PERTURBATIONS OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS PRESERVING THE ABSOLUTELY CONTINUOUS SPECTRUM

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ABSTRACT. The stability of the absolutely continuous spectrum of one-dimensional Schrödinger operators,

$$[Hu](x) = -u''(x) + q(x)u(x),$$

under perturbations of the potential is discussed. The focus is on demonstrating this stability under minimal assumptions on how fast the perturbation decays at infinity.

A general technique is presented together with sample applications. These include the following: for an operator with a periodic potential, any perturbation $V \in L^2$ preserves the a.c. spectrum. For the Stark operator, the same is true for pertubations with $\int |V(t^2)|^2\,dt < \infty$. Both of these results are known to be optimal, in the sense that the integrability index cannot be increased.
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1. Introduction

The main object we shall study, is the Schrödinger operator on $\mathbf{R}^+ = [0,\infty)$ with a Dirichlet boundary condition at the origin:

$$[Hu](x) = -u''(x) + q(x)u(x) \quad \text{with} \quad u(0) = 0. \tag{1}$$

Except in Section 6, where $q(x) = -x$, we consider only potentials $q \in L^2_{\text{loc}}$ for which

$$\sup_n \int_n^{n+1} |q_-(x)|^2\,dx < \infty. \tag{2}$$

Here $q_-(x) = \min\{0, q(x)\}$. For the questions we address, operators with other choices of boundary condition or on the whole line can be reduced to (1). This is explained near the end of this section.

Much is known about the spectral analysis of Sturm-Liouville operators [7, 40]. For example, there is a measure $d\mu$ on $\mathbf{R}$ so that the operator H is unitarily equivalent to

$$f(E) \mapsto Ef(E) \text{ acting in } L^2(\mathbf{R}; d\mu).$$

This measure, known as the spectral measure, expresses all the spectral properties of H. In particular, the the absolutely continuous spectrum of H, $\sigma_{\text{ac}}(H)$, is equal to the support of the absolutely continuous part of $d\mu$, $\operatorname{supp}(d\mu_{\text{ac}})$. The measure $d\mu$ also permits a simple definition of the essential support of the absolutely continuous spectrum of H, $\Sigma_{\text{ac}}(H)$. It is the unique, up to sets of Lebesgue measure zero, set Σ such that $\chi_\Sigma(E)dE$ and $d\mu_{\text{ac}}$ are mutually absolutely continuous.

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The problem we wish to discuss is the following: which perturbations of the potential, $q(x) \mapsto q(x) + V(x)$, preserve $\Sigma_{\mathrm{ac}}(H)$? More generally, which perturbations preserve some subset of $\Sigma_{\mathrm{ac}}(H)$? This question has generated considerable activity recently. Attracting most attention, is the question of how slowly V may decay at infinity, if no smoothness assumptions are imposed.

The stimulus for these studies was a dichotomy that arose in the study of sparse and random decaying potentials. Let H_0 denote the operator H with $q \equiv 0$ then, H_0 has purely absolutely continuous spectrum with $\sigma_{\mathrm{ac}} = \Sigma_{\mathrm{ac}} = [0, \infty)$. A number of authors (e.g. [22, 23, 29, 30]) have studied the spectral properties of $H_0 + V$ when V decays at infinity and is either sparse or random. It was noticed that as the decay rate was reduced, the absolutely continuous spectrum disappeared and was replaced by point or singular continuous spectrum. But in each case, the transition point was the same. Namely, for $V \in L^2$, the a.c. spectrum remained, $\Sigma_{\mathrm{ac}} = [0, \infty)$. While for $V \notin L^2$, it dissolved completely, $\Sigma_{\mathrm{ac}} = \emptyset$. Building on these results, Simon [36] showed that if $p > 2$, then $\sigma_{\mathrm{ac}}(H_0 + V) = \emptyset$ for a dense G_δ of $V \in L^p$. A similar ℓ^2-dichotomy was observed for the descretized operator

$$[hu](n) = u(n+1) + u(n-1) + q(n)u(n) \quad \text{with} \quad u(0) = 0 \tag{3}$$

acting in $\ell^2(\mathbf{N})$. The relevant publications are [10, 11, 35, 36].

Naturally, this phenomenon led to the following statement, originally conjectured by Kiselev, Last and Simon [22] and later proved by Deift and Killip:

Theorem 1. [8] *If $V \in L^2$ then $\Sigma_{\mathrm{ac}}(H_0 + V) = \Sigma_{\mathrm{ac}}(H_0) = [0, \infty)$.*

From the results described earlier, this is optimal in the sense that L^2 cannot be replaced by L^p for any $p > 2$. One should also note that the spectrum may be far from purely absolutely continuous. For any function $f(x)$ with $\lim_{x\to\infty} f(x) = \infty$, there are examples of potentials $V(x) \le f(x)/x$ (for large x) which have point spectrum dense in the interval $[0, \infty)$ [27, 38]. For potentials with power-law decay, Remling [32] has given a bound on the possible Hausdorff dimension of any embedded singular continuous spectrum. However, the question of whether such singular continuous spectrum may exist is still open.

The initial work in the direction of Theorem 1, [2, 3, 19, 20, 21, 26, 31], employed the following potent theorem from subordinacy theory [14, 16, 37]: suppose that for almost every energy E, in a Borel set S, all solutions of

$$-u''(x) + V(x)u(x) = Eu(x) \tag{4}$$

are uniformly bounded, then $S \subseteq \Sigma_{\mathrm{ac}}(H_0 + V)$. Showing that solutions of (4) are bounded is a difficult problem. By employing Prüfer variables (Remling), or the Harris–Lutz transformation (Christ–Kiselev), and a careful analysis of the resulting (multi-dimensional) Volterra operators a weakener version of Theorem 1 was proved. Specifically, if there exists $\epsilon > 0$ such that $x^\epsilon V(x) \in L^2$ then $\Sigma_{\mathrm{ac}}(H_0 + V) = [0, \infty)$. Quite recently, Christ and Kiselev extended their method to treat $V \in L^p$, $p < 2$ [4] and announced a proof of the existence of wave operators in a related setting [6].

While it may seem natural to prove the existence of absolutely continuous spectrum directly by showing the existence of modified wave operators, no one has succeeded in doing so under such weak assumptions. The strongest known results on the existence of wave operators require that V or one of it's derivatives is at least integrable [15]. In a similar vein, is the following result of Weidmann [41]: If

V is the sum of two terms, one integrable and the other tending to zero at infinity and of bounded variation, then $\Sigma_{\mathrm{ac}}(H_0 + V) = [0, \infty)$. Extending this result, Stolz showed that if q is periodic then $\Sigma_{\mathrm{ac}}(H + V) = \Sigma_{\mathrm{ac}}(H)$ for such V [39].

The method of Christ and Kiselev is quite robust and can be used to prove results for perturbations of H when $q \not\equiv 0$. For example:

Theorem 2. [2] *Let S be a Borel subset of $\mathbf{R}$ and let V be a potential which obeys $x^{\epsilon}V(x) \in L^2$ for some $\epsilon > 0$. Suppose that for each $E \in S$, there is a solution, $\theta(x, E)$, to*

$$-u''(x) + q(x)u(x) = Eu(x) \tag{5}$$

for which

i) *$\theta(x; E)$ is a uniformly bounded function of x;*
ii) *$\theta(x; E)$ and its complex conjugate, $\bar{\theta}(x; E)$, are linearly independent; and*
iii) *the integral operator*

$$[Tf](E) = \int_0^{\infty} \exp\left\{\tfrac{i}{\operatorname{Im}\theta\bar{\theta}'} \int_0^x V(t)\,dt\right\}\theta(x, E)^2 f(x)\,dx \tag{6}$$

is a bounded mapping $L^2(\mathbf{R}^+; dx) \to L^2(S; dE)$.

Then for Lebesgue almost every $E \in S$, all solutions of

$$-u'' + qu + Vu = Eu \tag{7}$$

are bounded. By subordinacy theory, this implies $S \subseteq \Sigma_{\mathrm{ac}}(H + V)$.

Remarks. 1. Notice that conditions i) and ii) imply $S \subseteq \Sigma_{\mathrm{ac}}(H)$, by the subordinacy theorem.
2. The methods from [4], suggest a similar result for $V \in L^p$, when $p < 2$.
3. The hypotheses of this theorem have been verified for background potentials, q, which are periodic [2].

Our first result is an analogue of this theorem for $V \in L^2$. Or, to say it differently, an analogue of Theorem 1 in the presence of a background potential q. Before we state this theorem, we wish to recall a definition.

For each $z \in \mathbf{C} \setminus \mathbf{R}$, the resolvent $R(z) = (H - z)^{-1}$ can be represented as an integral operator

$$[R(z)f](x) = \int G(x, y; z) f(y)\,dy.$$

The kernel of this operator, G, is known as the Green function and for each x, y it is an analytic function of $z \in \mathbf{R} \setminus \mathbf{C}$.

Theorem 3. *Suppose $q \in L^2_{\mathrm{loc}}$ obeys (2), $V \in L^2(\mathbf{R}^+)$ and let I be a closed interval contained in $\Sigma_{\mathrm{ac}}(H)$. If*

i) *for all $x, y \in \mathbf{R}^+$, $G(x, y; z)$ extends continuously from $z \in \mathbf{C}^+$ to $z \in I$; and*
ii) *for some $r \in \mathbf{N}$ and all functions $w \in C^r(\mathbf{R})$, supported by I, the integral*

$$\int_I w(E)\,\operatorname{Re}\bigl[G(x, x; E)\bigr]\,dE \tag{8}$$

defines a function in $L^2(\mathbf{R}^+; dx)$;

then $I \subseteq \Sigma_{\mathrm{ac}}(H + V)$.

Remarks. 1. Assumption i) is equivalent to the condition that $G(x, y; z)$ extends continuously for *some* pair $x, y \in \mathbf{R}^+$.

2. For assumption ii), it is sufficient that (8) defines a function in $L^2(\mathbf{R}^+;dx)$ for a very specific family of functions w. The functions we will use originate from the harmonic measure of tents sitting over I.
3. The absence of the exponentiated integral, makes (8) easier to control than (6).
4. The assumptions on q are not intended to be optimal. However a trace ideal estimate we use (22) is only known in slightly more generality ($q \in K^1_{\mathrm{loc}}$ and $q_- \in K^1$ see [34]). In particular, it is not known for general classes when H is unbounded below. We will treat one such example, $q(x) = -x$, by ad hoc methods.

Hypothesis ii) above, is quite close to the third hypothesis of Theorem 2. Compare $\int [Tf](E)w(E)dE$ with (8) noting that $\operatorname{Re} G(x,x;E)$ is the product of two solutions of (5). Note also that there is no real freedom in the choice for the solution $\theta(x,E)$ in Theorem 2. For example, in the free case ($q \equiv 0$), any choice for θ, other that multiples of $\exp(\pm ix\sqrt{E})$, makes T unbounded. Similarly, while (8) holds in all the examples examined later, it does not if one considers the imaginary part rather than the real part. This is because $\operatorname{Im} G(x,y;z)$ is the kernel of a positive operator. We would hazard to conjecture that for energies E in the interior of $\Sigma_{\mathrm{ac}}(H)$, $\operatorname{Re} G(x,x;E)$ is an oscillatory function of x. If true, this would suggest that hypothesis ii) above may be redundant.

