\frac{1}{2m}\{(p-\alpha A)\cdot a - a\cdot(p-\alpha A)\} + \frac{1}{2m}a^2 - \frac{\alpha}{2m}\sigma\cdot \mathrm{curl}\ A,$$

and

$$\mathrm{curl}\ A := \frac{-i}{\sqrt{2}}\sum_{r=1}^{2}\left\{a^{\dagger r}\left((k\times e^r)^r_\mu\tilde{\hat{\lambda}}\right) + a^r\left((-k\times e^r)_\mu\hat{\lambda}\right)\right\}.$$

For simplicity we neglect H_{I}. Then our Hamiltonian is defined by

$$H = \frac{1}{2m}(p-\alpha A)^2 + H_{\mathrm{f}} + V - \frac{g}{2m}\sigma\cdot \mathrm{curl}\ a.$$

Let $l=1$. Then the scaled Hamiltonian is given by

$$H(\kappa) := \frac{1}{2m}(p-\kappa\alpha A)^2 + \kappa^2 H_{\mathrm{f}} + V - \frac{g}{2m}\sigma\cdot \mathrm{curl}\ a.$$

Corollary 5.1 *Suppose that a is sufficiently smooth and that* curl a *is infinitesimally small with respect to p^2. Then $H(\kappa)$ is self-adjoint on $\mathcal{D}$ and bounded from below. Furthermore for $z\in\mathbb{C}\setminus\mathbb{R}$*

$$s-\lim_{\kappa\to\infty}(H(\kappa)-g(m/\kappa^2)-z)^{-1} = (H_{\mathrm{eff}}-z)\otimes P_1.$$

Here

$$H_{\mathrm{eff}} = \frac{1}{2(m+\delta m)}p^2 + \frac{g_{\mathrm{eff}}}{2(m+\delta m)}\sigma\cdot \mathrm{curl}\ a + V,$$

and

$$g_{\mathrm{eff}} := \frac{m+\delta m}{m}g > 2.$$

Proof: The self-adjointness follows from Proposition 1.1. Since σ_j and $\mathcal{U}_k$ are commutative, by Theorem 4.11, we have

$$s-\lim_{\kappa\to\infty}(H(\kappa)-g(m/\kappa^2)-z)^{-1} = \left(\frac{1}{2(m+\delta m)}p^2 - \frac{g}{4m}\sigma\cdot \mathrm{curl}\ a + V - z\right)^{-1}\otimes P_1.$$

Hence the corollary follows. $\square$

6 Appendix

6.1 A proof of Proposition 3.2

6.1.1 Construction of canonical transformations

We recall

$$D_{ij} := m_{ij} - \epsilon\alpha_i\alpha_j\frac{K}{2}S_{d-1}\left(\lim_{\eta\downarrow 0}\int_{|\omega(k)^2-x|>\eta, x>0}\frac{\varphi_{ij}(x)}{\omega(k)^2-x}dx - 2\pi i\varphi_{ij}(\omega(k)^2)\right).$$

Lemma 6.1 *Suppose* I_λ *or* II_λ. *Then* $\|D_{ij}\|_\infty < \infty$.

Proof: Set

$$\widetilde{\varphi}_{ij}(s) := \lim_{\eta\downarrow 0}\int_{|s-x|>\eta, x>0} \frac{\varphi_{ij}(x)}{s-x}dx.$$

Then by (A.3) and [20] we see that for $s,t \geq 0$

$$|\widetilde{\varphi}_{ij}(s) - \widetilde{\varphi}_{ij}(t)| \leq C_{ij}|s-t|.$$

Hence $\widetilde{\varphi}_{ij}$ is continuous. Since

$$\lim_{|s|\to\infty} \widetilde{\varphi}_{ij}(s) = 0,$$

it follows that $\|D_{ij}\|_\infty < \infty$. $\square$

Lemma 6.2 *Suppose* I_λ. *Then there exists*

$$Q = \begin{pmatrix} Q_1 \\ \vdots \\ Q_N \end{pmatrix} \in \oplus^N L^2(\mathbb{R}^d)$$

such that (1) $DQ = \sqrt{\omega}\begin{pmatrix} \hat{\lambda}_1 \\ \vdots \\ \hat{\lambda}_N \end{pmatrix}$, *(2)* $\|\omega^{(d-1)/2}Q_j\|_\infty < \infty$ *and* $\|\omega^{(d-3)/2}Q_j\|_\infty < \infty$, *(3)* $Q_j/\omega^{3/2} \in L^2(\mathbb{R}^d)$.

Proof: We set

$$D(k) := M - \epsilon K g(k) A(\alpha_1, \cdots, \alpha_N),$$

where

$$M := \mathrm{diag}\{m_1, \cdots, m_N\},$$

$$A(\alpha_1, \cdots, \alpha_N) := \begin{pmatrix} \alpha_1\alpha_1 & \cdots & \alpha_1\alpha_N \\ \vdots & & \vdots \\ \alpha_N\alpha_1 & \cdots & \alpha_N\alpha_N \end{pmatrix},$$

and

$$g(k) := \lim_{\eta\downarrow 0}\int_{\mathbb{R}^d} \frac{\omega(k')|\hat{\lambda}(k')|^2}{\omega(k)^2 - \omega(k')^2 + i\eta}dk'.$$

A is a positive definite matrix and we denote its eigenvalues by

$$\{a_1, \cdots, a_M, 0, \cdots, 0\}, \quad a_j > 0, \quad j = 1, ..., M, \tag{6.1}$$

with some $M \leq N$. A_M denotes the M-dimensional space spanned by eigenvectors with eigenvalues $a_1, ..., a_M$, and its orthogonal complement by $A_M^\perp$. Set $m = \min\{m_1, \cdots, m_N\}$, $a = \min\{a_1, \cdots, a_M\}$. For $z \in A_M^\perp$ we have

$$(z, D(k)z) = (z, Mz) \geq m|z|^2.$$

For $z \in A_M$, we have

$$|(z, D(k)z)| \geq \epsilon K |\Im g(k)||(z, Az)| \geq \epsilon a K |\Im g(k)||z|^2.$$

