Asymptotic properties of self-avoiding paths on d-dimensional Sierpiński gasket from renormalization group analysis

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Abstract

Notion of self avoiding fixed point, domain of attraction, and critical point are defined for self-avoiding paths on d-dimensional pre-Sierpiński gaskets, existence of which imply existence of limit distributions of scaled path lengths of 'canonical ensemble', exponential growth of path numbers with respect to the length, and exponents for mean square displacement.

The definitions are written in terms of the flows of the associated renormalization-group dynamical system.

We apply the result to prove asymptotic behaviors of a certain class of self-avoiding paths on the 4-dimensional pre-Sierpiński gasket.

1 Introduction.

Self-avoiding walks on hypercubic lattices $\mathbb{Z}^n$ have been studied mathematically for half a century, but compared to random walks (and diffusion processes, their continuum limits), amazingly little is known. For random walks, nice properties such as Markov properties enabled deep and accurate studies, many of which are effective for spaces with any dimension n. On the other hand, self-avoiding walks seem to have little such strong general methods. In fact, their behaviors are expected to vary drastically with the dimension n for small n, so that effective methods possibly vary for different spaces.

Turning our attention to the 2- and 3-dimensional Sierpiński gaskets, there are works on 'restricted' self-avoiding walk (a subset of self-avoiding walks) in the physics literature [1], and mathematically rigorous studies for the full self-avoiding walk (including a proof that the restriced self-avoiding walk of [1] are in the same universality classes – i.e., have the same exponents – as the full self-avoiding walk), with further precise asymptotic results, exist for both the 2-dimensional Sierpiński gasket [5, 4, 7] and the 3-dimensional Sierpiński gasket [6], but since then, no further development appeared, to the authors' knowledge, in the direction of generalizing the methods and results to arbitrary d-dimensional Sierpiński gaskets.

We are therefore interested in the possibility of general theory of asymptotic behaviors of self-avoiding paths on the d-dimensional Sierpiński gasket (dSG) for all d. (It is known that self-avoiding walks on $\mathbb{Z}^n$ with $n \geqq 5$ have same asymptotic behavior as the corresponding simple random walks [2, 3, 9]. The lace expansion is effective for these cases. However, dSG are, from renormalization group point of view, spaces 'between $\mathbb{Z}^1$ and $\mathbb{Z}^2$'.)

It is the purpose of this paper to formulate a generalization of studies on the 2SG and 3SG to dSG for all d.

Previous rigorous studies for $d = 2$ and $d = 3$ (and 'philosophy' of renormalization group in physics) suggest that a study in self-avoiding paths on a d-dimensional Sierpiński gasket is composed of two main ingredeients:

Analysis of flows of renormalization group: To formulate the renormalization group (RG), a dynamical system determined by a recursion map in the scale parameter n, for the generating functions

of the number of steps of the self-avoiding paths with endpoints on vertices of a simplex of size 2^n, and then derive nice properties of the fixed points and trajectories of the RG flows, such as uniqueness of fixed point and convergence of flows.

Asymptotic properties of paths: To derive asymptotic behaviors of the self-avoiding paths from the properties of RG flows.

The explicit form of the RG map becomes rapidly complex as d is increased (for example, see [6, Appendix] for the explicit form of the map for 3SG), and new complexities appear as d is increased. Hence in studying asymptotic behaviors of self-avoiding paths on the dSG, one must first clarify the conditions for asymptotic behaviors of self-avoiding paths, in terms of the properties of RG flows. This corresponds to the latter step in the above classification.

In the studies of random walks and diffusion processes, well-developed machineries such as a theory of Drichlet form does the work. However, self-avoiding walks are non-Markovian, and methods based on Markov properties do not work. On the other hand, the arguments in [5, 4, 7] relied heavily on explicit forms of the renormalization map for $d = 2$ or $d = 3$, and are insufficient for a general theory.

In this paper, we formulate conditions in terms of the renormalization group (RG) flows (Section 2.5), under which, we prove that a set of self-avoiding paths on a d-dimensional pre-Sierpiński gasket has limit distribution of scaled path length, exponential growth of path numbers with respect to the length, and exponent for mean square displacement. Section 2, Section 3, Section 4 are devoted to definitions, main results, and proofs, respectively.

We prove in Section 5 that the assumptions for our main results are satisfied for a certain set of self-avoiding paths in $d = 4$. We note that the previous works for $d = 2$ and $d = 3$ are necessary steps in guiding the present study, as well as those for $d = 4$.

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2 Renormalization group.

2.1 Self-avoiding paths on d-dimensional pre-Sierpiński gasket.

Let $d \geqq 2$ be an integer. We define a d-dimensional pre-Sierpiński gasket (pre-dSG) as follows. Consider a d-simplex of a unit side length embedded in $\mathbb{R}^d$, and let $G_0 = \{v_0, v_1, v_2, \cdots, v_d\}$ be the set of vertices of the d-simplex, where $v_0 = O = (0, 0, \cdots, 0)$ is the origin of $\mathbb{R}^d$. (We may occasionally also write $v_{0,i} = v_i$, $i = 1, \cdots, d$.) Let $B_0 = \{(v_i, v_j) \mid 0 \leqq i < j \leqq d\}$ be the set of non-ordered pairs of vertices, and we denote the pair (G_0, B_0) by F_0.

We define a sequence $F_n = (G_n, B_n)$, $n = 1, 2, 3, \cdots$, of finite pre-dSG inductively by

$$G_{n+1} = \bigcup_{i=0}^{d} (G_n + 2^n v_i), \ B_{n+1} = \bigcup_{i=0}^{d} (B_n + 2^n v_i), \quad n = 1, 2, 3, \cdots, \tag{1}$$

where we write $A + v = \{x + v \mid x \in A\}$ for a set A and a point v.

F_n is a d-simplex of side length 2^n, composed of $d + 1$ copies of F_{n-1}, with $d + 1$ outmost points being $v_{n,0} = O$ and $v_{n,i} = 2^n v_i$, $i = 1, 2, 3, \cdots, d$. G_n is a collection of vertices in the copies of G_{n-1}, and B_n is a collection of bonds in the copies of B_{n-1}.

We call

$$F = (G, B); \quad G = \bigcup_{n=0}^{\infty} G_n, \ B = \bigcup_{n=0}^{\infty} B_n, \tag{2}$$

the d-dimensional pre-Sierpiński gasket (pre-dSG). We identify $(v, v') \in B$ with line segments $\overline{vv'}$ whenever it would be natural to do so.

Denote the set of non-negative integers by $\mathbb{Z}_+$, and for $w : \mathbb{Z}_+ \to G$, denote by $L(w) \in \mathbb{Z}_+ \cup \{\infty\}$ ('the length of w') the smallest integer satisfying

$$w(i) = w(L(w)), \quad i \geqq L(w).$$

Define the set of self-avoiding paths W_0 to be the set of maps $w:\ \mathbb{Z}_+ \to G$, such that

$$\begin{array}{ll} w(i_1) \neq w(i_2), & 0 \leqq i_1 < i_2 \leqq L(w), \\ |w(i) - w(i+1)| = 1, & 0 \leqq i \leqq L(w) - 1, \\ \overline{w(i)w(i+1)} \in B, & 0 \leqq i \leqq L(w) - 1\,. \end{array}$$

2.2 Classification of self-avoiding paths.

Denote by $\mathcal{T}_b$ the family of all the translations of B_0 that are subsets of B. Namely, $\mathcal{T}_b$ contains all the unit d-simplices which compose the pre-dSG. (with each simplex regarded as a collection of bonds). Put

$$\begin{aligned} \mathcal{I}_d = \{(i_1, i_2, \cdots, i_k) \in \mathbb{Z}_+^k \mid k = 1, 2, 3, \cdots,\ 0 < i_1 \leqq i_2 \leqq \cdots \leqq i_k\,, \\ i_1 + i_2 + \cdots + i_k + k \leqq d + 1\}, \end{aligned} \tag{3}$$

and denote the number of elements of $\mathcal{I}_d$ by $f_d = \sharp\mathcal{I}_d$.

Proposition 1 *Let $w \in W_0$ and $\Delta \in \mathcal{T}_b$, and consider a set of bonds*

$$A = \{\overline{w(i)w(i+1)} \in \Delta \mid i = 0, 1, 2, \cdots, L(w)\}.$$

If A is not empty, then there exists $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_d$ such that A is congruent to

$$\Delta_I = \{\overline{Ov_1v_2\cdots v_{i_1-1}v_{i_1}}, \overline{v_{i_1+1}\cdots v_{i_1+i_2}}, \cdots, \overline{v_{i_1+\cdots i_{k-1}+1}\cdots v_{i_1+\cdots i_k}}\}, \tag{4}$$

where we used an abbreviation such as

$$\overline{Ov_1v_2\cdots v_{i_1-1}v_{i_1}} = \overline{Ov_1}, \overline{v_1v_2}, \overline{v_2v_3}, \cdots, \overline{v_{i_1-1}v_{i_1}}\,.$$

Example.

- $\mathcal{I}_2 = \{(1), (2)\}$: $A \neq \emptyset$ is congruent to either $\{\overline{Ov_1}\}$ or $\{\overline{Ov_1v_2}\}$.
- $\mathcal{I}_3 = \{(1), (2), (3), (1,1)\}$: There is a possibility that a path enters a unit tetrahedron twice, as $\{\overline{Ov_1}, \overline{v_2v_3}\}$.
- $\mathcal{I}_4 = \{(1), (2), (3), (4), (1,1), (1,2)\}$.

Correspondingly, $f_2 = 2$, $f_3 = 4$, $f_4 = 6$. ◇

Proof of Proposition 1. If $A \neq \emptyset$, namely, if the path w enters the unit d-simplex specified by Δ, then A is composed of one or more connected clusters. That is, w may pass through Δ and may come back and reenter Δ. Since w is self-avoiding, the second passage does not intersect with the first one. Thus we can classify A by the size of the connected segments. One may rearrange the segments in an increasing order of size, hence each class is determined by an increasing finite sequence of positive integers, $i_1 \leqq i_2 \leqq i_3 \leqq \cdots \leqq i_k$ for some $k \geqq 1$. The meaning of the conditions in the definition of $\mathcal{I}_d$ should now be obvious. Since Δ is a translation of B_0 which is the set of bonds in the unit d-simplex $Ov_1v_2\cdots v_d$, the statement follows. □

In analogy with Proposition 1 we can classify the set of self-avoiding paths on F_n by $\mathcal{I}_d$, and also generalize to two or more self-avoiding paths.

For $n \in \mathbb{Z}_+$ and $u, v \in G_n$, define $W^{(n,u,v)}$ by

$$W^{(n,u,v)} = \{w \in W_0 \mid w(0) = u,\ w(L(w)) = v, \quad w(i) \in G_n\,,\ i \in \mathbb{Z}_+\}.$$

For $n \in \mathbb{Z}_+$ and $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_d$, define $W_I^{(n)}$ by

$$\begin{aligned} W_I^{(n)} = \{(w_1, w_2, \cdots, w_k) \in W^{(n,O,v_{n,i_1})} \times W^{(n,v_{n,i_1+1},v_{n,i_1+i_2+1})} \times W^{(n,v_{n,i_1+i_2+2},v_{n,i_1+i_2+i_3+2})} \\ \times \cdots \times W^{(n,v_{n,i_1+i_2+\cdots+i_{k-1}+k-1},v_{n,i_1+i_2+\cdots+i_k+k-1})} \mid \\ \text{if } i \neq j \text{ then } w_i \text{ and } w_j \text{ do not hit common points, and for each } j \\ w_j \text{ hits points } v_{n,i_1+i_2+\cdots+i_{j-1}+j-1},\ v_{n,i_1+i_2+\cdots+i_{j-1}+j},\ v_{n,i_1+i_2+\cdots+i_{j-1}+j+1}, \\ \cdots\ v_{n,i_1+i_2+\cdots+i_{j-1}+i_j+j-1},\ \text{in this order,} \\ \text{but hits no other points in } \{v_{n,\ell} \mid \ell = 0, 1, 2, \cdots, d\}\}. \end{aligned} \tag{5}$$

Obviously, k is equal to the number of path segments that form an element in $W_I^{(n)}$.

Example. For $d = 4$, there are $f_4 = 6$ types of sets $W_I^{(n)}$, which are

$\{(1)\}$: Set of paths from O to $v_{n,1}$ which do not hit $v_{n,2}$, $v_{n,3}$, $v_{n,4}$.

$\{(2)\}$: Set of paths from O to $v_{n,2}$ passing through $v_{n,1}$ which do not hit $v_{n,3}$, $v_{n,4}$.

$\{(3)\}$: Set of paths from O to $v_{n,3}$ passing through $v_{n,1}$ and $v_{n,2}$ in this order and avoiding $v_{n,4}$.

$\{(4)\}$: Set of paths from O to $v_{n,4}$ passing through $v_{n,1}$, $v_{n,2}$, and $v_{n,3}$ in this order.

$\{(1,1)\}$: Set of pair of (self- and mutually-avoiding) paths, one from O to $v_{n,1}$ and the other from $v_{n,2}$ to $v_{n,3}$ neither hitting $v_{n,4}$.

$\{(1,2)\}$: Set of pair of paths, one from O to $v_{n,1}$ and the other from $v_{n,2}$ to $v_{n,4}$ via $v_{n,3}$.

$\diamond$

For $w \in \bigcup_{n\in\mathbb{Z}_+} \bigcup_{I\in\mathcal{I}_d} W_I^{(n)}$ denote by $\hat{w}$ the set of bonds which w passes. Namely, for $n \in \mathbb{Z}_+$ and $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_d$, and for $w = (w_1, w_2, \cdots, w_k) \in W_I^{(n)}$,

$$\hat{w} = \{\overline{w_j(i)w_j(i+1)} \in B \mid i = 0, 1, 2, \cdots, L(w_j) - 1,\ j = 1, 2, \cdots, k\}.$$

Also define $S_I(w)$, $I \in \mathcal{I}_d$, by,

$$S_I(w) = \{\Delta \in \mathcal{T}_b \mid \hat{w} \cap \Delta \ \text{ is congruent to } \ \Delta_I \ \text{ of (4)}\}, \tag{6}$$

and denote by $s_I(w) = \sharp S_I(w)$, the cardinality of $S_I(w)$. $s_I(w)$ is the number of unit d-simplices in F such that the trajectory of the path (or the paths) w is congruent to Δ_I. It is a generalized notion of the length of the path in the sense that

$$\sum_{i=1}^{k} L(w_i) = \sum_{J\in\mathcal{I}_d} |J| s_J(w), \quad w = (w_1, w_2, \cdots, w_k) \in W_I^{(n)},\ I = (i_1, \cdots, i_k) \in \mathcal{I}_d,\ n \in \mathbb{Z}_+, \tag{7}$$

where, for $J \in \mathcal{I}_d$ we define $|J|$, the length of J, by

$$|J| = j_1 + \cdots + j_\ell, \quad \text{if} \quad J = (j_1, \cdots, j_\ell). \tag{8}$$

2.3 Parameter space and renormalization group.

Assumptions of the main results are stated in terms of the flows of the associated renormalization group (RG), which is a map (discrete-time dynamical system) in a parameter space of variables in the generating function of generalized path length $(s_J,\ J \in \mathcal{I}_d)$. The dynamical system is derived as the response in the parameter space to the change in n. A graphical property of dSG called finite ramifideness emphasizes that the RG is a finite dimensional dynamical system.

Define the generating function

$$\vec{X}_n = (X_{n,I},\ I \in \mathcal{I}_d) :\ \mathbb{C}^{\mathcal{I}_d} \to \mathbb{C}^{\mathcal{I}_d}$$

of $(s_J,\ J \in \mathcal{I}_d)$ for a family of paths sets $(W_I^{(n)},\ I \in \mathcal{I}_d)$, by,

$$X_{n,I}(\vec{x}) = \sum_{w\in W_I^{(n)}} \prod_{J\in\mathcal{I}_d} x_J^{s_J(w)}, \quad \vec{x} = (x_J,\ J \in \mathcal{I}_d) \in \mathbb{C}^{\mathcal{I}_d},\ n = 0, 1, 2, \cdots. \tag{9}$$

The right hand side is a finite summation, so $X_{n,I}$ is defined on $\mathbb{C}^{\mathcal{I}_d}$.

The starting point of our analysis is the following.

Proposition 2 *$\vec{X}_n = (X_{n,I},\ I \in \mathcal{I}_d)$, $n = 0, 1, 2, \cdots$, satisfy the following recursion relations.*

$$\vec{X}_0(\vec{x}) = \vec{x},\ \vec{x} \in \mathbb{C}^{\mathcal{I}_d},$$

and

$$\vec{X}_{n+1} = \vec{\Phi} \circ \vec{X}_n \,, \tag{10}$$

where

$$\vec{\Phi} = (\Phi_I,\ I \in \mathcal{I}_d) = \vec{X}_1 \,,$$

is a f_d dimensional vector valued function whose components are polynomials in f_d variables with positive integer coefficients. In particular, $\mathbb{R}_+^{\mathcal{I}_d}$ is an invariant set of $\vec{\Phi}$.