The technique we will employ to prove Theorem 3 is completely different from that of Christ and Kiselev. By their own admission [4], extending their methods to $V \in L^2(\mathbf{R}^+)$ would be extremely difficult.

As discussed above, Theorem 2 was proved by showing that for almost every energy $E \in S$, all solutions of (7) are uniformly bounded (in x). The secret to [8], expressed more directly by Molchanov, Novitskii and Vainberg [25], is the following: it suffices to control the average (in energy) behaviour of solutions at each x. Moreover, this is equivalent to the control of a scattering theoretic quantity.

In both [8] and [25] this is done using the the Buslaev–Faddeev–Zakharov trace formulae [28]. These formulae express the first integrals of the KdV equation in terms of scattering data, specifically, the transmission co-efficient and the bound-state eigenvalues. In this way, the first integrals for KdV can be used to bound the transmission co-efficient. This in turn, can be used to control the behaviour of solutions [25], or may be used directly to obtain the necessary spectral information [8]. Examples of first integrals are $\int V$, $\int V^2$, $\int [V']^2 + 2V^3$. The second of these was used to prove Theorem 1. Molchanov, Novitskii and Vainberg used the higher order formulae to prove that the condition $V \in L^2$ in this theorem may be replaced by $V \in L^{n+1}$ and $V^{(n-1)} \in L^2$ for some integer $n \geq 1$.

Recent work by Denisov [12] has shown that if $q \equiv 0$ and the integral of V, $\int_x^\infty V(t)dt$, is square integrable then the absolutely continuous spectrum is preserved. This result rests on work of Krein with intimate connections to the forward and inverse spectral theory of the Dirac system. As this suggests, the result may also be derived by applying the methods of [8] to the non-linear Schrödinger equation.

Unfortunately, the trace formula method is restricted to the case when there is no background potential, $q \equiv 0$, and so does not provide a method for proving Theorem 3. As we shall also see, the trace formula method does not always give the best results: While the L^2 is optimal in the L^p setting, for $V \in L^3$ (or L^4) the condition $V' \in L^2$ (respectively $V'' \in L^2$) of [25] can be improved. See Corollaries 4.3 and 4.5.

A third, though perhaps purely aesthetic, fault the trace formula method is the necessity of using whole-line operators to study a half-line problem. (For results on whole-line operators, one must then glue together results on each half-line.)

In order to state the main theorem, let us introduce the following notation:

$$K(x_1, \dots, x_n; z) = \operatorname{Re} \prod_{j=1}^{n} G(x_j, x_{j+1}; z) \tag{9}$$

with the identification $x_{n+1} = x_1$.

Theorem 4. *Suppose $q \in L^2_{\rm loc}$ obeys (2) and $I \subseteq \Sigma_{\rm ac}(H)$ is a compact interval. If there is an integer $p \geq 2$ so that*

i) *$V \in L^p(\mathbf{R}^+)$;*

ii) *for any $x, y \in \mathbf{R}^+$, $G(x, y; z)$ extends continuously from $z \in \mathbf{C}^+$ to $z \in I$; and*

iii) *there is a positive integer, r, and a sequence, V_n, of compactly supported functions, converging to V in L^p, such that*

$$\sup_n \left| \int_{[0,\infty)^\ell} K(x_1, \dots, x_\ell; E + i0) V_n(x_1) \cdots V_n(x_\ell)\, dx_1 \dots dx_\ell\, w(E)\, dE \right| < \infty$$

for all integers $1 \leq \ell \leq (p-1)$ and all functions $w \in C^r(\mathbf{R})$ which are supported by I;

then $I \subseteq \Sigma_{\rm ac}(H + V)$.

Theorem 3 is a direct consequence of the above with $p = 2$. This is because $K(x; E + i0) = \operatorname{Re} G(x, x; E)$.

Here are some sample applications. The details and further applications are given in Sections 4 and 5.

Corollary 4.3. *Suppose $q \equiv 0$ and $V \in L^3(\mathbf{R}^+)$. If V can be written as a finite sum of functions each of which belongs to a homogeneous Sobolev space $\dot{\mathsf{H}}^s$ ($s \in \mathbf{R}$) then $\Sigma_{\rm ac}(H + V) = [0, \infty)$.*

As mentioned earlier, this improves upon the results obtained by the trace formula method. A still stronger statement is given as Corollary 4.2. Also improving on [25] is

Corollary 4.5. *Suppose $q \equiv 0$. If for some integer, $N \geq 0$, V can be written as $V(x) = \sum_{\nu=-1}^{N} U_\nu(x)$ such that $U_\nu \in L^4$ and $U_\nu^{(\nu)} \in L^2$ then $\Sigma_{\rm ac}(H + V) = [0, \infty)$. For $\nu \geq 0$, $U^{(\nu)}$ denotes the $\nu^{\rm th}$ derivative while $U^{(-1)}(x) = \int_x^\infty U(t)\, dt$.*

We consider three examples of non-zero background potentials. For q periodic, we not only extend the result of [2] to $V \in L^2$ but also obtain a result for $V \in L^3$:

Corollary 4.8. *Suppose $q \in L^2_{\rm loc}$ is periodic with period one. If $V \in L^2$, or both $V \in L^3$ and $\hat{V} \in \ell^\infty(L^2)$, then $\Sigma_{\rm ac}(H + V) = \sigma_{\rm ac}(H)$.*

We also treat a case where q is slowly varying:

Corollary 4.7. *If $q(x) \to 0$ as $x \to \infty$, $\int_x^\infty |q'(t)|^2 dt \in L^2(dx)$ and $\int_x^\infty |q''(t)| dt \in L^2(dx)$ then for each $V \in L^2$, $\Sigma_{\rm ac}(H + V) = [0, \infty)$.*

Corollaries 4.3 and 4.7 should be compared to a recent result of Christ and Kiselev [5]: Suppose $q \equiv 0$ and $V = V_0 + V_1$ where for some $p < 2$, $V_0 \in L^p$, $V_1^{(n)} \in L^p$ and $V_1(x) \to 0$ at infinity, then $\Sigma_{\rm ac}(H + V) = [0, \infty)$.

Our third example is the Stark operator, $q(x) = -x$. This is not within the scope of Theorem 4. Unable to treat any suitably broad class of potentials, we develop an analogue of Theorem 4 for this specific example: Theorem 5 in Section 5. In a recent pre-print, Christ and Kiselev showed that the absolutely continuous spectrum is preserved for perturbations which obey $|V(x)| \lesssim (1+|x|)^{-\epsilon-1/4}$ for some $\epsilon > 0$. They further prove that there are perturbations $\lesssim (1+|x|)^{-1/4}$ under which there is no absolutely continuous spectrum. The following therefore, is optimal in the sense that $\int |V(t^2)|^2\,dt$ cannot be replaced by $\int |V(t^2)|^p\,dt$ with any $p > 2$.

Corollary 5.2. *If* $\int |V(t^2)|^2\,dt < \infty$ *then* $\Sigma_{\mathrm{ac}}(-\frac{d^2}{dx^2} - x + V(x)) = \mathbf{R}$.

As mentioned at the beginning of this Introduction, the results given above can easily be extended to whole-line operators, or to operators with other boundary conditions at the origin. The reasoning is as follows: A change of boundary condition is a relatively trace-class perturbation. The Birman–Kuroda Theorem thus implies that the absolutely continuous parts of the two operators are unitarily equivalent. That is, the essential support of the absolutely continuous spectrum is unchanged.

In the case of a whole-line operator, we can enforce a Dirichlet boundary condition at the origin. This is also a relatively trace class perturbation and it decomposes the whole-line operator into the direct sum of two half-line operators. Not only does the Birman–Kuroda Theorem then show that the essential support of the absolutely continuous spectrum is the union of those of the direct summands; it also shows that on their intersection, the absolutely continuous spectrum of whole-line operator has multiplicity two. For a further discussion of the use of Dirichlet decoupling, particularly in higher dimensions, see [9].

Much of the material presented here appeared in a some-what different form in the author's PhD thesis [18].

The remainder of this article is arranged as follows: in the next section we discuss the Weyl theory of one-dimensional Schrödinger operators and relate it to the perturbation determinant or Jost function. Section 3 ties this together with the theory of regularized determinants to prove Theorem 4. Sections 4 and 5 give sample applications.

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2. Weyl theory and the Perturbation Determinant

As in the Introduction, we define H by

$$[Hu](x) = -u''(x) + q(x)u(x) \quad \text{with} \quad u(0) = 0. \tag{10}$$

The material presented in this section requires very little of q. We will assume that $q \in L^2_{\mathrm{loc}}$ and that (10) is limit point at infinity. This ensures that H is essentially self-adjoint. For (10) to be limit point at infinity, it is sufficient that $q \in L^2_{\mathrm{loc}}$ obey (2) or $q(x) \geq -C - Cx^2$ for some $C > 0$.

For each $z \in \mathbf{C}$ we let $\theta(x;z), \phi(x;z)$ denote the solutions of

$$-u'' + qu = zu \tag{11}$$

which obey $\theta(0;z) = \phi'(0;z) = 1$, $\theta'(0;z) = \phi(0;z) = 0$. Here u' denotes the derivative with respect to x. For each $z \in \mathbf{C} \setminus \mathbf{R}$ there is a unique solution $\psi(x;z)$

of (11) which obeys

$$\int_0^\infty |\psi(x;z)|^2\,dx < \infty \quad\text{and}\quad \psi(0;z) = 1. \tag{12}$$

This solution is known as the Weyl solution. The Weyl m-function is defined by $\psi(x;z) = \theta(x;z) + m(z)\phi(x;z)$ and is an analytic function of $z \in \mathbf{C}\setminus\mathbf{R}$. In fact, $m(z)$ admits a unique representation

$$m(z) = B + \int \Big[\frac{1}{\omega - z} - \frac{\omega}{1+\omega^2}\Big] d\mu(\omega) \qquad \forall z \in \mathbf{C}\setminus\mathbf{R} \tag{13}$$

where $B \in \mathbf{R}$ and $d\mu$ is a positive measure with $\int (1+\omega^2)^{-1} d\mu$ finite.

Given $m(z)$ one may recover $d\mu$ as weak limit:

$$\int f(E)d\mu(E) = \lim_{\epsilon\downarrow 0} \tfrac{1}{\pi}\int f(E)\,\operatorname{Im} m(E+i\epsilon)\,dE$$

for any continuous function f of compact support. Moreover, if we write $m(E+i0)$ for $\lim_{\epsilon\downarrow 0} m(E+i\epsilon)$ then $m(E+i0)$ exists for Lebesgue-a.e. $E\in\mathbf{R}$ and $\operatorname{Im} m(E+i0) = \pi\frac{d\mu}{dE}$. The measure, $d\mu$, is called the spectral measure for H and, as mentioned in the Introduction, H is unitarily equivalent to $f(E)\mapsto Ef(E)$ acting in $L^2(d\mu)$. This implies $\Sigma_{\mathrm{ac}}(H) = \{E : \operatorname{Im} m(E+i0) > 0\}$.