Hence for $\epsilon_0 < |k| < \epsilon_1$ with $0 < \epsilon_0$ and $0 < \epsilon_1$ there exists $d' = d'(\epsilon_0, \epsilon_1) > 0$ such that

$$|(z, D(k)z)| \geq d'|z|^2.$$

For $k = 0$ it follows that

$$(z, D(0)z) = (z, Mz) - \epsilon K g(0)(z, Az) \geq m|z|^2, \tag{6.2}$$

and as $|k| \to \infty$

$$(z, D(k)z) \to (z, Mz) \geq m|z|^2. \tag{6.3}$$

Combining (6.2) with (6.3) we see that for $|k| \leq \epsilon_0$ or $|k| \geq \epsilon_1$ with some sufficiently small ϵ_0 and sufficiently large ϵ_1 there exists d'' such that

$$|(z, D(k)z)| > d''|z|^2.$$

Hence for all $k \in \mathbb{R}^d$

$$|(z, D(k)z)| > \min\{d', d''\}|z|^2.$$

In particular the inverse D^{-1} of D exists as a bounded operator of $\oplus^N L^2(\mathbb{R}^d)$. Putting $Q := \sqrt{\omega} D^{-1} \begin{pmatrix} \hat{\lambda}_1 \\ \vdots \\ \hat{\lambda}_N \end{pmatrix}$, we see that Q satisfies (1). It is seen that

$$\|\omega^n Q_j\|_\infty \leq \|D^{-1}\| \|\omega^{n+(1/2)} \hat{\lambda}_j\|_\infty.$$

From $\|\hat{\lambda}_j\omega^{(d-2)/2}\|_\infty < \infty$ and $\|\hat{\lambda}_j\omega^{d/2}\|_\infty < \infty$ (see (A.4)), (2) follows. Moreover form

$$\|\omega^n Q_j\|_{L^2(\mathbb{R}^d)} \le \|D^{-1}\|\|\omega^{n+(1/2)}\hat{\lambda}_j\|_{L^2(\mathbb{R}^d)},$$

(3) follows, since $\hat{\lambda}/\omega \in L^2(\mathbb{R}^d)$ (see (A.2)). Thus the lemma follows. □

Lemma 6.3 *Suppose* II_λ. *Then there exists* Q *such that (1)–(3) in Lemma 6.2 are satisfied.*

Proof: Note that

$$D_{jj}(k) \neq 0, \text{ for } k \in \{k \in \mathbb{R}^d | \hat{\lambda}_j(k) \neq 0\}.$$

Hence we can define $Q = (Q_1, \cdots, Q_N)$ by

$$Q_j(k) := \begin{cases} \frac{\sqrt{\omega(k)}\hat{\lambda}_j(k)}{D_{jj}(k)}, & k \in \{k \in \mathbb{R}^d | \hat{\lambda}_j(k) \neq 0\}, \\ 0, & k \notin \mathrm{supp}\hat{\lambda}_j. \end{cases}$$

Since $\left|\frac{\hat{\lambda}_j(k)}{D_{jj}(k)}\right| \to 0$ as k converges to the boundary of $\mathrm{supp}\hat{\lambda}_j$, $Q(k)$ is well defined for all $k \in \mathbb{R}^d$, and we can easily check that Q satisfies (1)–(3). Thus the lemma follows. □

Let

$$R_j := \sqrt{\omega} D(0)^{-1} \begin{pmatrix} \hat{\lambda}_1 \\ \vdots \\ \hat{\lambda}_N \end{pmatrix}.$$

Define

$$Gf(k) := \lim_{\eta \downarrow 0} \int_{\mathbb{R}^d} \frac{f(k')}{(\omega(k)^2 - \omega(k')^2 + i\eta)(\omega(k)\omega(k'))^{(d-2)/2}} dk'.$$

G is a skew symmetric bounded operator of $L^2(\mathbb{R}^{dN})$. Let

$$T_{\mu\nu} f := \delta_{\mu\nu} f + \epsilon \sum_{j=1}^N \alpha_j^2 Q_j \omega^{(d-2)/2} G \omega^{(d-2)/2} d_{\mu\nu} \sqrt{\omega} \hat{\lambda}_j f.$$

Lemma 6.4 *Suppose* I_λ *or* II_λ. *Then the following algebraic relations hold:*

(1) $\sum_{\beta=1}^d T^*_{\mu\beta} d_{\beta\nu} (1/\omega^2) Q_j = d_{\mu\nu} R_j / \omega^{3/2}$.

(2) $[\omega^2, T^*_{\mu\nu}] f = -\sum_{j=1}^N \alpha_j^2 \left(Q_j, f\right)_{L^2(\mathbb{R}^d)} d_{\mu\nu} \sqrt{\omega} \hat{\lambda}_j$.

(3) $T_{\mu\nu}\sqrt{\omega}\hat{\lambda}_j = \delta_{\mu\nu} m_j Q_j$.

(4) $T^*_{\mu\nu} d_{\nu\beta} T_{\beta\gamma} = d_{\mu\gamma}$.

(5) $e^r_\mu T_{\mu\nu} d_{\nu\beta} T_{\beta\gamma} e^s_\gamma = \delta_{rs}$.

Proof: By (1) of Lemmas 6.2 and 6.3, it is proven. See [13, Lemma 2.6] for details. □

We define the bounded operator $W_\pm = (W^{rs}_\pm)_{1\le r,s\le d-1} : W \to W$ by

$$W^{rs}_+ f := \frac{1}{2} e^r_\mu \left(\omega^{-1/2} T^*_{\mu\nu} \omega^{1/2} + \omega^{1/2} T^*_{\mu\nu} \omega^{-1/2}\right) e^s_\nu f,$$

$$W^{rs}_- f := \frac{1}{2} e^r_\mu \left(\omega^{-1/2} T^*_{\mu\nu} \omega^{1/2} - \omega^{1/2} T^*_{\mu\nu} \omega^{-1/2}\right) \widetilde{e^s_\nu f}.$$

It is checked by (A.4), Lemmas 6.2 and 6.3 that

$$\|W^{rs}_\pm f\| \le c\|f\|,$$

where

$$c = 1 + \frac{1}{2}\left(\|\omega^{(d-3)/2} Q_j\|_\infty \|\omega^{d/2}\hat{\lambda}_j\|_\infty + \|\omega^{(d-1)/2} Q_j\|_\infty \|\omega^{(d-2)/2}\|_\infty\right)\|G\|.$$

We define

$$\mathbb{W} := \begin{pmatrix} W_+ & \overline{W_-} \\ W_- & \overline{W_+} \end{pmatrix}.$$

Lemma 6.5 *Suppose* I_λ *or* II_λ. *Then* $\mathbb{W} \in \Sigma_{\mathrm{B}}$. *Moreover* $\mathbb{W}$ *is invariant under the scaling* $\omega \to \kappa^2\omega$, $\alpha_j \to \kappa^l \alpha_j$, *and* $\epsilon \to \kappa^{2l-2}\epsilon$.

