ENHANCED BINDING THROUGH COUPLING TO A QUANTUM FIELD

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Abstract

We consider an electron coupled to the quantized radiation field in the dipole approximation. Even if the Hamiltonian has no ground state at zero coupling, $\alpha = 0$, the interaction with the quantized radiation field may induce binding. Under suitable assumptions on the external potential, we prove that there exists a critical constant $\alpha_{\mathrm{c}} \geq 0$ such that the Hamiltonian has a unique ground state for *arbitrary* $|\alpha| > \alpha_{\mathrm{c}}$.

1 Introduction

Atoms consist of charged particles and they are necessarily coupled to the quantized radiation field. In the lowest approximation, this interaction can be ignored and one is led to a Schrödinger operator of the form $\frac{1}{2m}p^2 + V$ for the particles only. Under suitable conditions on V the Schrödinger operator has a state of the lowest energy, the ground state of the atom. There has been renewed interest within mathematical physics to understand whether this ground state persists when the coupling to the radiation field is included [3, 5, 6, 8, 9, 10, 13, 14, 12, 18, 19]. We will investigate here a related, but distinct problem. To formulate it properly, we first have to introduce our Hamiltonian.

In the non-relativistic approximation, the coupling to the radiation field is described by the Pauli-Fierz Hamiltonian. We will simplify through the dipole approximation

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which neglects the variation of the vector potential over the size of the atoms. Thereby the Hamiltonian becomes

$$H = \frac{1}{2m}\left(p \otimes I - \alpha I \otimes A\right)^2 + V \otimes I + I \otimes H_{\mathrm{f}} \tag{1.1}$$

acting on the Hilbert space $L^2(\mathbb{R}^3) \otimes \mathcal{F}$. Here $p = -i\nabla$ is the momentum operator canonically conjugate to the position operator x in $L^2(\mathbb{R}^3)$, V is an external potential for which precise conditions will be specified below. In essence, V is short ranged and sufficiently shallow, i.e. $\lim_{|x|\to\infty} V(x) = 0$, $V \leq 0$. A is the quantized electromagnetic vector potential at the origin defined by

$$A = \sum_{j=1,2} \int \frac{1}{\sqrt{2\omega(k)}} e_j(k) \left\{ \hat{\varphi}(k) a^*(k,j) + \hat{\varphi}(k)^* a(k,j) \right\} d^3k.$$

Here $\hat{\varphi}^*$ denotes the complex conjugate of $\hat{\varphi}$, $e_1(k)$, $e_2(k)$, $k/|k|$ are a standard dreibein, and $a^*(k,j), a(k,j)$, $j = 1,2$, are Bose fields with commutation relations

$$[a^*(k,j), a(k',j')] = \delta_{jj'}\delta(k-k'), \quad [a^*(k,j), a^*(k',j')] = 0, \quad [a(k,j), a(k',j')] = 0 \tag{1.2}$$

and acting on the Boson Fock space $\mathcal{F}$ over $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$. $\hat{\varphi}$ cuts the high frequencies. The dispersion relation of the photons is

$$\omega(k) = |k| \tag{1.3}$$

and therefore the field energy

$$H_{\mathrm{f}} = \sum_{j=1,2} \int \omega(k) a^*(k,j) a(k,j) d^3k.$$

For H to be a well defined self-adjoint operator, $\hat{\varphi}$ has to satisfy certain integrability conditions which will be stated in Section 2. Also, if clear from the context, we omit factors $\otimes I$ and $I\otimes$ in the following.

The problem of the existence of the ground state for H is usually regarded as a stability property. One assumes that H has a ground state for $\alpha = 0$, which amounts to the existence of a ground state for $\frac{1}{2m}p^2 + V$, and proves that H has also a ground state for $\alpha \neq 0$. It is then necessarily unique, since e^{-tH} has a positivity improving kernel in a suitable function space [14].

In contrast, in our contribution, we assume that H has no ground state for $\alpha = 0$. In fact, this will be the case for a sufficiently shallow V. We expect the interaction with the quantized radiation field to enhance binding. The non-binding potential should become binding at a sufficiently strong coupling strength. This is precisely our main result. We prove that there is a critical coupling $\alpha_c \geq 0$ such that H has a unique ground state for *arbitrary* $|\alpha| > \alpha_c$.

The physical reasoning behind such a result is simple. As the particle binds photons it acquires an effective mass $m_{\text{eff}} = m_{\text{eff}}(\alpha^2)$ which is increasing in $|\alpha|$. Roughly speaking, H may be replaced by

$$H_{\text{eff}} = \frac{p^2}{2m_{\text{eff}}} + V,$$

which binds for sufficiently large α.

Our proof is based on a scaling argument from [2]. Let us start with

$$H_{\text{el}} = \frac{p^2}{2m} + V,$$

where $V(x) \leq 0$, $\lim_{|x|\to\infty} V(x) = 0$, such that H_{el} has only the absolutely continuous spectrum $[0, \infty)$. Then the dilated H_{el},

$$H_{\text{el}}(\kappa) = \kappa^2 \frac{p^2}{2m} + V(\frac{x}{\kappa}), \tag{1.4}$$

will have the same spectrum. We couple to the Bose field as

$$\kappa^2 \left\{ \frac{1}{2m}(p - \alpha A)^2 + \frac{1}{\kappa^2} V(\frac{x}{\kappa}) + H_{\text{f}} \right\} \cong \frac{1}{2m}(p - \kappa\alpha A)^2 + V + \kappa^2 H_{\text{f}} = H(\kappa), \tag{1.5}$$

$\cong$ denoting unitary equivalence. Let

$$m_{\text{eff}} = m_{\text{eff}}(\alpha^2) = m + \alpha^2 \frac{2}{3} \int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^2} d^3k \tag{1.6}$$

and g be a suitable energy reset. We will prove that, for sufficiently large $z > 0$,

$$s - \lim_{\kappa\to\infty} \left(H(\kappa) - \kappa^2\alpha^2 g + z \right)^{-1} = \left(\frac{p^2}{2m_{\text{eff}}} + V + z \right)^{-1} \otimes P_{\text{g}} \tag{1.7}$$

with P_{g} the projection onto the ground state of

$$H_{\text{f}} + \frac{\alpha^2}{2m} A^2.$$

Note that m_{eff} rather than m appears in the limit Hamiltonian. For sufficiently large α, the limit Hamiltonian has a ground state. If we can prove that this ground state persists for large κ, we are done.

This paper is organized as follows. In Section 2 we introduce our assumptions on V, $\hat{\varphi}$ and consider the κ-dependence of the ground state in case of massive photons. In Section 3 we establish the main Theorem 3.1. As corollaries, we provide examples for potentials V such that the zero-coupling Hamiltonian has no ground state, but for $|\alpha| > \alpha_{\text{c}} > 0$, with some α_{c}, H has a unique ground state.

2 The Pauli-Fierz Hamiltonian in the dipole approximation

2.1 Hamiltonian

Let us assume that the electron moves in d dimensions. We set $L^2 = L^2(\mathbb{R}^d)$ and denote by $\mathcal{F}$ the Boson Fock space over $L^2 \otimes \mathbb{C}^{d-1}$. The Hilbert space $\mathcal{H}$ of the coupled system is then

$$\mathcal{H} = L^2 \otimes \mathcal{F}.$$

Let $\mathcal{F}_{\text{fin}}$ denote the finite particle subspace of $\mathcal{F}$. The Fock representation of the Bose field is denoted by $a^*(k,j), a(k,j)$, $j = 1, \ldots, d-1$, which satisfy the commutation relations (1.2). We use the shorthand $a^*(\lambda, j) = \int \lambda(k) a^*(k,j) d^d k$ and $a(\lambda, j) = \int \lambda(k)^* a(k,j) d^d k$. Note that $[a(f,j), a^*(g,j')] = \delta_{jj'}(f,g)$ on $\mathcal{F}_{\text{fin}}$, where (f,g) is the scalar product on L^2. The d-dimensional polarization vectors are written as $e_j = e_j(k) = (e_j^1(k), \ldots, e_j^d(k))$, $j = 1, \ldots, d-1$, which satisfy $e_i(k) \cdot e_j(k) = \delta_{ij}$ and $e_j(k) \cdot k = 0$ almost everywhere on $\mathbb{R}^d$.

The quantized vector potential is defined by

$$A = \sum_{j=1}^{d-1} \int \frac{1}{\sqrt{2\omega(k)}} e_j(k) \left\{ \hat{\varphi}(k) a^*(k,j) + \hat{\varphi}(k)^* a(k,j) \right\} d^d k,$$

and the quantized electric field, as its canonically conjugate, by

$$\Pi = i \sum_{j=1}^{d-1} \int \frac{\sqrt{\omega(k)}}{\sqrt{2}} e_j(k) \left\{ \hat{\varphi}(k) a^*(k,j) - \hat{\varphi}(k)^* a(k,j) \right\} d^d k.$$

Let H_f be the field Hamiltonian and N_f be the number operator in $\mathcal{F}$,

$$H_\mathrm{f} = \sum_{j=1}^{d-1} \int \omega(k) a^*(k,j) a(k,j) d^d k,$$

$$N_\mathrm{f} = \sum_{j=1}^{d-1} \int a^*(k,j) a(k,j) d^d k.$$

The Hamiltonian H acting in $\mathcal{H}$ is then given by

$$H = \frac{1}{2m} (p - \alpha A)^2 + V + H_\mathrm{f}.$$

We first state the self-adjointness of H, which is established in [1, 15, 16] for arbitrary $\alpha \in \mathbb{R}$.