The degree of each term in the polynomials are no less than 2 and no greater than $d+1$, and $\Phi_{(1)}$ contains terms ${x_{(1)}}^2$ and ${x_{(1)}}^{d+1}$.

Proof. Let $n \in \mathbb{Z}_+$. F_1 is composed of $d+1$ d-simplices congruent to F_0. Similarly, F_{n+1} is composed of $d+1$ d-simplices F_n. The similarity of the two compositions leads to a natural map

$$\pi :\ W_I^{(n+1)} \to W_I^{(1)} \,, \quad I \in \mathcal{I}_d.$$

For each $X_{n+1,I}$, classify the summation in the right hand side of (9) (with $n+1$ in place of n) by $\pi(w) \in W_I^{(1)}$ to find

$$X_{n+1,I}(\vec{x}) = \sum_{w \in W_I^{(n+1)}} \prod_{J \in \mathcal{I}_d} x_J^{s_J(w)} = \sum_{w' \in W_I^{(1)}} \sum_{w \in W_I^{(n+1)};\, \pi(w)=w'} \prod_{J \in \mathcal{I}_d} x_J^{s_J(w)}$$

$$= \sum_{w' \in W_I^{(1)}} \prod_{I' \in \mathcal{I}_d} \left(\sum_{w'' \in W_{I'}^{(n)}} \prod_{J \in \mathcal{I}_d} x_J^{s_J(w'')} \right)^{s_{I'}(w')} = \sum_{w' \in W_I^{(1)}} \prod_{I' \in \mathcal{I}_d} (X_{n,I'}(\vec{x}))^{s_{I'}(w')} = X_{1,I}(\vec{X}_n(\vec{x})).$$

By definition (9), each term in $\Phi_I = X_{1,I}$ has a form $\displaystyle\prod_{J \in \mathcal{I}_d} x_J^{s_J(w)}$, hence its degree $\displaystyle\sum_{J \in \mathcal{I}_d} s_J(w)$ is, by definition (6), the number of unit simplices in F_1 that a path w passes through. This is bounded from above by the total number of unit simpices in F_1, which is $d+1$, and from below by 2, because any two (outmost) vertices of F_1 is apart by length 2.

Positivity of coefficients of $X_{1,I}$ are obvious. Existence of terms $x_{(1)}^2$ and $x_{(1)}^{d+1}$ in $\Phi_{(1)} = X_{1,(1)}$ follows from the paths $\overline{Ov_{0,1}v_{1,1}}$ and $\overline{Ov_{0,d}(v_{0,d}+v_{0,d-1})(v_{0,d-1}+v_{0,d-2})\cdots(v_{0,2}+v_{0,1})v_{1,1}}$ in $W_{(1)}^{(1)}$. □

Large n means that the endpoints of the paths are far apart, hence it corresponds to large path length L. Intuitively speaking Proposition 2 therefore gives a response to the change in the length scale of the system in consideration, the sets of self-avoiding paths, in terms of the parameter space of variables in the generating functions of s_I, the generalized path length. Global properties of the trajectories of the map $\vec{\Phi}$ therefore is expected to give (and we will show that it does) large length asymptotic behaviors of self-avoiding paths on dSG.

In analogy to the (mathematically misleading) terminology in physics literature, we call the discrete-time dynamical system on $\mathbb{R}_+^{\mathcal{I}_d}$ defined by the map $\vec{\Phi}$, the renormalization group (RG).

2.4 Invariant sets.

If there is a subset of $\mathbb{R}_+^{\mathcal{I}_d}$ which is an invariant set of the RG map $\vec{\Phi}$, then the recursion (9) is naturally regarded as a recursion equation on the subset.

For $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_\ell)$ in $\mathcal{I}_d$, denote by $I \oplus J$ the rearrangement of $i_1, \cdots, i_k, j_1, \cdots, j_\ell$ in non-decreasing order, and define $\Xi_d \subset \mathbb{R}_+^{\mathcal{I}_d}$ by

$$\Xi_d = \{\vec{x} \in \mathbb{R}_+^{\mathcal{I}_d} \mid x_{I \oplus J} \leqq x_I x_J \ \text{ for all } \ I, J \in \mathcal{I}_d \ \text{ such that } \ I \oplus J \in \mathcal{I}_d\}. \tag{11}$$

Example. $\Xi_3 = \{\vec{x} \in \mathbb{R}_+^{\mathcal{I}_3} \mid x_{(11)} \leqq x_{(1)}^2\}$, $\Xi_4 = \{\vec{x} \in \mathbb{R}_+^{\mathcal{I}_4} \mid x_{(11)} \leqq x_{(1)}^2,\ x_{(12)} \leqq x_{(1)} x_{(2)}\}$. ◇

Proposition 3 *Ξ_d is an invariant set of $\vec{\Phi}$.*

Proof. Let $I, J, I \oplus J \in \mathcal{I}_d$. Note that there is a natural one-to-one into map $W^{(1)}_{I\oplus J} \to W^{(1)}_I \times W^{(1)}_J$. For $w \in W^{(1)}_{I\oplus J}$, let $(w_1, w_2) \in W^{(1)}_I \times W^{(1)}_J$ be the corresponding pair. Then, for each $\Delta \in \mathcal{T}_b$, $\hat{w} \cap \Delta$ may be regarded as a composition of $\hat{w}_1 \cap \Delta$ and $\hat{w}_2 \cap \Delta$, hence if $\hat{w} \cap \Delta$ is congruent to Δ_K of (4) for some $K \in \mathcal{I}_d$, then there exists $K_1, K_2 \in \mathcal{I}_d$ (allowing an emptyset) such that $K = K_1 \oplus K_2$ and such that $\hat{w}_i \cap \Delta$, $i = 1, 2$, is congruent to Δ_{K_i}, $i = 1, 2$, respectively. Note also that $\vec{x} \in \Xi_d$ implies $x_{K_1 \oplus K_2} \leqq x_{K_1} x_{K_2}$.

Therefore by definition (Proposition 2 and (9)),

$$\Phi_{I\oplus J}(\vec{x}) = \sum_{w \in W^{(1)}_{I\oplus J}} \prod_{K \in \mathcal{I}_d} {x_K}^{s_K(w)} \leqq \sum_{w_1 \in W^{(1)}_I} \sum_{w_2 \in W^{(1)}_J} \prod_{K_1 \in \mathcal{I}_d} {x_{K_1}}^{s_{K_1}(w_1)} \prod_{K_2 \in \mathcal{I}_d} {x_{K_2}}^{s_{K_2}(w_2)} = \Phi_I(\vec{x})\Phi_J(\vec{x}).$$

□

In the following, for $\mathcal{K} \subset \mathcal{I}_d$, we use a (somewhat irregular) notation

$$\mathbb{R}^{\mathcal{K}}_+ = \{\vec{x} \in \mathbb{R}^{\mathcal{I}_d}_+ \mid x_J = 0,\ J \notin \mathcal{K}\} \ \subset\ \mathbb{R}^{\mathcal{I}_d}_+. \tag{12}$$

We also write $\mathbb{C}^{\mathcal{K}} \subset \mathbb{C}^{\mathcal{I}_d}$, $\mathbb{Z}^{\mathcal{K}}_+ \subset \mathbb{Z}^{\mathcal{I}_d}_+$, etc.

Define

$$\mathcal{K}_{res} = \{(1), (11), \cdots, (1\cdots 1)\}. \tag{13}$$

The indices in $\mathcal{K}_{res}$ correspond to those paths which go out of a simplex after single step passage each time they enter the simplex.

Proposition 4 *$\mathbb{R}^{\mathcal{K}_{res}}_+$ is an invariant subset of $\vec{\Phi}$.*

Proof. This is proved by generalizing the arguments in the proof of [6, Proposition 2.1 (4)(5)]. □

2.5 Fixed point and domain of attraction.

Based on experiences with dSG for $d = 2, 3, 4$, we define notions which are relevant for asymptotic behaviors of self-avoiding paths on dSG.

Denote the Jacobi matrix of $\vec{\Phi}$ in Proposition 2 by $\mathcal{J} = (\mathcal{J}_{IJ})$:

$$\mathcal{J}_{IJ}(\vec{x}) = \frac{\partial \Phi_I}{\partial x_J}(\vec{x}), \quad I, J \in \mathcal{I}_d,\ \vec{x} \in \mathbb{C}^{\mathcal{I}_d}. \tag{14}$$

We say that $\vec{x}_c$ is a *self-avoiding fixed point*, if the following hold.

(FP1) $\vec{\Phi}(\vec{x}_c) = \vec{x}_c$.

(FP2) $\mathcal{J}(\vec{x}_c)$ in (14) is diagonalizable by an invertible matrix. The eigenvalue λ which is largest in absolute value satisfies $\lambda > 1$ and all the other eigenvalues have absolute values strictly less than 1.

Denote by $\vec{v}_L = (v_{L,I},\ I \in \mathcal{I}_d)$ a left eigenvector of $\mathcal{J}(\vec{x}_c)$ corresponding to λ;

$$\sum_{I \in \mathcal{I}_d} v_{L,I} \mathcal{J}_{IJ}(\vec{x}_c) = \lambda v_{L,J}, \quad J \in \mathcal{I}_d,$$

which we chose to have non-negative components (possible, thanks to Fröbenius' theorem). Then $v_{L,J} > 0$, $J \in \mathcal{I}_d$.

Similarly, denote by $\vec{v}_R$ a right eigenvector corresponding to λ with non-negative components;

$$\sum_{J \in \mathcal{I}_d} \mathcal{J}_{IJ}(\vec{x}_c) v_{R,J} = \lambda v_{R,I}, \quad I \in \mathcal{I}_d.$$

Then $v_{R,(1)} > 0$.

(FP3) For all $I \in \mathcal{I}_d$ such that $x_{c,I} \neq 0$, there exists $m \in \mathbb{Z}^{\mathcal{I}_d}_+$, satisfying $m_{(1)} > 0$ and $m_J = 0$ if $x_{c,J} = 0$, such that there is a term $\prod_{J \in \mathcal{I}_d} {x_J}^{m_J}$ in Φ_I.

(FP4) $\vec{x}_c \in \Xi_d \setminus \{\vec{0}\}$.

We first note some simple consequences of the definitions.

Proposition 5 *Let $\vec{x}_c$ be a self-avoiding fixed point and λ be as in (FP2). Then $2 < \lambda < d+1$.*

Proof. Proposition 2 and (14) imply

$$2\Phi_I(\vec{x}) < \sum_{J \in \mathcal{I}_d} \mathcal{J}_{IJ}(\vec{x}) x_J < (d+1)\Phi_I(\vec{x}), \quad \vec{x} \in \mathbb{R}_+^{\mathcal{I}_d};\ x_{(1)} > 0\,.$$

This, with (FP1) and (FP2), further implies

$$2\sum_{I \in \mathcal{I}_d} v_{L,I} x_{c,I} < \lambda \sum_{J \in \mathcal{I}_d} v_{L,J} x_{c,J} < (d+1) \sum_{I \in \mathcal{I}_d} v_{L,I} x_{c,I}\,,$$

which implies the statement.

□

Note that if $\vec{x}_c$ is a self-avoiding fixed point then Proposition 2 and (FP1) imply

$$\vec{X}_n(\vec{x}_c) = \vec{x}_c\,, \quad n \in \mathbb{Z}_+, \tag{15}$$

For $n \in \mathbb{Z}_+$, denote the Jacobi matrices of $\vec{X}_n$ by $\mathcal{J}_n$:

$$\mathcal{J}_{n,IJ}(\vec{x}) = \frac{\partial X_{n,I}}{\partial x_J}(\vec{x}), \quad I, J \in \mathcal{I}_d\,,\ \vec{x} \in \mathbb{C}^{\mathcal{I}_d}\,. \tag{16}$$

In particular, we have, from (14), $\mathcal{J}_1 = \mathcal{J}$. Proposition 2 implies

$$\mathcal{J}_n(\vec{x}) = \mathcal{J}(\vec{X}_{n-1}(\vec{x})) \cdot \mathcal{J}(\vec{X}_{n-2}(\vec{x})) \cdots \mathcal{J}(\vec{X}_1(\vec{x})) \cdot \mathcal{J}(\vec{x}), \quad n \in \mathbb{Z}_+, \tag{17}$$

which further implies with (15),

$$\mathcal{J}_n(\vec{x}_c) = \mathcal{J}(\vec{x}_c)^n, \quad n \in \mathbb{Z}_+. \tag{18}$$

We note a relation between 0 components of a fixed point and an invariant subset of $\vec{\Phi}$. In the known case of $d = 2$ and $d = 3$, the self-avoiding fixed points have 0 components. We will write

$$\mathcal{K}_{\vec{x}_c} = \{I \in \mathcal{I}_d \mid x_{c,I} \neq 0\}, \tag{19}$$

and, as in (12),

$$\mathbb{R}_+^{\mathcal{K}_{\vec{x}_c}} = \{\vec{x} \in \mathbb{R}_+^{\mathcal{I}_d} \mid x_J = 0,\ J \notin \mathcal{K}_{\vec{x}_c}\}\,.$$

Proposition 6 *$\mathbb{R}_+^{\mathcal{K}_{\vec{x}_c}}$ is an invariant subset of $\vec{\Phi}$.*

Proof. If $x_{c,I} = 0$ then $\Phi_I(\vec{x}_c) = x_{c,I} = 0$. On the other hand Φ_I is a polynomial with positive coefficients. Therefore, each term in $\Phi_I(\vec{x})$ contains one of x_J's such that $x_{c,J} = 0$. In other words, each term in $\Phi_I(\vec{x})$ contains x_J such that $J \notin \mathcal{K}_{\vec{x}_c}$.

Therefore, if $\vec{x} \in \mathbb{R}_+^{\mathcal{K}_{\vec{x}_c}}$, then $\Phi_I(\vec{x}) = 0$ for those I satisfying $x_{c,I} = 0$, or equivalently, $I \notin \mathcal{K}_{\vec{x}_c}$. □

Remark. For $d = 2$ and $d = 3$ it is known that the self-avoiding fixed point $\vec{x}_c$ is unique and that $\mathcal{K}_{res} = \mathcal{K}_{\vec{x}_c}$. ◇

Let $\vec{x}_c$ be a self-avoiding fixed point. We say that $\vec{x} \in \mathbb{R}_+^{\mathcal{I}_d}$ is *in the domain of attraction of $\vec{x}_c$*, if the following hold.

(DA1) $\lim_{n\to\infty} \vec{X}_n(\vec{x}) = \vec{x}_c$.

(DA2) If $x_{c,I} \neq 0$ then $x_I \neq 0$.

We denote by $\mathcal{D}om(\vec{x}_c)$ the set of $\vec{x} \in \mathbb{R}_+^{\mathcal{I}_d}$ which are in the domain of attraction of $\vec{x}_c$.

Example. If $\vec{x}_c$ is a self-avoiding fixed point, then $\vec{x}_c \in \mathcal{D}om(\vec{x}_c)$, i.e., a self-avoiding fixed point satisfies (DA1) – (DA2).

$\diamond$

Let $\mathcal{K} \subset \mathcal{I}_d$. Instead of (5), we may consider a set of walks $W_{\mathcal{K},I}^{(n)}$ by restricting to those paths in $W_I^{(n)}$ which satisfy $s_J(w) = 0$ if $J \notin \mathcal{K}$:

$$W_{\mathcal{K},I}^{(n)} = \{w \in W_I^{(n)} \mid s_J(w) = 0,\ J \notin \mathcal{K}\}. \tag{20}$$

If $\mathcal{K} = \mathcal{I}_d$, then we are dealing with the original (full) model; $W_{\mathcal{I}_d,I}^{(n)} = W_I^{(n)}$. We define the corresponding generating functions by

$$X_{\mathcal{K},n,I}(\vec{x}) = \sum_{w \in W_{\mathcal{K},I}^{(n)}} \prod_{J \in \mathcal{K}} x_J^{s_J(w)}, \quad \vec{x} = (x_J,\ J \in \mathcal{K}) \in \mathbb{C}^{\mathcal{K}},\ n = 0, 1, 2, \cdots,\ I \in \mathcal{I}_d. \tag{21}$$

If $\mathbb{R}_+^{\mathcal{K}}$ is an invariant subset of $\mathbb{R}_+^{\mathcal{I}_d}$, then $\vec{X}_{\mathcal{K},n}$ satisfy (10), with a convention that the components corresponding to $J \notin \mathcal{K}$ are 0.