Given a perturbation V, one may similarly write down a Weyl theory for the operator $H+V$. Actually, in this section, we will not deal with the true perturbation, but rather, obtain some *a priori* information. To this end, let V_n be a sequence of square integrable functions of compact support and let m_n denote the Weyl m-function for the operator $H+V_n$.

The resolvent of H, $R(z) = (H-z)^{-1}$, can be represented as an integral operator whose kernel is the Green function:

$$G(x,x';z) = \phi(x^<;z)\psi(x^>;z) \tag{14}$$

where $x^> = \max\{x,x'\}$ and $x^< = \min\{x,x'\}$. As V_n has compact support, the operator $R(z)V_n$ is trace class for any $z\in\mathbf{C}\setminus\mathbf{R}$. In this way,

$$a_n(z) = \det\big[1 + R(z)V_n\big] \tag{15}$$

defines an analytic function on $\mathbf{C}\setminus\mathbf{R}$, known as the perturbation determinant. For such z, $a_n(z)$ is non-zero: if $a_n(z) = 0$ then $R(z)(H+V_n-z) = 1 + R(z)V_n$ is not invertible which means $z\in\sigma(H+V_n)$. This is impossible because $\sigma(H+V_n)\subseteq\mathbf{R}$.

Lemma 2.1. *Suppose the Green function, G, extends continuously from $\mathbf{C}^+$ to a compact set $I\subseteq\mathbf{R}$ then $a_n(z)$ also extends continuously to I. Moreover, if* $\operatorname{Im} m(E+i0)$ *is non-zero on I then* $\operatorname{Im} m_n(E+i0)$ *is also non-zero and*

$$\big|a_n(E+i0)\big|^{-2} = \frac{\operatorname{Im}[m_n(E+i0)]}{\operatorname{Im}[m(E+i0)]} \qquad \forall E\in I. \tag{16}$$

Proof. Because the Green function $G(x,y;z)$ extends continuously from $\mathbf{C}^+$ to I, so does the Weyl solution $\psi(x;z)$. As V_n is square integrable and of compact support, one may solve the Volterra equation

$$u_n(x;z) = \psi(x;z) + \int_x^\infty \big[\theta(x;z)\phi(y;z) - \phi(x;z)\theta(y;z)\big] V_n(y)u_n(y;z)\,dy$$

for any $z\in\mathbf{C}^+\cup I$. Moreover, $u_n(x;z)$ depends continuously on z, solves $-u_n'' + qu_n + Vu_n = zu_n$ and $u_n(x;z) = \psi(x;z)$ for all $x > L$, $L = \sup[\operatorname{supp} V_n]$.

The function u_n is known as the Jost solution and $u(0;z)$ as the Jost function. The latter is equal to the perturbation determinant: $u_n(0;z) = a_n(z)$ [17]. The stated properties of u_n prove that a_n extends continuously from $\mathbf{C}^+$ to I. In addition, the limiting value $u(0;E+i0)$ cannot be zero because $\operatorname{Im} m(E+i0) \neq 0$. Therefore a_n is continuous and non-zero on $\mathbf{C}^+ \cup I$.

Notice that $u_n(x;z)/a_n(z)$ is the Weyl solution for $H+V_n$—it is square integrable for $z \in \mathbf{C}^+$ and $u_n(0;z) = a_n(z)$. To prove (16), recall that

$$\operatorname{Im} m(E+i\epsilon) = \epsilon \int_0^\infty \big|\psi(x;E+i\epsilon)\big|^2 dx$$

and similarly for m-function and Weyl solution for $H+V_n$. Namely m_n and $u_n(x;z)/a_n(z)$. Thus for $E \in I$,

$$\begin{aligned}\operatorname{Im} m(E+i0) &= \lim_{\epsilon\to 0} \epsilon \int_0^\infty \big|\psi(x;E+i\epsilon)\big|^2 dx \\ &= \lim_{\epsilon\to 0} \epsilon \int_0^\infty \big|u_n(x;E+i\epsilon)\big|^2 dx \\ &= \lim_{\epsilon\to 0} |a_n(E+i\epsilon)|^2 \operatorname{Im} m_n(E+i\epsilon). \end{aligned} \tag{17}$$

While $\psi(x;z) = u_n(x;z)$ only for $x \geq L$, the continuity of ψ, u (as functions of ϵ) shows that $\int |u_n|^2 - |\psi|^2 dx$ is uniformly bounded as $\epsilon \to 0$.

As $a_n(z)$ is known to be continuous, (17) implies that $\operatorname{Im} m_n$ extends continuously to I and that (16) holds there. □

Remark. The equation (16) holds under much less stringent conditions—it is just an expression of the fact that wave operators are isometries.

Now we come to the main result of this section.

Proposition 2.2. *Suppose $V \in L^p(\mathbf{R}^+)$ (for any $1 \leq p \leq \infty$) and V_n is a sequence of square integrable functions of compact support which converge to V in L^p. Let I be a compact subset of $\mathbf{R}$ and let $w: I \to \mathbf{R}$ be continuous and almost everywhere positive. If the Green function extends continuously from $\mathbf{C}^+$ to I,* $\operatorname{Im} m(E+i0) > 0$ *for all $E \in I$ and there is a constant C such that*

$$\int_I \log\big|a_n(E+i0)\big| w(E)dE \leq C \tag{18}$$

for all n, then $I \subseteq \Sigma_{\mathrm{ac}}(H+V)$.

Proof. Let $n(z)$ denote the Weyl function for $H+V$ and let $d\nu$ denote the corresponding spectral measure. As $V_n \to V$ in L^p, $m_n(z) \to n(z)$ uniformly on compact subsets of $\mathbf{C}^+$. This implies that $d\mu_n$, the spectral measures for $H+V_n$, converge weakly to $d\nu$. In particular, $\mu_n(I)$ is uniformly bounded.

Let us write $\log^\pm|z| = \max\{0, \pm\log|z|\}$ so that $\log|z| = \log^+|z| - \log^-|z|$. As I is compact, the continuous function, $\log[\operatorname{Im} m(E+i0)]$, is uniformly bounded from both below and above. Employing the lower bound,

$$\begin{aligned}\int_I \log^- \Big|\frac{\operatorname{Im} m(E+i0)}{\operatorname{Im} m_n(E+i0)}\Big| w(E)dE &\leq c + \int_I \log^+ \big|\operatorname{Im} m_n(E+i0)\big| w(E)dE \\ &\leq c + \int_I \operatorname{Im} m_n(E+i0)\, w(E)dE \\ &\leq c + \int_I w(E)d\mu_n(E) \\ &\leq C'\end{aligned}$$

independently of n. Here we have used $\log^+|z| \leq |z|$, $\operatorname{Im} m_n = \frac{d\mu_n}{dE}$, $w \in L^\infty$ and the fact that $\mu_n(I)$ is uniformly bounded.

Using the above estimate, (18) implies

$$\int_I \log^+ \Big|\frac{\operatorname{Im} m(E+i0)}{\operatorname{Im} m_n(E+i0)}\Big| w(E)dE \leq C + C'$$

and so, as $\log[\operatorname{Im} m(E+i0)]$ is bounded above on I, there is a constant C'' such that

$$\int_I \log^- \big|\operatorname{Im} m_n(E+i0)\big| w(E)dE \leq C''.$$

Employing Jensen's inequality and $\operatorname{Im} m_n = \frac{d\mu_n}{dE}$, we have

$$\int_K w(E)d\mu_n(E) \geq w(K)\exp\big\{-C''/w(K)\big\} \tag{19}$$

for any compact $K \subset I$ of positive Lebesgue measure. Here $w(K) = \int_K w(E)dE$. The measure $w(E)d\nu(E)$ is the weak limit of $w(E)d\mu_n(E)$ and so

$$\int_K w(E)d\nu(E) \geq \limsup_{n\to\infty} \int_K w(E)d\mu(E)$$

follows from the inner regularity of Borel measures (c.f. Lemma 2 of [8]). This combined with (19), shows that $\nu(K) > 0$ for all compact $K \subseteq I$ of positive Lebesgue measure. Hence $I \subseteq \text{ess-supp}(d\nu) = \Sigma_{\rm ac}(H+V)$. □

In Section 4 it will be helpful to have an expression for the Green function other than that given in (14).

Lemma 2.3. *Suppose the Weyl solution, $\psi(x;z)$, extends continuously from $z \in \mathbf{C}^+$ to $E \in \mathbf{R}$. If $\psi(\,\cdot\,;E)$ and $\bar\psi(\,\cdot\,;E)$ are linearly independent then the Green function also has a continuous extension to E. Moreover, it is given by*

$$\begin{aligned} G(x,y) &= \frac{\big[\psi(x)-\bar\psi(x)\big]\psi(y)}{W[\bar\psi,\psi]} \qquad \forall\ x\leq y, \\ 2\operatorname{Re} G(x,x) &= \frac{\psi(x)^2-\bar\psi(x)^2}{W[\bar\psi,\psi]}\end{aligned}$$

Here $W[\bar\psi,\psi](x) = \bar\psi(x)\psi'(x) - \psi(x)\bar\psi'(x)$ is the Wronskian of $\bar\psi(x;E)$ and $\psi(x;E)$ and so is independent of x. By homogeneity, the same formula holds if ψ is merely a multiple of the Weyl solution.

Proof. This is a purely computational consequence of $\phi(x;z)\operatorname{Im} m(z) = \operatorname{Im}\psi(x;z)$ and $2i\operatorname{Im} m(z) = \bar\psi(0)\psi'(0) - \psi(0)\bar\psi'(0)$. □

3. Regularized determinants

Fredholm determinants are only defined for trace class operators, however it is possible to extend them to higher trace ideals, $\mathfrak{I}_p = \{A : \operatorname{tr}(|A|^p) < \infty\}$, by adding convergence factors. Specifically, for each $p \in \mathbf{N}$ we write

$$\det_p(1+A) = \det(1+A)\,\exp\Big\{\sum_{k=1}^{p-1} \tfrac{(-1)^k}{k}\operatorname{tr}\big(A^k\big)\Big\}. \tag{20}$$

Whilst this definition only works for elements of trace class, $A \in \mathfrak{I}_1$, the resulting function does extend continuously to $\mathfrak{I}_p$. Indeed one has

$$\big|\det_p(1+A)\big| \leq \exp\{C_p\|A\|_p^p\} \qquad \forall p \in \mathbf{N}, A \in \mathfrak{I}_p \tag{21}$$

where the C_p are universal constants and $\|A\|_p^p = \operatorname{tr}(|A|^p)$ denotes the pth power of the norm on $\mathfrak{I}_p$. For more on regularized determinants, see Section 9 of [33].