Proposition 2.1 *Let $\hat{\varphi}/\omega, \hat{\varphi}/\sqrt{\omega}, \hat{\varphi}, \sqrt{\omega}\hat{\varphi} \in L^2$. Moreover suppose that V is relatively bounded with respect to $p^2 = -\Delta$ with a sufficiently small relative bound. Then H is self-adjoint on $D(p^2) \cap D(H_\mathrm{f})$ and bounded below for arbitrary $\alpha \in \mathbb{R}$.*

We need in addition some technical assumptions on $\hat{\varphi}$ which we list as

Definition 2.2 *We say that $\hat{\varphi}$ is in $\mathcal{E}$ if*
(1) $\hat{\varphi}(k)^* = \hat{\varphi}(-k)$ *and $\hat{\varphi}$ is rotation invariant, i.e.* $\hat{\varphi}(k) = \hat{\varphi}(|k|)$,
(2) $\omega^{n/2}\hat{\varphi} \in L^2$ *for* $n = -5, -4, -3, -2, -1, 0, 1$,
(3) $|\hat{\varphi}(\sqrt{s})| s^{(d-1)/2} \in L^\epsilon([0,\infty), ds)$, $0 < \epsilon < 1$, *and Lipschitz continuous on* $[0,\infty)$ *with an order strictly less than one,*
(4) $\|\hat{\varphi}\omega^{(d-2)/2}\|_\infty < \infty$ *and* $\|\hat{\varphi}\omega^{(d-1)/2}\|_\infty < \infty$, *where* $\|f\|_\infty = \sup_{k \in \mathbb{R}^d} |f(k)|$,
(5) $\hat{\varphi}(k) \neq 0$ *for* $k \neq 0$.

H for $V = 0$ is quadratic and can therefore be diagonalized explicitly, which is carried out in the Appendix. We also need the corresponding unitary U so as to control the error $\delta V = U^{-1} V U - V$ in the unitarily transformed potential.

Proposition 2.3 *Let $\hat{\varphi} \in \mathcal{E}$ and V be relatively bounded with respect to $-\Delta$ with a sufficiently small relative bound. Then, for each $\alpha \in \mathbb{R}$, there exists a unitary operator U on $\mathcal{H}$ such that U maps $D(p^2) \cap D(H_\mathrm{f})$ onto itself and*

$$U^{-1} H U = H_\mathrm{eff} + H_\mathrm{f} + \delta V + \alpha^2 g,$$

where

$$H_{\text{eff}} = \frac{p^2}{2m_{\text{eff}}} + V, \quad \delta V = U^{-1}VU - V,$$

$$m_{\text{eff}} = m_{\text{eff}}(\alpha^2) = m + \alpha^2 q \|\hat{\varphi}/\omega\|^2, \quad q = \frac{d-1}{d},$$

$$g = \frac{d-1}{2\pi} \int_{-\infty}^{\infty} \frac{t^2 \|\hat{\varphi}/(t^2+\omega^2)\|^2}{m + \alpha^2 q \|\hat{\varphi}/\sqrt{t^2+\omega^2}\|^2} dt.$$

Proof: See Theorem 4.8 of the Appendix. □

Proposition 2.4 *We have $\delta V = T^{-1}VT - V$, where*

$$T = \exp\left(-i\frac{\alpha}{m_{\text{eff}}} p \cdot K\right),$$

$$K = \frac{1}{\sqrt{2}} \sum_{j=1}^{d-1} \int \{\Lambda_j(k) a^*(k,j) + \Lambda_j(k)^* a(k,j)\} \, d^d k, \quad \Lambda_j \in L^2(\mathbb{R}^d; \mathbb{C}^d),$$

and Λ_j satisfies

$$\|\omega^{n/2}\Lambda_j\| \le C_1 \|\omega^{(n-3)/2}\hat{\varphi}\|, \quad n = -2, -1, 0,$$

with some constant C_1. In particular,

$$\|K_\mu \Psi\| \le C_2 (2\|\hat{\varphi}/\omega^2\| + \|\hat{\varphi}/\omega^{3/2}\|)(\|H_{\text{f}}{}^{1/2}\Psi\| + \|\Psi\|) \tag{2.1}$$

with $C_2 = C_1/\sqrt{2}$.

Proof: See Theorem 4.9 of the Appendix. □

2.2 External potentials

We introduce the following conditions on V.

Condition 2.5

(1) *V is relatively bounded with respect to $-\Delta$ with a sufficiently small relative bound.*

(2) *$V \in C^1(\mathbb{R}^d)$ and $\nabla V \in L^\infty(\mathbb{R}^d)$.*

(3) *There exist $\mu_0 \ge 1$ and $r_0 > 0$ such that, for all $\eta > \mu_0$,*

$$\inf \sigma\left(\frac{p^2}{2m} + \eta V\right) \le -r_0, \quad \sigma_{\text{ess}}\left(\frac{p^2}{2m} + \eta V\right) = [0, \infty).$$

Let V satisfy Condition 2.5 and let

$$\alpha_{\rm c} = \sqrt{m(\mu_0 - 1)}\|\hat{\varphi}/\omega\|^{-1}.$$

Then $H_{\rm eff}$ has a ground state for $|\alpha| > \alpha_{\rm c}$, since

$$H_{\rm eff} = \frac{m}{m_{\rm eff}}\left(\frac{p^2}{2m} + \frac{m_{\rm eff}}{m}V\right), \quad \frac{m_{\rm eff}}{m} > \mu_0.$$

If we set

$$\Sigma = \inf\sigma(H_{\rm eff}), \quad V_\infty = \inf_{x\in\mathbb{R}^d} V(x),$$

then

$$V_\infty \le \Sigma \le -\frac{mr_0}{m_{\rm eff}} \quad \text{for } |\alpha| > \alpha_{\rm c} \tag{2.2}$$

and

$$\lim_{|\alpha|\to\infty} \Sigma = V_\infty \ge -\infty, \tag{2.3}$$

since $H_{\rm eff} \to V$ as $|\alpha| \to \infty$ in the norm resolvent sense. In what follows we assume $\hat{\varphi} \in \mathcal{E}$ and V to satisfy Condition 2.5.

2.3 Scaled Hamiltonian

Let $D(\kappa)$ be the dilation unitary on L^2 defined by

$$D(\kappa)f(k) = \kappa^{d/2}f(\frac{k}{\kappa}).$$

We introduce the scaled Hamiltonian by

$$\begin{aligned} H(\kappa) &= \kappa^2 D(\kappa)^{-1}\left\{\frac{1}{2m}(p-\alpha A)^2 + H_{\rm f} + \frac{1}{\kappa^2}V(\frac{x}{\kappa})\right\}D(\kappa) \\ &= \frac{1}{2m}(p-\kappa\alpha A)^2 + V + \kappa^2 H_{\rm f}, \end{aligned}$$

which means that $H(\kappa)$ can be obtained from H by the substitution

$$\omega \to \kappa^2\omega, \quad \hat{\varphi} \to \kappa^2\hat{\varphi}. \tag{2.4}$$

If we make the corresponding substitution in U, δV, denoted by $U(\kappa)$, $\delta V(\kappa)$, then

$$\widehat{H}(\kappa) = U(\kappa)^{-1}H(\kappa)U(\kappa) = H_{\rm eff} + \kappa^2 H_{\rm f} + \delta V(\kappa) + \kappa^2\alpha^2 g.$$

Note that $H_{\rm eff}$ and g are independent of κ, whereas $\alpha K/m_{\rm eff}$ is scaled as

$$\frac{\alpha}{m_{\rm eff}}K \to \frac{1}{\kappa}\frac{\alpha}{m_{\rm eff}}K,$$

see (4.19) of the Appendix.

We prove the existence of a ground state of $\widehat{H}(\kappa)$.

Lemma 2.6 *Let $\Psi \in D(p^2)\cap D(H_{\rm f}^{1/2})$. Then $\Psi \in D(\delta V(\kappa))$ for all $\kappa \in \mathbb{R}$ and*

$$\|\delta V(\kappa)\Psi\| \leq \frac{D}{\kappa}(\|H_{\rm f}{}^{1/2}\Psi\| + \|\Psi\|), \tag{2.5}$$

where

$$D = D(\alpha) = \frac{C_2|\alpha|}{m_{\rm eff}(\alpha^2)}(2\|\hat{\varphi}/\omega^2\| + \|\hat{\varphi}/\omega^{3/2}\|)\|\nabla V\|_\infty,$$

and C_2 is given in (2.1).