Proof: It follows from Lemma 6.4 (4) and (5). □

We define

$$\begin{pmatrix} b(f) \\ b^\dagger(f) \end{pmatrix} := {}^T\mathbb{W} \begin{pmatrix} a(f) \\ a^\dagger(f) \end{pmatrix}.$$

Let

$$L(q) := \sum_{j=1}^N \sum_{\mu=1}^d q_{j\mu} L^\mu_j \in W$$

with

$$L^\mu_j := \frac{\alpha_j}{\sqrt{2}} \oplus_{r=1}^{d-1} \frac{e^r_\mu Q_j}{\omega^{3/2}}.$$

L_j^μ is well defined by Lemmas 6.2 and 6.3. We define

$$\begin{pmatrix} B_q(f) \\ B_q^\dagger(f) \end{pmatrix} := \begin{pmatrix} b(f) \\ b^\dagger(f) \end{pmatrix} + \begin{pmatrix} (L(q), f)_W \\ (\overline{L(q)}, f)_W \end{pmatrix}.$$

By (2.2) we have

$$\begin{pmatrix} a(f) \\ a^\dagger(f) \end{pmatrix} = {}^T(\mathbb{W}^{-1}) \begin{pmatrix} B_q(f) \\ B_q^\dagger(f) \end{pmatrix} - \begin{pmatrix} \left(W_+ L(q) - \overline{W_-} \overline{L(q)}, f\right)_W \\ \left(\overline{W_+ L(q) - \overline{W_-} \overline{L(q)}}, f\right)_W \end{pmatrix}.$$

By Proposition 2.2 there exists a unitary operator $U(\mathbb{W})$ such that

$$U(\mathbb{W})^{-1} b^\sharp(f) U(\mathbb{W}) = a^\sharp(f).$$

Recall

$$\theta(q) := \exp\left(\sum_{j=1}^N \sum_{r=1}^{d-1} \sum_{\mu=1}^d \frac{\alpha_j q_{j\mu}}{\sqrt{2}} \left\{ a^r\left(\overline{\frac{e_\mu^r R_j}{\omega}}\right) - a^{\dagger r}\left(\frac{e_\mu^r R_j}{\omega}\right) \right\} \right)$$

and

$$\mathcal{U}(q) := \theta(q) U(\mathbb{W}).$$

A proof of Proposition 3.2 (1)

By Lemma 6.4 (1), we have

$$W_- L(q) - \overline{W_+ L(q)} = -\sum_{j=1}^N \sum_{\mu=1}^d q_{j\mu} \frac{\alpha_j}{\sqrt{2}} \oplus_{r=1}^{d-1} \frac{e_\mu^r R_j}{\omega}.$$

Hence $\theta(q, \mathbb{W}, L) = \theta(q)$ follows. $\square$

A proof of Proposition 3.2 (2)

It is seen that by Lemma 6.4 (2) and (3), for $f \in W$,

$$[H_\epsilon^0(q), B_q(f)] = -B_q((\oplus^{d-1}\omega) f),$$

$$[H_\epsilon^0(q), B_q^\dagger(f)] = B_q^\dagger((\oplus^{d-1}\omega) f)$$

on some dense domain. Using these commutation relations we can prove that

$$e^{itH_\epsilon^0(q)} B^\dagger(f) e^{-itH_\epsilon^0(q)} = B^\dagger(e^{it\oplus^{d-1}\omega} f), \tag{6.4}$$

$$e^{itH_\epsilon^0(q)} B(f) e^{-itH_\epsilon^0(q)} = -B(e^{it\oplus^{d-1}\omega} f). \tag{6.5}$$

Then we have

$$B(f)e^{itH_\epsilon^0(q)}\mathcal{U}(q)\Omega = 0,$$

for all $f \in W$. It implies that $e^{itH_\epsilon^0(q)}\mathcal{U}(q)\Omega = \text{constant} \times \mathcal{U}(q)\Omega$. Since $e^{itH_\epsilon^0(q)}$ is one parameter unitary group,

$$e^{itH_\epsilon^0(q)}\mathcal{U}(q)\Omega = e^{itE(q)}\mathcal{U}(q)\Omega \tag{6.6}$$

with some constant $E(q)$. Hence by (6.4),(6.5) and (6.6) we obtain that

$$\mathcal{U}(q)^{-1}e^{itH_\epsilon^0(q)}\mathcal{U}(q)a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega = e^{it(H_{\mathrm{f}}-E(q))}a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega.$$

Then

$$\mathcal{U}(q)^{-1}e^{itH_\epsilon^0(q)}\mathcal{U}(q) = e^{it(H_{\mathrm{f}}-E(q))}.$$

Since $H_\epsilon^0(q)$ is self-adjoint on $D(H_{\mathrm{f}})$, we see that $\mathcal{U}(q)$ maps $D(H_{\mathrm{f}})$ onto itself and $\mathcal{U}(q)^{-1}H_\epsilon^0(q)\mathcal{U}(q) = H_{\mathrm{f}} - E(q)$ holds by Stone's theorem. □

6.1.2 Analytic continuation

Lemma 6.6 *Suppose* I_λ. *Let* $f \in L^2(\mathbb{R}^d)$. *Then* $Q_j f$ *can be strongly analytically continued to a neighborhood* $\mathcal{O}$ *of* $\mathbb{R}$ *in each* α_i*'s.*

Proof: We fix j. We can find a neighborhood $\mathcal{O}$ of $\mathbb{R}$ such that, for $z_j \in \mathcal{O}$,

$$D(z_j) = D(z_j, k) := M - \epsilon K g(k) A(\alpha_1, \cdots, z_j, \cdots, \alpha_N)$$

is a strongly analytic bounded operator on $\oplus^N L^2(\mathbb{R}^d)$, and

$$\inf_{k\in\mathbb{R}^d} |(w, D(z_j, k)w)| > \delta |w|^2, \quad \text{for } w \in \mathbb{C}^N,$$

with some positive δ. Hence $D(z_j)^{-1}$ exists as a bounded operator for $z_j \in \mathcal{O}$ and

$$\|D(z_j)^{-1}\| \le \frac{1}{\delta}.$$

We can take a sufficiently small neighborhood $\mathcal{O}(z_j) \subset \mathcal{O}$ of z_j such that

$$\inf_{z_j \in \mathcal{O}(z_j)} \|D(z_j)^{-1}\| < \infty. \tag{6.7}$$

Since
$$\frac{D(z_j)^{-1}-D(z_j')^{-1}}{z_j-z_j'}=D(z_j)^{-1}\left(\frac{D(z_j')-D(z_j)}{z_j-z_j'}\right)D(z_j')^{-1},$$
by (6.7) we conclude that $D(z_j)^{-1}$ is strongly differentiable in $\mathcal{O}$. Thus $D(z_j)^{-1}$ is a strongly analytic bounded operator. Hence the lemma follows. □