For $\mathcal{K} \subset \mathcal{I}_d$ and $\beta \in \mathbb{R}$, define $\vec{x}_{can,\mathcal{K}}(\beta) = (x_{can,\mathcal{K},I}(\beta),\ I \in \mathcal{I}_d)$ by

$$x_{can,\mathcal{K},I}(\beta) = \begin{cases} e^{-\beta|I|}, & I \in \mathcal{K}, \\ 0, & I \notin \mathcal{K}, \end{cases} \tag{22}$$

where $|I|$ is defined in (8). Following the notions in statistical mechanics, the partition function for a set of self-avoiding paths specified by $\mathcal{K}$ is defined by $\vec{Z}_{\mathcal{K},n} = (Z_{\mathcal{K},n,I},\ I \in \mathcal{I}_d)$, with

$$Z_{\mathcal{K},n,I}(\beta) = \sum_{w \in W_{\mathcal{K},I}^{(n)}} e^{-\beta L(w)}, \quad \beta \in \mathbb{R},\ n = 0, 1, 2, \cdots. \tag{23}$$

With (7), we see that

$$Z_{\mathcal{K},n,I}(\beta) = X_{\mathcal{K},n,I}(\vec{x}_{can,\mathcal{K}}(\beta)). \tag{24}$$

If $\mathcal{K} = \mathcal{I}_d$ we also use an abbreviation $\vec{x}_{can}(\beta) = \vec{x}_{can,\mathcal{I}_d}(\beta)$ and $\vec{Z}_n(\beta) = \vec{Z}_{\mathcal{I}_d,n}(\beta)$. Hence

$$Z_{n,I}(\beta) = X_{n,I}(\vec{x}_{can}(\beta)) = \sum_{w \in W_I^{(n)}} e^{-\beta L(w)}, \quad \beta \in \mathbb{R},\ n = 0, 1, 2, \cdots. \tag{25}$$

In the following, the set of paths in (20) with $\mathcal{K} = \mathcal{K}_{res}$ will be called the restricted self-avoiding paths, the corresponding generating function (21), the generating function for the restricted model, and so on.

We need the following additional definitions for Theorem 9.

(CS1) We say that $\beta_c \in \mathbb{R}$ is a critical point of the full model if $\vec{x}_{can}(\beta_c) \in \mathcal{D}om(\vec{x}_c)$ for a self-avoiding fixed point $\vec{x}_c$.

(CS2) We say that $\beta_{c,res} \in \mathbb{R}$ is a critical point of the restricted model if $\vec{x}_{can,\mathcal{K}_{res}}(\beta_{c,res}) \in \mathcal{D}om(\vec{x}_c)$ for a self-avoiding fixed point $\vec{x}_c$.

Note that by definition (11),

$$\vec{x}_{can}(\beta_c) \in \Xi_d, \quad \text{and} \quad \vec{x}_{can,\mathcal{K}_{res}}(\beta_{c,res}) \in \Xi_d. \tag{26}$$

3 Main results.

First we note the following characterization of a critical point β_c.

Theorem 7 *If $\beta_c \in \mathbb{R}$ is a critical point of the full model and $\vec{x}_c$ the corresponding fixed point (implicit in the definition (CS1)), then for $I \in \mathcal{I}_d$,*

$$\lim_{n\to\infty} Z_{n,I}(\beta) = \begin{cases} 0, & \beta > \beta_c\,, \\ x_{c,I}\,, & \beta = \beta_c\,. \end{cases}$$

Moreover,

$$\lim_{n\to\infty} \sum_{i=1}^{d} Z_{n,(i)}(\beta) = \infty, \quad \beta < \beta_c\,.$$

In particular, critical point (if exists) is unique.
Similar result holds also for the restricted model (CS2).

Remark. Since the critical point (if exists) is unique, there is one and only one self-avoiding fixed point that is related by (CS1) to the critical point.

Though it is not trivial to prove the uniqueness of self-avoiding fixed point, we therefore can (and we will, in the proof of Theorem 9) talk about the unique self-avoiding fixed point that is related to the critical point, under the asumption that the critical point exists. ◇

For $I \in \mathcal{I}_d$, $n \in \mathbb{Z}_+$, and $\vec{x} \in \mathbb{R}_+^{\mathcal{I}_d}$, define a probability measure $\mu_{\vec{x},n,I}$ on the finite set $W_I^{(n)}$ by

$$\mu_{\vec{x},n,I}[\{w\}] = \frac{1}{X_{n,I}(\vec{x})} \prod_{J\in\mathcal{I}_d} x_J^{s_J(w)}, \quad w \in W_I^{(n)}, \tag{27}$$

whenever $X_{n,I}(\vec{x}) \neq 0$.

Note that if $x_{c,I} \neq 0$ and $\vec{x} \in \mathcal{D}om(\vec{x}_c)$, then (DA1) implies that $X_{n,I}(\vec{x}) > 0$ for sufficiently large n, hence if $\vec{x} \in \mathcal{D}om(\vec{x}_c)$ then $\mu_{\vec{x},n,I}$ is well defined.

Theorem 8 *Let $\vec{x}_c$ be a self-avoiding fixed point and $\vec{x} \in \mathcal{D}om(\vec{x}_c)$. Then the following hold.*

(i) There exists $f_d \times f_d$ matrix $\Lambda(\vec{x})$ whose elements are non-negative such that

$$\Lambda(\vec{x}) = \lim_{n\to\infty} \lambda^{-n} \mathcal{J}_n(\vec{x}), \tag{28}$$

where $\mathcal{J}_n$ is as in (14).

(ii) For $I \in \mathcal{K}_{\vec{x}_c}$, the joint distribution of scaled generalized path lengths $(\lambda^{-n} s_J\,,\ J \in \mathcal{K}_{\vec{x}_c})$ under $\mu_{\vec{x},n,I}$ converges weakly to a Borel probability measure $p^*_{\vec{x},I}$ on $\mathbb{R}^{\mathcal{I}_d}$ as $n \to \infty$. Here, λ is as in (FP2). $p^*_{\vec{x},I}$ is supported on $\mathbb{R}_+^{\mathcal{I}_d}$.

The generating function $\varphi^*_I = \varphi^*_{\vec{x},I}$, as a function of $(t_J,\ J \in \mathcal{K}_{\vec{x}_c})$, defined by

$$\varphi^*_{\vec{x},I}(\vec{t}) = \int_0^\infty e^{\vec{t}\cdot\vec{\xi}} p^*_{\vec{x},I}[d\vec{\xi}], \quad \vec{t} \in \mathbb{C}^{\mathcal{K}_{\vec{x}_c}}\,,$$

is an entire function in $\vec{t}$.

(iii) The set of functions $\varphi^*_I = \varphi^*_{\vec{x},I}$, $I \in \mathcal{K}_{\vec{x}_c}$, are uniquely determined by

$$\begin{aligned} &x_{c,I} \frac{\partial \varphi^*_I}{\partial t_J}(\vec{0}) = \Lambda_{IJ}(\vec{x})\, x_J\,, \quad \text{if } I,\, J \in \mathcal{K}_{\vec{x}_c}\,, \\ &x_{c,I} \varphi^*_I(\lambda \vec{t}) = \Phi_I(\vec{x}_c\, \vec{\varphi}^*(\vec{t})), \quad \vec{t} \in \mathbb{C}^{\mathcal{K}_{\vec{x}_c}}, \quad \text{if } I \in \mathcal{K}_{\vec{x}_c}\,, \end{aligned} \tag{29}$$

where we define $\varphi^*_J = 0$ for $J \notin \mathcal{K}_{\vec{x}_c}$, and in the variable for $\vec{\Phi}$ we used an (irregular) notation

$$(\vec{x}\,\vec{\varphi}^*(\vec{t}))_J = x_J\, \varphi^*_J(\vec{t}), \quad J \in \mathcal{I}_d\,.$$

(iv) If $\vec{x} \in \mathcal{D}om(\vec{x}_c) \cap \Xi_d$ and $I \in \mathcal{K}_{\vec{x}_c}$, then the distribution of $\lambda^{-n} L(w)$, the scaled length of $w \in W_I^{(n)}$, under $\mu_{\vec{x},n,I}$ converges weakly to a Borel probability measure $\bar{p}^*_{\vec{x},I}$, which has a C^∞ density $\bar{\rho}^*_{\vec{x},I}$.

In particular, $\bar{\rho}^*_{\vec{x},(1)}(\xi) > 0$, $\xi > 0$.

Remark. (i) If $x_{c,I} = 0$, then the right hand sides in (29) are 0 because of Proposition 6 and (28), hence the equations in (29) are trivially correct for if $I \notin \mathcal{K}_{\vec{x}_c}$.

(ii) Existence and positivity of density is used in Theorem 9.

$\diamond$

We move on to the results on paths with step numbers fixed, instead of paths with endpoints fixed.

We denote the self-avoiding paths starting from origin O by $W^{(0)}$: $W^{(0)} = \{w \in W_0 \mid w(0) = O\}$. Also, we define, in analogy with (20), $W_{\mathcal{K}}^{(0)} = \{w \in W^{(0)} \mid s_J(w) = 0,\ J \notin \mathcal{K}\}$, for $\mathcal{K} \subset \mathcal{I}_d$.

For each $k \in \mathbb{Z}_+$, let

$$N(k) = \sharp\{w \in W^{(0)} \mid L(w) = k\}$$

be the number of self-avoiding paths of length k starting from O, and for $\mathcal{K} \subset \mathcal{I}_d$,

$$N_{\mathcal{K}}(k) = \sharp\{w \in W_{\mathcal{K}}^{(0)} \mid L(w) = k\}.$$

Theorem 9 *If there exists a critical point $\beta_c \in \mathbb{R}$ of the full model, then*

$$\lim_{k\to\infty} \frac{1}{k} \log N(k) = \beta_c .$$

Similarly, if there exists a critical point $\beta_{c,res} \in \mathbb{R}$ of the restricted model, then $\lim_{k\to\infty} \frac{1}{k} \log N_{\mathcal{K}_{res}}(k) = \beta_{c,res}$.

For each positive integer k, let $\tilde{P}_k$ be a distribution on $W^{(0)}$, defined by

$$\tilde{P}_k[A] = \frac{1}{N(k)} \sharp\{w \in A \mid L(w) = k\}, \quad A \subset W^{(0)}.$$

The next result shows the existence of the exponent for mean square displacement, which indicates (in a log ratio sense) that a typical self-avoiding path w of length $L(w) = k$ deviates from the starting point by $|w(k)| \asymp k^{1/d_w}$, where

$$d_w = \frac{\log \lambda}{\log 2}. \tag{30}$$

Theorem 10 *If there exists a critical point $\beta_c \in \mathbb{R}$ of the full model, then*

$$\lim_{k\to\infty} \frac{1}{\log k} \log E_k[|w(k)|^{s\, d_w}] = s, \quad s \geqq 0,$$

where E_k denotes expectation with respect to $\tilde{P}_k$, and $|\cdot|$ denotes the (Euclidean) length in $\mathbb{R}^d \supset F$.

Similar result holds also for the restricted model.

4 Proofs.

4.1 Phase structure.

Here we will prove Theorem 7.

Proposition 11 *$\Phi_{(1)}(\vec{x})$ contains terms ${x_{(i)}}^2$, $1 \leqq i \leqq d$,*

Proof. This is proved by a similar graphical consideration as that in the proof of Proposition 2 that $\Phi_{(1)}(\vec{x})$ contains a term ${x_{(1)}}^2$.

$\square$

Proposition 12 *If $\vec{x}_c$ is a self-avoiding fixed point, then*

(i) $x_{c,(1)} > 0$. Namely, $(1) \in \mathcal{K}_{\vec{x}_c}$.

(ii) $v_{R,I} > 0$, $I \in \mathcal{K}_{\vec{x}_c}$, where $\vec{v}_R$ is as in (FP2).

Proof. (i) $\vec{x}_c \neq \vec{0}$, by (FP4). Hence there exists $I \in \mathcal{I}_d$ such that $x_{c,I} > 0$. With (FP4), we further see that there exists $1 \leqq i \leqq d$ such that $x_{c,(i)} > 0$. Proposition 12 then implies that $x_{c,(1)} = \Phi_{(1)}(\vec{x}_c) \geqq x_{c,(i)}^2 > 0$.

(ii) (FP3) and $I \in \mathcal{K}_{\vec{x}_c}$ impy $\mathcal{J}(\vec{x}_c)_{I,(1)} > 0$. With (FP2), this further implies $v_{R,I} \geqq \dfrac{1}{\lambda} J(\vec{x}_c)_{I,(1)}\, v_{R,(1)} > 0$.

□

Let

$$D = \{\vec{x} \in \Xi_d \mid \sup_{n \in \mathbb{Z}_+} \max_{I \in \mathcal{I}_d} X_{n,I}(\vec{x}) < \infty\}, \tag{31}$$

and denote its exterior, boundary, interior in Ξ_d by D^c, ∂D, and D^o, respectively. (Namely, $D^c = \Xi_d \setminus \overline{D}$, $\partial D = \overline{D} \cap \overline{D^c}$, $D^o = D \setminus \partial D$.) Let also

$$\tilde{D} = \{\vec{x} \in \Xi_d \mid \lim_{n \to \infty} \max_{I \in \mathcal{I}_d} X_{n,I}(\vec{x}) = 0\}.$$

Theorem 13 *(i) It holds that*

$$D = \{\vec{x} \in \Xi_d \mid \sup_{n \in \mathbb{Z}_+} \max_{I \in \mathcal{I}_d} X_{n,I}(\vec{x}) \leqq 1\}. \tag{32}$$

In particular, D is a closed subset of Ξ_d.

(ii) Let $\vec{x} \in D$ and $\vec{x}' \in \Xi_d$. If, for each $I \in \mathcal{I}_d$ either $x'_I < x_I$ or $x'_I = x_I = 0$ holds, then $\vec{x}' \in \tilde{D}$.

(iii) It holds that

$$D^o = \tilde{D} \tag{33}$$

(iv) D^c, ∂D, and D^o are non-empty invariant sets of $\vec{\Phi}$.

Remark. Note that this theorem holds independently of the notion of self-avoiding fixed points and critical points. ◇

Proof. (i) Denote the right hand side of (32) by D'. Obviously $D' \subset D$. To prove $D'^c \subset D^c$ (in Ξ_d), let $\vec{x} \in D'^c \subset \Xi_d$. Then the definition of D' implies that there exists $n \in \mathbb{Z}_+$ and $I \in \mathcal{I}_d$ such that $X_{n,I}(\vec{x}) > 1$, and the definition of Ξ_d therefore implies $X_{n,(i)}(\vec{x}) > 1$ for some $1 \leqq i \leqq d$. Then Proposition 11 implies $X_{n+1,(1)}(\vec{x}) > 1$, which, with Proposition 2, further implies $\lim_{n \to \infty} X_{n,(1)}(\vec{x}) = \infty$. Hence $D'^c \subset D^c$.

(ii) The case $\vec{x}' = \vec{0}$ is trivial. For $\vec{x}$ and $\vec{x}'$ in $\mathbb{R}_+^{\mathcal{I}_d} \setminus \{0\}$, let $r = \max_{I \in \mathcal{I}_d;\, x_I \neq 0} \dfrac{x'_I}{x_I}$. Since Φ_I, $I \in \mathcal{I}_d$, are polynomials with positive coefficients and each term of degree no less than 2 (Proposition 2), we have $\max_{I \in \mathcal{I}_d} \Phi_I(\vec{x}') \leqq r^2 \max_{I \in \mathcal{I}_d} \Phi_I(\vec{x})$ if $0 \leqq r \leqq 1$. By induction, $\max_{I \in \mathcal{I}_d} X_{n,I}(\vec{x}') \leqq r^{2^n} \max_{I \in \mathcal{I}_d} X_{n,I}(\vec{x})$, $n \in \mathbb{Z}_+$. If $\vec{x} \in D \setminus \{\vec{0}\}$ and $\vec{x}' \in \Xi_d$, and if for each $I \in \mathcal{I}_d$ either $x'_I < x_I$ or $x'_I = x_I = 0$ holds, then $0 \leqq r < 1$. Since $\{X_{n,I}(\vec{x})\}$ is bounded, we have $\vec{x}' \in \tilde{D}$.

(iii) For $\epsilon > 0$ let $D_\epsilon = \{\vec{x} \in \Xi_d \mid \sum_{I \in \mathcal{I}_d} x_I < \epsilon\}$. Since Φ_I is a polynomial with each term of order no less than 2 (Proposition 2), there exists a constant $M > 0$ such that

$$\sum_{I \in \mathcal{I}_d} \Phi_I(\vec{x}) \leqq M\epsilon \sum_{I \in \mathcal{I}_d} x_I\,, \quad \vec{x} \in D_\epsilon,\ 0 < \epsilon < 1\,.$$

Let $\epsilon = \frac{1}{2M} \wedge \frac{1}{2}$. Then by induction in n with

$$\sum_{I \in \mathcal{I}_d} X_{n+1,I}(\vec{x}) = \sum_{I \in \mathcal{I}_d} \Phi_I(\vec{X}_n(\vec{x})),$$

we have, for $\vec{x} \in D_\epsilon$, $\vec{X}_n(\vec{x}) \in D_\epsilon$, $n \in \mathbb{Z}_+$. Then we further have

$$\sum_{I \in \mathcal{I}_d} X_{n+1,I}(\vec{x}) \leqq M\epsilon \sum_{I \in \mathcal{I}_d} X_{n,I}(\vec{x}) \leqq \frac{1}{2} \sum_{I \in \mathcal{I}_d} X_{n,I}(\vec{x}), \quad n \in \mathbb{Z}_+,$$

which implies $\vec{x} \in \tilde{D}$. Hence $D_\epsilon \subset \tilde{D}$ for $\epsilon = \frac{1}{2M} \wedge \frac{1}{2}$. On the other hand, the definition of $\tilde{D}$ implies that if $\vec{x} \in \tilde{D}$ then $\vec{X}_n(\vec{x}) \in D_\epsilon$ for sufficiently large n. $\vec{X}_n$ is a continuous function and D_ϵ is an open set, hence there exists an open neighborhood U of $\vec{x}$ such that $\vec{X}_n(U) \subset D_\epsilon$. Hence $\vec{X}_n(U) \subset \tilde{D}$. This implies that $\tilde{D}$ is an open subset of D. Since D^o is the largest open set of D, this impleis $\tilde{D} \subset D^o$.