Proof: Let $\phi_\mu = \alpha K_\mu/(\kappa m_{\rm eff})$ for instance. Taking the Q-space representation of [15], we see that $\mathcal{F}$ can be identified with the probability measure space $L^2(Q)$ and ϕ_μ with a multiplication operator in $L^2(Q)$. We set $\phi = \phi(q) = (\phi_1(q), ..., \phi_d(q))$, $q \in Q$. Moreover we regard $L^2 \otimes L^2(Q)$ as the set of $L^2(Q)$-valued L^2-function i.e. for $\Psi \in L^2 \otimes L^2(Q)$, $\Psi(x) \in L^2(Q)$ for almost all $x \in \mathbb{R}^d$. Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be such that $\rho(x) \geq 0$, supp$\rho \subset \{x \in \mathbb{R}^d||x| \leq 1\}$ and $\int_{\mathbb{R}^d}\rho(x)dx = 1$. Define $\rho_\epsilon(x) = \rho(x/\epsilon)/\epsilon^d$, $\epsilon > 0$, and

$$V_\epsilon = \rho_\epsilon * V,$$

where $*$ denotes the convolution. Then it follows that $V_\epsilon \in C_0^\infty(\mathbb{R}^d)$ and $V_\epsilon \to V$, $\nabla V_\epsilon \to \nabla V$ uniformly on compact sets. Let $\Phi \in C_0^\infty(\mathbb{R}^d) \otimes_{\rm alg} [\mathcal{F}_{\rm fin} \cap D(H_{\rm f})]$, where $\otimes_{\rm alg}$ denotes the algebraic tensor product. Since

$$T^{-1}\rho_\epsilon T = \rho_\epsilon(\cdot + \phi)$$

as a bounded operator in $L^2 \otimes L^2(Q)$, we have

$$T^{-1}V_\epsilon T\Phi = V_\epsilon(\cdot + \phi)\Phi.$$

Fix $q \in Q$. We may assume $\phi_\mu(q) \geq 0$, $\mu = 1, ..., d$, without loss of generality. It then follows that

$$V_\epsilon(x + \phi(q)) - V_\epsilon(x) = \phi(q)\cdot\nabla V_\epsilon(\xi)$$

with some $\xi_\mu \in (x_\mu, x_\mu + \phi(q))$ for arbitrary $x \in \mathbb{R}^d$. Thus

$$\|V_\epsilon(\cdot + \phi(q)) - V_\epsilon\|_{L^\infty(C)} \leq \|\nabla V_\epsilon\|_{L^\infty(C)} |\phi(q)|$$

for any compact set $C \subset \mathbb{R}^d$. Hence we have

$$\|(V_\epsilon(\cdot + \phi) - V_\epsilon)\Phi\|_{L^2 \otimes L^2(Q)} \leq \|\nabla V_\epsilon\|_{L^\infty(C)} \||\phi|\Psi\|_{L^2 \otimes L^2(Q)},$$

where $C = \mathrm{supp}\|\Phi(\cdot)\|_{L^2(Q)}$ is a compact set. Taking $\epsilon \to 0$ on the both sides, we conclude

$$\|(V(\cdot + \phi) - V)\Phi\|_{L^2 \otimes L^2(Q)} \leq \|\nabla V\|_{L^\infty(C)} \||\phi|\Psi\|_{L^2 \otimes L^2(Q)} \leq \|\nabla V\|_{L^\infty(\mathbb{R}^d)} \||\phi|\Psi\|_{L^2 \otimes L^2(Q)}.$$

Thus (2.5) holds by (2.1) for $\Phi \in C_0^\infty(\mathbb{R}^d) \otimes_{\mathrm{alg}} [\mathcal{F}_{\mathrm{fin}} \cap D(H_{\mathrm{f}})]$. By a simple limiting argument, the lemma follows for arbitrary $\Phi \in D(p^2) \cap D(H_{\mathrm{f}})$. $\square$

From the definition of m_{eff} it follows that

$$\lim_{|\alpha| \to \infty} D(\alpha) = 0. \tag{2.6}$$

We write $A \leq B$, if $D(B) \subset D(A)$ and if for $\psi \in D(B)$, $(\psi, A\psi) \leq (\psi, B\psi)$. Thus, by Lemma 2.6, we conclude the inequalities

$$-\frac{1}{\kappa}\left(\frac{D}{2}H_{\mathrm{f}} + \frac{3D}{2}\right) \leq \delta V(\kappa) \leq \frac{1}{\kappa}\left(\frac{D}{2}H_{\mathrm{f}} + \frac{3D}{2}\right). \tag{2.7}$$

In particular we have

$$\inf\sigma(\widehat{H}(\kappa)) \leq \Sigma + \frac{1}{\kappa}\frac{3D}{2} + \kappa^2\alpha^2 g. \tag{2.8}$$

2.4 Estimates for massive ground states

The ground state of our massless model is approximated through the ground state of massive models and we first develop some preparatory lemmas. Let $\nu > 0$. We set

$$m_{\mathrm{eff}}^\nu = m + \alpha^2 q \|\hat{\varphi}/(\omega + \nu)\|^2$$

and D^ν equal to D with m_{eff} replaced by m_{eff}^ν. As can be easily seen

$$m_{\mathrm{eff}}^\nu \uparrow m_{\mathrm{eff}}, \quad \nu \to 0_+,$$

and

$$D^\nu \downarrow D, \quad \nu \to 0_+. \tag{2.9}$$

Let $\Gamma(l,a)$, $l=(l_1,\cdots,l_d)\in\mathbb{Z}^d$, $a>0$, be the momentum lattice with spacing $1/a$ $\Gamma(l,a)=[l_1/a,(l_1+1)/a)\times\cdots\times[l_d/a,(l_d+1)/a)$ and $\chi_{\Gamma(l,a)}$ be the characteristic function of $\Gamma(l,a)$. We define $L_a:L^2\to L^2$ by $(L_af)(k)=\sum_{l\in\mathbb{Z}^d,|l|<L}f(l)\chi_{\Gamma(l,a)}(k)$. Set $L_aL^2=\ell_a$ and denote by $\mathbb{L}_a:\mathcal{F}\to\mathcal{F}$ the second quantization of L_a. Let $\mathbb{L}_a\mathcal{F}=\mathcal{F}_a\subset\mathcal{F}$ and $\mathcal{H}_a=L^2\otimes\mathcal{F}_a\subset\mathcal{H}$. We set

$$\widetilde{H}_a(\kappa)=H_{\rm eff}+\kappa^2H_{{\rm f}\,a}^\nu+\delta V_a(\kappa),$$

where

$$\delta V_a(\kappa)=T_a^{-1}VT_a-V,$$
$$T_a=\exp\left(-i\frac{\alpha}{m_{{\rm eff}\,a}^\nu}p\cdot K_a\right),$$
$$m_{{\rm eff}\,a}^\nu=m+\alpha^2q\|(L_a\hat{\varphi})/L_a(\omega+\nu)\|^2,$$
$$K_a=\sum_{j=1}^{d-1}\frac{1}{\sqrt{2}}\int\left\{L_a\Lambda_j(k)a^*(k,j)+L_a\Lambda_j(k)^*a(k,j)\right\}d^dk,$$
$$H_{{\rm f}\,a}^\nu=\sum_{j=1}^{d-1}\int L_a(\omega+\nu)(k)a^*(k,j)a(k,j)d^dk.$$

Lemma 2.7 *$\widetilde{H}_a(\kappa)$ is reduced by $\mathcal{H}_a$ and* $\inf\sigma(\widetilde{H}_a(\kappa)\lceil_{\mathcal{H}_a^\perp})>\nu+\inf\sigma(\widetilde{H}_a(\kappa))$.

Proof: The first statement is proved easily. We shall prove the second statement. Let P_a be the projection onto the vacuum state of $\mathcal{F}(\ell_a^\perp\otimes\mathbb{C}^{d-1})$, where $\mathcal{F}(\ell_a^\perp\otimes\mathbb{C}^{d-1})$ is the Boson Fock space over $\ell_a^\perp\otimes\mathbb{C}^{d-1}$. Under the identification $\mathcal{H}_a^\perp\cong\mathcal{H}_a\otimes P_a^\perp\mathcal{F}(\ell_a^\perp\otimes\mathbb{C}^{d-1})$, we have

$$\widetilde{H}_a(\kappa)\lceil_{\mathcal{H}_a^\perp}\cong\widetilde{H}_a(\kappa)\lceil_{\mathcal{H}_a}\otimes I+I\otimes H_{{\rm f}\,a}^\nu\ge\inf\sigma(\widetilde{H}_a(\kappa))+\nu,$$

which implies the desired results. □

Lemma 2.8 *Let a be sufficiently large, and κ, α, and ν be such that*

$$|\Sigma|>\frac{mr_0}{m_{{\rm eff}\,a}^\nu}>\frac{3D^\nu}{2\kappa}+\nu. \tag{2.10}$$

Then $\widetilde{H}_a(\kappa)$ has a ground state.

Proof: By Lemma 2.7 it is enough to prove that the dimension of the spectrum of $\widetilde{H}_a(\kappa)\lceil_{\mathcal{H}_a}$ on the interval $[\inf\sigma(\widetilde{H}_a(\kappa)), \inf\sigma(\widetilde{H}_a(\kappa))+\nu)$ is finite. On $\mathcal{H}_a$, by (2.7),

$$\widetilde{H}_a(\kappa) - \inf\sigma(\widetilde{H}_a(\kappa)) - \nu \geq \overline{H_{\rm eff}} + \theta' H_{\rm f}{}^{\nu}_{a} - \theta,$$

where $\overline{H_{\rm eff}} = H_{\rm eff} - \Sigma$, $\theta = 3D_a^\nu/(2\kappa)+\nu$ and $\theta' = \kappa^2 - D_a^\nu/(2\kappa)$. As before D_a^ν is defined by D^ν with $\omega+\nu$ and $\hat{\varphi}$ replaced by $L_a(\omega+\nu)$ and $L_a\hat{\varphi}$, respectively. Let E_A be the spectral projection of $\overline{H_{\rm eff}}$ on a measurable set $A\subset\mathbb{R}^d$. Then

$$\begin{aligned}\widetilde{H}_a(\kappa) - \inf\sigma(\widetilde{H}_a(\kappa)) - \nu &\geq (|\Sigma|-\theta)E_{[|\Sigma|,\infty)} + E_{[0,|\Sigma|)}\otimes(\theta' H_{\rm f}{}^{\nu}_{a} - \theta)\\ &\geq E_{[0,|\Sigma|)}\otimes(\theta' H_{\rm f}{}^{\nu}_{a} - \theta).\end{aligned}$$

Here we used $|\Sigma|-\theta>0$. Since $H_{\rm f}{}^{\nu}_{a}$ and $E_{[0,|\Sigma|)}$ have purely discrete spectrum, the lemma follows. $\square$

As can be proved directly, $\widetilde{H}_a(\kappa)$ converges to $\widehat{H}(\kappa)+\nu N_{\rm f} - \kappa^2\alpha^2 g_\nu$ as $a\to\infty$ and $L\to\infty$ in the norm resolvent sense, where g_ν is defined by g with ω replaced by $\omega+\nu$. Combined with Lemma 2.8, we see that $\widehat{H}(\kappa)+\nu N_{\rm f} - \kappa^2\alpha^2 g_\nu$ has a ground state, i.e. $\widehat{H}_\nu(\kappa) = \widehat{H}(\kappa)+\nu N_{\rm f}$ has a ground state, which is denoted by $\Psi_\nu = \Psi_\nu(\kappa)$.