Lemma 6.7 *Suppose* II_λ. *Let* $f\in L^2(\mathbb{R}^d)$. *Then* Q_jf *can be strongly analytically continued to a neighborhood* $\mathcal{O}$ *of* $\mathbb{R}$ *in each* α_i*'s.*

Proof: Note
$$Q_j(k)=\frac{1}{\epsilon K\alpha_j^2}\frac{1}{\frac{m_j}{\epsilon K\alpha_j^2}-g_j(k)}.$$
Here
$$g_j(k):=\lim_{\eta\downarrow 0}\int_{\mathbb{R}^d}\frac{\omega(k')|\hat{\lambda}_j(k')|^2}{\omega(k)^2-\omega(k')^2+i\eta}dk'.$$
It is seen that
$$(1)\ g_j(0)<0,\quad (2)\ \Im g_j(k)<0 \text{ for } k\in\mathrm{supp}\hat{\lambda}_j,$$
$$(3)\ g(k)=0 \text{ for } k\notin\mathrm{supp}\hat{\lambda}_j,\quad (4)\ g_j \text{ is continuous.}$$
Hence, for arbitrary $\delta>0$,
$$\mathrm{dist}(\mathrm{Ran}g_j,[\delta,\infty))>M$$
with some positive M. Thus Q_jf can be analytically continued to
$$\mathcal{O}':=\{a+ib,a\in\mathbb{R},b\in\mathbb{R}||a|>\delta,|b|<M\}$$
in $z_j:=m_j/(\epsilon K\alpha_j^2)$. Thus Q_jf can be strongly analytically continued in a neighborhood $\mathcal{O}$ of $\mathbb{R}$ in each α_j. □

A proof of Lemma 3.2 (3)

Note that
$$E(q)=\frac{(\Omega,H_\epsilon(q)\mathcal{U}(q)\Omega)}{(\Omega,\mathcal{U}(q)\Omega)}. \tag{6.8}$$
It is proven in [11, 13] that by Lemma 2.5 $E(q)$ can be expressed as
$$E(q)=\sum_{j=1}^N\frac{1}{2m_j}(q_j+\tilde{q}_j)^2+\Delta,$$

where

$$\tilde{q}_{j\nu} := \sum_{i=1}^{N}\sum_{r=1}^{d-1}\sum_{s=1}^{d-1}\sum_{\mu=1}^{d}\frac{1}{2}q_{\mu i}\alpha_i\alpha_j\left(e_\mu^r R_j/\omega, (1+W_-W_+^{-1})_{rs}e_\nu^s\hat{\lambda}_i\right),$$

$$\Delta = \sum_{j=1}^{N}\sum_{r=1}^{d-1}\sum_{\mu=1}^{d}\frac{\alpha_j^2}{4m_j}\left(e_\mu^r\hat{\lambda}_j, (1-W_-W_+^{-1})_{rs}e_\mu^s\hat{\lambda}_j\right).$$

By Lemmas 6.6 and 6.7 we see that $T_{\mu\nu}$ can be strongly analytically continued to some $\mathcal{O}\subset\mathbb{R}$ in each α_j's. Moreover $W_\pm$ also can be done. By the identities derived from the symplectic structure,

$$W_+^*W_+ - \overline{W_-^*W_-} = I,$$

$$W_+W_+^* - \overline{W_-W_-^*} = I,$$

taking a sufficiently small neighborhood $\mathcal{O}'\subset\mathcal{O}$, we see that W_+^{-1} can be extended to $\alpha_j\in\mathcal{O}'$ as a bounded operator $W_+^{-1}(\alpha_j)$. Moreover it is seen that $\sup_{\alpha_j\in\mathcal{O}(\alpha_j)}\|W_+^{-1}(\alpha_j)\|<\infty$ for sufficiently small neighborhood $\mathcal{O}(\alpha_j)$ of α_j. Thus $W_+^{-1}(\alpha_j)$ is strongly differentiable in α_j, and hence strongly analytic on $\mathcal{O}'$. □

6.2 A proof of Lemma 3.3

A proof of Lemma 3.3 (1)

This proof is a slight extension of [19] to the N particle case as we have been suggested. We take a momentum lattice approximation. The momentum lattice approximated Hamiltonian (6.9) below can be identified with harmonic oscillators. Then its ground state energy can be explicitly obtained. By a limiting argument we get $E(q)$. First we assume that $|\alpha_j|$'s are sufficiently small and

$$\omega_\delta(k) := \omega(k)+\delta, \quad \delta>0.$$

Let $1/a>0$ be the width of the momentum lattice, and some $L>0$ be fixed. For

$$l=(l^1,\cdots,l^d)\in(2\pi\mathbb{Z}/a)^d$$

we define

$$\Gamma(l) := \left[l^1, l^1+\frac{2\pi}{a}\right)\times\cdots\times\left[l^d, l^d+\frac{2\pi}{a}\right).$$

Let $|l| := \max_r |l^r|$. Suppose

$$|l|\le 2\pi L.$$

Set

$$D := (2[aL]+1)^d,$$

where $[z]$ denotes the integer part of z. We named the lattice points of the rectangle with width $2[aL]$ centered at the origin as

$$l_1, l_2, \cdots, l_D.$$

We set

$$\vec{q} := \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\omega_\delta(l)}} \left\{ a^{\dagger r}\left(\chi_{\Gamma(l)}\right) + a^r\left(\chi_{\Gamma(l)}\right)\right\}_{1\le r\le d-1, |l|\le 2\pi L},$$

$$\vec{p} := \frac{i}{\sqrt{2}} \sqrt{\omega_\delta(l)} \left\{ a^{\dagger r}\left(\chi_{\Gamma(l)}\right) - a^r\left(\chi_{\Gamma(l)}\right)\right\}_{1\le r\le d-1, |l|\le 2\pi L},$$

where χ_I denotes the characteristic function of $I \subset \mathbb{R}^d$. Define

$$H_\epsilon(q,\delta,a,L) := \sum_{j=1}^N \left\{ \frac{1}{2m_j} q_j^2 - \frac{\alpha_j}{m_j} \sum_{\mu=1}^d (q_{j\mu} v_\mu^j, \vec{q}) + \epsilon \sum_{\mu=1}^d \frac{\alpha_j^2}{2m_j} (\vec{q}, v_\mu^j)(v_\mu^j, \vec{q}) \right\}$$

$$+\frac{1}{2}\left\{(\vec{p},\vec{p}) + (\vec{q}, A_0\vec{q})\right\} - \frac{1}{2}\mathrm{tr}\sqrt{A_0}. \tag{6.9}$$