Lemma 3.1. *Suppose $q \in L^2_{\mathrm{loc}}$ obeys (2) and V is of compact support. For each integer $p \geq 2$ there is a constant C so that*

$$\big\|R(E+i\epsilon)V\big\|_p^p \leq C\epsilon^{-p}(1+\epsilon^2)^{1/2}(1+E^2)^{1/2}\|V\|_{L^p}^p \tag{22}$$

for all $E \in \mathbf{R}$ and all $\epsilon > 0$. Consequently, for a new constant C,

$$\Big|\log\big|\det_p\big[1+R(E+i\epsilon)V\big]\big|\Big| \leq C\epsilon^{-p}(1+\epsilon)(1+E^2)^{1/2}\|V\|_{L^p}^p. \tag{23}$$

Proof. The first estimate follows directly from Theorem B.9.3 of [34] using $f(x) = (1+|x|)^{-1/2}$.

For the remainder of this proof, we will write R for $(H-E-i\epsilon)^{-1}$ and $\tilde{R}$ for $(H+V-E-i\epsilon)^{-1}$. To prove the second inequality we use

$$\frac{d}{d\epsilon}\log\big[\det_p(1+RV)\big] = (-1)^p\operatorname{tr}\big(i\tilde{R}(RV)^p\big) \tag{24}$$

which may be checked by induction. By applying the trace ideal version of Hölder's inequality and (22), this implies

$$\left|\frac{d}{d\epsilon}\log\big[\det_p(1+RV)\big]\right| \leq \|RV\|_p^p\,\|\tilde{R}\| \leq C\epsilon^{-p-1}(1+\epsilon)(1+E^2)^{1/2}\|V\|_{L^p}^p.$$

Here $\|\tilde{R}\|$ is the operator norm of $\tilde{R}$ and so bounded by ϵ^{-1}.

As V is of compact support, it is easy to show that

$$\lim_{\epsilon\to\infty}\det_p\big[1+R(E+i\epsilon)V\big] = 1 \quad \forall E \in \mathbf{R}.$$

Equation (23) now follows from our derivative bound and the Fundamental Theorem of Calculus. ☐

This lemma shows how to control the regularized determinant, $\det_p[1+R(z)V]$ for $\operatorname{Im} z > 0$. But Proposition 2.2 suggests we need an *a priori* estimate of the normal determinant for real energies. The key to studying real energies is the fact that $f_n(z) = \log\big|\det_p[1+R(z)V_n]\big|$ is harmonic. The regularization is controlled by fiat: assumptions ii) and iii) of Theorems 3 and 4 respectively. In the next section we will see that that these assumptions do in fact hold in applications.

If $I = [a,b]$, we define the tent $\triangle_\alpha$ over I to be the solid triangle with vertices a, b and $\frac{a+b}{2} + i\frac{b-a}{2}\tan(\alpha)$. We also define z_α to be the centroid (centre of mass) of this triangle and partition $\partial\triangle_\alpha$ into $I \cup \Lambda_\alpha$. This construction is depicted more clearly in Figure 1.

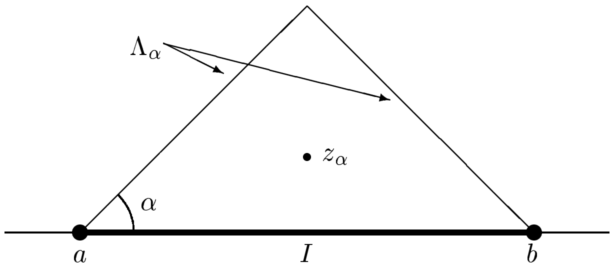

Figure 1: The definition of $\triangle_\alpha, \Lambda_\alpha$ and z_α.

Let $\mathcal{P}_\alpha(z; z_\alpha)$ denote the Poisson kernel for the triangle $\triangle_\alpha$ evaluated at the point z_α. That is to say, for any function F, continuous and harmonic throughout the solid triangle $\triangle_\alpha$,

$$F(z_\alpha) = \int_{I\cup\Lambda_\alpha} F(z)\mathcal{P}_\alpha(z; z_\alpha)\,|dz|. \tag{25}$$

We need the following information about $\mathcal{P}_\alpha(z; z_\alpha)$.

Lemma 3.2. *Given I, α, there is a constant C such that*

$$\mathcal{P}_\alpha(z; z_\alpha) \le C\big[\operatorname{Im} z\big]^{(\pi-\alpha)/\alpha} \quad \forall z \in \Lambda. \tag{26}$$

While E is in the interior of I, $\mathcal{P}_\alpha(E; z_\alpha)$ is C^∞ and positive. At the ends of I, $\mathcal{P}_\alpha(E; z_\alpha)$ vanishes to order $(\pi - \alpha)/\alpha$.

We will not prove this here. It is a fairly standard result which can be proved by conformal transplantation to the disk.

Proof of Theorem 4. As $I \subseteq \Sigma_{\rm ac}(H)$, we know that $\operatorname{Im} m(E + i0) > 0$ for a.e. $E \in I$. Also, by assumption i), it is a continuous function on I. Therefore, in order to prove that $I \subseteq \Sigma_{\rm ac}(H + V)$, it will suffice to show that $J \subseteq \Sigma_{\rm ac}(H + V)$ for all closed intervals $J \subseteq I$ on which $\operatorname{Im} m(E + i0)$ is positive. By this argument, we may freely assume that $\operatorname{Im} m(E + i0) > 0$ for all $E \in I$. We will do so for the remainder of this proof.

The statement of the theorem provides a sequence of compactly supported functions, V_n, which converges to V in L^p. We define $a_n(z) = \det[1 + R(z)V_n]$, as in Section 2, and

$$\begin{aligned} f_n(z) &= \log\big|\det\nolimits_p[1 + R(z)V_n]\big| \\ &= \log\big|a_n(z)\big| - \operatorname{Re}\sum_{\ell=1}^{p-1} \operatorname{tr}\Big\{\big[R(z)V_n\big]^\ell\Big\}. \end{aligned} \tag{27}$$

Our goal is to apply Proposition 2.2 for which we need to obtain the bound (18) on $\log|a_n|$. We will do so by estimating the contribution of f_n and each of the traces separately. For the function w of (18), we choose

$$w(E) = \begin{cases} 0 & E \notin I \\ \mathcal{P}_\alpha(E; z_\alpha) & E \in I \end{cases}$$

with $\alpha = \pi/\max\{p + 2, r + 2\}$ (r is the integer from the third assumption). Notice that Lemma 3.2 implies $w \in C^r(\mathbf{R})$.

Because f_n is harmonic on $\triangle_\alpha$,

$$\int_I f_n(E)w(E)dE = f_n(z_\alpha) - \int_{\Lambda_\alpha} f_n(E)\mathcal{P}_\alpha(z;z_\alpha)|dz|. \tag{28}$$

Lemma 3.1 immediately gives a (n-independent) bound on $f_n(z_\alpha)$. Combining this Lemma with Lemma 3.2 shows that the integral over Λ_α is also bounded (the ϵ^{-p} growth of $f_n(E+i\epsilon)$ is cancelled by $\epsilon^{(\pi-\alpha)/\alpha}$ decay of $\mathcal{P}_\alpha$). Thus, we see that $\int f_n(E)w(E)dE$ is uniformly bounded.

Consider now $\operatorname{Re}\operatorname{tr}\{[R(z)V_n]^\ell\}$. The integral kernel for the resolvent, $R(z)$, is just the Green function, $G(x,y;z)$ which extends continuously from $z\in\mathbf{C}^+$ to I (by assumption). Therefore, for each ℓ, $\operatorname{tr}\{[R(z)V_n]^\ell\}$ also has a continuous extension to $z\in I$ and

$$\operatorname{Re}\int \operatorname{tr}\Big\{\big[R(E+i0)V_n\big]\Big\}w(E)\,dE = \iint \big[\operatorname{Re} G(x,x;E+i0)\big]V_n(x)w(E)\,dx\,dE$$

or, more generally,

$$\operatorname{Re}\int \operatorname{tr}\Big\{\big[R(E+i0)V_n\big]^\ell\Big\}w(E)\,dE$$
$$= \iint K(x_1,\dots,x_\ell;E+i0)\,V_n(x_1)\cdots V_n(x_\ell)\,dx_1\dots dx_\ell\,w(E)\,dE$$

The kernel K is exactly the same one defined in (9). In this way, we see that assumption iii) has been contrived precisely to bound the integrals of the traces.

To recap, we have shown that $\int f_n(E)w(E)\,dE$ and each

$$\int \operatorname{Re}\operatorname{tr}\{[R(E+i0)V_n]^\ell\}w(E)\,dE \qquad 1\le \ell\le p-1$$

is bounded uniformly in n. Combining these with (27) demonstrates that

$$\sup_n \int \log|a_n(E+i0)|w(E)\,dE < \infty$$

With this, we apply Proposition 2.2 to conclude that $I\subseteq\Sigma_{\mathrm{ac}}(H+V)$. □

4. Examples

We begin with the simplest case. Namely, when there is no background potential, $q\equiv 0$. In this case, the Green function is given by

$$G(x,x';k^2) = \tfrac{1}{k}\sin(kx^<)e^{ikx^>} \qquad k\in\mathbf{C}^+. \tag{29}$$

(The notation $x^{\gtrless}$ was introduced for (14).) This extends continuously to positive energies. We now begin the more arduous task of checking hypothesis iii) of Theorem 4. By simple computation,

$$K(x;k^2+i0) = \tfrac{1}{2k}\sin(2kx), \tag{30}$$

$$K(x,x';k^2+i0) = \tfrac{1}{2k^2}[1-\cos(2kx^<)]\cos(2kx^>). \tag{31}$$

As these formulae suggest, the Fourier transform will be important. We adopt the convention $\hat f(k) = \int f(x)e^{ikx}\,dx$.

Lemma 4.1. *Given $a > 0$, $1 \le p \le \infty$ and any $w \in C^3(\mathbf{R})$, which is supported on the interval $[a^2, b^2]$,*

$$\int_{a^2}^{b^2} \int K(x; E + i0) V(x)\, dx\, w(E) dE \lesssim \|\hat{V}\|_{L^2([2a,2b])} \tag{32}$$

$$\int_{a^2}^{b^2} \iint K(x, y; E + i0) V(x) V(y)\, dx\, dy\, w(E) dE \lesssim \|\hat{V}\|^2_{L^2([2a,2b])} + \|V\|^2_{L^p(\mathbf{R})} \tag{33}$$

for all real-valued Schwartz functions V.