To prove $\tilde{D} \supset D^o$, assume $\vec{x}' \in D^o$. Since D^o is an open set, there exists $\vec{x} \in D^o$ such that $x'_I < x_I$, $I \in \mathcal{I}_d$. Then, the above claim imples $\vec{x}' \in \tilde{D}$.

(iv) By (33) and (32), we see that D and D^o are invariant sets of $\vec{\Phi}$. Since $D = \overline{D}$, D^c is also invariant, hence ∂D also is. Obviously, $\vec{x}$ with small enough components is in D^o and that with sufficiently large components is in D^c, hence the boundary ∂D is also non-empty.

□

Proof of Theorem 7. The case $\beta = \beta_c$ holds by the definition (CS1).

By the definitions (CS1) and (FP4), $\lim_{n\to\infty} \vec{X}_n(\vec{x}_{can}(\beta_c)) = \vec{x}_c \neq \vec{0}$, which, with (26), implies $\vec{x}_{can}(\beta_c) \in \partial D$. Monotonicity property in Theorem 13 then implies $\vec{x}_{can}(\beta) \in \tilde{D}$ if $\beta > \beta_c$, hence, in particular, $\lim_{n\to\infty} \vec{Z}_n(\beta) = 0$.

Finally, if $\beta < \beta_c$ and $\vec{x}_{can}(\beta) \in D = \overline{D}$, then the monotonicity property in Theorem 13 implies $\vec{x}_{can}(\beta_c) \in \tilde{D} = D^o$, which contradicts $\vec{x}_{can}(\beta_c) \in \partial D$. Hence $\vec{x}_{can}(\beta) \in D^c$ and in particular, with (11), we have $\lim_{n\to\infty} \sum_{i=1}^{d} Z_{n,(i)}(\beta) = \infty$.

The case of restricted model is similarly proved, if we note (CS2) in place of (CS1).

□

We will use the following in the proof of Theorem 9.

Proposition 14 *If $\vec{x} \in D^o$ then $\overline{\lim_{n\to\infty}} \max_{I \in \mathcal{I}_d} 2^{-n} \log X_{n,I}(\vec{x}) < 0$.*

Proof. Write $Y_n = \max_{I \in \mathcal{I}_d} \log X_{n,I}$, $n \in \mathbb{Z}_+$. Since Φ_I are polynomials of positive coefficients with each term of degree no less than 2 (Proposition 2), there exists a polynomial P_1 with positive coefficients such that

$$Y_{n+1}(\vec{x}) \leqq 2Y_n(\vec{x}) + \log P_1(\exp(Y_n(\vec{x}))), \quad n \in \mathbb{Z}_+, \ \vec{x} \in \mathbb{R}_+^{\mathcal{I}_d}.$$

If $\vec{x} \in D^o$, Theorem 13 implies that $\vec{X}_n(\vec{x}) \in D^o$, $n \in \mathbb{Z}_+$. Hence, $\{\vec{X}_n(\vec{x}) \mid n \in \mathbb{Z}_+\}$ is bounded, and there exists $M \in \mathbb{R}$, a constant in n, such that

$$Y_{n+1}(\vec{x}) \leqq 2Y_n(\vec{x}) + M, \quad n \in \mathbb{Z}_+. \tag{34}$$

In particular, $2^{-n}(Y_n(\vec{x}) + M)$ is decreasing, hence

$$\overline{\lim_{n\to\infty}} \, 2^{-n}(Y_n(\vec{x}) + M) \leqq 2^{-n_0}(Y_{n_0}(\vec{x}) + M), \quad n_0 \in \mathbb{Z}_+.$$

Also $\vec{x} \in D^o$ implies

$$\lim_{n\to\infty} Y_n(\vec{x}) = -\infty.$$

Hence

$$\overline{\lim_{n\to\infty}} \, 2^{-n} Y_n(\vec{x}) = \overline{\lim_{n\to\infty}} \, 2^{-n}(Y_n(\vec{x}) + M) < 0,$$

which implies the statement.

□

4.2 Distribution of path length.

Here we will prove Theorem 8.

Proposition 15 *Let $\vec{x}_c$ be a self-avoiding fixed point and $\vec{x} \in \mathcal{D}om(\vec{x}_c)$ then there exists $f_d \times f_d$ matrix $\Lambda(\vec{x})$ such that (28) holds. The elements of the matrix are non-negative, and*

$$\Lambda(\vec{x})_{I,(1)} > 0, \quad I \in \mathcal{K}_{\vec{x}_c}\,. \tag{35}$$

Proof. First note that (FP2) and (18) imply an existence of the limit in (28) for $\vec{x} = \vec{x}_c$.

Next recall the following two results.

Lemma 16 ([6, Lemma 4.2], [5, Lemma (3.1)]) *Let N be a positive integer, and A and A_n, $n \in \mathbb{N}$, be $N \times N$ matrices. Assume that there is an invertible $N \times N$ matrix Q such that $Q^{-1}AQ$ is a diagonal matrix whose elements are non-negative, and whose eigenvalue λ that is largest in absolute value, is positive. Assume further that $\sum_{n=1}^{\infty} \|A_n - A\| < \infty$. Then there exists a matrix R such that*

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \left\| \lambda^{-n} A_{n+m} A_{n+m-1} \cdots A_{m+1} \; - \; R \right\| = 0\,.$$

$Q^{-1}RQ$ is a diagonal matrix whose diagonal elements satisfy $(Q^{-1}RQ)_{ii} = 1$ if $(Q^{-1}AQ)_{ii} = \lambda$, and $(Q^{-1}RQ)_{ii} = 0$ otherwise. Moreover $\lim_{n\to\infty} \lambda^{-n} A_n A_{n-1} \cdots A_1$ exists.

($\|A\|$ is the standard operator norm defined as the square root of largest eigenvalue of A^*A.)

Lemma 17 ([6, Lemma 3.8]) *If $\vec{x}_c$ is a self-avoiding fixed point and $\vec{x} \in \mathcal{D}om(\vec{x}_c)$, then there exist positive constants ρ and C, satisfying $\rho < 1$, such that*

$$|X_{n,I}(\vec{x}) - x_{c,I}| \leqq C\rho^n, \quad n \in \mathbb{Z}_+,\ I \in \mathcal{I}_d\,.$$

Remark. (i) The proof in [6, Proposition 3.8] is for $d = 3$, but the proof is directly applicable to general cases. In fact, the proof is an application of a general theory for diffeomorphisms [8].

(ii) Though we do not use this, the proof in [6, Proposition 3.8] implies that Lemma 17 holds for any ρ satisfying

$$\max\{\lambda^{-1}, |\lambda_2|\} < \rho < 1,$$

where λ_2 is the eigenvalue of $\mathcal{J}(\vec{x}_c)$ which is second largest in absolute value.

◇

Let us proceed with the proof of existence of the limit in Proposition 15. By the mean-value theorem,

$$\mathcal{J}_{IJ}(\vec{x}) - \mathcal{J}_{IJ}(\vec{x}_c) = \sum_{K\in\mathcal{I}_d} \frac{\partial \mathcal{J}_{IJ}}{\partial x_K}(\vec{u})\,(x_K - x_{c,K}), \quad I, J \in \mathcal{I}_d\,. \tag{36}$$

where $\vec{u} = \vec{x}_c + (\vec{x} - \vec{x}_c)\theta$ for some $\theta \in (0,1)$.

Since $\vec{x} \in \mathcal{D}om(\vec{x}_c)$, $\vec{X}_m(\vec{x})$ is in a neghborhood of $\vec{x}_c$ for sufficiently large m. Therefore $\vec{X}_m(\vec{x})$ is in a bounded domain. In particular, there exists a positive constant M such that

$$\left| \frac{\partial \mathcal{J}_{IJ}}{\partial x_K}(\vec{u}) \right| \leqq M, \quad I, J, K \in \mathcal{I}_d\,,\ \vec{u} = \vec{x}_c + (\vec{X}_m(\vec{x}) - \vec{x}_c)\theta,\ 0 < \theta < 1\,,\ m \in \mathbb{N}.$$

This with (36) and Lemma 17 implies

$$\left\| \mathcal{J}(\vec{X}_m(\vec{x})) - \mathcal{J}(\vec{x}_c) \right\| \leqq C'\rho^m,\ m \in \mathbb{Z}_+,$$

for some $C' > 0$ and $0 < \rho < 1$. Therefore we see that the assumptions in Lemma 16 are satisfied with

$$A_n = \mathcal{J}(\vec{X}_n(\vec{x})) \tag{37}$$

and $A = \mathcal{J}(\vec{x}_c)$. Hence, the limit (28) exists.

To prove (35), let i be an index satisfying $(Q^{-1}AQ)_{ii} = \lambda$. Then the collum i of the matrix Q in Lemma 16, applied to the present case, can be chosen to be proportional to the right eigenvector $\vec{v}_R$ of $A = \mathcal{J}(\vec{x}_c)$ in (FP2). Similarly, the row i of matrix Q^{-1}, applied to the present case, can be chosen to be the left eigenvector $\vec{v}_L$ in (FP2). Therefore Lemma 16 implies that if $v_{R,I} > 0$ then for sufficiently large m, $\lim_{n\to\infty} \lambda^{-n} A_{n+m} A_{n+m-1} \cdots A_{m+1}$ for A_n given by (37) has positive (I,J) element for all $J \in \mathcal{I}_d$, in particular, for $J = (1)$. Also, since $\Phi_{(1)}(\vec{x})$ contains a term ${x_{(1)}}^2$ (Proposition 2), A_n has positive $((1),(1))$ element, which further implies that $A_m A_{m-1} \cdots A_1$ has positive $((1),(1))$ element. Finally, we know from Proposition 12 that $v_{R,I} > 0$ for $I \in \mathcal{K}_{\vec{x}_c}$. Hence (35) holds.

□

For $\vec{x} = (x_I,\ I \in \mathcal{I}_d) \in \mathbb{C}^{\mathcal{I}_d}$ and $\vec{t} = (t_I,\ I \in \mathcal{I}_d) \in \mathbb{C}^{\mathcal{I}_d}$, we use an (irregular) notation

$$\vec{x}(\vec{t}) = (x_I \exp(\lambda^{-n} t_I),\ I \in \mathcal{I}_d). \tag{38}$$

Proposition 18 *If $\vec{x} \in \mathcal{D}om(\vec{x}_c)$ then $\vec{X}_n(\vec{x}(\vec{t}))$ converges uniformly in $\vec{t} \in \mathbb{C}^{\mathcal{I}_d}$ for any compact subset of $\mathbb{C}^{\mathcal{I}_d}$. Furthermore,*

(i) If $I \in \mathcal{I}_d$ satisfies $x_{c,I} = 0$ (i.e., if $I \notin \mathcal{K}_{\vec{x}_c}$), then

$$\lim_{n\to\infty} X_{n,I}(\vec{x}(\vec{t})) = 0, \quad \vec{t} \in \mathbb{C}^{\mathcal{I}_d}.$$

(ii) Define $\vec{\varphi}^* = \vec{\varphi}^*_{\vec{x}} = (\varphi^*_{\vec{x},I},\ I \in \mathcal{I}_d)$ by

$$\varphi^*_{\vec{x},I}(\vec{t}) = \begin{cases} \frac{1}{x_{c,I}} \lim_{n\to\infty} X_{n,I}(\vec{x}(\vec{t})), & \vec{t} \in \mathbb{C}^{\mathcal{I}_d}, \quad \text{if } I \in \mathcal{K}_{\vec{x}_c}, \\ 0, & \text{if } I \notin \mathcal{K}_{\vec{x}_c}. \end{cases}$$

Then $\vec{\varphi}^*$ is entire and satisfies

$$x_{c,I} \varphi^*_I(\lambda \vec{t}) = \Phi_I(\vec{x}_c \vec{\varphi}^*(\vec{t})), \quad \vec{t} \in \mathbb{C}^{\mathcal{I}_d},\ I \in \mathcal{I}_d.$$

where, the notation of the variable for Φ_I is as in Theorem 8. It also holds that

$$x_{c,I} \frac{\partial \varphi^*_I}{\partial t_J}(\vec{0}) = x_J\, \Lambda_{IJ}(\vec{x}), \quad I, J \in \mathcal{I}_d,$$

where Λ is defined in (28).

Proof. For $\vec{z} \in \mathbb{C}^{\mathcal{I}_d}$, define

$$|\vec{z}|_* = \sum_{I \in \mathcal{I}_d} v_{L,I}\, |z_I|,$$

where $v_{L,I}$ is as in (FP2). It is easy to see that (FP2) implies that $|\cdot|_*$ is a norm. (Note that $v_{L,I} > 0$, $I \in \mathcal{I}_d$, implies uniqueness, i.e., that $|a|_* = 0$ implies $a = 0$.)

Using non-negativity of elements of $\mathcal{J}$, we have

$$|(\mathcal{J}(\vec{x}_c)\, \vec{z})_I| \leqq \sum_{J \in \mathcal{I}_d} \mathcal{J}(\vec{x}_c)_{IJ}\, |z_J|, \quad I \in \mathcal{I}_d.$$

Hence

$$|\mathcal{J}(\vec{x}_c)\, \vec{z}|_* \leqq \lambda |\vec{z}|_*, \quad \vec{z} \in \mathbb{C}^{\mathcal{I}_d}. \tag{39}$$

We now see that Proposition 18 is a consequence of the following, which is proved in the proof of [6, Proposition 4.4].

Lemma 19 *Let $p \in \mathbb{N}$, and let $H^{(n)} : \mathbb{C}^p \to \mathbb{C}^p$, $n \in \mathbb{N}$, be a sequence of polynomials with positive coefficients, and each term with degree 2 or larger, which satisfies a recursion relation*

$$H^{(n+1)} = H^{(1)} \circ H^{(n)}, \quad n \in \mathbb{N}. \tag{40}$$

Let $\partial H^{(1)}$ be the Jacobi matrix of $H^{(1)}$:

$$\partial H^{(1)}_{ij}(z) = \frac{\partial H^{(1)}_i}{\partial z_j}(z), \quad i, j \in 1, \cdots, p,\ z \in \mathbb{C}^p.$$

Assume that $a \in \mathbb{R}^p_+$ is a fixed point of $H^{(1)}$, and that there exists a norm $|\cdot|_*$ on $\mathbb{C}^p$ and $\lambda > 1$ such that

$$|\partial H^{(1)}(a)z|_* \leqq \lambda |z|_*, \quad z \in \mathbb{C}^p, \tag{41}$$

and that $x \in \mathbb{R}^p_+$ is a point for which there exist positive constants ρ and C, satisfying $\rho < 1$, such that

$$|H_i^{(n)}(x) - a_i| \leqq C\rho^n, \quad i = 1, \cdots, p, \ n \in \mathbb{Z}_+, \tag{42}$$

and for which the limit

$$\Lambda(x) = \lim_{n\to\infty} \lambda^{-n} \partial H^{(n)}(x) \tag{43}$$

exists.

Then there exists an entire function $H^* = (H_1^*, \cdots, H_p^*) : \ \mathbb{C}^p \to \mathbb{C}^p$ such that

$$\lim_{n\to\infty} H^{(n)}(x_1 \exp(\lambda^{-n} t_1), \cdots, x_p \exp(\lambda^{-n} t_p)) = H^*(t),$$

uniformly in $t = (t_1, \cdots, t_p)$ on any compact set of $\mathbb{C}^p$.

H^* satisfies

$$H^*(\lambda t) = H^{(1)}(H^*(t)), \quad t \in \mathbb{C}^p,$$

and

$$\frac{\partial H_i^*}{\partial t_j}(0) = x_j \Lambda_{ij}(x), \quad i, j \in \{1, 2, \cdots, p\}.$$

Furthermore, if $a_i = 0$ for some $i \in \{1, \cdots, p\}$, then $H_i^*(t) = 0$.

Remark. The proof of [6, Proposition 4.4] is for 3SG, but the proof is valid for the general case of Lemma 19. In the proof of [6, Proposition 4.4], [6, Theorem 3.5] is referred to, which should be replaced, in the proof of Lemma 19, by $\lim_{n\to\infty} H^{(n)}(x) = a$, which is a consequence of the assumption (42). Similarly, Proposition 3.8, Proposition 4.3, and (4.4) in [6] are replaced by (42), (43), and (40), respectively.

◇

To apply Lemma 19 to the proof of Proposition 18, we only need to note that Proposition 2, (39), Lemma 17, and Proposition 15, correspond to (40), (41), (42), and (43), respectively.