Lemma 2.9 *On $D(p^2)\cap D(H_{\rm f})$ we have*

$$[\delta V(\kappa), a(f,j)] = \frac{1}{\kappa}\frac{\alpha}{m_{\rm eff}}(f,\Lambda_j)\cdot T(\kappa)^{-1}(\nabla V)T(\kappa).$$

Proof: Since T maps $D(p^2)\cap D(H_{\rm f})$ onto itself, one can check that $T^{-1}VTa(f,j)\Psi$ and $a(f,j)T^{-1}VT\Psi$ for $\Psi\in D(p^2)\cap D(H_{\rm f})$ are well defined. Thus $[\delta V(\kappa), a(f,j)]$ is well defined on $D(p^2)\cap D(H_{\rm f})$. We have

$$[\delta V(\kappa), a(f,j)] = T(\kappa)^{-1}[V, T(\kappa)a(f,j)T(\kappa)^{-1}]T(\kappa).$$

Since

$$T(\kappa)a(f,j)T(\kappa)^{-1} = a(f,j) + i\alpha(f,\Lambda_j)\cdot\frac{p}{m_{\rm eff}\kappa},$$

the lemma follows. $\square$

Lemma 2.10 *We have*

$$\frac{\|H_{\rm f}{}^{1/2}\Psi_\nu\|}{\|\Psi_\nu\|} \leq \frac{1}{\kappa^3}\frac{|\alpha|}{m_{\rm eff}}\epsilon,$$

where $\epsilon = C\|\hat{\varphi}/\omega^{5/2}\|\|\nabla V\|_\infty$ with some constant C.

Proof: Note that $\Psi_\nu \in D(H_\nu) = D(p^2) \cap D(H_{\rm f})$. We have

$$0 \le (a(f,j)\Psi_\nu, (\widehat{H}_\nu(\kappa) - \inf\sigma(\widehat{H}_\nu(\kappa)))a(f,j)\Psi_\nu)$$

$$= -\kappa^2 (a^*((\omega+\nu)f,j)a(f,j)\Psi_\nu,\Psi_\nu) + (a(f,j)\Psi_\nu, [\delta V(\kappa), a(f,j)]\Psi_\nu).$$

By Lemma 2.9,

$$\kappa^2 (N_{\rm f}\Psi,\Psi) \le \frac{1}{\kappa}\frac{|\alpha|}{m_{\rm eff}} \left(a\left(\frac{\Lambda_j}{\omega+\nu}, j\right)\Psi_\nu, T(\kappa)^{-1}(\nabla V)T(\kappa)\Psi_\nu \right),$$

which is our claim. $\square$

Lemma 2.11 *Let P_Ω be the projection onto the vacuum state of $\mathcal{F}$. Let $Q = E_{[0,\infty)} \otimes P_\Omega$. Let κ, α and ν be such that*

$$\frac{mr_0}{m_{\rm eff}^\nu} - \frac{3D^\nu}{2\kappa} > 0. \tag{2.11}$$

Then

$$\frac{\|Q\Psi_\nu\|}{\|\Psi_\nu\|} \le \sqrt{\frac{3D^\nu/2}{\kappa^2(|\Sigma| - 3D^\nu/(2\kappa))}}.$$

Proof: We have

$$(\Psi_\nu, Q(H_{\rm eff} - \inf\sigma(\widehat{H}_\nu(\kappa)) + \kappa^2\alpha^2 g_\nu)\Psi_\nu) = -(\Psi_\nu, Q\delta V(\kappa)\Psi_\nu).$$

Hence

$$(\kappa^2\alpha^2 g_\nu - \inf\sigma(\widehat{H}_\nu(\kappa)))\|Q\Psi_\nu\|^2 \le \|\delta V(\kappa)Q\Psi_\nu\|\|\Psi_\nu\|.$$

Note that, by (2.7) and (2.8),

$$\kappa^2\alpha^2 g_\nu - \inf\sigma(\widehat{H}_\nu(\kappa)) \ge |\Sigma| - \frac{3D^\nu}{2\kappa} > \frac{mr_0}{m_{\rm eff}^\nu} - \frac{3D^\nu}{2\kappa} > 0,$$

$$\|\delta V(\kappa)Q\Psi_\nu\| \le \frac{3D^\nu}{2\kappa}\|\Psi_\nu\|.$$

Thus the lemma follows. $\square$

3 Binding

Our next task is to establish that for sufficiently large α and/or sufficiently large κ the ground state of $H(\kappa)$ exists.

3.1 Existence of a ground state for large coupling constant

Theorem 3.1 *There exists $\alpha_* \geq \alpha_c$ such that, for arbitrary $|\alpha| > \alpha_*$, the ground state of H exists and is unique.*

Proof: The uniqueness follows from the ergodic property of e^{-tH}, see [14]. Let $a > 0$ be sufficiently small, then $P = E_{[\Sigma,\Sigma+a)} \otimes P_\Omega$ is a finite rank operator. We denote by Ψ the weak-limit of the normalized $\Psi_{\nu'}$ as a subsequence ν' tends to infinity. Since $P \geq I - N_{\mathrm{f}} - Q$, we have, by Lemma 2.10 and 2.11,

$$(\Psi, P\Psi) \geq \|\Psi\|^2 - \left(\frac{|\alpha|\epsilon}{m_{\mathrm{eff}}}\right)^2 \|\Psi\|^2 - \frac{3D/2}{|\Sigma| - 3D/2}\|\Psi\|^2. \tag{3.1}$$

Since

$$\lim_{|\alpha|\to\infty} \frac{|\alpha|}{m_{\mathrm{eff}}} = 0, \quad \lim_{|\alpha|\to\infty} D = 0, \quad \lim_{|\alpha|\to\infty} \Sigma = V_\infty < 0,$$

we see that there exists $\alpha_* > 0$ such that, for arbitrary $|\alpha| > \alpha_*$, the right-hand side of (3.1) is strictly positive. Thus $\Psi \neq 0$ for $|\alpha| > \alpha_*$, which implies that Ψ is a ground state of $\widehat{H}(1)$. Since $\widehat{H}(1)$ is unitarily equivalent to H, the existence of the ground state of H follows. □

3.2 A lower bound of the critical coupling constant

By assumption H_{eff} has a ground state for $|\alpha| > \alpha_c$. We have to make sure that H shares the same property. Our key is

Lemma 3.2 *For arbitrary $\alpha \in \mathbb{R}$, $H_{\mathrm{f}} + \alpha^2 A^2/2m$ has a unique ground state, and*

$$\inf\sigma\left(H_{\mathrm{f}} + \frac{\alpha^2}{2m}A^2\right) = \alpha^2 g.$$

Let P_{g} be the projection onto the ground state of $H_{\mathrm{f}} + \alpha^2 A^2/2m$. Then, for sufficiently large $z > 0$,

$$s - \lim_{\kappa\to\infty} \left(H(\kappa) - \alpha^2\kappa^2 g + z\right)^{-1} = (H_{\mathrm{eff}} + z)^{-1} \otimes P_{\mathrm{g}}.$$

Proof: See Theorem 4.6 of the Appendix. □

Lemma 3.2 suggests that $H(\kappa)$ has a ground state for sufficiently large κ, since H_{eff} does so.

Lemma 3.3 *Fix a sufficiently large κ. Then, for arbitrary $|\alpha| > \alpha_{\rm c}$, the ground state of $\widehat{H}(\kappa)$ exists.*

Proof: As Theorem 3.1, we have

$$(\Psi, P\Psi) \geq \|\Psi\|^2 - \left(\frac{1}{\kappa^3}\frac{|\alpha|\epsilon}{m_{\rm eff}}\right)^2 \|\Psi\|^2 - \frac{3D/(2\kappa)}{\kappa^2(|\Sigma| - 3D/(2\kappa))}\|\Psi\|^2. \tag{3.2}$$

Since κ is sufficiently large, the right-hand side of (3.2) is strictly positive. Therefore Ψ is a ground state of $\widehat{H}(\kappa)$. □

Theorem 3.4 *Let κ be sufficiently large. Set*

$$V_\kappa(x) = \frac{1}{\kappa^2}V(\frac{x}{\kappa}).$$

Then the ground state of

$$H_\kappa = \frac{1}{2m}(p - \alpha A)^2 + V_\kappa + H_{\rm f}$$

exits for arbitrary $|\alpha| > \alpha_{\rm c}$. In particular, if $H_{\rm el}$ has a ground state, then H_κ has a unique ground state for all α.