Proof. It is important that w is supported away from zero. For (33), we break $K(x, y; E)$ into two parts using (31). The first part is bounded by employing

$$\tfrac{1}{4k^2} \iint \cos(2kx) \cos(2ky) V(x) V(y)\, dx\, dy \lesssim |\hat{V}(2k)|^2$$

and

$$\int_{a^2}^{b^2} |\hat{V}(2\sqrt{E})|^2 w(E) dE \lesssim \|\hat{V}\|^2_{L^2([2a,2b])}.$$

The second with

$$\int_{a^2}^{b^2} \tfrac{1}{2E} \cos(2y\sqrt{E}) w(E) dE \lesssim (1 + y)^{-3}$$

and

$$\int_0^\infty \int_0^y |V(x)|\, |V(y)|\, (1 + y)^{-3}\, dx\, dy \lesssim \|V\|^2_{L^p(\mathbf{R})}.$$

Using (30), the proof of (32) is straight-forward. □

Corollary 4.2. *Suppose $q \equiv 0$ and $V \in L^3(\mathbf{R}^+)$. If the distribution $\hat{V}$ restricts to an L^2 function on an interval $[2a, 2b] \subset (0, \infty)$ then $[a^2, b^2] \subset \Sigma_{\mathrm{ac}}(H + V)$.*

Proof. There is a sequence of compactly supported functions V_n which converge to V in L^3 and so that for each $J = [2c, 2d]$, properly contained in $[2a, 2b]$, $\int_J |\hat{V}_n(k)|^2 dk$ is uniformly bounded. The proof of this fact is not very edifying and will be postponed (Lemma 4.9). The preceding lemma then shows that the hypotheses of Theorem 4 hold with $I = [c^2, d^2]$, $p = 3$ and $r = 3$. Thus $[c^2, d^2] \subset \Sigma_{\mathrm{ac}}(H+V)$.

We have just shown that any proper sub-interval of $[a^2, b^2]$ is contained in $\Sigma_{\mathrm{ac}}(H + V)$. It follows that $[a^2, b^2] \subset \Sigma_{\mathrm{ac}}(H + V)$. □

We say that a distribution V belongs to the homogeneous Sobolev space $\dot{\mathsf{H}}^s$ if $\int |\hat{V}^2(k)|^2 |k|^{2s} dk < \infty$.

Corollary 4.3. *Suppose $q \equiv 0$ and $V \in L^3(\mathbf{R}^+)$. If V can be written as a finite sum of functions each of which belongs to a homogeneous Sobolev space $\dot{\mathsf{H}}^s$ ($s \in \mathbf{R}$) then $\Sigma_{\mathrm{ac}}(H + V) = [0, \infty)$.*

Proof. The previous corollary shows that each compact sub-interval of $(0, \infty)$ is contained in $\Sigma_{\mathrm{ac}}(H + V)$. This implies that $\Sigma_{\mathrm{ac}}(H + V) \supset [0, \infty)$. Conversely, $\Sigma_{\mathrm{ac}}(H + V)$ is a subset of the essential spectrum. By Weyl's Theorem on relatively compact perturbations, we know that $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H + V) = [0, \infty)$. □

Treating $V \in L^p$ becomes more involved as p grows. We have just proved a result for $p = 3$ and will content ourselves with treating just one more case: $p = 4$. For this we use the tri-linear estimate below. The main idea is due to Kovrijkine [24].

Lemma 4.4. *Let $\nu \geq 0$ be an integer. Suppose $\rho \in C^{\nu+2}(\mathbf{R})$ is compactly supported with $0 \notin \operatorname{supp}(\rho)$. If for each pair of compactly supported functions $f \in C^\nu$ and $g \in L^4$*

$$I(f,g) = \iiint_x^z f(y)\,dy\; g(x)g(z) \int \sin[2k(z-x)]\rho(k)\,dk\,dx\,dz$$

then $|I(f,g)| \lesssim \{\|f^{(\nu)}\|_{L^2} + \|f\|_{L^4} + \|g\|_{L^4} + \|\hat g\|_{L^2(\Omega)}\}^3$ where $\Omega = \{k : \pm k/2 \in \operatorname{supp}\rho\}$. Further, if we write $F(x) = \int_x^\infty f(t)\,dt$ then $|I(f,g)| \lesssim \|F\|_{L^2}\|g\|_{L^4}^2$.

Remark. Both $\int_x^z f(y)dy$ and $\sin[2k(z-x)]$ have odd parity under the interchange of x and z. This makes the full integrand even.

Proof. If $r(\xi) = \int \sin[2k\xi]\rho(k)\,dk$ then $\rho \in C^{\nu+2}(\mathbf{R})$ implies $\xi^j r(\xi) \in L^1$ for all $0 \leq j \leq \nu+1$. Moreover, for each such j the Fourier transform of $\xi^j r(\xi)$ is uniformly bounded and is supported by Ω. This means that for $0 \leq \ell \leq \nu$,

$$\Big\| \int (z-x)^{\ell+1} r(z-x) f(z)\,dz \Big\|_{L^2(dx)} \lesssim \|\hat g\|_{L^2(\Omega)}. \tag{34}$$

Let $[Mh](x) = \sup_z \frac{1}{|z-x|}\int_x^z |h(y)|\,dy$ that is, the maximal function of $|h|$. Then $\|Mh\|_{L^2} \lesssim \|h\|_{L^2}$ and

$$\sup_z \Big| \int_x^z f^{(\nu)}(t) \frac{(z-t)^\nu}{(z-x)^{\nu+1}}\,dt \Big| \leq [M f^{(\nu)}](x). \tag{35}$$

Consider the case $\nu = 0$. Using the above and Young's inequality,

$$\begin{aligned}
|I(f,g)| &= \Big| \iint r(z-x) \int_x^z f(y)\,dy\; g(x)g(z)\,dx\,dz \Big| \\
&\leq \iint \big|(z-x)r(z-x)\big|\, [Mf](x)\, |g(x)|\, |g(z)|\,dx\,dy \\
&\lesssim \|g[Mf]\|_{L^{4/3}}\|g\|_{L^4} \lesssim \|f\|_{L^2}\|g\|_{L^4}^2.
\end{aligned}$$

For $\nu > 0$, we employ the following consequence of Taylor's formula (with the integral form of the remainder):

$$\int_x^z f(y)\,dy = \sum_{\ell=0}^{\nu-1} \tfrac{1}{(\ell+1)!}(z-x)^{\ell+1} f^{(\ell)}(x) + \tfrac{1}{\nu!}(z-x)^\nu \int_x^z f^{(\nu)}(t)\frac{(z-t)^\nu}{(z-x)^\nu}\,dt.$$

Substituting it into the definition of $I(f)$,

$$\begin{aligned}
I(f,g) = \sum_{\ell=0}^{\nu-1} \tfrac{1}{(\ell+1)!} \int g(x)\, f^{(\ell)}(x) \int (z-x)^{\ell+1}\, r(z-x)\, g(z)\,dz\,dx \\
+ \tfrac{1}{\nu!} \iiint_x^z f^{(\nu)}(t)\frac{(z-t)^\nu}{(z-x)^{\nu+1}}\,dt\; (z-x)^{\nu+1}\, r(z-x)\, g(x)\, g(z)\,dx\,dz.
\end{aligned}$$

The summands from the first line are bounded using (34), Hölder's inequality and the fact that for all $0 \leq \ell < \nu$, $\|f^{(\ell)}\|_{L^4} \lesssim \|f\|_{L^4} + \|f^{(\nu)}\|_{L^2}$:

$$\begin{aligned}
&\int g(x)\, f^{(\ell)}(x) \int (z-x)^{\ell+1}\, r(z-x)\, g(z)\,dz\,dx \\
&\lesssim \|g f^{(\ell)}\|_{L^2}\|\hat g\|_{L^2(\Omega)} \lesssim \{\|f^{(\nu)}\|_{L^2} + \|f\|_{L^4}\}\|g\|_{L^4}\|\hat g\|_{L^2(\Omega)}.
\end{aligned}$$

The term on the second line is estimated by using (35) much as in the $\nu = 0$ case:

$$\begin{aligned}
&\Big|\iiint_x^z f^{(\nu)}(t)\frac{(z-t)^\nu}{(z-x)^{\nu+1}}\,dt\;(z-x)^{\nu+1}\,r(z-x)\,g(x)\,g(z)\,dx\,dz\Big| \\
&\le \iint [Mf^{(\nu)}](x)\,|g(x)|\,\big|(z-x)^{\nu+1}r(z-x)\big|\,|g(z)|\,dx\,dz \\
&\lesssim \|f^{(\nu)}\|_{L^2}\|g\|_{L^4}^2
\end{aligned}$$

Adding together these estimates does indeed show that $|I(f,g)| \lesssim \{\|f^{(\nu)}\|_{L^2} + \|f\|_{L^4} + \|g\|_{L^4} + \|\hat{g}\|_{L^2(\Omega)}\}^3$.

The second estimate stated in the lemma, the one involving F, is easier to prove:

$$\begin{aligned}
I(f,g) &= \iint r(z-x)\int_x^z f(y)\,dy\;g(x)g(z)\,dx\,dz \\
&= \iint r(z-x)[F(x)-F(z)]g(x)g(z)\,dx\,dz.
\end{aligned}$$

Thus, by Young's and Hölder's inequalities, $|I(f,g)| \lesssim \|g\|_{L^4}^2\|F\|_{L^2}$. □

Corollary 4.5. *Suppose $q \equiv 0$. If for some integer, $N \ge 0$, V can be written as $V(x) = \sum_{\nu=-1}^N U_\nu(x)$ such that $U_\nu \in L^4$ and $U_\nu^{(\nu)} \in L^2$ then $\Sigma_{\mathrm{ac}}(H+V) = [0,\infty)$. For $\nu \ge 0$, $U^{(\nu)}$ denotes the ν^{th} derivative while $U^{(-1)}(x) = \int_x^\infty U(t)\,dt$.*

Proof. We will show that Theorem 4 (with $p = 4$ and $r = N + 100$) is applicable to any interval $[a^2, b^2]$ where $0 < a < b$. As described in the preceding Corollaries, this is suffices to prove the present result.

To apply Theorem 4 we must check the third hypothesis. By choosing V_n to be V multiplied by a smooth cut-off function, this amounts to three ($\ell = 1, 2, 3$) *a priori* estimates for compactly supported functions, V, which can be written as $\sum_{\nu=-1}^N U_\nu(x)$ with $U_\nu \in L^4$ and $U_\nu^{(\nu)} \in L^2$. The first two follow immediately from Lemma 4.1:

$$\left|\int_{a^2}^{b^2}\int K(x;E+i0)V(x)\,dx\,w(E)dE\right| \lesssim \sum_\nu \|U_\nu^{(\nu)}\|_{L^2}$$

$$\left|\int_{a^2}^{b^2}\iint K(x,y;E+i0)V(x)V(y)\,dx\,dy\,w(E)dE\right| \lesssim \sum_\nu \|U_\nu^{(\nu)}\|_{L^2}^2 + \|V\|_{L^4}^2$$

For the $\ell = 3$ case we will prove that for each $w \in C^{N+100}(\mathbf{R})$ that is supported by $[a^2, b^2]$,

$$\begin{aligned}
&\left|\int_{a^2}^{b^2}\iiint K(x,y,z;E+i0)V(x)V(y)V(z)\,dy\,dx\,dz\,w(E)dE\right| \\
&\lesssim \|V\|_{L^4}^3 + \sum_\nu \|U_\nu^{(\nu)}\|_{L^2}^3.
\end{aligned} \tag{36}$$

From its definition, $K(x,y,z;k^2+i0)$ is a symmetric function of x, y and z. As a result, we can freely restrict the spatial integral in (36) to the region $x < y < z$.