□

Proof of Theorem 8. Assume that $x_{c,I} \neq 0$ and $\vec{x} \in \mathcal{D}om(\vec{x}_c)$.

Denote by $p_{\vec{x},n,I}$, the joint distribution of $(\lambda^{-n} s_J, \ J \in \mathcal{I}_d)$ under $\mu_{\vec{x},n,I}$. Its generating function is expressed, with the definitions (9) and (27), as

$$\int_0^\infty e^{\vec{t}\cdot\vec{\xi}} p_{\vec{x},n,I}[d\vec{\xi}] = \frac{X_{n,I}(\vec{x}(\vec{t}))}{X_{n,I}(\vec{x})}, \quad \vec{t} \in \mathbb{C}^{\mathcal{I}_d}, \tag{44}$$

where the notation $\vec{x}(\vec{t})$ is as in (38). Since the probability measure in question is a measure on a finite set, the integration over $\vec{\xi}$ in (44) is actually a finite summation, hence the quantity is defined for all complex $\vec{t}$.

Using Proposition 18 and (DA1) in (44), we see that the right hand side of (44) converges to $\varphi_I^*(\vec{t}) = \varphi_{\vec{x},I}^*(\vec{t})$ of Proposition 18, uniformly in $\vec{t}$ on compact sets. Hence $p_{\vec{x},n,I}$ converges weakly to a probability measure $p_{\vec{x},I}^*$ whose generating function is $\varphi_I^*(\vec{t})$.

We therefore see that the claims in Theorem 8, except those on the existence and positivity of the density of $\bar{p}_{\vec{x},(1)}^*$, are direct consequences of Proposition 15 and Proposition 18. That (29) determines $\vec{\varphi}^*$ follows from the fact that its Taylor coefficients are determined recursively by (29). (Note that Φ_I is a polynomial with each term having degree 2 or larger.)

To prove that $\bar{p}_{\vec{x},I}^*$ has a C^∞ density, we prepare the following.

Denote by $\bar{p}_{\vec{x},n,I}$, the distribution of $\lambda^{-n} L$ under $\mu_{\vec{x},n,I}$. Its generating function is, from (7) and (44),

$$\int_0^\infty e^{t\xi} \bar{p}_{\vec{x},n,I}[d\xi] = \frac{X_{n,I}(\vec{x}(t\vec{e}))}{X_{n,I}(\vec{x})}, \quad t \in \mathbb{C}, \tag{45}$$

where $\vec{e} = (e_J,\ J \in \mathcal{I}_d)$ is given by

$$e_J = |J|. \tag{46}$$

According to what has been proved below (44), (45) converges to an entire function

$$g_I(t) = g_{\vec{x},I}(t) = \varphi^*_I(t\vec{e}) \tag{47}$$

uniformly in t on compact sets of $\mathbb{R}$, and $\bar{p}_{\vec{x},n,I}$ converges weakly to a probability measure $\bar{p}^*_{\vec{x},I}$ whose generating function is g_I.

Proposition 18 also implies that

$$x_{c,I}\,\frac{dg_I}{dt}(0) = x_{c,I}\int_0^\infty \xi \bar{p}^*_{\vec{x},I}[d\xi] = \sum_{J\in\mathcal{I}_d} e_J\, x_J\, \Lambda_{IJ}(\vec{x}),\quad I \in \mathcal{I}_d\,, \tag{48}$$

and

$$x_{c,I} g_I(\lambda t) = \Phi_I(\vec{x}_c \vec{g}(t)),\quad I \in \mathcal{I}_d,\ t \in \mathbb{C}, \tag{49}$$

where, the notation of the variable for Φ_I is as in Theorem 8, and we also defined

$$g_J = 0,\quad \text{if } x_{c,J} = 0\,.$$

Lemma 20 *(i)* $M_I = M_I(\vec{x}) := \displaystyle\int_0^\infty \xi \bar{p}^*_{\vec{x},I}[d\xi] > 0,\quad I \in \mathcal{K}_{\vec{x}_c}\,,\ \vec{x} \in \Xi_d \cap \mathcal{D}om(\vec{x}_c).$

(ii) If $\vec{x} \in \Xi_d \cap \mathcal{D}om(\vec{x}_c)$ and $I \in \mathcal{K}_{\vec{x}_c}$ then $\bar{p}^*_{\vec{x},I}$ is not concentrated on a single point (i.e., has non-zero variance). Namely,

$$V_I = V_I(\vec{x}) := \int_0^\infty (\xi - M_I(\vec{x}))^2 \bar{p}^*_{\vec{x},I}[d\xi] > 0,\quad I \in \mathcal{K}_{\vec{x}_c}\,,\ \vec{x} \in \Xi_d \cap \mathcal{D}om(\vec{x}_c).$$

(iii) If $\vec{x} \in \Xi_d \cap \mathcal{D}om(\vec{x}_c)$ then there exist positive constants C_1 and C_2 such that

$$\max_{I\in\mathcal{K}_{\vec{x}_c}} |g_I(\sqrt{-1}t)| \leqq C_2 \exp(-C_1 |t|^{1/d_w}),\quad t \in \mathbb{R},$$

where d_w is as in (30).

Proof. (i) Using (48) we have

$$\int_0^\infty \xi \bar{p}^*_{\vec{x},I}[d\xi] \geqq e_{(1)} \frac{x_{(1)}}{x_{c,I}}\, \Lambda_{I\,(1)}(\vec{x}) > 0,$$

where we used (46) for $e_{(1)} > 0$, $\vec{x} \in \Xi_d \cap \mathcal{D}om(\vec{x}_c)$ for $x_{(1)} > 0$, and (35) for $\Lambda_{I\,(1)}(\vec{x}) > 0$.

(ii) By definition (Proposition 2 and (9)), we have

$$\Phi_I(\vec{x}) = \sum_{m\in\mathbb{Z}_+^{\mathcal{I}_d}} b_{I,m} \prod_{J\in\mathcal{I}_d} {x_J}^{m_J},\quad \vec{x} \in \mathbb{C}^{\mathcal{I}_d},\ I \in \mathcal{I}_d\,,$$

where

$$b_{I,m} = \sharp\{w \in W_I^{(1)} \mid \vec{s}(w) = m\},\quad I \in \mathcal{I}_d\,,\ m \in \mathbb{Z}_+^{\mathcal{I}_d}.$$

Then (49) implies

$$x_{c,I} g_I(\lambda t) = \Phi_I(\vec{x}_c\, \vec{g}(t)) = \sum_{m\in\mathbb{Z}_+^{\mathcal{I}_d}} C_{I,m} \prod_{J\in\mathcal{I}_d} g_J(t)^{m_J},\quad t \in \mathbb{C}, \tag{50}$$

where we wrote

$$C_{I,m} = b_{I,m} \prod_{J\in\mathcal{I}_d} x_{c,J}^{m_J}\,.$$

Now assume that $\bar{p}_{\vec{x},I}$ is concentrated on a single point for some $I \in \mathcal{K}_{\vec{x}_c}$. Then its generating function is

$$g_I(t) = e^{M_I t}. \tag{51}$$

Differentiating (50) by t up to twice, putting $t=0$, and using (51) on the left hand side, we obtain equations, which eventually lead to

$$(0 \leqq) \quad \sum_m C_{I,m}(\lambda M_I - \sum_J m_J M_J)^2 = -\sum_m C_{I,m} \sum_J m_J V_J \,.$$

Since $C_{I,m}$, m_J, and V_J are non-negative, it follows that $V_J = 0$ for all $J \in \mathcal{I}_d$ such that there exists $m \in \mathbb{Z}_+^{\mathcal{I}_d}$ satisfying $m_J > 0$ and $m_K = 0$ if $K \notin \mathcal{K}_{\vec{x}_c}$. In other words, $V_J = 0$ for all $J \in \mathcal{I}_d$ such that $\Phi_I(\vec{x})$ has a term which contains a positive power of x_J and composed of those x_K with $K \in \mathcal{K}_{\vec{x}_c}$. With (FP3) we see that $J = (1)$ has such property for all $I \in \mathcal{K}_{\vec{x}_c}$. Hence $V_{(1)} = 0$, and $\bar{p}_{\vec{x},(1)}$ is concentrated on a single point.

The arguments after (51) can now be repeated with $I = (1)$. Using also (FP3) again, we eventually see that (50) implies

$$x_{c,(1)} \exp(\lambda\, M_{(1)}\, t) = \sum_{m \in \mathbb{Z}_+^{\mathcal{K}_{\vec{x}_c}}} C_{(1),m} \exp(\sum_{J \in \mathcal{K}_{\vec{x}_c}} M_J\, m_J\, t), \quad t \in \mathbb{C}.$$

But $\Phi_{(1)}$ is a polynomial, hence the right hand side is actually a finite summation with positive coefficients, so this can hold only if

$$\lambda\, M_{(1)} = \sum_{J \in \mathcal{K}_{\vec{x}_c}} M_J\, m_J,$$

for all m such that $C_{(1),m} > 0$. Proposition 2 implies that $C_{(1),m} > 0$ for m such that;

(a) $m_{(1)} = 2$ and $m_J = 0$, $J \neq (1)$,

(b) $m_{(1)} = d+1$ and $m_J = 0$, $J \neq (1)$.

Therefore $2M_{(1)} = (d+1)M_{(1)}$, hence $M_{(1)} = 0$ for $d \geqq 2$, which is a contradiction, because Proposition 12 implies $(1) \in \mathcal{K}_{\vec{x}_c}$, which, with the preceding result, implies $M_{(1)} > 0$.

This proves that if $I \in \mathcal{K}_{\vec{x}_c}$ and $\vec{x} \in \mathcal{D}om(\vec{x}_c) \cap \Xi_d$, then $\bar{p}_{\vec{x},I}$ is not concentrated on a single point.

(iii) The preceeding result implies, by a standard argument relating positivity of variances and the absolute values of characteristic functions, that there exists $a > 0$ such that

$$|g_I(\sqrt{-1}t)| < 1, \quad -a < t < a,\ t \neq 0,\ I \in \mathcal{K}_{\vec{x}_c}\,.$$

Since characteristic functions are continuous, this further implies that there exist $a' > 0$ and $0 < C < 1$ such that

$$|g_I(\sqrt{-1}t)| \leqq C, \quad \frac{a'}{\lambda} \leqq |t| \leqq a',\ t \in \mathbb{R},\ I \in \mathcal{K}_{\vec{x}_c}\,. \tag{52}$$

Also (50) and (FP1) and the fact that each term in Φ_I has degree 2 or more (Proposition 2) imply (together with the fact that absolute values of characteristic functions are no greater than 1),

$$\max_{I \in \mathcal{K}_{\vec{x}_c}} |g_I(\sqrt{-1}\lambda t)| \leqq \max_{I \in \mathcal{K}_{\vec{x}_c}} |g_I(\sqrt{-1}t)|^2, \quad t \in \mathbb{R}.$$

This and (52) imply

$$|g_I(\sqrt{-1}t)| \leqq C_2 \exp(-C_1 |t|^{1/d_w}), \quad t \in \mathbb{R},\ I \in \mathcal{K}_{\vec{x}_c}\,,$$

for some positive constants C_1 and C_2.

This completes a proof of Lemma 20.

□

The last statement in Lemma 20 implies the existence of C^∞ density $\bar{\rho}^*_{\vec{x},(1)}$ of $\bar{p}^*_{\vec{x},(1)}$.

To prove the positivity of $\bar{\rho}^*_{\vec{x},(1)}$, substitute $g_{(1)}(t) = \displaystyle\int_0^\infty e^{t\xi} \bar{\rho}^*_{\vec{x},(1)}(\xi)\, d\xi$ in (49), to find

$$\lambda^{-1} \bar{\rho}^*_{\vec{x},(1)}(\lambda^{-1}\xi) = x_{c,(1)} (\bar{\rho}^*_{\vec{x},(1)} * \bar{\rho}^*_{\vec{x},(1)})(\xi) + {x_{c,(1)}}^d (\bar{\rho}^*_{\vec{x},(1)} * \bar{\rho}^*_{\vec{x},(1)} * \cdots * \bar{\rho}^*_{\vec{x},(1)})(\xi) + \text{ non-negative terms}, \quad \xi \geqq 0\,,$$

where the operation $*$ denotes convolution, and we also used Proposition 2. Note first that since $\bar{\rho}^*_{\vec{x},(1)}$ is continuous and $M_{(1)} > 0$, there is a $\xi_0 > 0$ for which $\bar{\rho}^*_{\vec{x},(1)}(\xi) > 0$ in a neighborhood of ξ_0. Furthermore, the above relation implies that if $\bar{\rho}^*_{\vec{x},(1)}$ is positive in a neighborhood of $\xi_1, \xi_2 \geqq 0$, then it also is in a neighborhood of $\lambda^{-1}(\xi_1 + \xi_2)$. With Proposition 5 and the continuity of $\bar{\rho}^*_{\vec{x},(1)}$, we therefore see that $\bar{\rho}^*_{\vec{x},(1)}$ is positive in a neighborhood of 0. The above relation also implies that if $\bar{\rho}^*_{\vec{x},(1)}$ is positive in a neighborhood of $\xi_1, \xi_2, \cdots, \xi_{d+1} \geqq 0$, then it also is in a neighborhood of $\lambda^{-1}(\xi_1 + \xi_2 + \cdots + \xi_{d+1})$. With Proposition 5, we inductively see that $\bar{\rho}^*_{\vec{x},(1)}$ is positive on $(0, \infty)$.

This completes a proof of Theorem 8.

□

4.3 Exponent for mean square displacement.

Here we prove Theorem 9 and Theorem 10. Since the proofs are similar for the full model and the restricted model, we will concentrate on the full model.

We introduce the following notation which we use throughout this section.

Define $\nu : W^{(0)} \to \mathbb{Z}_+ \cup \{\infty\}$ by

$$\nu(w) = \min\{n \in \mathbb{Z}_+ \cup \{\infty\} \mid w(k) \in G_n\,,\ k = 0, 1, 2, \cdots, L(w)\}, \quad w \in W^{(0)}. \tag{53}$$

Note the obvious relation

$$2^{\nu(w)-1} < L(w) \leqq d(d+1)^{\nu(w)}, \quad w \in W^{(0)}. \tag{54}$$

For $w \in W^{(0)}$, let $\|w\| = \max\{|w(k)| \mid k = 0, 1, 2, \cdots, L(w)\}$, where $|\cdot|$ denotes the (Euclidean) length in $\mathbb{R}^d \supset F$. By definition,

$$2^{\nu(w)-1} < \|w\| \leqq 2^{\nu(w)}. \tag{55}$$

4.3.1 Tauberian type estimates and number of paths.

Here we prove Theorem 9 for the full model.

Let λ be as before, β_c be a critical point of the full model, and $\vec{x}_{can}(\beta_c) = \vec{x}_{can,\mathcal{I}_d}(\beta_c)$ be as in (22).

Proposition 21 *Define, for $b > 0$, $n \in \mathbb{Z}_+$, $\xi \in \mathbb{R}$, $c_n = b\lambda^{-n}\sqrt{n}$ and $h_n(\xi) = \dfrac{1}{\sqrt{2\pi}c_n} e^{-\xi^2/(2c_n^2)}$. Then for sufficiently large b it holds that*

$$\lim_{n\to\infty} (\bar{p}_{\vec{x}_{can}(\beta_c),n,I} * h_n)(\xi) = \bar{\rho}^*_{\vec{x}_{can}(\beta_c),I}(\xi), \quad I \in \mathcal{K}_{\vec{x}_c},$$

uniformly in $\xi \in \mathbb{R}$, where

$$(\bar{p}_{\vec{x},n,I} * h_n)(\xi) = \int_{\mathbb{R}} h_n(\xi - \eta)\, \bar{p}_{\vec{x},n,I}(d\eta)$$

is a convolution.