Proof: We have

$$H_\kappa = \frac{1}{\kappa^2}D(\kappa)H(\kappa)D(\kappa)^{-1}.$$

Since $H(\kappa)$ is unitarily equivalent to $\widehat{H}(\kappa)$, H_κ has a unique ground state by Lemma 3.3. If $H_{\rm eff}$ has a ground state, then $\alpha_{\rm c} = 0$ and the zero-coupling H_κ has a ground state. Thus H_κ has a ground state for all α. □

3.3 Examples

We provide some examples of shallow external potentials. Let $d \geq 3$. We assume that V is nonpositive ($V \not\equiv 0$) and satisfies Condition 2.5. Moreover

$$N(V) = a_d \int_{\mathbb{R}^d} |mV(x)|^{d/2} dx < 1, \tag{3.3}$$

where a_d is a universal constant [17]. Note that, for all $\kappa > 0$,

$$N(V) = N(V_\kappa).$$

Thus $\frac{1}{2m}p^2 + V_\kappa$ has *no* ground state for all $\kappa > 0$.

Example 3.5 *We assume that κ is sufficiently large. By virtue of Theorem 3.4, we see that H_κ has a unique ground state for $|\alpha| > \alpha_c$. On the other side (3.3) says that H_κ has no ground state for $\alpha = 0$.*

Let

$$H_{\rm el}(\gamma) = \frac{1}{2m}p^2 + \gamma V$$

and

$$\gamma_* = \sup \{\gamma \in \mathbb{R} | \sigma (H_{\rm el}(\gamma)) = [0, \infty)\}.$$

Since $E(\gamma) = \inf\sigma(H_{\rm el}(\gamma))$ is continuous in γ, we have $E(\gamma_*) = 0$. Thus, for any $\epsilon > 0$, we have $E(\gamma+\epsilon) < 0$ and then $H_{\rm el}(\gamma+\epsilon)$ has a ground state with a spectral gap, which yields the following

Example 3.6 *Let arbitrary $\epsilon > 0$ be given. We assume that κ is sufficiently large. Then*

$$\frac{1}{2m}(p - \alpha A)^2 + \gamma_* V_\kappa + H_{\rm f}$$

has a unique ground state for arbitrary $|\alpha| > \epsilon$, but $H_{\rm el}(\gamma)$ has no spectral gap.

4 Appendix

4.1 Bogoliubov transformation

To estimate δV and $m_{\rm eff}$ explicitly, we need the exact form of the transformed Hamiltonian $H_{\rm eff} + H_{\rm f} + \delta V + \alpha^2 g$ for arbitrary $\alpha \in \mathbb{R}$. Let us define an operator in $\mathcal{H}$ by

$$H_{\rm dip} = \frac{1}{2m}(p - \alpha A)^2 + H_{\rm f}.$$

Arai diagonalized $H_{\rm dip}$ exactly in [1]. However this is not enough for our purposes and we have to improve the results in [1]. Following [1] we review the Bogoliubov transform of $H_{\rm dip}$. The operator $H_{\rm dip}$ is decomposable for all $\alpha \in \mathbb{R}$, i.e.

$$H_{\rm dip} = \int_{\mathbb{R}^d}^{\oplus} H_{\rm dip}(p)dp,$$

where

$$H_{\rm dip}(p) = \frac{1}{2m}(p - \alpha A)^2 + H_{\rm f}, \quad p \in \mathbb{R}^d.$$

We shall prove that there exists a unitary operator $U(p)$ such that

$$U(p)^{-1}H_{\rm dip}(p)U(p) = E(p) + H_{\rm f},$$

where $E(p) = E(p, \alpha^2)$ is a constant. We will see that $E(p)$ can be analytically continued to a domain $\mathcal{O}$ such that $(0, \infty) \subset \mathcal{O} \subset \mathbb{C}$ in α^2.

Let

$$D_\pm(s) = m - \alpha^2 q \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{s - \omega(k)^2 \pm i0} d^dk, \quad s \in [0, \infty).$$

Then we have

$$D_\pm(s) = m - \frac{\alpha^2 q |S_{d-1}|}{2} \left(\lim_{\epsilon \downarrow 0} \int_{|s-x|>\epsilon, x\geq 0} \frac{|\hat{\varphi}(\sqrt{x})|^2 |x|^{(d-2)/2}}{s-x} dx \mp 2\pi i |\hat{\varphi}(\sqrt{s})|^2 s^{(d-2)/2} \right), \tag{4.1}$$

where $|S_{d-1}|$ denotes the volume of the $d-1$-dimensional unit sphere S_{d-1}. In particular, there exists $\epsilon > 0$ such that $\sup_{s\in[0,\infty)} D_\pm(s) > \epsilon$ by Definition 2.2 (3), (4). Define

$$Gf(k) = \int_{\mathbb{R}^d} \frac{f(k')}{(\omega(k)^2 - \omega(k')^2 + i0)(\omega(k)\omega(k'))^{(d-2)/2}} d^dk'.$$

It is seen that

$$Gf(k) = \frac{1}{2\omega(k)^{(d-2)/2}} \times$$

$$\times \left(\lim_{\epsilon\downarrow 0} \int_{|\omega(k)^2 - x|>\epsilon, x\geq 0} \frac{[f](\sqrt{x})}{\omega(k)^2 - x} x^{(d-2)/4} dx - 2\pi i [f](\omega(k)) \omega(k)^{(d-2)/2} \right), \tag{4.2}$$

where $[f](r) = \int_{S_{d-1}} f(r, \phi) d\phi$ and ϕ is the volume element of S_{d-1}. Define

$$T_{\mu\nu} f = T_{\mu\nu}(\alpha^2) f = \delta_{\mu\nu} f + \alpha^2 Q \omega^{(d-2)/2} G \omega^{(d-2)/2} d_{\mu\nu} \hat{\varphi} f,$$

where $d_{\mu\nu}(k) = \delta_{\mu\nu} - k_\mu k_\nu / |k|^2$ and

$$Q(k) = Q(k, \alpha^2) = \frac{\hat{\varphi}(k)}{D_+(\omega(k)^2)}.$$

Since G is a bounded operator on L^2, we see that, by (2) of Definition 2.2, $\|\omega^{n/2} T_{\mu\nu} f\| \leq C \|\omega^{n/2} f\|$, $n = -1, 0, 1$, with some constant C. Let $T^*_{\mu\nu} = (T_{\mu\nu})^*$. We define

$$B_p(f, j) = \frac{1}{\sqrt{2}} \left\{ \widetilde{A}_\mu \left(\frac{1}{\sqrt{\omega}} T^*_{\mu\nu} \sqrt{\omega} e^\nu_j f \right) + i \widetilde{\Pi}_\mu \left(\sqrt{\omega} T^*_{\mu\nu} \frac{1}{\sqrt{\omega}} e^\nu_j f \right) - \alpha \left(\frac{p \cdot e_j}{\omega^{3/2}} Q, f \right) \right\},$$

$$B_p^*(f,j)=\frac{1}{\sqrt{2}}\left\{\widetilde{A}_\mu\left(\frac{1}{\sqrt{\omega}}\overline{T}^*_{\mu\nu}\sqrt{\omega}\widetilde{e_j^\nu f}\right)-i\widetilde{\Pi}_\mu\left(\sqrt{\omega}\overline{T}^*_{\mu\nu}\frac{1}{\sqrt{\omega}}\widetilde{e_j^\nu f}\right)-\alpha\left(\frac{p\cdot e_j}{\omega^{3/2}}\bar{Q},f\right)\right\},$$

where $\overline{X}f=\overline{X\bar{f}}$, $\bar{f}$ denotes the complex conjugate of f, and

$$\widetilde{A}(\lambda)=\sum_{j=1}^{d-1}\int\frac{e_j(k)}{\sqrt{2\omega(k)}}\left\{\widetilde{\lambda}(k)a^*(k,j)+\lambda(k)a(k,j)\right\}d^dk,$$

$$\widetilde{\Pi}(\lambda)=\sum_{j=1}^{d-1}\int\frac{\sqrt{\omega(k)}}{\sqrt{2}}e_j(k)\left\{\widetilde{\lambda}(k)a^*(k,j)-\lambda(k)a(k,j)\right\}d^dk.$$

Here $\widetilde{\lambda}(k)=\lambda(-k)$. Then $\widetilde{A}(\lambda)$ and $\widetilde{\Pi}(\lambda)$ are complex linear in λ. We have the following commutation relations on $\mathcal{F}_{\rm fin}\cap D({H_{\rm f}}^{3/2})$

$$[\widetilde{A}_\mu(\eta),\widetilde{\Pi}_\nu(\rho)]=i\int d_{\mu\nu}(k)\eta(k)\rho(k)d^dk,\quad[\widetilde{A}_\mu(\eta),\widetilde{A}_\nu(\rho)]=0,\quad[\widetilde{\Pi}_\mu(\eta),\widetilde{\Pi}_\nu(\rho)]=0,$$

$$[H_{\rm f},\widetilde{A}_\mu(\eta)]=-i\widetilde{\Pi}_\mu(\eta),\quad[H_{\rm f},\widetilde{\Pi}_\mu(\eta)]=i\widetilde{A}_\mu(\omega\eta).\tag{4.3}$$

We have

$$B_p(f,j)=a^*(W_{+ij}f,i)+a(\overline{W_{-ij}f},i)-\alpha\left(\frac{p\cdot e_j}{\sqrt{2}\omega^{3/2}}Q,f\right),$$

$$B_p^*(f,j)=a^*(\overline{W}_{+ij}f,i)+a(\overline{\overline{W}_{-ij}f},i)-\alpha\left(\frac{p\cdot e_j}{\sqrt{2}\omega^{3/2}}\bar{Q},f\right),$$

where

$$W_{+ij}f=\frac{1}{2}e_i^\mu\left(\omega^{-1/2}T^*_{\mu\nu}\omega^{1/2}+\omega^{1/2}T^*_{\mu\nu}\omega^{-1/2}\right)e_j^\nu f,$$

$$W_{-ij}f=\frac{1}{2}e_i^\mu\left(\omega^{-1/2}T^*_{\mu\nu}\omega^{1/2}-\omega^{1/2}T^*_{\mu\nu}\omega^{-1/2}\right)\widetilde{e_j^\nu f}.$$