When $x<y<z$

$$\begin{aligned} K(x,y,z;k^2+i0) = \tfrac{1}{8k^3}\Big\{ & 2\sin[2k(y+z)] - 2\sin[2kz] + \sin[2k(x+z)] \\ & - \sin[2k(x+y+z)] - \sin[2k(y+z-x)] \\ & - \sin[2k(z-x)]\Big\}. \end{aligned}$$

Naturally, it suffices to prove (36) with K replaced by each of the summands given above. For the terms appearing in the first two rows, this presents no difficulty. We demonstrate with the last of these terms, the others may be dealt with similarly: As $w\in C^{100}$, applying Hölder directly, and after one hundred integrations by parts, shows that for $x<y<z$,

$$\left|\int_a^b \tfrac{1}{4k^2}\sin[2k(z+y-x)]w(k^2)\,dk\right| \lesssim \min\{1,(z+y-x)^{-100}\} \lesssim (1+z)^{-100}.$$

The required estimate then follows from

$$\int_0^\infty\!\!\int_0^z\!\!\int_x^z (1+z)^{-100}\big|V(x)V(y)V(z)\big|\,dy\,dx\,dz \lesssim \|V\|_{L^4}^3.$$

Treating the last summand, $-\frac{1}{8k^3}\sin[2k(z-x)]$, is more subtle and will use Lemma 4.4. We wish to bound

$$\int\!\!\int_0^\infty\!\!\int_0^z \sin[2k(z-x)]\int_x^z U_\nu(y)\,dy\,V(x)V(z)\,dx\,dz\,k^{-2}w(k^2)\,dk$$

Let $\rho=k^{-2}w(k^2)$ then $\rho\in C^{N+100}$ and $0\notin\operatorname{supp}\rho$ because $0\notin\operatorname{supp}w$. Using symmetry considerations, we can remove the restriction $x<z$ from the region of integration. In this way, we need only bound

$$\sum_\nu \iiint_x^z U_\nu(y)\,dy\;V(x)V(z)\int \sin[2k(z-x)]\rho(k)\,dk\,dx\,dz.$$

This can be done by applying Lemma 4.4 to each summand. For $\nu=-1$, this is obvious. For $\nu\geq 0$, let us point out that because $V=\sum U_\nu$ and $U_\nu^{(\nu)}\in L^2$ so $\hat V$ is square integrable on any compact interval excluding 0.

This completes the demonstration that hypothesis iii) of Theorem 4 holds. As the other hypotheses are easily seen to be valid, this completes the proof. $\square$

In the free case just treated, we had an explicit expression for the Green function. However, as our next example will show, all that is required, is its asymptotics at large distances from the origin. We need a traditional WKB-type result:

Lemma 4.6. *Suppose $q(x)\to 0$ as $x\to\infty$ and both $\int_x^\infty |q'(t)|^2dt$ and $\int_x^\infty |q''(t)|dt$ belong to $L^2(dx)$. Given I, a compact sub-interval of $(0,\infty)$, there exist $x_0>0$ and smooth functions $A(E),\vartheta(E)$ so that writing*

$$\operatorname{Re} G(x,x;E+i0) = A(E)[E-q(x)]^{-1/2}\sin\left\{\vartheta(E)+2\int_{x_0}^x\sqrt{E-q(t)}\,dt\right\}+\zeta(x;E)$$

for all $x>x_0$ and all $E\in I$ implies that $\int_{x_0}^\infty \zeta(x;E)^2\,dx$ is a uniformly bounded function of $E\in I$.

Proof. Let $\Delta(x;E) = \frac{5}{16}[q'(x)]^2[E-q(x)]^{-2} + \frac{1}{4}q''(x)[E-q(x)]^{-1}$ and choose x_0 so that for all $E \in I$, both $E > 2q(x)$ for all $x > x_0$ and $\int_{x_0}^\infty |\Delta(x;E)|dx < 10^{-6} \min_{x \geq x_0} \sqrt{E-q(x)}$.

Let us define operators $\tilde{H}^D, \tilde{H}^N$ as the restriction of H to the interval $[x_0,\infty)$ with a Dirichlet, respectively Neumann, boundary condition at x_0. We begin by constructing the Green functions for these operators. For each $z \in \mathbf{C}^+$, define

$$\tilde{u}_\pm(x) = \big[z-q(x)\big]^{-1/4} \exp\Big\{ \pm i \int_{x_0}^x \sqrt{z-q(t)}\, dt\Big\}$$

where the branch of the square root is chosen to make $\tilde{u}_+$ exponentially decaying. Next write $\tilde{G}_0^D(x,x';z) = \frac{1}{2i}\big[\tilde{u}_+(x^<) - \tilde{u}_-(x^<)\big]\tilde{u}_+(x^>)$ and $\tilde{G}_0^N(x,x';z) = \frac{1}{2i}\big[\alpha(z)\tilde{u}_+(x^<) - \tilde{u}_-(x^<)\big]\tilde{u}_+(x^>)$ where $\alpha(z) = [q'(0) - 4i(z-q(0))^{3/2}]/[q'(0) + 4i(z-q(0))^{3/2}]$. These formulae are Green functions for an operator with a 'potential' other that $q-z$. The difference between this and $q-z$ is given by $\Delta(x;z)$. These approximate Green functions, $\tilde{G}_0^{D/N}$, clearly extend continuously from $z \in \mathbf{C}^+$ to $z \in I$. However, because Δ is so small, we can write the true Green functions in a Neumann series

$$\tilde{G}^{D/N} = \tilde{G}_0^{D/N} + \sum_{k=1}^\infty \tilde{G}_0^{D/N}\Big\{G_0^{D/N}\Delta G_0^{D/N}\Big\}^k$$

which is convergent for z in a neighbourhood of $I + i[0,1) \subset \overline{\mathbf{C}^+}$. This shows that $\tilde{G}^{D/N}$ have continuous extensions from $z \in \mathbf{C}^+$ to the interval I. It also permits us to calculate the large x asymptotics of $\operatorname{Re}\tilde{G}^{D/N}(x,x;E)$. We will do this for $\tilde{G}^D$, the calculation for $\tilde{G}^N$ is similar.

From the Neumann series,

$$\tilde{G}^D(x,x;E) = \tilde{G}_0^D(x,x;E) + \int \tilde{G}_0^D(x,y;E)\Delta(y)\tilde{G}_0^D(y,x;E)\, dy$$
$$+ \iint \tilde{G}_0^D(x,y)\Delta(y)\tilde{G}^D(y,z;E)\Delta(z)\tilde{G}_0^D(z,x;E)\, dy\, dz.$$

We will calculate the asymptotics of each term on the right hand side separately. For the the second term we have,

$$-\tfrac{1}{4}\, \tilde{u}_+(x)^2 \int_{x_0}^\infty [\tilde{u}_+(y) - \tilde{u}_-(y)]^2 \Delta(y)\, dy + O\Big\{\int_x^\infty \Delta(y)\, dy\Big\}$$

and for the third,

$$-\tfrac{1}{4}\, \tilde{u}_+(x)^2 \int_{x_0}^\infty \int_{x_0}^\infty [\tilde{u}_+(y) - \tilde{u}_-(y)][\tilde{u}_+(z) - \tilde{u}_-(z)]\tilde{G}^D(y,z;E)\Delta(y)\Delta(z)\, dy\, dz$$
$$+ O\Big\{\int_x^\infty \Delta(y)\, dy\Big\}.$$

By assumption, $\int_x^\infty \Delta(y)dy$ is a square integrable function of x. Thus the real part of each of the previous two terms may be written as

$$\tilde{A}(E)\sin\Big\{\tilde{\vartheta}(E) + 2\int_{x_0}^x \sqrt{E-q(t)}\, dt\Big\} + \text{square integrable error}$$

for some smooth amplitude $\tilde{A}$ and phase $\tilde{\vartheta}$. Moreover, the L^2 norm of the error is a uniformly bounded function of $E \in I$. The first term has the same asymptotics.

In fact, in this case, $\tilde{A} = 1/2$, $\tilde{\vartheta} = 0$ and there is no error term. Because the sum of sinusoids is a sinusoid, adding the three terms proves that

$$\tilde{G}^D(x,x;E) = \tilde{A}(E)\sin\Big\{\tilde{\vartheta}(E) + 2\int_{x_0}^x \sqrt{E-q(t)}\,dt\Big\} + \text{square integrable error}$$

for some smooth functions $\tilde{A}(E), \tilde{\vartheta}(E)$. Similar arguments show that the same is true for $\tilde{G}^N(x,x;E)$.

We have shown that $\tilde{G}^{D/N}$ have continuous extensions from $\mathbf{C}^+$ to I and that the values on I have the right asymptotic form. The result for the original operator now follows because, for $x > x_0$, the Green function $G(x,x;E)$ is a smoothly-E-dependent linear combination of $\tilde{G}^D(x,x;E)$ and $\tilde{G}^N(x,x;E)$. $\square$

Corollary 4.7. *If $q(x) \to 0$ as $x \to \infty$, $\int_x^\infty |q'(t)|^2 dt \in L^2(dx)$ and $\int_x^\infty |q''(t)| dt \in L^2(dx)$ then for each $V \in L^2$, $\Sigma_{\mathrm{ac}}(H+V) = [0,\infty)$.*

Proof. That $\Sigma_{\mathrm{ac}}(H+V)$ is a subset of $[0,\infty)$, follows from Weyl's theorem, as in Corollary 4.3. To prove the converse it suffices to show that Theorem 3 is applicable to each compact interval $I \subset (0,\infty)$. To this end, let $w(E)$ be a $C^1(\mathbf{R})$ function supported on such an interval. We wish to show that

$$\int_I \operatorname{Re} G(x,x;E+i0) w(E) dE \in L^2(\mathbf{R}^+;dx)$$

By referring to the preceding lemma, we see that the following calculation suffices: Let $\varphi(x,E) = 2\int_{x_0}^x \sqrt{E-q(t)}\,dt + \vartheta(E)$, $\varphi'(x,E) = \frac{\partial}{\partial E}\varphi(x,E)$ and $\tilde{w}(E) = w(E)A(E)$. For x_0 sufficiently large and each $x > x_0$,

$$\Big|\int e^{i\varphi(x,E)} w(E)\,dE\Big| = \Big|\int e^{i\varphi(x,E)}\Big[\tfrac{w'(E)}{\varphi'(x,E)} - \tfrac{w(E)\varphi''(x,E)}{\varphi'(x,E)^2}\Big]\,dE\Big| \lesssim x^{-1}.$$

The first step is integration by parts after first multiplying and dividing by φ'. The second step is the combination of that facts $x \lesssim \varphi'$ and $\varphi'' \lesssim x$ for x sufficiently large and $E \in I$. $\square$

The final example for this section is the case of a periodic background potential. We assume that q is locally square integrable and, without loss of generality, has period one. Much is known about such operators, but we shall only describe those parts of Floquet Theory, as it is called, that we will require. For more information see [13].