Proof. For $n \in \mathbb{Z}_+$ and $I \in \mathcal{I}_d$, denote the characteristic function of $(\bar{p}_{\vec{x}_{can}(\beta_c),n,I} * h_n)(\xi)\, d\xi$ by $\phi_{n,I}$. Then

$$\phi_{n,I}(t) = \int e^{\sqrt{-1}\xi t} (\bar{p}_{\vec{x}_{can}(\beta_c),n,I} * h_n)(\xi)\, d\xi = \bar{\varphi}_{\vec{x}_{can}(\beta_c),n,I}(t) e^{-c_n^2 t^2/2}, \tag{56}$$

where, using (22), (24), (46), (38) in (45), we find

$$\bar{\varphi}_{\vec{x}_{can}(\beta_c),n,I}(t) = \frac{Z_{n,I}(\beta_c - \sqrt{-1}\lambda^{-n}t)}{Z_{n,I}(\beta_c)}\,, \quad t \in \mathbb{C}.$$

According to what is proved for (47), we have

$$\lim_{n\to\infty} \bar{\varphi}_{\vec{x}_{can}(\beta_c),n,I}(t) = g_{\vec{x}_{can}(\beta_c),I}(t), \tag{57}$$

uniformly in t on compact sets. Let $A = \{t \in \mathbb{C} \mid \mathbf{Im}t \geqq 0,\ \lambda^{-1} \leqq |t| \leqq 1\}$. Then Lemma 20 and (CS1) implies $\sup_{t\in A} |g_{\vec{x}_{can}(\beta_c),I}(\sqrt{-1}t)| < 1$ for $I \in \mathcal{K}_{\vec{x}_c}$. With (CS1) we therefore see that for sufficiently small positive $\epsilon > 0$ there exists an integer n_1 such that

$$|Z_{n,I}(\beta_c - \sqrt{-1}\lambda^{-n}t)| = Z_{n,I}(\beta_c)|g_{\vec{x}_{can}(\beta_c),n,I}(t)| < x_{c,I} - \epsilon, \quad I \in \mathcal{K}_{\vec{x}_c}\,,\ n = n_1, n_1+1, \cdots,\ I \in \mathcal{K}_{\vec{x}_c}\,,\ t \in A. \tag{58}$$

Define $\vec{x}' = (x'_I;\ I \in \mathcal{I}_d)$ by

$$x'_I = \begin{cases} x_{c,I} - \epsilon, & I \in \mathcal{K}_{\vec{x}_c}\,, \\ 0 & I \notin \mathcal{K}_{\vec{x}_c}\,. \end{cases}$$

Then $\vec{x}' \in D^o \setminus \{0\}$ by Theorem 13. Since D^o is an open set in Ξ_d, there exists $\delta > 0$ such that if we define $\vec{x}'' = (x''_I;\ I \in \mathcal{I}_d)$ by

$$x''_I = \begin{cases} x_{c,I} - \epsilon, & I \in \mathcal{K}_{\vec{x}_c}\,, \\ \delta & I \notin \mathcal{K}_{\vec{x}_c}\,, \end{cases}$$

then $\vec{x}'' \in D^o$. Also, by the definition of $\mathcal{K}_{\vec{x}_c}$ and (CS1), there exists $n_0 \geqq n_1$ such that

$$|Z_{n,I}(\beta_c - \sqrt{-1}\lambda^{-n}t)| \leqq Z_{n,I}(\beta_c) < \delta, \quad I \notin \mathcal{K}_{\vec{x}_c}\,,\ n = n_0, n_0+1, \cdots,\ t \in \mathbb{R}. \tag{59}$$

Proposition 14, (58), and (59) then imply, with $\vec{x}'' \in D^o$, that there exists positive constants C_1 and C_2 such that

$$\begin{aligned} &|Z_{m+n,I}(\beta_c - \sqrt{-1}\lambda^{-n}t)| = |X_{m,I}(\vec{Z}_n(\beta_c - \sqrt{-1}\lambda^{-n}t)| \leqq X_{m,I}(\vec{x}'') \leqq C_1 e^{-C_2 2^m}, \\ &\quad m \in \mathbb{Z}_+,\ n = n_0, n_0+1, \cdots,\ I \in \mathcal{I}_d,\ t \in A. \end{aligned} \tag{60}$$

Let n be an integer satisfying $n > n_0$, and t be a real satisfying $|t| \in [1, \lambda^{n-n_0-1})$, and $m = \left[\dfrac{\log|t|}{\log\lambda}\right] + 1$, where $[x]$ is the largest integer not exceeding x. Then $n - m \geqq n_0$ and $\lambda^{-m}t \in A$. Hence (60) and (56) imply

$$\begin{aligned} &|\phi_{n,I}(t)| \leqq \frac{|Z_{m+n-m,I}(\beta_c - \sqrt{-1}\lambda^{-(n-m)}(\lambda^{-m}t))|}{Z_{n,I}(\beta_c)} \leqq \frac{C_1}{Z_{n,I}(\beta_c)}\, e^{-C_2 2^m} \leqq \frac{C_1}{Z_{n,I}(\beta_c)}\, e^{-C_2|t|^{1/d_w}}, \\ &\quad n > n_0,\ I \in \mathcal{I}_d\,,\ |t| \in [1, \lambda^{n-n_0-1}). \end{aligned}$$

On the other hand, we have $\lim_{n\to\infty} Z_{n,I}(\beta_c) = x_{c,I}$, and $\lim_{n\to\infty} \phi_{n,I}(t) = g_{\vec{x}_{can}(\beta_c),I}(t)$, $t \in \mathbb{R}$, by (57). Hence the dominated convergence theorem implies

$$\lim_{n\to\infty} \int_{\mathbb{R}} |\chi_{[-\lambda^{n-n_0-1}, \lambda^{n-n_0-1}]}(t)\phi_{n,I}(t) - g_{\vec{x}_{can}(\beta_c),I}(t)|\, dt = 0, I \in \mathcal{K}_{\vec{x}_c}\,,$$

where χ_A denotes the indicator function of a set A.

On the other hand,

$$\begin{aligned} &\int_{\mathbb{R}} |\chi_{[-\lambda^{n-n_0-1}, \lambda^{n-n_0-1}]}(t)\phi_{n,I}(t) - \phi_{n,I}(t)|\, dt \leqq 2\int_{\lambda^{n-n_0-1}}^{\infty} e^{-c_n^2 t^2/2}\, dt \\ &\leqq 2\int_{\lambda^{n-n_0-1}}^{\infty} te^{-c_n^2 t^2/2}\, dt = \frac{2}{b^2 n} e^{-n(b^2\lambda^{-2n_0-2}/2 - 2\log\lambda)} \ \to\ 0, \quad n \to \infty, \end{aligned}$$

for sufficiently large b. Therefore

$$\lim_{n\to\infty} \int_{\mathbb{R}} |\phi_{n,I}(t) - g_{\vec{x}_{can}(\beta_c),I}(t)|\, dt = 0\,.$$

Using the inversion formula we therefore have

$$\sup_{\xi\in\mathbb{R}} |(\bar{p}_{\vec{x}_{can}(\beta_c),n,I} * h_n)(\xi) - \bar{\rho}^*_{\vec{x}_{can}(\beta_c),I}(\xi)| \leqq \int_{\mathbb{R}} e^{-\sqrt{-1}\xi t} |\phi_{n,I}(t) - \bar{\varphi}_{\vec{x}_{can}(\beta_c),I}(t)|\, dt \ \to\ 0, \quad n \to \infty.$$

□

We will use the following in Section 4.3.2.

Corollary 22 *(i) There exists positive constants C_1 and C_2 such that*

$$Z_{m+n,I}(\beta_c + \lambda^{-n}) \leqq C_1 e^{-C_2 2^m}, \quad m, n \in \mathbb{Z}_+,\ I \in \mathcal{I}_d\,.$$

(ii) There exists a positive constant C such that

$$Z_{n,I}(\beta_c - \lambda^{-n}) \leqq C, \quad n \in \mathbb{Z}_+,\ I \in \mathcal{I}_d\,.$$

Proof. (i) Since $\vec{x}_{can}(\beta_c + \lambda^{-n}t) \in D^o \cap \Xi_d$ for $t > 0$, we may use Proposition 14 in a similar way as in the proof of (60) to prove the first claim.

(ii) The second claim is a direct consequence of Proposition 18 for $\vec{t} = \vec{e}$ (defined in (46)), combined with (24).

□

Theorem 9 for the full model is an obvious consequence of the following.

Proposition 23 *If β_c is the critical point, then there exist positive constants C_i, $i = 1, 2$, and real constants C_i, $i = 3, 4$, such that*

$$C_1 k^{C_3} e^{\beta_c k} \leqq N(k) \leqq C_2 k^{C_4} e^{\beta_c k}, \quad k = 1, 2, 3, \cdots.$$

Proof. To prove the upper bound, define

$$\zeta_n = \sum_{w \in W^{(0)};\, \nu(w) \leqq n} e^{-\beta_c L(w)}, \quad n \in \mathbb{Z}_+. \tag{61}$$

Then a consideration similar to that in the proof of Proposition 2 proves that there exists a polynomial $f_1 : \mathbb{R}^{\mathcal{I}_d} \to \mathbb{R}$ with positive coefficients, of maximal degree $d+1$, satisfying $f_1(\vec{0}) = 1$, such that

$$\zeta_{n+1} \leqq f_1(\vec{Z}_n(\beta_c))\zeta_n, \quad n \in \mathbb{Z}_+. \tag{62}$$

The assumption (CS1) implies that $\vec{Z}_n(\beta_c)$ is bounded, hence there exist positive constants A_1 and $A_2 \geqq 1$ such that

$$\zeta_n \leqq A_1 A_2^n, \quad n \in \mathbb{Z}_+. \tag{63}$$

$L > 2^{\nu-1}$ in (54) therefore implies

$$e^{-\beta_c k} N(k) \leqq \sum_{w \in W^{(0)};\, L(w) \leqq k} e^{-\beta_c L(w)} \leqq \zeta_{[\log_2 k]+1} \leqq A_1 A_2\, k^{\log_2 A_2},$$

which proves the upper bound.

To prove the lower bound, let $b > 0$ be a constant such that the bound in Proposition 21 holds, and let $d_n = \sqrt{2n \log \lambda} c_n = \sqrt{2 \log \lambda}\, b\, n\, \lambda^{-n}$, $n \in \mathbb{Z}_+$. Then Proposition 21 implies

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{\xi \in \mathbb{R}} \Big| \int_{[\xi - d_n, \xi + d_n]} h_n(\xi - \eta)\, \bar{p}_{\vec{x}_{can}(\beta_c),n,(1)}(d\eta) - \bar{\rho}^*_{\vec{x}_{can}(\beta_c),(1)}(\xi) \Big| \\
&\leqq \lim_{n\to\infty} \sup_{\xi \in \mathbb{R}} \int_{\mathbb{R} \setminus [\xi - d_n, \xi + d_n]} h_n(\xi - \eta)\, \bar{p}_{\vec{x}_{can}(\beta_c),n,(1)}(d\eta) \\
&\leqq \lim_{n\to\infty} h_n(d_n) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi b^2 n}} = 0\,.
\end{aligned}$$

With the positivity of $\bar{\rho}^*_{\vec{x}_{can}(\beta_c),(1)}$ (Theorem 8), this implies that there exists an integer n_2 and a positive constant ϵ such that

$$\bar{p}_{\vec{x}_{can}(\beta_c),n,(1)}([\xi - d_n, \xi + d_n]) \geqq c_n \epsilon, \quad n \geqq n_2\,,\ \xi \in [\lambda^{-1}, \lambda].$$

Let k be a positive integer, and let n be a positive integer satisfying $\lambda^{-n} k \in [1, \lambda)$. Assume that k is sufficiently large so that $n \geqq n_2$ and $d_n \leqq 1 - \lambda^{-1}$. Then

$$\bar{p}_{\vec{x}_{can}(\beta_c),n,(1)}([\lambda^{-n}k - 2d_n, \lambda^{-n}k]) \geqq c_n \epsilon.$$

Note that we can construct an injection $\{w \in W^{(n)}_{(1)} \mid L(w) \leqq k\} \to \{w \in W^{(0)} \mid L(w) = k\}$ by extending the path. Hence

$$Z_{n,(1)}(\beta_c) c_n \epsilon \leqq \sum_{w \in W^{(n)}_{(1)};\, k - 2d_n \lambda^n \leqq L(w) \leqq k} e^{-\beta_c L(w)} \leqq e^{2\beta_c d_n \lambda^n} e^{-\beta_c k} N(k).$$

Since $\lambda^{-n} k \geqq 1$ implies $n \leqq \dfrac{\log k}{\log \lambda}$, we have

$$N(k) \geqq e^{\beta_c k} k^{-1 - 2b\beta_c \sqrt{2/\log\lambda}} \sqrt{\log k}\, Z_{n,(1)}(\beta_c) \frac{b}{\sqrt{\log \lambda}} \epsilon,$$

which implies the lower bound.

□

4.3.2 Large deviation type estimates on long paths and short paths.

Define, for $n, m \in \mathbb{Z}_+$,

$$U_{n,m} = \sum_{\substack{w \in W^{(0)}; \\ \nu(w) \leqq n, \\ L(w) \geqq \lambda^{n+(m/2)}}} e^{-\beta_c L(w)} \qquad \text{and} \qquad V_{n,m} = \sum_{\substack{w \in W^{(0)}; \\ \nu(w) = n+1, \\ L(w) \leqq \lambda^{n-m}}} e^{-\beta_c L(w)}.$$

Proposition 24 *There exist positive constants C' and C'' such that*

$$U_{n,m} \leqq C' A_2^n e^{-C'' \lambda^{m/2}}, \quad \textit{and} \quad V_{n,m} \leqq C' A_2^n e^{-C'' 2^m}, \quad n, m \in \mathbb{Z}_+,$$

where A_2 is as in (63).

Proof. Let C'' be a positive constant satisfying $C'' \leqq \dfrac{\lambda - \sqrt{\lambda}}{d+1}$.

To prove the bound for $U_{n,m}$, define

$$S_{n,m,I} = \sum_{w \in W_I^{(n)},\ L(w) \geqq C'' \lambda^{n+(m/2)}} e^{-\beta_c L(w)}, \quad n, m \in \mathbb{Z}_+,\ I \in \mathcal{I}_d\,.$$

Then, in a similar way as deriving (62), we find

$$U_{n+1,m} \leqq f_1(\vec{Z}_n(\beta_c)) U_{n,m+1} + \sum_{I \in \mathcal{I}_d} S_{n,m,I}\, \frac{\partial f}{\partial x_I}(\vec{Z}_n(\beta_c))\, \zeta_n\,, \quad n, m \in \mathbb{Z}_+.$$

This with (63) implies that there exists $C_1 > 0$ such that

$$A_2^{-n-1} U_{n+1,m} \leqq A_2^{-n} U_{n,m+1} + C_1 \sum_{I \in \mathcal{I}_d} S_{n,m,I}\,, \quad n, m \in \mathbb{Z}_+.$$

Note that $L(w) \leqq d\,(d+1)^{\nu(w)}$ in (54) implies

$$U_{n,m} = 0, \quad \lambda^{n+(m/2)} > d\,(d+1)^n,$$

which holds if

$$m > \frac{2 \log d}{\log \lambda} + 2\alpha n,$$

where we wrote $\alpha = \dfrac{\log(d+1)}{\log \lambda} - 1$. (Note that Proposition 5 implies $\alpha > 0$.) Therefore, if we write

$$k_0(n,m) = \left[\frac{2\alpha n - m}{2\alpha + 1} + \frac{2 \log d}{(2\alpha+1) \log \lambda} \right],$$

then

$$U_{n,m} \leqq C_1 A_2^n \sum_{k=0}^{k_0(n,m)} \sum_{I \in \mathcal{I}_d} S_{n-k-1, m+k, I}\,, \quad n, m \in \mathbb{Z}_+. \tag{64}$$

On the other hand,

$$\begin{aligned} S_{n,m,I} &\leqq \sum_{w \in W_I^{(n)},\ L(w) \geqq C'' \lambda^{n+(m/2)}} e^{-(\beta_c - \lambda^{-n}) L(w)} e^{-C'' \lambda^{m/2}} \leqq Z_{n,I}(\beta_c - \lambda^{-n}) e^{-C'' \lambda^{m/2}} \\ &\leqq C e^{-C'' \lambda^{m/2}}, \quad n, m \in \mathbb{Z}_+,\ I \in \mathcal{I}_d\,, \end{aligned}$$

for a positive constant C, where we used Corollary 22. This and (64) imply

$$U_{n,m} \leqq C C_1 \sharp \mathcal{I}_d\, A_2^n \sum_{k=0}^{\infty} e^{-C'' \lambda^{(m+k)/2}}.$$

Note that there exists a constant k_0 such that

$$\lambda^{(m+k)/2} \geqq \lambda^{(m-k_0)/2} + \lambda^{(k-k_0)/2}, \quad m,k \in \mathbb{Z}_+.$$

(This is because

$$\lambda^{(m+k)/2} - (\lambda^{m/2-k_0} + \lambda^{k/2-k_0}) = (\lambda^{m/2} - \lambda^{-k_0/2})(\lambda^{k/2} - \lambda^{-k_0/2}) - \lambda^{-k_0}$$

is increasing in m and k, hence it is sufficient to choose k_0 such that $1 \geqq 2\lambda^{-k_0/2}$.) Therefore there exists a positive constant C' such that

$$U_{n,m} \leqq CC_1 \sharp\mathcal{I}_d \sum_{k=0}^{\infty} e^{-C''\lambda^{(k-k_0)/2}} A_2^n e^{-C''\lambda^{(m-k_0)/2}} \leqq C' A_2^n e^{-C''\lambda^{m/2}}, \quad n,m \in \mathbb{Z}_+.$$

To prove the bound for $V_{n,m}$, define

$$T_{n,m,I} = \sum_{w \in W_I^{(n)},\ L(w) \leqq \lambda^{n-m}} e^{-\beta_c L(w)}, \quad n,m \in \mathbb{Z}_+;\ m \leqq n,\ I \in \mathcal{I}_d\,,$$

and write $\vec{T}_{n,m} = (T_{n,m,I}\,,\ I \in \mathcal{I}_d)$. Then, again in a similar way as deriving (62), we find

$$V_{n,m} \leqq (f_1(\vec{T}_{n,m}) - 1)\zeta_n = (f_1(\vec{T}_{n,m}) - f_1(\vec{0}))\zeta_n\,.$$

On the other hand, the condition $L(w) \leqq \lambda^{n-m}$ in the definition of $T_{n,m,I}$ implies

$$T_{n,m,I} \leqq \sum_{w \in W_I^{(n)}} e^{-(\beta_c + \lambda^{m-n})L(w)+1} = eZ_{n,I}(\beta_c + \lambda^{m-n}).$$

With Corollary 22 and (62) we have the statement.