Here A_0 is the $(d-1)D \times (d-1)D$ symmetric matrix defined by

$$A_0 := \left(1_{(d-1)}\delta_{ll'}\omega_\delta(l)^2\right)_{|l|,|l'|\le 2\pi L}$$

$$= \begin{pmatrix} \omega_\delta(l_1)1_{(d-1)} & & & \\ & \omega_\delta(l_2)1_{(d-1)} & & \\ \vdots & \cdots & \ddots & \vdots \\ & & & \omega_\delta(l_D)1_{(d-1)} \end{pmatrix}, \tag{6.10}$$

and

$$v_\mu^j := v_\mu := \begin{pmatrix} \sqrt{\omega_\delta(l_1)}\hat{\lambda}(l_1)e_\mu^1(l_1) \\ \vdots \\ \sqrt{\omega_\delta(l_1)}\hat{\lambda}(l_1)e_\mu^{(d-1)}(l_1) \\ \vdots \\ \vdots \\ \sqrt{\omega_\delta(l_D)}\hat{\lambda}(l_D)e_\mu^1(l_D) \\ \vdots \\ \sqrt{\omega_\delta(l_D)}\hat{\lambda}(l_D)e_\mu^{(d-1)}(l_D) \end{pmatrix} \in \mathbb{R}^{(d-1)D}.$$

Here $1_{(d-1)}$ denotes the $(d-1)\times(d-1)$ identity matrix. It is seen that

$$(H_\epsilon(q,\delta,a,L)-z)^{-1}\to(H_\epsilon(q,\delta)-z)^{-1} \tag{6.11}$$

uniformly as $a\to\infty$ and then $L\to\infty$. Here $H_\epsilon(q,\delta)$ denotes $H_\epsilon^0(q)+\delta N$. We define the projection P by

$$P:=\epsilon\sum_{j=1}^N\sum_{\mu=1}^d\frac{\alpha_j^2}{m_j}|v_\mu\rangle\langle v_\mu|.$$

Then

$$H_\epsilon(q,\delta,a,L)=\sum_{j=1}^N\left\{\frac{1}{2m_j}q_j^2-\frac{\alpha_j}{m_j}\sum_{\mu=1}^d(q_{j\mu}v_\mu,\vec{q})\right\}+\frac{1}{2}(\vec{q},P\vec{q})$$
$$+\frac{1}{2}(\vec{p},\vec{p})+\frac{1}{2}(\vec{q},A_0\vec{q})-\frac{1}{2}\mathrm{tr}\sqrt{A_0}.$$

Set

$$A:=A_0+P.$$

Since $\omega_\delta(k)>\delta>0$, the inverse of A_0 exists, and hence A also has the inverse, and it is symmetric. Then we have

$$=\sum_{j=1}^N\frac{1}{2m_j}q_j^2-\sum_{j=1}^N\sum_{\mu=1}^d\frac{\alpha_j}{m_j}(AA^{-1}q_{j\mu}v_\mu,\vec{q})+\frac{1}{2}(\vec{q},A\vec{q})+\frac{1}{2}(\vec{p},\vec{p})-\frac{1}{2}\mathrm{tr}\sqrt{A_0}$$

$$=\sum_{j=1}^N\frac{1}{2m_j}q_j^2+\frac{1}{2}((\vec{q}-\vec{f}),A(\vec{q}-\vec{f}))-\frac{1}{2}(\vec{f},A\vec{f})+\frac{1}{2}(\vec{p},\vec{p})-\frac{1}{2}\mathrm{tr}\sqrt{A_0}.$$

Here we put

$$f:=\sum_{j=1}^N\sum_{\mu=1}^d\frac{\alpha_j}{m_j}A^{-1}q_{j\mu}v_\mu.$$

By the commutation relation on $e^{is\vec{p}}$ and $e^{it\vec{q}}$:

$$e^{is\vec{p}_{rl_n}}e^{it\vec{q}_{r'l'_n}}=e^{its\delta_{rr'}\delta_{l_nl'_n}}e^{it\vec{q}_{r'l'_n}}e^{is\vec{p}_{rl_n}},$$

and the von Neumann uniqueness theorem, we can identify $\vec{p}$ and $\vec{q}$ with the momentum and position operators, respectively,

$$\vec{p}_{t,l_n}\cong-i\frac{\partial}{\partial x_{r,n}},\quad \vec{q}_{r,l_n}\cong x_{r,n},$$

for $r = 1, ..., d-1$ and $n = 1, ..., D$. Hence we identify $H_\epsilon(q, \delta, a, L)$ as a harmonic oscillator in $L^2(\mathbb{R}^{(d-1)D})$. Generally for the harmonic oscillator

$$H_T := \frac{1}{2}(\vec{p}, \vec{p}) + \frac{1}{2}(\vec{q}, T\vec{q})$$

with nonnegative symmetric matrix T, we have

$$\inf \sigma(H_T) = \mathrm{tr}\sqrt{T}.$$

Then it follows that

$$E(q) = \inf \sigma(H_\epsilon(q, \delta, a, L))$$

$$= \inf \sigma \left(\frac{1}{2}(\vec{p}, \vec{p}) + \frac{1}{2}((\vec{q} - \vec{f}), A(\vec{q} - \vec{f})) \right) + \sum_{j=1}^{N} \frac{1}{2m_j} q_j^2 - \frac{1}{2}(\vec{f}, A\vec{f}) - \frac{1}{2}\mathrm{tr}\sqrt{A_0}$$

$$= \sum_{j=1}^{N} \frac{1}{2m_j} q_j^2 - \frac{1}{2}(f, Af) + \frac{1}{2}\mathrm{tr}(\sqrt{A} - \sqrt{A_0}).$$

Separately we calculate

(1) $\sum_{j=1}^{N} \frac{1}{2m_j} q_j^2 - \frac{1}{2}(f, Af)$,

(2) $\frac{1}{2}\mathrm{tr}(\sqrt{A} - \sqrt{A_0})$.