Let

$$W_\pm=\begin{pmatrix}W_{\pm11}&\cdots&W_{\pm1\,d-1}\\ \vdots&\cdots&\vdots\\ W_{\pm d-1\,1}&\cdots&W_{\pm d-1\,d-1}\end{pmatrix}.$$

Remark 4.1 *From the substitution $\omega\to\kappa^2\omega$ and $\hat{\varphi}\to\kappa\hat{\varphi}$, we infer that $T_{\mu\nu}$, W_+, and W_- are independent of κ.*

By using (4.1) and (4.2), as operator equations on $L^2\otimes\mathbb{C}^{d-1}$, we see a symplectic structure

$$W_+^*W_+-W_-^*W_-=I,\quad\overline{W}_+^*W_--\overline{W}_-^*W_+=0,\tag{4.4}$$

$$W_+W_+^* - \overline{W}_-\overline{W}_-^* = I, \quad W_-W_+^* - \overline{W}_+\overline{W}_-^* = 0. \tag{4.5}$$

By (4.4) we have

$$[B_p(f,j), B_p^*(g,j')] = \delta_{jj'}(\bar{f},g), \quad [B_p^*(f,j), B_p^*(g,j')] = 0, \quad [B_p(f,j), B_p(g,j')] = 0,$$

and by (4.5)

$$a(f,j) = -B_p^*(\overline{W}_{-ij}^*\overline{f}, i) + B_p(W_{+ij}^*\overline{f}, i) - \frac{\alpha}{m_{\text{eff}}}\left(\frac{p\cdot e_j\hat{\varphi}}{\sqrt{2}\omega^{3/2}}, \overline{f}\right), \tag{4.6}$$

$$a^*(f,j) = B_p^*(\overline{W}_{+ij}^* f, i) - B_p(W_{-ij}^* f, i) - \frac{\alpha}{m_{\text{eff}}}\left(\frac{p\cdot e_j\hat{\varphi}}{\sqrt{2}\omega^{3/2}}, f\right). \tag{4.7}$$

Lemma 4.2 *We have on a dense domain*

$$[H_{\text{dip}}(p), B_p(f,j)] = -B_p(\omega f, j), \quad [H_{\text{dip}}(p), B_p^*(f,j)] = B_p^*(\omega f, j), \quad \omega f, f \in L^2.$$

Proof: A direct calculation, see [1]. □

We define $b = B_0$ and $b^* = B_0^*$.

Lemma 4.3 *There exists a unitary operator R of $\mathcal{F}$ such that*

$$R^{-1}b^*(f,j)R = a^*(f,j), \quad R^{-1}b(f,j)R = a(\overline{f},j).$$

Proof: See [1]. □

Let

$$S(p) = \exp\left(-i\frac{\alpha}{m_{\text{eff}}}p\cdot\widetilde{\Pi}\left(\frac{\hat{\varphi}}{\omega^{3/2}}\right)\right)$$

and

$$U(p) = S(p)R.$$

Lemma 4.4 *For all $\alpha \in \mathbb{R}$ and all $p \in \mathbb{R}^d$, $U(p)$ maps $D(H_{\text{f}})$ onto itself and*

$$U(p)^{-1}B_p^*(f,j)U(p) = a^*(f,j), \quad U(p)^{-1}B_p(f,j)U(p) = a(\overline{f},j). \tag{4.8}$$

There exists $E(p) = E(p,\alpha^2) \in \mathbb{R}$ such that

$$U(p)^{-1}H_{\text{dip}}(p)U(p) = E(p) + H_{\text{f}}. \tag{4.9}$$

Moreover $E(p,\alpha^2)$ can be analytically continued to a complex domain $\mathcal{O} \supset [0,\infty)$ in α^2.

Proof: As is easily seen $U(p)$ maps $\mathcal{F}_{\rm fin} \cap D(H_{\rm f})$ to $D(H_{\rm f})$. By a limiting argument $U(p)$ maps $D(H_{\rm f})$ onto itself. (4.8) easily follows from Lemma 4.3. By Lemma 4.2, $U(p)^{-1}H_{\rm dip}(p)U(p) = E(p) + H_{\rm f}$ with some real number $E(p)$ and by (4.4) we obtain $W_+^*W_+ = I + W_-^*W_- \geq I$. Therefore W_+^{-1} exists as a bounded operator. For simplicity we set $\lambda_j = e_j\hat{\varphi}/\sqrt{\omega} : \mathbb{R}^d \to \mathbb{C}^d$, and $U(p)\Omega = \Omega(p)$. Since $\Omega(p)$ is the ground state of $H_{\rm dip}(p)$, we have

$$E(p) = (\Omega, H_{\rm dip}(p)\Omega(p))/(\Omega, \Omega(p)).$$

Note that $a(\lambda_j, j)R\Omega = -a^*((W_-W_+^{-1})_{ji}\lambda_i, j)R\Omega$. Then we have

$$\begin{aligned} a(\lambda_j, j)\Omega(p) &= -S(p)\langle a^\dagger|(|a^\dagger\rangle(W_-W_+^{-1})_{ji}\lambda_i, j)R\Omega + \frac{1}{\sqrt{2}}\frac{\alpha}{m_{\rm eff}}\delta_{ij}(p\cdot\lambda_j/\omega, \lambda_i)\Omega(p) \\ &= -a^*((W_-W_+^{-1})_{ji}\lambda_i, j)\Omega(p) + \frac{1}{\sqrt{2}}\frac{\alpha}{m_{\rm eff}}(p\cdot\lambda_j/\omega, (I + W_-W_+^{-1})_{ji}\lambda_i)\Omega(p). \end{aligned}$$

Hence

$$(\Omega, a(\lambda_j, j)\Omega(p)) = \alpha B\cdot p(\Omega, \Omega(p)), \tag{4.10}$$

where $B = (B_{\mu\nu})_{1\leq\mu,\nu\leq d}$ and $B_{\mu\nu} = (\lambda_j^\nu/\omega, (I + W_-W_+^{-1})_{ji}\lambda_i^\mu)/(\sqrt{2}m_{\rm eff})$. We also have

$$\begin{aligned} &(\Omega, a(\lambda_j, j)a(\lambda_i, i)\Omega(p)) \\ &= -(\Omega, a(\lambda_j, j)a^*((W_-W_+^{-1})_{ik}\lambda_k, i)U\Omega) + \alpha B\cdot p(\Omega, a(\lambda_j, j)\Omega(p)) \\ &= -(\lambda_j, (W_-W_+^{-1})_{ji}\lambda_i)(\Omega, \Omega(p)) + \alpha^2(B\cdot p, B\cdot p)(\Omega, \Omega(p)). \end{aligned}$$

Then

$$\begin{aligned} E(p) &= \frac{1}{2m}\left(p^2 - \frac{2\alpha}{\sqrt{2}}\frac{(\Omega, pa(\lambda_j, j)\Omega(p))}{(\Omega, \Omega(p))} + \frac{\alpha^2}{2}\frac{(\Omega, a(\lambda_j, j)a(\lambda_j, j)\Omega(p))}{(\Omega, \Omega(p))} + \frac{\alpha^2}{2}(\lambda_j, \lambda_j)\right) \\ &= \frac{1}{2m}\left(p^2 - \frac{2\alpha^2}{\sqrt{2}}(p, B\cdot p) + \frac{\alpha^4}{2}(B\cdot p, B\cdot p)\right) + \frac{\alpha^2}{4m}(\lambda_j, (I - W_-W_+^{-1})_{ji}\lambda_i). \end{aligned}$$

In [11, Appendix], it is proved that $(f, (W_-W_+^{-1})_{ji}g)$, $f, g \in L^2$, can be analytically continued to a complex domain with respect to α^2. Thus the lemma follows. $\square$

4.2 Effective mass and ground state energy

In the previous subsection it is established that $U(p)^{-1}H_{\rm dip}(p)U(p) = E(p) + H_{\rm f}$ for all $\alpha \in \mathbb{R}$ and that $E(p)$ can be analytically continued to $\mathcal{O}$ in α^2.