□

Proposition 25 *For sufficiently large positive constant α, it holds that*

$$\underline{\lim_{k\to\infty}} (\log k)^{s\alpha} k^{-s} E_k[\|w\|^{sd_w}] > 0, \qquad \textit{and} \qquad \overline{\lim_{k\to\infty}} (\log k)^{-s\alpha} k^{-s} E_k[\|w\|^{sd_w}] < \infty,$$

for all $s \geqq 0$, where $E_k[\cdot]$ is the expectation defined in Theorem 10.

Proof. For $k \in \mathbb{N}$, define $\tilde{n}(k) = \left[\dfrac{\log k}{\log \lambda}\right]$. Obviously, $\lambda^{\tilde{n}(k)} \leqq k < \lambda^{\tilde{n}(k)+1}$.

For positive integers m and k satisfying $m \leqq \tilde{n}(k)$, Proposition 24 implies

$$\begin{aligned}
&\sharp\{w \in W^{(0)} \mid L(w) = k,\ \nu(w) \leqq \tilde{n}(k) - m\} \leqq e^{\beta_c k} U_{\tilde{n}(k)-m,2m} \\
&\leqq C' \exp(\beta_c k + (\tilde{n}(k) - m)\log A_2 - C''\lambda^m) \\
&\leqq C' \exp(\beta_c k + \log_{\log\lambda} A_2 \log k - C''\lambda^m).
\end{aligned}$$

This and Proposition 23 imply that for sufficiently large α and for any real ϵ, there exists $C > 0$ such that

$$\begin{aligned}
&\tilde{P}_k[\nu(w) \leqq \tilde{n}(k) - \alpha \log\log k + \epsilon] \leqq C'C_1^{-1} \exp((\log_{\log\lambda} A_2 - C_3)\log k - C''\lambda^{-\epsilon}(\log k)^{\alpha \log\lambda}) \\
&\leqq Ce^{-(\log k)^3}, \quad k \in \mathbb{N},
\end{aligned}$$

where $\tilde{P}_k[\cdot]$ is as defined in Theorem 10. This, with (55), implies for sufficiently large α,

$$\lim_{k\to\infty} \tilde{P}_k[\|w\| < (\log k)^{-\alpha/d_w} k^{1/d_w}]\, e^{(\log k)^2} \leqq \lim_{k\to\infty} \tilde{P}_k[2^{\nu(w)} < 2(\log k)^{-\alpha/d_w} k^{1/d_w}]\, e^{(\log k)^2} = 0\,.$$

Chebishev's inequality implies, for $s \geqq 0$,

$$E_k[\|w\|^{sd_w}] \geqq ((\log k)^{-\alpha} k)^s (1 - \tilde{P}_k[\|w\| \leqq (\log k)^{-\alpha/d_w} k^{1/d_w}]).$$

Therefore,

$$\varliminf_{k\to\infty} (\log k)^{s\alpha} k^{-s} E_k[\|w\|^{sd_w}] > 0, \quad s \geqq 0\,.$$

Next, for positive integers m, k, and ℓ, Proposition 24 implies, with an obvious inequality $2^{m+\ell} \geqq 2^{m-1} + 2^{\ell-1}$, $m, \ell \in \mathbb{Z}_+$,

$$\begin{aligned}
&\sharp\{w \in W^{(0)} \mid L(w) = k,\ \nu(w) = \tilde{n}(k) + m + \ell + 2\} \leqq e^{\beta_c k} V_{\tilde{n}(k)+m+\ell+1,m+\ell} \\
&\leqq C' \exp(\beta_c k + (\tilde{n}(k) + m + \ell + 1)\log A_2 - C''2^{m+\ell}) \\
&\leqq C' \exp(\beta_c k + \log_{\log\lambda} A_2 \log k + m \log A_2 - C''2^{m-1}) A_2^{\ell+1} \exp(-C''2^{\ell-1}).
\end{aligned}$$

This and Proposition 23 imply that there exists a positive constant C such that

$$\begin{aligned}
&\tilde{P}_k[\nu(w) \geqq \tilde{n}(k) + \frac{\alpha}{\log 2}\log\log k] = \sum_{\ell=0}^{\infty} \tilde{P}_k[\nu(w) = \tilde{n}(k) + \ell + \frac{\alpha}{\log 2}\log\log k] \\
&\leqq \frac{C'}{C_1 A_2} \sum_{\ell=0}^{\infty} A_2^{\ell} e^{-C''2^{\ell-1}} \exp((\log_{\log\lambda} A_2 - C_3)\log k + \alpha \log_2 A_2 \log\log k - \frac{C''}{8}(\log k)^{\alpha}) \\
&\leqq C \exp((\log_{\log\lambda} A_2 - C_3)\log k + \alpha \log_2 A_2 \log\log k - \frac{C''}{8}(\log k)^{\alpha}), \quad k \in \mathbb{N}.
\end{aligned}$$

This, with (55), implies for sufficiently large α,

$$\lim_{k\to\infty} \tilde{P}_k[\|w\| \geqq (\log k)^{\alpha/d_w} k^{1/d_w}]\, e^{(\log k)^2} \leqq \lim_{k\to\infty} \tilde{P}_k[2^{\nu(w)} \geqq (\log k)^{\alpha/d_w} k^{1/d_w}]\, e^{(\log k)^2} = 0\,.$$

Let $s \geqq 0$. Note that $\|w\| \leqq L(w)$, which implies

$$E_k[\|w\|^{sd_w}] \leqq ((\log k)^{\alpha} k)^s + k^{sd_w} \tilde{P}_k[\|w\| \geqq (\log k)^{\alpha} k].$$

Therefore,

$$\varlimsup_{k\to\infty} (\log k)^{-s\alpha} k^{-s} E_k[\|w\|^{sd_w}] < \infty,$$

□

4.3.3 Reflection principle.

The definition of pre-dSG in (1) and (2) induces a natural coordinate system on G which is an onto map $\pi : \{0,1,2,\cdots,d\}^{\mathbb{Z}_+} \to G$ defined as follows. For each $v_{0,i} = v_i \in G_0$ $(i = 0,1,2,\cdots,d)$ we assign a coordinate $(i,0,0,0,\cdots)$;

$$\pi(i,0,0,0,\cdots) = v_{0,i},\ i = 0,1,2,3,\cdots,d.$$

For $n = 0,1,2,\cdots$ and $i = 0,1,2,3,\cdots,d$, put $G_{n,i} = G_n + 2^n v_{0,i}$, and define a $1:1$ onto map $\iota_{n,i} : G_{n,i} \to G_n$ by $\iota_{n,i}(v) = v - 2^n v_{0,i}$. $\iota_{n,i}$ naturally induces a $1:1$ onto map $B_{n,i} \to B_n$, which we also denote by $\iota_{n,i}$.

We proceed with by induction in n and assume that a coordinate system π on G_{n-1} $(= G_{n-1,0})$ has been defined for an $n \geqq 1$, in such a way that $\pi(v) \in G_{n-1}$ holds for any $v = (i_0, i_1, i_2, \cdots, i_{n-1}, 0, 0, 0, \cdots)$ with $i_k \in \{0,1,2,\cdots,d\}$, $k = 0,1,2,\cdots,n-1$. For $v \in G_{n-1,j}$, with $j \in \{0,1,2,\cdots,d\}$, define

$$\pi(i_0, i_1, i_2, \cdots, i_{n-1}, j, 0, 0, \cdots) = v, \quad \text{if} \quad \pi(i_0, i_1, i_2, \cdots, i_{n-1}, 0, 0, 0, \cdots) = \iota_{n-1,j}(v).$$

Note that this definition is compatible with $G_{n-1,0} = G_{n-1} \subset G_n$.

Each point in $G \setminus \{O\}$ has exactly two coordinate representations, because

$$\pi(j, j, \cdots, j, i, 0, \cdots) = \pi(i, i, \cdots, i, j, 0, \cdots) \in G_{m,i} \cap G_{m,j}\,, \quad 0 \leqq i < j \leqq d,\ m \in \mathbb{Z}_+.$$

Note also that if $\pi(i_0, i_1, i_2, \cdots) \in G_n$ then $i_k = 0$, $k = n+1, n+2, \cdots$.

We now define a reflection map with which we formulate a reflection principle (Theorem 27). For each $i = 1,2,3,\cdots,d$ define $R_{0,i} : \{0,1,2,\cdots,d\} \to \{0,1,2,\cdots,d\}$ by

$$\begin{cases} R_{0,i}(0) = i, \\ R_{0,i}(i) = 0, \\ R_{0,i}(j) = j,\ j \neq 0, i. \end{cases}$$

For $n = 1, 2, 3, \cdots$ and $i = 1, 2, \cdots, d$, we define $1:1$ maps $R_{n,i} : G \to G$ and $\tilde{R}_{n,i} : G \to G$ ('partial reflections' with respect to a hyperplane containing $\pi(j, j, \cdots, j, i, 0, 0, \cdots) \in G_{n-1,i}$, $j \neq 0, i$, and 'perpendicular to i-th axis'), by:

$$\begin{aligned} &R_{n,i}(\pi(x_0, x_1, x_2, \cdots)) \\ &= \begin{cases} \pi(x_0, x_1, x_2, \cdots), \text{ if } \pi(x_0, x_1, x_2, \cdots, x_n, \cdots) \notin G_{n-1,i}, \\ \pi(R_{0,i}(x_0), R_{0,i}(x_1), \cdots, R_{0,i}(x_{n-1}), x_n, 0, 0, 0, \cdots), \\ \qquad\qquad \text{if } \pi(x_0, x_1, x_2, \cdots, x_n, \cdots) \in G_{n-1,i} \ \text{ and } \ x_n = i, \end{cases} \end{aligned} \tag{65}$$

and

$$\begin{aligned} &\tilde{R}_{n,i}(\pi(x_0, x_1, x_2, \cdots)) \\ &= \begin{cases} \pi(x_0, x_1, x_2, \cdots), \text{ if } \pi(x_0, x_1, x_2, \cdots, x_n, \cdots) \notin G_{n-1} \cup \iota_{n,i}^{-1}(G_{n-1}), \\ \pi(R_{0,i}(x_0), R_{0,i}(x_1), \cdots, R_{0,i}(x_{n-1}), x_n, R_{0,i}(x_{n+1}), 0, 0, \cdots), \\ \qquad\qquad \text{if } \pi(x_0, x_1, x_2, \cdots, x_n, \cdots) \in G_{n-1} \cup \iota_{n,i}^{-1}(G_{n-1}) \ \text{ and } x_n = 0\,. \end{cases} \end{aligned} \tag{66}$$

Note that G_{n-1}, $G_{n-1,i}$, and $\iota_{n,i}^{-1}(G_{n-1})$ are three copies of G_{n-1} aligned in 'i-th axis' direction, such that

$$G_{n-1} \cap G_{n-1,i} = \{\pi(0, 0, 0, \cdots, 0, \overset{n}{i}, 0, 0, \cdots)\} = \{\pi(\overset{0,1,2}{i, i, i}, \cdots, \overset{n-1}{i}, 0, 0, 0, \cdots)\}, \tag{67}$$

and

$$G_{n-1,i} \cap \iota_{n,i}^{-1}(G_{n-1}) = \{\pi(0, 0, 0, \cdots, 0, \overset{n+1}{i}, 0, 0, \cdots)\} = \{\pi(\overset{0,1,2}{i, i, i}, \cdots, \overset{n}{i}, 0, 0, \cdots)\}. \tag{68}$$

Note also that, by construction all the points in $G_{n-1,i}$ can be written as $\pi(x_0, x_1, \cdots, x_n, 0, 0, \cdots)$ with $x_n = i$, those in G_{n-1} as $\pi(x_0, \cdots, x_n, 0, 0, \cdots)$ with $x_n = 0$, and those in $\iota_{n,i}^{-1}(G_{n-1})$ as $\pi(x_0, \cdots, x_n, x_{n+1}, 0, \cdots)$ with $x_n = 0$ and $x_{n+1} = i$.

Proposition 26 *$R_{n,i}$ and $\tilde{R}_{n,i}$ are $1:1$ maps. Moreover, the following hold.*

(i) *If $x \in G_{n-1}$ then $\tilde{R}_{n,i}(x) \in \iota_{n,i}^{-1}(G_{n-1})$.*

(ii) *If $(x, y) \in B_{n-1}$, then $(\tilde{R}_{n,i}(x), \tilde{R}_{n,i}(y)) \in B$.*

(iii) *If $x \in G_{n-1} \cap G_{n-1,i}$ then $R_{n,i}(x) = \tilde{R}_{n,i}(x)$.*

(iv) *If $x \in G_n$ then $R_{n,i}(x) \in G_n$.*

(v) *If $x \in G_{n-1,i} \cap \bigcup_{j \neq 0,i} G_{n-1,j}$ then $R_{n,i}(x) = x$.*

(vi) *If $(x, y) \in B_{n-1,i}$ then $(R_{n,i}(x), R_{n,i}(y)) \in B$.*

Proof. By definition, $R_{n,i}^2(x) = \tilde{R}_{n,i}^2(x) = x$, $x \in G$, hence, in particular, $R_{n,i}$ and $\tilde{R}_{n,i}$ are $1:1$ maps.

(i) If $x \in G_{n-1}$ then x has a coordinate of the form $x = \pi(x_0, x_1, \cdots, x_{n-1}, 0, 0, 0, \cdots)$, which, with (66), implies $\tilde{R}_{n,i}(x) = \pi(x_0, x_1, \cdots, x_{n-1}, 0, i, 0, \cdots)$, which is in $\iota_{n,i}^{-1}(G_{n-1})$.

(ii) let $(x, y) \in B_{n-1}$. Then $x, y \in G_{n-1}$, hence their coordinates can be written as

$$x = \pi(x_0, x_1, \cdots, x_{n-1}, 0, 0, 0, \cdots), \quad y = \pi(y_0, y_1, \cdots, y_{n-1}, 0, 0, 0, \cdots).$$

With (66), we have

$$\tilde{R}_{n,i}(x) = \pi(R_{0,i}(x_0), R_{0,i}(x_1), \cdots, R_{0,i}(x_{n-1}), 0, i, 0, 0, \cdots) \in \iota_{n,i}^{-1}(G_{n-1}),$$

and a similar expression holds for y. Noting that $\iota_{n,i}^{-1}(G_{n-1})$ is a copy of G_{n-1}, it then suffices to prove

$$(\pi(R_{0,i}(x_0), \cdots, R_{0,i}(x_{n-1}), 0, 0, \cdots), \pi(R_{0,i}(y_0), \cdots, R_{0,i}(y_{n-1}), 0, 0, \cdots)) \in B_{n-1}\,.$$

Noting the definition of B_{n-1}, we may assume, from $(x, y) \in B_{n-1}$, that $x_{n-1} = y_{n-1}$ and

$$(\pi(x_0, x_1, \cdots, x_{n-2}, 0, 0, 0, \cdots), \pi(y_0, y_1, \cdots, y_{n-2}, 0, 0, 0, \cdots)) \in B_{n-2}\,.$$

Inductively, it eventually turns out that it suffices to prove

$$(\pi(R_{0,i}(x), 0, 0, \cdots), \pi(R_{0,i}(y), 0, 0, \cdots)) \in B_0 \quad \text{if} \quad (\pi(x, 0, 0, \cdots), \pi(y, 0, 0, \cdots)) \in B_0\,,$$

which is obvious from the definition of $F_0 = (G_0, B_0)$.

(iii) Applying (67) in (65) and (66), we have

$$R_{n,i}(x) = R_{n,i}(\pi(0,0,0,\cdots,0,\overset{n}{i},0,0,\cdots)) = \pi(\overset{0,1,2}{i,i,i},\cdots,i,\overset{n}{i},0,0,\cdots),$$

and

$$\tilde{R}_{n,i}(x) = \tilde{R}_{n,i}(\pi(\overset{0,1,2}{i,i,i},\cdots,\overset{n-1}{i},0,0,0,\cdots)) = \pi(0,0,0,\cdots,0,0,\overset{n+1}{i},0,\cdots)$$
$$= \pi(\overset{0,1,2}{i,i,i},\cdots,i,\overset{n}{i},0,0,\cdots).$$

Hence $R_{n,i}(x) = \tilde{R}_{n,i}(x)$.

(iv) If $x \in G_n$ then x has a coordinate $x = \pi(x_0, x_1, \cdots, x_n, 0, 0, \cdots)$. Then (65) implies that $R_{n,i}(x)$ has a similar coordinate, hence $R_{n,i}(x) \in G_n$.