First we calculate (1). We give a remark. Let

$$d_{\mu\nu}(k) := \sum_{r=1}^{d-1} e_\mu^r(k) e_\nu^r(k) = \left(1 - \frac{k}{|k|} \otimes \frac{k}{|k|} \right)_{\mu\nu} = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}.$$

For rotation invariant functions f and g it follows that

$$(d_{\mu\nu} f, g) = \delta_{\mu\nu} K(f, g).$$

We have

$$(f, Af) = \sum_{i,j=1}^{N} \frac{\alpha_i}{m_i} \frac{\alpha_j}{m_j} q_{i\mu} q_{j\nu} (A^{-1} v_\mu, v_\nu).$$

Note that we assume that $|\alpha_j|^2$ are suffieictly small. Then

$$A^{-1} = \sum_{n=1}^{\infty} (-1)^{n-1} A_0^{-1} (P A_0^{-1})^{n-1}.$$

Hence

$$
\begin{aligned}
(A^{-1}v_\mu, v_\nu) &= \sum_{n=1}^{\infty}(-1)^{n-1}(A_0^{-1}(PA_0^{-1})^{n-1}v_\mu, v_\nu) \\
&= \sum_{n=1}^{\infty}(-1)^{n-1}\sum_{j_1,\ldots,j_{n-1}=1}^{N}\sum_{\mu_1,\ldots,\mu_{n-1}=1}^{d}\epsilon^{n-1}\frac{\alpha_{j_1}^2}{m_{j_1}}\cdots\frac{\alpha_{j_{n-1}}^2}{m_{j_{n-1}}}\times \\
&\times(v_\mu A_0^{-1}v_{\mu_1})(v_{\mu_1}, A_0^{-1}v_{\mu_2})(v_{\mu_2}, A_0^{-1}v_{\mu_3})\cdots(v_{\mu_{n-2}}, A_0^{-1}v_{\mu_{n-1}})(v_{\mu_{n-1}}, A_0^{-1}v_\nu) \\
&= \delta_{\mu\nu}\sum_{n=1}^{\infty}(-1)^{n-1}\sum_{j_1,\ldots,j_{n-1}=1}^{N}\sum_{\mu_1,\ldots,\mu_{n-1}=1}^{d}\epsilon^{n-1}K^n\frac{\alpha_{j_1}^2}{m_{j_1}}\cdots\frac{\alpha_{j_{n-1}}^2}{m_{j_{n-1}}}\xi^n,
\end{aligned}
$$

where

$$
\xi := \sum_{|l|\le 2\pi L}\frac{|\hat{\lambda}(l)|^2}{\omega_\delta(l)}.
$$

Then it follows that

$$
\begin{aligned}
&= \delta_{\mu\nu}\sum_{n=1}^{\infty}(-1)^{n-1}K\xi\left\{\epsilon K\xi\sum_{j=1}^{N}\frac{\alpha_j^2}{m_j}\right\}^{n-1} \\
&= \delta_{\mu\nu}\frac{K\xi}{1+\epsilon K\xi\sum_{j=1}^{N}\frac{\alpha_j^2}{m_j}}.
\end{aligned}
$$

We have

$$
\begin{aligned}
(f, Af) &= \sum_{i,j=1}^{N}\delta_{\mu\nu}\frac{\alpha_i}{m_i}\frac{\alpha_j}{m_j}q_{i\nu}q_{j\mu}\frac{K\xi}{1+\epsilon K\xi\sum_{j=1}^{N}\frac{\alpha_j^2}{m_j}} \\
&= \frac{K\xi}{1+\epsilon K\xi\sum_{j=1}^{N}\frac{\alpha_j^2}{m_j}}\left|\sum_{j=1}^{N}\frac{\alpha_j}{m_j}q_j\right|^2.
\end{aligned}
$$

Hence we have

$$
(1) = \frac{1}{2}\sum_{j=1}^{N}\frac{q_j^2}{m_j} - \frac{1}{2}\frac{K\xi}{1+\epsilon K\xi\sum_{j=1}^{N}\frac{\alpha_j^2}{m_j}}\left|\sum_{j=1}^{N}\frac{\alpha_j}{m_j}q_j\right|^2.
$$

Next we calculate (2). Using the formula for nonnegative self-adjoint operator B,

$$
\sqrt{B} = \frac{1}{\pi}\int_{-\infty}^{\infty}B(s^2+B)^{-1}ds,
$$

we see that

$$
(2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{tr}\left\{A(s^2+A)^{-1} - A_0(s^2+A_0)^{-1}\right\}ds.
$$

Since

$$(s^2+A)^{-1} = (s^2+A_0)^{-1}\sum_{n=1}^{\infty}(-1)^{n-1}\left\{P(s^2+A_0)^{-1}\right\}^{n-1},$$

we have

$$A(s^2+A)^{-1} - A_0(s^2+A_0)^{-1} = (\mathrm{I}) + (\mathrm{II}).$$

Here we put

$$(\mathrm{I}) := A_0(s^2+A_0)^{-1}\sum_{n=1}^{\infty}(-1)^{n}\left\{P(s^2+A_0)^{-1}\right\}^{n}, \tag{6.12}$$

$$(\mathrm{II}) := P(s^2+A)^{-1}. \tag{6.13}$$

It follows that

$$\mathrm{tr}(\mathrm{I}) = \sum_{n=1}^{\infty}(-1)^n \sum_{\varphi:\mathrm{CONS}}(\varphi, A_0(s^2+A_0)^{-1}\left\{P(s^2+A_0)^{-1}\right\}^{n}\varphi),$$

where $\sum_{\varphi:\mathrm{CONS}}$ means to sum up vectors in a complete orthonormal system. Then

$$= \sum_{n=1}^{\infty}(-1)^n \sum_{\varphi:\mathrm{CONS}}(\varphi, A_0(s^2+A_0)^{-1}\underbrace{P(s^2+A_0)^{-1}P(s^2+A_0)^{-1}\cdots P(s^2+A_0)^{-1}}_{n\ \mathrm{times}}\varphi)$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty}(-1)^n \sum_{\varphi:\mathrm{CONS}}\sum_{j_1,\ldots,j_n=1}^{N}\sum_{\mu_1,\ldots,\mu_n=1}^{d}\epsilon^n\frac{\alpha_{j_1}^2}{m_{j_1}}\cdots\frac{\alpha_{j_n}^2}{m_{j_n}}\times\\
&\times(\varphi, A_0(s^2+A_0)^{-1}v_{\mu_1})(v_{\mu_1},(s^2+A_0)^{-1}v_{\mu_2})\times\cdots\\
&\cdots\times(v_{\mu_{n-1}},(s^2+A_0)^{-1}v_{\mu_n})(v_{\mu_n},(s^2+A_0)^{-1}\varphi).
\end{aligned}$$

We take as a CONS

$$\left\{\varphi_1 := \frac{(s^2+A_0)^{-1}v_{\mu_n}}{\|(s^2+A_0)^{-1}v_{\mu_n}\|}, \varphi_2, \varphi_3, \ldots,\right\}.$$