There is a continuous function $\gamma : \mathbf{R} \to \overline{\mathbf{C}^+}$, so that, with the exception of the discrete set where $e^{i\gamma(E)} = \pm 1$, there is a basis of solutions $\psi_\pm(x;E)$ of $-\psi'' + q\psi = E\psi$ with $\psi_\pm(x+1) = e^{\pm\gamma(E)}\psi_\pm(x)$. (The function $\gamma(E)$ is typically called the quasi-momentum and $\psi_\pm$ the Bloch (or Floquet) solutions.) The absolutely continuous spectrum of H is equal to the set where $|e^{i\gamma(E)}| = 1$. Because of the Dirichlet boundary condition at the origin, eigenvalues appear in the spectral gaps. That is in the intervals where $|e^{i\gamma(E)}| < 1$.

Let Σ denote the set $\{E \in \sigma_{\mathrm{ac}} : e^{i\gamma(E)} \neq \pm 1\}$. While the closure of Σ is equal to $\sigma_{\mathrm{ac}}(H)$, it need not be the case that Σ is equal to the interior of the spectrum. This is because of the possibility of two bands touching. The function $\gamma(E)$ is analytic and strictly monotone on Σ. For $E \in \Sigma$, it is possible to define the Bloch solutions uniquely by adding the constraint $\psi_\pm(0) = 1$. This normalization makes ψ_+ equal to the Weyl solution. It also follows that $\psi_+(x;E) = \overline{\psi_-(x;E)}$.

It is well known that the Green function extends continuously (indeed analytically) from $\mathbf{C}^+$ to Σ. Indeed, we have the following formula

$$G(x,x';E+i0) = \tfrac{1}{W(E)}\psi_-(x^<;E)\psi_+(x^>;E) - \tfrac{1}{W(E)}\psi_+(x;E)\psi_+(x';E) \tag{37}$$

where $W(E)$ is the Wronskian of ψ_+ and ψ_-.

For part of the following Corollary, we will require that $\hat{V}$ be uniformly locally square integrable. To this end, we introduce the Banach space $\ell^\infty(L^2)$ with norm

$$\|f\|^2_{\ell^\infty(L^2)} = \sup_n \int_{-2\pi}^{2\pi} \left|f(x+2\pi n)\right|^2 dx. \tag{38}$$

Corollary 4.8. *Suppose $q \in L^2_{\mathrm{loc}}$ is periodic with period one. If $V \in L^2$, or both $V \in L^3$ and $\hat{V} \in \ell^\infty(L^2)$, then $\Sigma_{\mathrm{ac}}(H+V) = \sigma_{\mathrm{ac}}(H)$.*

Proof. As in the previous results, $\Sigma_{\mathrm{ac}}(H+V) \subset \sigma_{\mathrm{ac}}(H)$ follows from Weyl's Theorem. We will prove the converse inclusion by showing that each compact interval $I \subset \Sigma$ is contained in $\Sigma_{\mathrm{ac}}(H+V)$. This will be a consequence of Theorem 3, for $V \in L^2$, and of Theorem 4 in the other case. As mentioned above, the Green function extends continuously to I. It remains, however, to estimate the behaviour of the Green function. We use an idea from "The Proof of Theorem 1.8" in the preprint [20] (the published version [21] uses a different method). It is the following: The function $\sigma(x;E) = \exp[-i\gamma(E)x]\psi_+(x;E)$ is periodic, with period 1, a $C^\infty(I)$ function of E and a $C^1(\mathbf{R})$ function of x. Moreover, $\sup_{E\in I}\|\sigma(\cdot;E)\|_{C^1} < \infty$ and $\sup_x \|\sigma(x;\cdot)\|_{C^2} < \infty$.

Let I be as described above, and let $w \in C^2(\mathbf{R})$ be supported by I. We begin by showing that

$$\left|\mathrm{Re}\iint_I G(x,x;E+i0)V(x)\,w(E)dE\,dx\right| \lesssim \|V\|_{L^p} \tag{39}$$

for any $p > 1$. Because $\psi_\pm$ are complex conjugates, the first term on right of (37) is purely imaginary. As a result,

$$\begin{aligned}\left|\int \mathrm{Re}\,G(x,x;E+i0)\,w(E)dE\right| &= \left|\int e^{2i\gamma(E)x}\,\tfrac{\sigma(x;E)^2 w(E)}{W(E)}\,dE\right| \\ &\le x^{-1}\int \left|\tfrac{\partial}{\partial E}\tfrac{\sigma(x;E)w(E)}{2\gamma'(E)W(E)}\right| dE\end{aligned}$$

by multiplying and dividing by $\gamma'(E)$ and then using integration by parts. The last integral is finite because both $W(E)$ and $\gamma'(E)$ are bounded away from zero on I and because all terms are C^1 functions of E. Furthermore, since σ and its derivative are bounded independently of x, this proves (39).

With $p=2$, (39) shows that Theorem 3 is applicable when $V \in L^2$. This proves the first case of the theorem. The second, $V \in L^3$ and $\hat{V} \in \ell^\infty(L^2)$, follows from Theorem 4 with $p=3$ and $r=2$. The third hypothesis of this theorem is satisfied because the calculation performed above and because

$$\iiint \mathrm{Re}\,[G(x,y;E+i0)^2]V(x)V(y)\,dx\,dy\,w(E)dE \lesssim \|\hat{V}\|^2_{\ell^\infty(L^2)} + \|V\|^2_{L^3} \tag{40}$$

for every $w \in C^2(\mathbf{R})$ supported by I. The remainder of this proof is devoted to the justification of this inequality.

Using (37), we can expand $G(x,x';E+i0)^2$ as three terms: $\frac{1}{W^2}\psi_+(x^<)^2\psi_+(x^>)^2$, $\frac{2}{W^2}\psi_-(x^<)\psi_+(x^<)\psi_+(x^>)^2$ and $\frac{1}{W^2}\psi_-(x^<)^2\psi_+(x^>)^2$. We will deal with them in this order. For the first, we have

$$\begin{aligned}&\Big|\int \tfrac{1}{W(E)^2}\psi_+(x;E)^2\psi_+(y;E)^2\,w(E)dE\Big|\\ &=\Big|\int \tfrac{1}{W(E)^2}\sigma(x;E)^2\sigma(y;E)^2e^{2i\gamma(E)[x+y]}\,w(E)dE\Big|\\ &\lesssim (x+y)^{-2}.\end{aligned}$$

The last line follows by integrating by parts twice, each time multiplying and dividing by $\gamma'(E)$. (Compare the proof of (39) given above.) This calculation, together with an easy $L^\infty(dx\,dy)$ estimate, shows that

$$\Big|\iiint \tfrac{1}{W(E)^2}\psi_+(x;E)^2\psi_+(y;E)^2V(x)V(y)\,dx\,dy\,w(E)dE\Big| \lesssim \|V\|^2_{L^3}.$$

Consequently, we have controlled the first of the three terms. By the same arguments,

$$\begin{aligned}&\Big|\int \tfrac{2}{W(E)^2}\psi_+(x;E)\psi_-(x;E)\psi_+(y;E)^2\,w(E)dE\Big|\\ &=\Big|\int \tfrac{2}{W(E)^2}|\sigma(x;E)|^2\sigma(y;E)^2e^{2i\gamma(E)y}\,w(E)dE\Big|\\ &\lesssim y^{-2}\end{aligned}$$

shows that the contribution the second term makes to (40) is $\lesssim \|V\|^2_{L^3}$. This leaves the third and most interesting term. This time, we will perform the spatial integrals first (rather than the energy integral, as above). Because ψ_+ and ψ_- are complex conjugates, the Wronskian is purely imaginary and taking the complex conjugate of $\psi_+(x)^2\psi_-(y)^2$ is equivalent to interchanging x,y. Combining these facts,

$$2\mathrm{Re}\iint_{x<y}\frac{\psi_+(y)^2\psi_-(x)^2}{W(E)^2}V(x)V(y)\,dx\,dy = \frac{-1}{|W(E)|^2}\left|\int\psi_+(x)^2V(x)\,dx\right|^2. \tag{41}$$

As $\sup_{E\in I}\|\sigma(\cdot,E)\|_{C^1}<\infty$, when we develop σ^2 as a Fourier series,

$$\sigma(x;E)^2=\sum_n c_n(E)e^{2\pi inx}$$

the sequence $\tilde c_n=\sup_{E\in I}|c_n(E)|$ is bounded in ℓ^1. To employ this observation, we write

$$\begin{aligned}\int\psi_+(x;E)^2V(x)dx &= \int\sigma(x;E)^2e^{2i\gamma(E)x}V(x)\,dx\\ &=\int\sum_n c_n(E)\exp\big\{i\big[2\pi n+2\gamma(E)\big]x\big\}V(x)\,dx\\ &=\sum_n c_n(E)\hat V\big(2\pi n+2\gamma(E)\big).\end{aligned}$$

so that, by Hölder's and Minkowski's inequalities,

$$\left\|\int\psi_+(x)^2V(x)\,dx\right\|_{L^2(w(E)dE)}\lesssim \varepsilon_I^{-1}\|\tilde c_n\|_{\ell^1}\|\hat V\|_{\ell^\infty(L^2)}.$$

Here $\varepsilon_I = \inf_{E \in I} |\gamma'(E)|$, which is a positive number. As $W(E)$ is bounded away from zero for $E \in I$, this shows that (41) is bounded by a multiple of $\|\hat{V}\|_{\ell^\infty(L^2)}$. This completes the proof of (40) and so also, of the result. $\square$

Lemma 4.9. *Suppose $V \in L^p(\mathbf{R})$ with $p \geq 2$ and the distribution $\hat{V}$ is square-integrable on an interval I. If $\varphi(x)$ is a $C^\infty(\mathbf{R})$ function which takes the value one for $|x| < 1$ and the value zero for $|x| > 2$, then the sequence $V_n(x) = \phi(x/n)V(x)$ converges to V in L^3. Moreover, for any interval J properly contained in I, the integral $\int_J |\widehat{V_n}(k)|^2 dk$ is uniformly bounded.*

Proof. We will write $\varphi_n(x) = \varphi(x/n)$. That $V_n \to V$ in L^3 is well known, the difficulty lies in the second part. Let J be a proper sub-interval of I. It suffices to show that for any function g, whose Fourier transform is C^∞ and supported by J,

$$\int_J \hat{V}_n(k)\hat{g}(k)dk \lesssim \|\hat{g}\|_{L^2}. \tag{42}$$

The implicit constant depends on the distance from J to the complement of I.

To prove (42), choose a $\rho \in C^\infty(\mathbf{R})$ which is supported by I and equal to one in a neighbourhood of J. Then write $\varphi_n(x)g(x) = G_1(x) + G_2(x)$ with $\hat{G}_1(k) = \rho(k)\widehat{\varphi_n g}(k)$ and $\hat{G}_2(k) = [1-\rho(k)]\widehat{\varphi_n g}(k)$. Let p' denote the conjugate exponent of p: $1/p + 1/p' = 1$. Using Parseval and Hölder,

$$\begin{aligned}\int_J \hat{V}_n(k)\hat{g}(k)dk &\lesssim \int V(x)\phi_n(x)g(x)dx = \int V(x)[G_1(x) + G_2(x)]dx \\ &\lesssim \|\hat{V}\|_{L^2(I)}\|\hat{G}_1\|_{L^2} + \|V\|_{L^p}\|G_2\|_{L^{p'}}\end{aligned}$$

so it suffices to bound $\|\hat{G}_1\|_{L^2}$ and $\|G_2\|_{L^{p'}}$ by an n-independent multiple of $\|\hat{g}\|_{L^2}$.