Lemma 4.5 *We have, for all $\alpha \in \mathbb{R}$,*

$$E(p) = \frac{p^2}{2m_{\rm eff}} + \alpha^2 g. \tag{4.11}$$

Proof: First we assume that $|\alpha|$ is sufficiently small and that $\omega(k) = |k| + \nu$, $\nu > 0$. Define $\Gamma(\vec{n}) = [2\pi n_1/a, 2\pi(n+1)/a) \times ... \times [2\pi n_d/a, 2\pi(n_d+1)/a)$. For $l = (l_1, \cdots, l_d) \in \mathbb{Z}^d$, let $|l| = \max_j |l_j|$. Let

$$\vec{q} = q_{jl} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{\omega(l)}}\left\{a^*(\chi_{\Gamma(l)}, j) + a(\chi_{\Gamma(l)}, j)\right\},$$

$$\vec{p} = p_{jl} = \frac{i}{\sqrt{2}}\sqrt{\omega(l)}\left\{a^*(\chi_{\Gamma(l)}, j) - a(\chi_{\Gamma(l)}, j)\right\}.$$

Then the Weyl relations hold,

$$e^{itp_{jl}} e^{isq_{j'l'}} = e^{its\delta_{l_1 l_1'}\cdots\delta_{l_d l_d'}\delta_{jj'}} e^{isq_{j'l'}} e^{itp_{jl}}, \quad t, s \in \mathbb{R}. \tag{4.12}$$

Let $D = (d-1)(2L+1)^d$ and $L \in \mathbb{N}$. We define a $D \times D$-matrix by

$$A_0 = (\delta_{jj'}\delta_{ll'}\omega(l)^2)_{1 \le j,j' \le d-1, 0 \le |l|,|l'| \le L}.$$

We denote by $(f,g)_D$ the D-dimensional scalar product. Let $v^\mu_{jl} = \hat{\varphi}(l)e^\mu_j(l)$ and $\vec{v}_\mu = (v^\mu_{jl}) \in \mathbb{R}^D$, $\mu = 1, ..., d$. Define

$$H^\nu_{L,a}(p) = \frac{1}{2m}\sum_{\mu=1}^d \{p_\mu - \alpha(\vec{v}_\mu, \vec{q})_D\}^2 + \frac{1}{2}\{(\vec{p}, \vec{p})_D + (\vec{q}, A_0\vec{q})_D\} - {\rm tr}\sqrt{A_0}.$$

It is known and easily proved that $H^\nu_{L,a}(p) \to H_{\rm dip}(p) + \nu N_{\rm f}$ uniformly in the sense of resolvent as $a \to \infty$ and $L \to \infty$. Define a $D \times D$-matrix by $P = |\vec{v}_\mu\rangle\langle\vec{v}_\mu|$ and set

$$\lambda = \frac{\alpha^2}{m}.$$

Then let us define

$$A = A_0 + \lambda P.$$

Note that A is a strictly positive symmetric matrix, since $|\alpha|$ is sufficiently small and $\nu > 0$. In particular, $(A+a)^{-1}$ exists for $a \ge 0$. Let

$$\vec{f} = \vec{f}(p) = \frac{\alpha}{m}A^{-1}p_\mu\vec{v}_\mu \in \mathbb{R}^D.$$

For linear operator T, let $\langle T\rangle_D = (\vec{v}_\mu, T\vec{v}_\mu)_D$. Then we have

$$H^\nu_{L,a}(p) = \frac{1}{2}(\vec{p},\vec{p})_D + \frac{1}{2}((\vec{q}-\vec{f}), A(\vec{q}-\vec{f}))_D + \frac{1}{2m}p^2 - \frac{1}{2}(\vec{f}, A\vec{f})_D - \frac{1}{2}\mathrm{tr}\sqrt{A_0}.$$

By (4.12) and the von Neumann uniqueness theorem, there exists a unitary operator $\vartheta : \mathcal{F} \to L^2(\mathbb{R}^D)$ implementing

$$\vartheta p_{jl}\vartheta^{-1} = -i\frac{\partial}{\partial x_{jl}}, \quad \vartheta q_{jl}\vartheta^{-1} = x_{jl}.$$

Then $H^\nu_{L,a}(p)$ is unitarily equivalent with a harmonic oscillator in $L^2(\mathbb{R}^D)$. Hence

$$\mathrm{inf}\sigma\Big(\frac{1}{2}(\vec{p},\vec{p})_D + \frac{1}{2}((\vec{q}-\vec{f}), A(\vec{q}-\vec{f}))_D\Big) = \mathrm{inf}\sigma\Big(\frac{1}{2}(\vec{p},\vec{p})_D + \frac{1}{2}(\vec{q}, A\vec{q})_D\Big) = \frac{1}{2}\mathrm{tr}\sqrt{A}.$$

Thus the ground state energy of $H^\nu_{L,a}(p)$ is given by

$$\mathrm{inf}\sigma(H^\nu_{L,a}(p)) = \frac{1}{2m}p^2 - \frac{1}{2}(\vec{f}, A\vec{f})_D + \frac{1}{2}\mathrm{tr}(\sqrt{A} - \sqrt{A_0}).$$

We calculate $(\vec{f}, A\vec{f})_D$ and $\mathrm{tr}(\sqrt{A} - \sqrt{A_0})$ as follows. We have

$$\begin{aligned}(\vec{f}, A\vec{f})_D &= \frac{\lambda}{m}p_\mu p_\nu \sum_{n=1}^\infty ((A_0^{-1}P)^{n-1}A_0^{-1}\vec{v}_\mu, \vec{v}_\nu)_D\,(-\lambda)^{n-1} \\ &= \frac{\lambda}{m}\sum_{n=1}^\infty (-\lambda)^{n-1}\,p^2\theta^n = \frac{\lambda\theta}{1+\lambda\theta}\frac{p^2}{m},\end{aligned}$$

where

$$\theta = \theta_{L,a} = \langle A_0^{-1}\rangle_D = q\sum_{|l|\le L}\frac{|\hat{\varphi}(l)|^2}{\omega(l)^2}.$$

Hence

$$\frac{1}{2m}p^2 - \frac{1}{2}(f, Af)_D = \frac{p^2}{2m}\frac{1}{1+\lambda\theta}.$$

Next we see that

$$\mathrm{tr}\sqrt{A} - \mathrm{tr}\sqrt{A_0} = \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{tr}\left\{A(s^2+A)^{-1} - A_0(s^2+A_0)^{-1}\right\}ds.$$

Let $A_\infty = \sum_{n=1}^\infty \{-\lambda P(s^2+A_0)^{-1}\}^n$. We have

$$A(s^2+A)^{-1} - A_0(s^2+A_0)^{-1} = \lambda P(s^2+A_0)^{-1} + A(s^2+A_0)^{-1}A_\infty.$$

It follows that

$$\mathrm{tr}\lambda P(s^2+A_0)^{-1} = \lambda\langle(s^2+A_0)^{-1}\rangle_D = \lambda\xi, \tag{4.13}$$

where

$$\xi = \xi_{L,a} = q \sum_{|l|\le L} \frac{|\hat{\varphi}(l)|^2}{s^2+\omega(l)^2}.$$

We see that

$$A(s^2+A_0)^{-1}A_\infty = A_0(s^2+A_0)^{-1}A_\infty + \lambda P(s^2+A_0)^{-1}A_\infty.$$

It follows that

$$\mathrm{tr}A_0(s^2+A_0)^{-1}A_\infty = \sum_{n=1}^\infty (-\lambda)^n \langle ((s^2+A_0)^{-1}P)^{n-1}(s^2+A_0)^{-2}A_0\rangle_D$$

$$= \sum_{n=1}^\infty (-\lambda)(-\lambda\xi)^{n-1} \langle (s^2+A_0)^{-2}A_0\rangle_D = -\lambda \frac{\langle (s^2+A_0)^{-2}A_0\rangle_D}{1+\lambda\xi}, \tag{4.14}$$

and

$$\mathrm{tr}\lambda P(s^2+A_0)^{-1}A_\infty = \sum_{n=1}^\infty \lambda(-\lambda\xi)^n \langle (s^2+A_0)^{-1}\rangle_D^{n+1} = d\lambda\frac{-\lambda\xi^2}{1+\lambda\xi}. \tag{4.15}$$

Hence we have by (4.13),(4.14) and (4.15),

$$\mathrm{tr}\left\{A(s^2+A)^{-1} - A_0(s^2+A_0)^{-1}\right\} = \frac{\langle (s^2+A_0)^{-2}A_0\rangle_D}{1+\lambda\xi}(-\lambda) - d\frac{(\lambda\xi)^2}{1+\lambda\xi} + d\lambda\xi$$

$$= \lambda\frac{dq}{1+\lambda\xi}\sum_{|l|\le L}\left\{\frac{|\hat{\varphi}(l)|^2}{(s^2+\omega(l)^2)} - \frac{\omega(l)^2|\hat{\varphi}(l)|^2}{(s^2+\omega(l)^2)^2}\right\} = \frac{dq\lambda s^2}{1+\lambda\xi}\sum_{|l|\le L}\frac{|\hat{\varphi}(l)|^2}{(s^2+\omega(l)^2)^2}.$$

We conclude that

$$\mathrm{inf}\sigma(H^\nu_{L,a}(p)) = \frac{p^2}{2(m+\alpha^2\theta)} + \frac{d-1}{2\pi}\int_{-\infty}^\infty \frac{\alpha^2 s^2}{m+\alpha^2\xi}\sum_{|l|\le L}\frac{|\hat{\varphi}(l)|^2}{(s^2+\omega(l)^2)^2}ds.$$

Taking $a\to\infty$ and $L\to\infty$, we see that $H^\nu_{L,a}(p)$ uniformly converges to $H_{\rm dip}(p)+\nu N_{\rm f}$ in the resolvent sense, which yields that $\mathrm{inf}\sigma(H^\nu_{L,a}(p)) \to \mathrm{inf}\sigma(H_{\rm dip}(p)+\nu N_{\rm f})$. Hence $E(p) = p^2/(2m_{\rm eff})+\alpha^2 g$ for $\nu>0$. Since $H_{\rm dip}(p)+\nu N_{\rm f}\to H_{\rm dip}(p)$ strongly on $D(H_{\rm f})$ and $\liminf_{\nu\to 0}\mathrm{inf}\sigma(H_{\rm dip}(p)+\nu N_{\rm f}) \ge \mathrm{inf}\sigma(H_{\rm dip}(p))$, $E(p) = p^2/(2m_{\rm eff})+\alpha^2 g$ holds for $\nu=0$ by (4.9). $E(p)=E(p,\alpha^2)$ is analytically continued in α^2 to $\mathcal{O}$. On the other hand $p^2/(2m_{\rm eff})+\alpha^2 g$ is clearly analytically continued on a complex domain $\mathcal{O}'\supset[0,\infty)$. Since $\mathcal{O}\cap\mathcal{O}'\supset[0,\infty)$, $E(p)=p^2/(2m_{\rm eff})+\alpha^2 g$ is valid for all $\alpha\in\mathbb{R}$ and we obtain the desired results. □

The following corollary is proved in [1].