Then

$$\begin{aligned}
&= \sum_{n=1}^{\infty}(-1)^n \sum_{j_1,\ldots,j_n=1}^{N}\sum_{\mu_1,\ldots,\mu_n=1}^{d}\epsilon^n\frac{\alpha_{j_1}^2}{m_{j_1}}\cdots\frac{\alpha_{j_n}^2}{m_{j_n}}\times\\
&\times(v_{\mu_n}, A_0(s^2+A_0)^{-2}v_{\mu_1})(v_{\mu_1},(s^2+A_0)^{-1}v_{\mu_2})\times\cdots\\
&\cdots\times(v_{\mu_{n-1}},(s^2+A_0)^{-1}v_{\mu_n})\frac{(v_{\mu_n},(s^2+A_0)^{-2}v_{\mu_n})}{\|(s^2+A_0)^{-1}v_{\mu_n}\|^2}\\
&= d\sum_{n=1}^{\infty}(-1)^n\left(\sum_{j=1}^{N}\frac{\alpha_j^2}{m_j}\right)^n\epsilon^n K^n\xi(s)^{n-1}\eta(s),
\end{aligned}$$

where

$$\xi(s) := \sum_{|l|\leq 2\pi L} \frac{\omega_\delta(l)|\hat{\lambda}(l)|^2}{s^2+\omega_\delta(l)^2},$$

and

$$\eta(s) := \sum_{|l|\leq 2\pi L} \frac{\omega_\delta(l)^3|\hat{\lambda}(l)|^2}{(s^2+\omega_\delta(l)^2)^2}.$$

Hence

$$= d\sum_{n=1}^{\infty}(-1)^n\left(\epsilon K\xi(s)\sum_{j=1}^{N}\frac{\alpha_j^2}{m_j}\right)^n\frac{\eta(s)}{\xi(s)}$$

$$= d\frac{-\epsilon K\xi(s)\sum_{j=1}^{N}\frac{\alpha_j^2}{m_j}}{1+\epsilon K\xi(s)\sum_{j=1}^{N}\frac{\alpha_j^2}{m_j}}\frac{\eta(s)}{\xi(s)} = d\frac{-\epsilon\rho\eta(s)}{1+\epsilon\rho\xi(s)}.$$

Next we see that

$$\mathrm{tr(II)} = \sum_{\varphi:\mathrm{CONS}}(\varphi, P(s^2+A)^{-1}\varphi)$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\sum_{\varphi:\mathrm{CONS}}(\varphi, P(s^2+A_0)^{-1}\left(P(s^2+A_0)^{-1}\right)^{n-1}\varphi)$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\sum_{\varphi:\mathrm{CONS}}\sum_{j_1,\dots,j_n=1}^{N}\sum_{\mu_1,\dots,\mu_n=1}^{d}\frac{\alpha_{j_1}^2}{m_{j_1}}\cdots\frac{\alpha_{j_n}^2}{m_{j_n}}\epsilon^n\times$$

$$\times(\varphi, v_{\mu_1})(v_{\mu_1},(s^2+A_0)^{-1}v_{\mu_2})\cdots(v_{\mu_{n-1}},(s^2+A_0)^{-1}v_{\mu_n})(v_{\mu_n},(s^2+A_0)^{-1}\varphi).$$

Take as a CONS

$$\left\{\varphi_1 := \frac{(s^2+A_0)^{-1}v_{\mu_n}}{\|(s^2+A_0)^{-1}v_{\mu_n}\|}, \varphi_2, \varphi_3, \dots,\right\}.$$

Then we have

$$\mathrm{tr(II)} = \sum_{n=1}^{\infty}(-1)^{n-1}\sum_{j_1,\dots,j_n=1}^{N}\sum_{\mu_1,\dots,\mu_n=1}^{d}\frac{\alpha_{j_1}^2}{m_{j_1}}\cdots\frac{\alpha_{j_n}^2}{m_{j_n}}\epsilon^n\times$$

$$\times(v_{\mu_n},(s^2+A_0)^{-1}v_{\mu_1})(v_{\mu_1},(s^2+A_0)^{-1}v_{\mu_2})\times\cdots$$

$$\cdots\times(v_{\mu_{n-1}},(s^2+A_0)^{-1}v_{\mu_n})\frac{(v_{\mu_n},(s^2+A_0)^{-2}v_{\mu_n})}{\|(s^2+A_0)^{-1}v_{\mu_n}\|^2}$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\sum_{j_1,\dots,j_n=1}^{N}\sum_{\mu_1,\dots,\mu_n=1}^{d}\frac{\alpha_{j_1}^2}{m_{j_1}}\cdots\frac{\alpha_{j_n}^2}{m_{j_n}}\epsilon^n K^n d\xi(s)^n$$

$$= d\sum_{n=1}^{\infty}(-1)^{n-1}\left(\epsilon K\xi(s)\sum_{j=1}^{N}\frac{\alpha_j^2}{m_j}\right)^n = d\frac{\epsilon\rho\xi(s)}{1+\epsilon\rho\xi(s)}.$$

Hence

$$\mathrm{tr}((\mathrm{I})+(\mathrm{II})) = \frac{d\epsilon\rho}{1+\epsilon\rho\xi(s)}(\xi(s)-\eta(s)).$$

From

$$\xi(s)-\eta(s) = \sum_{|l|\le 2\pi L}\left\{\frac{|\hat\lambda(l)|^2\omega_\delta(l)}{s^2+\omega_\delta(l)^2} - \frac{\omega_\delta(l)^3|\hat\lambda(l)|^2}{(s^2+\omega_\delta(l)^2)^2}\right\} = \sum_{|l|\le 2\pi L}\frac{s^2|\hat\lambda(l)|^2\omega_\delta(l)}{(s^2+\omega_\delta(l)^2)^2} := \eta'(s),$$

it follows that

$$(2) = \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\frac{\epsilon\rho\eta'(s)}{1+\epsilon\rho\xi(s)}ds.$$

As was mentioned in (6.11), as $a\to\infty$ and then $L\to\infty$, $H_\epsilon(q,\delta,a,L)$ uniformly converges to $H_\epsilon^0(q)+\delta N$ in the resolvent sense. Thus as $a\to\infty$ and then $L\to\infty$, it follows that

$$E(q,\delta,a,L)\to\inf\sigma(H_\epsilon(q)+\delta N) := E(q,\delta).$$

Since $N\ge 0$, we have

$$\liminf_{\delta\to 0}E(q,\delta)\ge E(q)$$

and the strong resolvent convergence of $H_\epsilon^0(q)+\delta N$ to $H_\epsilon(q)$ as $\delta\to 0$ implies that

$$E(q)\ge\limsup_{\delta\to 0}E(q,\delta).$$

Thus

$$E(q,\delta)\to E(q)$$

as $\delta\to 0$. Thus the lemma follows for sufficiently small $|\alpha_j|$'s. By Proposition 3.2 (3), $\inf\sigma(H_\epsilon^0(q))$ can be analytically continued in each α_j's to $\mathcal{O}$, and clearly $E(q)$ also can be done. Hence the lemma follows for all α_j's in $\mathbb{R}$.