From $\|\varphi_n g\|_{L^2} \leq \|g\|_{L^2}$ it follows that both $\|\hat{G}_1\|_{L^2} \leq \|\hat{g}\|_{L^2}$ and $\|\hat{G}_2\|_{L^2} \leq \|\hat{g}\|_{L^2}$. This completes the study of G_1. For G_2 we continue with

$$\|G_2\|_{L^{p'}} \lesssim \|G_2\|_{L^2} + \|xG_2\|_{L^2} \lesssim \|\hat{G}_2\|_{L^2} + \|\hat{G}_2'\|_{L^2} \lesssim \|\hat{g}\|_{L^2} + \|\hat{G}_2'\|_{L^2}.$$

Showing that $\|\hat{G}_2'\|_{L^2} \lesssim \|\hat{g}\|_{L^2}$ is all that remains to be done. However, this is rather involved.

From its definition, $\hat{G}_2'(k) = (1-\rho(k))[\widehat{\varphi_n}' * \hat{g}](k) - \rho'(k)\widehat{\varphi_n g}(k)$. The L^2 norm of the second term is $\lesssim \|\rho'\|_{L^\infty}\|\varphi_n g\|_{L^2} \leq \|\rho'\|_{L^\infty}\|g\|_{L^2}$ as we need. If we denote the first term by $f(k)$ then

$$f(k) = [1-\rho(k)]\,[\widehat{\varphi_n}' * \hat{g}](k) = n^2[1-\rho(k)]\int_J g(k')\hat{\varphi}'(nk - nk')\,dk.$$

By employing Cauchy–Schwarz, this implies

$$|f(k)|^2 \leq n^4[1-\rho(k)]^2\,\|g\|_{L^2}^2 \int_J \left|\hat{\varphi}'(nk-nk')\right|^2 dk'.$$

As $1-\rho(k)$ is zero for $k \in J$, we may continue with

$$\begin{aligned}|f(k)|^2 &\leq [1-\rho(k)]^2\,\|g\|_{L^2}^2 \int_J \frac{|nk-nk'|^3}{\operatorname{dist}(k,J)^3}\left|\hat{\varphi}'(nk-nk')\right|^2 d(nk') \\ &\leq \frac{(1-\rho(k))^2}{\operatorname{dist}(k,J)^3}\,\|g\|_{L^2}^2 \int |y|^3|\hat{\varphi}'(y)|^2 dy.\end{aligned}$$

Thus the assumptions on ρ and φ are sufficient to conclude that $\|f\|_{L^2} \lesssim \|g\|_{L^2}$. $\square$

5. The Stark Operator

In this section we discuss perturbations of the Stark operator

$$[Hu](x) = -u''(x) - xu(x) \tag{43}$$

acting on $L^2(\mathbf{R}^+)$ with a Dirichlet boundary condition at the origin. This operator has purely absolutely continuous spectrum with $\Sigma_{\rm ac}(H) = \mathbf{R}$. Lemma 3.1 is not applicable because the potential $q(x) = -x$ is unbounded below. However, as solutions to the equation $-u'' - xu = zu$ are well understood—they are Airy functions—we are able to obtain the necessary estimates by direct computation. Throughout this section, the notation for special functions is that of Abramowitz and Stegun [1].

Lemma 5.1. *Suppose V is continuous and compactly supported. For each $2 \leq p < \infty$, we have the following trace ideal estimate:*

$$\big\|R(E+i\epsilon)V\big\|_p^p \leq C\epsilon^{-1}\int |V(t^2)|^p\,dt \qquad \forall 0<\epsilon<1. \tag{44}$$

The constant C depends on p and continuously on E. This implies that for a similar constant,

$$\Big|\log\big|\det{}_p\big[1+R(E+i\epsilon)V\big]\big|\Big| \leq C\epsilon^{-p}\int |V(x^2)|^2\,dx \qquad \forall 0<\epsilon<1. \tag{45}$$

Proof. The Green function is given by

$$G(x,y;z) = \frac{\sqrt{x+z}H^{(1)}_{1/3}(\frac{2}{3}(x+z)^{3/2})}{\pi\sqrt{z}H^{(1)}_{1/3}(\frac{2}{3}z^{3/2})}\big[Ai(-z-y)Bi(-z) - Ai(-z)Bi(-z-y)\big]$$

whenever $x > y$ and $\operatorname{Im} z > 0$. It is important to know that the branch of $(x+z)^{3/2}$ is chosen such that as $x\to\infty$, $\operatorname{Im}(x+E+i\epsilon)^{3/2} = \frac{3}{2}\epsilon\sqrt{x} + O(1)$. It follows from the asymptotics of the Airy and Hankel functions that if $x > y$ then

$$|G(x,y;E+i\epsilon)|^2 \lesssim x^{-1/2}y^{-1/2}\exp\{2\epsilon(\sqrt{y}-\sqrt{x})\} \qquad \forall 0<\epsilon<1.$$

The implicit constant depends on E. Employing this bound gives

$$\big\|R(E+i\epsilon)V\big\|_2^2 = \iint |G(x,y;E+i\epsilon|^2|V(x)|^2\,dx\,dy \lesssim \epsilon^{-1}\int |V(t^2)|^p\,dt.$$

(Note that $\int y^{-1/2}\exp\{\pm 2\epsilon\sqrt{y}\}\,dy = \epsilon^{-1}\exp\{\pm2\epsilon\sqrt{x}\}$.) This proves the $p=2$ case of (44). For the remaining values of p we employ interpolation as in Theorem 4.1 of [33]. To do so, one need only note that for V of compact support, $R(E+i\epsilon)V$ is compact and moreover, $\|R(E+i\epsilon)V\| \lesssim \epsilon^{-1}\|V\|_{L^\infty}$.

The argument given in Lemma 3.1 shows that (45) follows directly from (44). $\square$

With Lemma 5.1 replacing Lemma 3.1, the original proof of Theorem 4 extends to the Stark operator.

Theorem 5. *Suppose $\int |V(t^2)|^p\,dt < \infty$ and I is a compact interval. If there are integers $p \geq 2$, $r > 0$ and a sequence of compactly supported functions V_n so that*

i) *as $n\to\infty$, $\int |V(t^2) - V_n(t^2)|^p\,dt \to 0$; and*

ii) *for each $1 \leq \ell \leq (p-1)$ and every $w \in C^r(\mathbf{R})$ which is supported on I,*

$$\sup_n \left| \int_{[0,\infty)^\ell} K(x_1,\dots,x_\ell;E+i0)V_n(x_1)\cdots V_n(x_\ell)\,dx_1\dots dx_\ell\,w(E)\,dE \right| < \infty$$

then $I \subseteq \Sigma_{\mathrm{ac}}(H+V)$.

Corollary 5.2. *If* $\int |V(t^2)|^2\, dt < \infty$ *then* $\Sigma_{\mathrm{ac}}(H+V) = \mathbf{R}$.

Proof. Let $V_n = \chi_{[0,n]}(x)V(x)$ and let I be a compact interval in $\mathbf{R}$. To show that $I \subset \Sigma_{\mathrm{ac}}(H+V)$, we need only check hypothesis ii) above with $\ell = 1$.

For each $z \in \mathbf{C}^+$, the Weyl solution is proportional to $(x+z)^{-1/2}H^{(1)}_{1/3}\big(\frac{2}{3}(x+z)^{3/2}\big)$ so Lemma 2.3 gives

$$\operatorname{Re} G(x,x;E+i0) = \tfrac{\pi}{3}(x+E)\ \operatorname{Im}\left[H^{(1)}_{1/3}\big(\tfrac{2}{3}(x+E)^{3/2}\big)\right]^2.$$

Because I is compact, there is an $x_0 > 0$ such that Hankel function asymptotics permit us to write

$$\operatorname{Re} G(x,x;E+i0) = (x+E)^{-1/2}\sin\left\{\tfrac{4}{3}(x+E)^{3/2} - \tfrac{5\pi}{6}\right\} + \zeta(x,E) \tag{46}$$

with $\sup_{E\in I}|\zeta(x,E)| \leq Cx^{-3/2}$ for $x \geq x_0$. We also require that $x_0 > -2E$. Let us suppose that $w \in C^1$ then we may estimate

$$\iint V_n(x)\operatorname{Re} G(x,x;E+i0)\, dx\, w(E) dE \tag{47}$$

as follows. First, $\sup_{E\in I, x\in\mathbf{R}} |G(x,x;E+i0)| < \infty$ so

$$\begin{aligned}\iint_0^{x_0} V_n(x)\operatorname{Re} G(x,x;E+i0)\, dx\, w(E) dE &\lesssim \|w\|_{L^1}\int_0^{x_0} |V_n(x)|\, dx \\ &\lesssim \|w\|_{L^1}\left\{\int |V(t^2)|^2\, dt\right\}^{1/2}.\end{aligned}$$

By (46), we can write the remainder of (47) as

$$\begin{aligned}&\iint_{x_0}^{\infty} V_n(x)(x+E)^{-1/2}\sin\left\{\tfrac{4}{3}(x+E)^{3/2} - \tfrac{5\pi}{6}\right\} dx\, w(E) dE \\ &+ \iint_{x_0}^{\infty} V_n(x)\zeta(x,E)\, dx\, w(E) dE.\end{aligned}$$

The absolute value of the second summand is $\lesssim \|w\|_{L^1}\{\int |V_n(t^2)|^2\, dt\}^{1/2}$. For the first summand, we integrate by parts with respect to E using $\frac{d}{dE}\cos\left\{\frac{4}{3}(x+E)^{3/2} - \frac{5\pi}{6}\right\} = -(x+E)^{1/2}\sin\left\{\frac{4}{3}(x+E)^{3/2} - \frac{5\pi}{6}\right\}$ to obtain

$$\iint_{x_0}^{\infty} \cos\left\{\tfrac{4}{3}(x+E)^{3/2} - \tfrac{5\pi}{6}\right\}\frac{(x+E)w'(E) - w(E)}{(x+E)^2} V_n(x)\, dx\, dE.$$

which is $\lesssim \|w\|_{C^1}\int_{x_0}^{\infty}(x+E)^{-1}V_n(x)\, dx \lesssim \|w\|_{C^1}\{\int |V_n(t^2)|^2\, dt\}^{1/2}$ in absolute value. The second inequality used the fact that $x_0 > -2E$.

To recap, we have shown that the absolute value of (47) is bounded by a multiple of $\|w\|_{C^1}\{\int |V_n(t^2)|^2\, dt\}^{1/2}$ and thus verified hypothesis ii) of Theorem 5. This shows $I \subset \Sigma_{\mathrm{ac}}(H+V)$. Since I was arbitrary, $\Sigma_{\mathrm{ac}}(H+V) = \mathbf{R}$. ☐

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