Corollary 4.6 *$H_{\rm dip}(p)$ is self-adjoint on $D(H_{\rm f})$ for all $\alpha \in \mathbb{R}$ and $p \in \mathbb{R}^d$, and bounded below. Moreover $U(p)\Omega$ is a ground state of $H_{\rm dip}(p)$ with eigenvalue $\frac{1}{2m_{\rm eff}}p^2 + \alpha^2 g$.*

Proof: The corollary follows from Lemma 4.4 and 4.5. □

4.3 Effective potential and scaling limit

We define a unitary operator on $\mathcal{H} \cong \int_{\mathbb{R}^d}^{\oplus} \mathcal{F} dx$ by

$$U = \left(\int_{\mathbb{R}^d}^{\oplus} S(p) dp \right) R e^{i\frac{\pi}{2}N_{\rm f}}. \tag{4.16}$$

Lemma 4.7 *We define $U(\kappa)$ by the substitution $\omega \to \kappa^2\omega$ and $\hat{\varphi} \to \kappa\hat{\varphi}$. Then*

$$s - \lim_{\kappa\to\infty} U(\kappa) = I \otimes R.$$

Proof: By the definition of R, we see that R is independent of κ, while $S(p)$ is scaled as

$$S(p) \to S(p,\kappa) = \exp\left(-i \frac{1}{\kappa} \frac{\alpha}{m_{\rm eff}} p \cdot \Pi\left(\frac{\hat{\varphi}}{\omega^{3/2}} \right) \right), \quad p \in \mathbb{R}^d.$$

Hence

$$s - \lim_{\kappa\to\infty} S(p,\kappa) = I, \quad p \in \mathbb{R}^d.$$

Thus

$$s - \lim_{\kappa\to\infty} U(p,\kappa) = R, \quad p \in \mathbb{R}^d.$$

The desired result follows from the definition (4.16). □

Proof of Lemma 3.2. We apply [2, Theorem 2.2]. We check

(1) $D(\delta V(k)) \supset D(H_{\rm eff})$ and $\delta V(\kappa)(H_{\rm eff}+\lambda)^{-1}$ is bounded in $\mathcal{H}$ for large $\lambda > 0$ with $\lim_{\lambda\to\infty} \|\delta V(\kappa)(H_{\rm eff}+\lambda)^{-1}\|_{\mathcal{H}} = 0$ uniformly in κ,

(2) $\delta V(\kappa)(H_{\rm eff}+\lambda)^{-1}$ is strongly continuous in κ,

(3) $s - \lim_{\kappa\to\infty} \delta V(\kappa)(H_{\rm eff}+\lambda)^{-1} = 0$.

Thus (1) to (3) imply that

$$s - \lim_{\kappa\to\infty} \left(\widehat{H}(\kappa) - \alpha^2\kappa^2 g + z \right)^{-1} = (H_{\rm eff}+z)^{-1} \otimes P_\Omega.$$

Since, by Lemma 4.7, we have

$$s-\lim_{\kappa\to\infty}\left(H(\kappa)-\alpha^2\kappa^2 g+z\right)^{-1} = s-\lim_{\kappa\to\infty}U(\kappa)\left(\widehat{H}(\kappa)-\alpha^2\kappa^2 g+z\right)^{-1}U(\kappa)^{-1}$$
$$= s-\lim_{\kappa\to\infty}(H_{\mathrm{eff}}+z)^{-1}\otimes(RP_\Omega R^{-1}).$$

Corollary 4.6 tells us that $RP_\Omega R^{-1}=P_{\mathrm{g}}$. Thus the lemma follows. □

Theorem 4.8 *Let V be relatively bounded with respect to $-\Delta$ with a sufficiently small relative bound. Then U maps $D(p^2)\cap D(H_{\mathrm{f}})$ onto itself and*

$$U^{-1}(H_{\mathrm{dip}}+V)U = H_{\mathrm{eff}}+H_{\mathrm{f}}+\alpha^2 g+\delta V. \tag{4.17}$$

Proof: By Lemma 4.4, we have

$$U^{-1}H_{\mathrm{dip}}U = H_{\mathrm{eff}}+H_{\mathrm{f}}+\alpha^2 g \tag{4.18}$$

on a core of the right-hand side above, e.g., $C_0^\infty(\mathbb{R}^d)\otimes_{\mathrm{alg}}[\mathcal{F}_{\mathrm{fin}}\cap D(H_{\mathrm{f}})]$. Since H_{dip} is self-adjoint on $D(p^2)\cap D(H_{\mathrm{f}})$, a limiting argument tells us that U maps $D(p^2)\cap D(H_{\mathrm{f}})$ onto itself and (4.18) is valid on $D(p^2)\cap D(H_{\mathrm{f}})$. Thus the theorem follows. □

Theorem 4.9 *We have $\delta V := T^{-1}VT-V$, where*

$$T=\exp\left(-i\frac{\alpha}{m_{\mathrm{eff}}}pK\right),$$
$$K=\frac{1}{\sqrt{2}}\sum_{j=1}^{d-1}\int\{\Lambda_j(k)a^*(k,j)+\Lambda_j(k)^*a(k,j)\}\,d^dk,$$
$$\Lambda_j=\overline{\omega^{1/2}e_j^\alpha T_{\alpha\beta}d_{\beta\mu}\omega^{-3/2}\hat{\varphi}}. \tag{4.19}$$

Proof: We see that

$$U^{-1}e^{-ikx}U = e^{-ikx}e^{-i\frac{\pi}{2}N_{\mathrm{f}}}R^{-1}\exp\left(-i\frac{\alpha}{m_{\mathrm{eff}}}k\widetilde{\Pi}\left(\frac{\hat{\varphi}}{\omega^{3/2}}\right)\right)Re^{i\frac{\pi}{2}N_{\mathrm{f}}}.$$

Let $\hat{\varphi}/\omega = f$. By (4.6), (4.7), Lemma 4.2, and $\widetilde{f} = \bar{f} = f$, we have

$$
\begin{aligned}
R^{-1}\widetilde{\Pi}_\mu \left(\frac{\hat{\varphi}}{\omega^{3/2}}\right) R &= \frac{i}{\sqrt{2}} \sum_{j=1}^{d-1} R^{-1} \left\{ a^*(e_j^\mu \widetilde{f}, j) - a(e_j^\mu f, j) \right\} R \\
&= \frac{i}{\sqrt{2}} \sum_{j=1}^{d-1} R^{-1} \left\{ b^*(\overline{W}^*_{+ij} e_j^\mu \widetilde{f}, i) - b(W^*_{-ij} e_j^\mu \widetilde{f}, i) \right. \\
&\qquad \left. + b^*(\overline{W}^*_{-ij} e_j^\mu \bar{f}, i) - b(W^*_{+ij} e_j^\mu \bar{f}, i) \right\} R \\
&= \frac{i}{\sqrt{2}} \sum_{j=1}^{d-1} \left\{ a^*(\overline{W}^*_{+ij} e_j^\mu f + \overline{W}^*_{-ij} e_j^\mu f, i) - a(\overline{W^*_{-ij} e_j^\mu f + W^*_{+ij} e_j^\mu f}, j) \right\} \\
&= \frac{i}{\sqrt{2}} \sum_{j=1}^{d-1} \left\{ a^*(\overline{\omega^{1/2} e_i^\alpha T_{\alpha\beta} d_{\beta\mu} \omega^{-1/2} f}, i) - a(\overline{\omega^{1/2} e_i^\alpha T_{\alpha\beta} d_{\beta\mu} \omega^{-1/2} f}, i) \right\}.
\end{aligned}
$$

Since

$$
e^{-i\frac{\pi}{2}N_{\mathrm{f}}} i \left\{ a^*(g) - a(g) \right\} e^{i\frac{\pi}{2}N_{\mathrm{f}}} = a^*(g) + a(g),
$$

we have

$$
e^{-i\frac{\pi}{2}N_{\mathrm{f}}} R^{-1} \widetilde{\Pi}_\mu \left(\frac{\hat{\varphi}}{\omega^{3/2}}\right) R e^{i\frac{\pi}{2}N_{\mathrm{f}}} = K_\mu. \tag{4.20}
$$

Hence

$$
U^{-1} e^{-ikx} U = e^{-ikx} e^{-ik\alpha K/m_{\mathrm{eff}}} = T^{-1} e^{-ikx} T.
$$

Let ρ and V_ϵ be in the proof of Lemma 2.6. We see that

$$
U^{-1}\rho U = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \check{\rho}(k) U^{-1} e^{-ikx} U d^d k = \int_{\mathbb{R}^d} \check{\rho}(k) T^{-1} e^{-ikx} T d^d k = T^{-1}\rho T.
$$

Thus, for $\Psi \in C_0^\infty(\mathbb{R}^d) \otimes_{\mathrm{alg}} [\mathcal{F}_{\mathrm{fin}} \cap D(H_{\mathrm{f}})]$,

$$
U^{-1} V_\epsilon U \Psi = T^{-1} V_\epsilon T \Psi.
$$

Hence (4.17) holds on $C_0^\infty(\mathbb{R}^d) \otimes_{\mathrm{alg}} [\mathcal{F}_{\mathrm{fin}} \cap D(H_{\mathrm{f}})]$. By a limiting argument, we obtain (4.17) on $D(p^2) \cap D(H_{\mathrm{f}})$. $\square$

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