□

A proof of Lemma 3.3 (2)

The proof is similar to Lemma 3.3. Define the projection operator

$$P := \epsilon\sum_{j=1}^{N}\sum_{\mu=1}^{d}\frac{\alpha_j^2}{m_j}|v_\mu^j\rangle\langle v_\mu^j|,$$

and

$$A := A_0 + P,$$

where A_0 is defined in (6.10). Here

$$v_\mu^j := \begin{pmatrix} \sqrt{\omega_\delta(l_1)}\hat{\lambda}_j(l_1)e_\mu^1(l_1) \\ \vdots \\ \sqrt{\omega_\delta(l_1)}\hat{\lambda}_j(l_1)e_\mu^{(d-1)}(l_1) \\ \vdots \\ \vdots \\ \sqrt{\omega_\delta(l_D)}\hat{\lambda}_j(l_D)e_\mu^1(l_D) \\ \vdots \\ \sqrt{\omega_\delta(l_D)}\hat{\lambda}_j(l_D)e_\mu^{(d-1)}(l_D) \end{pmatrix} \in \mathbb{R}^{(d-1)D}.$$

Let

$$f := \sum_{j=1}^N \sum_{\mu=1}^d \frac{\alpha_j}{m_j} A^{-1} q_{j\mu} v_\mu^j.$$

Similarly to Lemma 3.3 it is enough to calculate

(1) $\frac{1}{2}\sum_{j=1}^N \frac{1}{m_j} q_j^2 - \frac{1}{2}(Af, f)$,

(2) $\frac{1}{2}\mathrm{tr}(\sqrt{A} - \sqrt{A_0})$.

First we calculate (1). We have

$$\begin{aligned} (A^{-1}v_\mu^i, v_\nu^j) &= \delta_{\mu\nu}\delta_{ij} \sum_{n=1}^\infty (-1)^{n-1} K^n \epsilon^{n-1} \left(\frac{\alpha_j^2}{m_j}\right)^{n-1} \xi_j^n \\ &= \delta_{\mu\nu}\delta_{ij} \frac{K\xi_j}{1 + \epsilon K \xi_j \frac{\alpha_j^2}{m_j}}, \end{aligned}$$

where

$$\xi_j := \sum_{|l| \le 2\pi L} \frac{|\hat{\lambda}_j(l)|^2}{\omega_\delta(l)}.$$

Note that, for function g,

$$(\hat{\lambda}_j, g\hat{\lambda}_i) = \delta_{ij}(\hat{\lambda}_j, g\hat{\lambda}_j).$$

Then

$$(Af, f) = \sum_{i,j=1}^N \sum_{\mu,\nu=1}^d (A^{-1}v_\mu^i, v_\nu^j)$$

$$= \sum_{i,j=1}^{N} \sum_{\mu,\nu=1}^{d} \delta_{\mu\nu}\delta_{ij} \frac{\alpha_i}{m_i} \frac{\alpha_j}{m_j} q_{i\nu} q_{j\mu} \frac{K\xi_j}{1+\epsilon K\xi_j \frac{\alpha_j^2}{m_j}}$$

$$= \sum_{j=1}^{N} \left(\frac{\alpha_j}{m_j}\right)^2 q_j^2 \frac{K\xi_j}{1+\epsilon K\xi_j \frac{\alpha_j^2}{m_j}}.$$

Then

$$\frac{1}{2}\sum_{j=1}^{N} \frac{q_j^2}{m_j} - \frac{1}{2}(Af,f) = \frac{1}{2}\sum_{j=1}^{N} \frac{q_j^2}{m_j} \left(\frac{m_j - (1-\epsilon)K\alpha_j^2\xi_j}{m_j + \epsilon K\alpha_j^2 \xi_j} \right).$$

Next we calculate (2). Let (I) and (II) be as in (6.12) and (6.13), respectively. We define

$$\xi_j(s) := \sum_{|l|\le 2\pi L} \frac{\omega_\delta(l)|\hat{\lambda}_j(l)|^2}{s^2+\omega_\delta(l)^2},$$

$$\eta_j(s) := \sum_{|l|\le 2\pi L} \frac{\omega_\delta(l)^3|\hat{\lambda}_j(l)|^2}{(s^2+\omega_\delta(l)^2)^2},$$

and

$$\eta_j'(s) := \sum_{|l|\le 2\pi L} \frac{s^2|\hat{\lambda}_j(l)|^2\omega_\delta(l)}{(s^2+\omega_\delta(l)^2)^2}.$$

We have

$$\mathrm{tr}(\mathrm{I}) = d\sum_{j=1}^{N}\sum_{n=1}^{\infty}(-1)^n \left(\frac{\alpha_j^2}{m_j}\right)^n \epsilon^n K^n \xi_j(s)^{n-1}\eta_j(s)$$

$$= -d\sum_{j=1}^{N} \eta_j(s)\epsilon\frac{\alpha_j^2}{m_j}K\frac{1}{1+\epsilon\xi_j(s)K\frac{\alpha_j^2}{m_j}} = -d\sum_{j=1}^{N}\frac{\epsilon\rho_j\eta_j(s)}{1+\epsilon\rho_j\xi_j(s)}.$$

Moreover

$$\mathrm{tr}(\mathrm{II}) = \sum_{j=1}^{N}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\alpha_j^2}{m_j}\epsilon^n K^n \xi_j(s)^n$$

$$= d\sum_{j=1}^{N}\frac{\epsilon\rho_j\xi_j(s)}{1+\epsilon\rho_j\xi_j(s)}.$$

Hence

$$(\mathrm{I})+(\mathrm{II}) = d\sum_{j=1}^{N}\frac{\epsilon\rho_j}{1+\epsilon\rho_j\xi_j(s)}(\xi_j(s)-\eta_j(s)) = d\sum_{j=1}^{N}\frac{\epsilon\rho_j}{1+\epsilon\rho_j\xi_j(s)}\eta_j'(s).$$

Then

$$\frac{1}{2}\mathrm{tr}(\sqrt{A}-\sqrt{A_0}) = \frac{1}{2}\mathrm{tr}((\mathrm{I})+(\mathrm{II})) = \sum_{j=1}^{N}\frac{d}{2\pi}\int_{-\infty}^{\infty}\frac{\epsilon\rho_j\eta_j'(s)}{1+\epsilon\rho_j\xi_j(s)}ds.$$

The same limiting argument on a, L, δ as in Lemma 3.3 leads to the desired results. □

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