(v) Noting (67) and (68), we see that x has a coordinate $x = \pi(j, j, \cdots, j, \overset{n}{i}, 0, 0, \cdots)$, $j \neq 0, i$. Then (65) implies $R_{n,i}(x) = x$.

(vi) This can be proved in a similar way as (ii).

□

For each self-avoiding path $w \in W^{(0)}$ satisfying $|w(L(w))| < 2^{\nu(w)-1}$, we want to assign a self-avoiding path $R(w) \in W^{(0)}$ ('reflected path') such that $|R(w)(L(R(w)))| > 2^{\nu(R(w))-1}$ (the right hand side stands for the Euclidean distance of the endpoints of $R(w)$). This is possible using (65) and (66), as follows.

Given $w = (w(0), w(1), \cdots, w(L(w))) \in W^{(0)}$ with the property

$$w(L(w)) \in G_{n-1} \setminus \bigcup_{i=1}^{d} G_{n-1,i}, \quad \text{where} \quad n = \nu(w), \tag{69}$$

we shall define a path $R(w)$ as follows.

Let $T(w)$ be a positive integer satisfying

$$w(k) \in G_{n-1},\ T(w) \leqq k \leqq L(w), \quad \text{and} \quad w(T(w)-1) \notin G_{n-1}.$$

The condition (69) implies that such an integer (uniquely) exists. Since $w(T(w)) \in G_{n-1}$, $w(T(w)-1) \in \bigcup_{i=1}^{d} G_{n-1,i}$. Let $i(w) \in \{1, 2, \cdots, d\}$ be such that $w(T(w)-1) \in G_{n-1,i(w)}$. Clearly, such an integer is also unique. Also the definitions of $T(w)$ and $i(w)$ imply

$$w(T(w)) \in G_{n-1} \cap G_{n-1,i(w)}. \tag{70}$$

Define $R(w)$ by

$$R(w)(k) = \begin{cases} R_{n,i(w)}(w(k)), & 0 \leqq k < T(w), \\ \tilde{R}_{n,i(w)}(w(k)), & T(w) \leqq k \leqq L(w). \end{cases}$$

Theorem 27 *For each $w \in W^{(0)}$ satisfying $|w(L(w))| < 2^{\nu(w)-1}$, $R(w) \in W^{(0)}$ (i.e., is a self-avoiding path starting from O) which satisfies $L(R(w)) = L(w)$, $\nu(R(w)) = \nu(w) + 1$, and $2^{\nu(w)} < |R(w)(L(w))|$.*

Proof. The non-trivial part of the claim is that $R(w)$ is a self-avoiding path. All the other properties are obvious from the definition of the reflections.

We first prove that $R(w)$ is self-avoiding, when w is self-avoiding. Let $n = \nu(w)$. Note that (70) and Proposition 26 (iii) imply

$$R_{n,i(w)}(w(T(w))) = \tilde{R}_{n,i(w)}(w(T(w))). \tag{71}$$

Since $R_{n,i}$ and $\tilde{R}_{n,i}$ are $1:1$ maps (Proposition 26), the path segments $w_1 = \{R(w)(k),\ 0 \leqq k \leqq T(w)\}$ and $w_2 = \{R(w)(k),\ T(w) \leqq k \leqq L(w)\}$ are self-avoiding.

The definition of $T(w)$ implies $w(k) \in G_{n-1}$, $T(w) \leqq k \leqq L(w)$, which, with Proposition 26 (i), implies $\tilde{R}_{n,i}(w(k)) \in \iota_{n,i}^{-1}(G_{n-1})$, hence, in particular, $w_2 \cap G_n = \{R_{n,i}(w(T(w)))\}$. On the other hand,

Proposition 26 (iv) implies $w_1 \subset G_n$. Therefore w_1 and w_2 are mutually avoiding. This proves that $R(w)$ is self-avoiding.

We are left with proving that $R(w)$ is a path, i.e.,

$$(R(w)(k), R(w)(k+1)) \in B,\ k = 0, 1, 2, \cdots, L(w) - 1\,. \tag{72}$$

Definition of $T(w)$ and Proposition 26 (ii) imply (72) for $T(w) \leqq k < L(w)$. Hence we may assume $0 \leqq k < T(w)$. Definition of $T(w)$ and (71) imply $R(w)(k) = R_{n,i(w)}(w(k))$, $0 \leqq k \leqq T(w)$. If $w(k) \notin G_{n-1,i(w)}$, then $(w(k), w(k+1)) \in B$ and $n = \nu(w)$ and (70) with self-avoiding property of w imply $w(k+1) \notin G_{n-1,i(w)}$ or $w(k+1) \in G_{n-1,i(w)} \cap \bigcup_{j \neq 0, i(w)} G_{n-1,j}$. The definition of $R_{n,i(w)}$ and Proposition 26(v) imply that in either case $R_{n,i(w)}(w(k+1)) = w(k+1)$. Also $w(k) \notin G_{n-1,i(w)}$ implies $R_{n,i(w)}(w(k)) = w(k)$. Hence in this case (72) holds.

The remaining case is when $0 \leqq k < T(w)$ and $w(k) \in G_{n-1,i(w)}$ hold. If $w(k+1) \notin G_{n-1,i(w)}$, then a similar reasoning as above applies. Therefore we may also assume $w(k+1) \in G_{n-1,i(w)}$. Hence $(w(k), w(k+1)) \in B_{n-1,i(w)}$, in which case Proposition 26 (vi) applies and we have (72).

□

Corollary 28 *If $s \geqq 0$ and $k \in \mathbb{N}$,*

$$E_k[2^{s(\nu(w)-1)};\ |w(k)| \leqq 2^{\nu(w)-1}] \leqq E_k[2^{s(\nu(w)-1)};\ |w(k)| \geqq 2^{\nu(w)-1}],$$

where $E_k[\cdot]$ is the expectation defined in Theorem 10.

Proof. Recalling the definition of the measure $\tilde{P}_k$, we find from Theorem 27,

$$\begin{aligned}
&E_k[2^{s(\nu(w)-1)};\ |w(k)| < 2^{\nu(w)-1}] = \sum_{n=0}^{\infty} 2^{s(n-1)} \tilde{P}_k[\nu(w) = n,\ |w(k)| < 2^{n-1}] \\
&\leqq \sum_{n=0}^{\infty} 2^{s(n-1)} \tilde{P}_k[\nu(w) = n+1,\ |w(k)| > 2^n] = 2^{-s} E_k[2^{s(\nu(w)-1)};\ |w(k)| > 2^{\nu(w)-1}] \\
&\leqq E_k[2^{s(\nu(w)-1)};\ |w(k)| > 2^{\nu(w)-1}].
\end{aligned}$$

Hence

$$\begin{aligned}
&E_k[2^{s(\nu(w)-1)};\ |w(k)| \leqq 2^{\nu(w)-1}] = E_k[2^{s(\nu(w)-1)};\ |w(k)| = 2^{\nu(w)-1}] + E_k[2^{s(\nu(w)-1)};\ |w(k)| < 2^{\nu(w)-1}] \\
&\leqq E_k[2^{s(\nu(w)-1)};\ |w(k)| \geqq 2^{\nu(w)-1}].
\end{aligned}$$

□

We are now ready to prove Theorem 10.

Proof of Theorem 10. Note an obvious bound

$$|w(L(w))| \leqq \|w\| \leqq 2^{\nu(w)}, \quad w \in W^{(0)}.$$

This and Corollary 28 imply

$$\begin{aligned}
&2^{-s-1} E_k[\|w\|^s] \\
&\leqq 2^{-s-1} E_k[2^{s\nu(w)}] = \frac{1}{2}(E_k[2^{s(\nu(w)-1)};\ |w(k)| \leqq 2^{\nu(w)-1}] + E_k[2^{s(\nu(w)-1)};\ |w(k)| > 2^{\nu(w)-1}]) \\
&\leqq E_k[2^{s(\nu(w)-1)};\ |w(k)| \geqq 2^{\nu(w)-1}] \leqq E_k[|w(k)|^s;\ |w(k)| \geqq 2^{\nu(w)-1}] \\
&\leqq E_k[|w(k)|^s] \leqq E_k[\|w\|^s], \quad s \geqq 0,\ k \in \mathbb{Z}.
\end{aligned}$$

This and Proposition 25 further imply, for sufficiently large α,

$$\underline{\lim_{k\to\infty}} (\log k)^{s\alpha} k^{-s} E_k[|w(k)|^{s\,d_w}] > 0, \qquad \text{and} \qquad \overline{\lim_{k\to\infty}} (\log k)^{-s\alpha} k^{-s} E_k[|w(k)|^{s\,d_w}] < \infty.$$

Namely, there exist constants k_0, C, and C' such that

$$s \log k - s\alpha \log\log k + C \leqq \log E_k[|w(k)|^{s\,d_w}] \leqq s \log k + s\alpha \log\log k + C', \quad k \geqq k_0,\ s \geqq 0.$$

This implies

$$\lim_{k\to\infty} \frac{1}{\log k} \log E_k[|w(k)|^{s\,d_w}] = s, \quad s \geqq 0,$$

which completes a proof of Theorem 10. □

5 Restricted model on 4-dimensional Sierpiński gasket.

In this section, we consider the case $d = 4$. The RG map $\vec{\Phi}$ (Proposition 2) is a map on 6 dimensional space $\mathbb{C}^{\mathcal{I}_4}$ where, as in Section 2, $\mathcal{I}_4 = \{(1), (1,1), (2), (3), (4), (1,2)\}$. (For convenience, we assign the second coordinate to $(1,1)$ in this section.)

We will consider the restricted self-avoiding walks, the self-avoding paths w starting from O with the property $s_J(w) = 0$, $J \notin \mathcal{K}_{res}$, where $\mathcal{K}_{res} = \{(1), (11)\}$ (see (13) and (20)). We regard $\mathbb{R}_+^{\mathcal{K}_{res}} = \{(x_{(1)}, x_{(11)}, 0, \cdots, 0) \mid x_{(1)}, x_{(11)} \in \mathbb{R}_+\} \subset \mathbb{R}_+^{\mathcal{I}_4}$.

To apply the results of previous sections for the restricted model, we use the following explicit properties of $\vec{\Phi}$.

Proposition 29 *The map $\vec{\Phi}$ satisfies the following.*

(i) *$\vec{\Phi}$ is a 6 dimensional vector valued function whose components are polynomials in 6 variables (1), (11), (2), (3), (4), (12) with positive integer coefficients. The degree of each term in the polynomials are no less than 2 and no greater than 5.*

(ii)

$$\begin{aligned}
\Phi_{(1)}(x,y,0,0,0,0) &= x^2 + 3x^3 + 6x^4 + 6x^5 + 12x^3y + 30x^4y + 18x^2y^2 \\
&\quad + 78x^3y^2 + 96x^2y^3 + 132xy^4 + 132y^5, \\
\Phi_{(1,1)}(x,y,0,0,0,0) &= x^4 + 2x^5 + 4x^3y + 13x^4y + 32x^3y^2 + 88x^2y^3 \\
&\quad + 22y^4 + 220xy^4 + 186y^5.
\end{aligned} \tag{73}$$

(iii) *There exist polynomials $\Phi_{I,i}$, $I = (1), (11)$, $i = 0, 1, 2, 3, 4$, of positive coefficients, such that $\Phi_{(1),0}$ contains a term $x_{(1)}^2$, $\Phi_{(11),0}$ contains a term $x_{(1)}^4$, and*

$$\begin{aligned}
\Phi_{(1)}(\vec{x}) &= \Phi_{(1),1}(\vec{x})\, x_{(1)} + \tfrac{1}{2}\Phi_{(1),2}(\vec{x})\, x_{(2)} + \tfrac{1}{3}\Phi_{(1),3}(\vec{x})\, x_{(3)} + \Phi_{(1),4}(\vec{x})\, x_{(11)} + \Phi_{(1),0}(\vec{x}), \\
\Phi_{(2)}(\vec{x}) &= \Phi_{(1),1}(\vec{x})\, x_{(2)} + \Phi_{(1),2}(\vec{x})\, x_{(3)} + \Phi_{(1),3}(\vec{x})\, x_{(4)} + \Phi_{(1),4}(\vec{x})\, x_{(12)}, \\
\Phi_{(11)}(\vec{x}) &= \Phi_{(11),1}(\vec{x})\, x_{(1)} + \tfrac{1}{2}\Phi_{(11),2}(\vec{x})\, x_{(2)} + \tfrac{1}{3}\Phi_{(11),3}(\vec{x})\, x_{(3)} + \Phi_{(11),4}(\vec{x})\, x_{(11)} + \Phi_{(11),0}(\vec{x}), \\
\Phi_{(12)}(\vec{x}) &= \Phi_{(11),1}(\vec{x})\, x_{(2)} + \Phi_{(11),2}(\vec{x})\, x_{(3)} + \Phi_{(11),3}(\vec{x})\, x_{(4)} + \Phi_{(11),4}(\vec{x})\, x_{(12)}.
\end{aligned} \tag{74}$$

(iv) *For each $I \notin \mathcal{K}_{res}$ there exist positive integers $m = m_I$ and $m' = m'_I$ such that $\Phi_{(1)}$ and $\Phi_{(11)}$ contain terms $x_{(1)}^m x_I$ and $x_{(1)}^{m'} x_I$, respectively.*

(v) *If $I \notin \mathcal{K}_{res}$, then each term in Φ_I contains a positive power of x_J for some $J \notin \mathcal{K}_{res}$. Furthermore, each term in $\Phi_{(3)}$ and $\Phi_{(4)}$ has total degree 2 or more of x_J's with $J \notin \mathcal{K}_{res}$. $\Phi_{(2)}$ contains a term $x_{(1)}^3 x_{(11)} x_{(12)}$ and $\Phi_{(12)}$ contains a term $x_{(1)}^4 x_{(2)}$.*

Proof. (i) This is already proved in Proposition 2.

(ii) The explicit forms (73) of $\Phi_I(x,y,0,0,0,0)$, $I = (1), (1,1)$, are obtained by explicit counting of paths in the right hand side of the definition (9) for $\vec{\Phi} = \vec{X}_1$, with aid of computer calculations.

(iii) The expression for $\Phi_{(2)}$ in (74) follows from an observation (from the definition (5)) that each path w in $W_{(2)}^{(1)}$, which starts from O and ends at $v_{1,2}$, must pass through $v_{1,1}$. w therefore has a path segment of length more than 1 in the unit simplex $F_0 + v_{0,1}$ (which contains $v_{1,1}$). The contribution x_I to $\Phi_{(2)}(\vec{x})$ from this simplex is therefore in $\{x_I \mid I \notin \mathcal{K}_{res}\}$. Classifying the terms in$\Phi_{(2)}(\vec{x})$ by this x_I, we obtain the claimed expression.

The expression for $\Phi_{(12)}$ follows in a similar way.

To relate $\Phi_{(2)}$ to $\Phi_{(1)}$, classify the terms again by the contribution from $F_0 + v_{0,1}$. Let $w \in W_{(2)}^{(1)}$. According to what is just proved, $\hat{w} \cap (B_0 + v_{0,1})$ is congruent to one of $\{\Delta_I \mid I \notin \mathcal{K}_{res}\}$, in the notation of Proposition 1.

If it is congruent to $\Delta_{(2)}$, then w has a form $\overline{v_a v_{1,1} v_b}$ in the simplex. To this w assign a path w' obtained by replacing $\overline{v_a v_{1,1} v_b}$ by $\overline{v_a v_b}$. Then $w' \in W_{(1)}^{(1)}$, and contributes a term to $\Phi_{(1)}$ similar to the contribution of w to $\Phi_{(2)}$ but with $x_{(2)}$ replaced by $x_{(1)}$.

If $w \cap (B_0 + v_{0,1})$ is congruent to $\Delta_{(3)}$, then w has a form $\overline{v_a v_{1,1} v_c v_b}$ or $\overline{v_a v_c v_{1,1} v_b}$ in the simplex. To this w assign a path w' obtained, as before, by shortcutting $v_{1,1}$, to obtain $\overline{v_a v_c v_b}$. Then $w' \in W_{(1)}^{(1)}$,

and contributes a term to $\Phi_{(1)}$ similar to the contribution of w to $\Phi_{(2)}$ but with $x_{(3)}$ replaced by $x_{(2)}$. This correspondence is 2 to 1, which explains the factor $\frac{1}{2}$ in the expression of $\Phi_{(1)}$ in (74).

The other possibilities (4) and (12) are similar.

That $x_{(1)}^2$ is contained in $\Phi_{(1)}(\vec{x})$ is proved in Proposition 2. This term can not be obtained by an above mentioned 'shortcutting' procedure on paths, so it must be contained in $\Phi_{(1),0}(\vec{x})$.

The expression for $\Phi_{(11)}$ follows in a similar way.

(iv) $W_{(1)}^{(1)}$ contains paths $\overline{Ov_1(v_1+v_2)v_{1,1}}$, $\overline{Ov_1(v_1+v_2)(v_1+v_3)v_{1,1}}$, $\overline{Ov_1(v_1+v_2)(v_1+v_3)(v_1+v_4)v_{1,1}}$, and $\overline{Ov_2(v_2+v_3)v_3v_4v_1v_{1,1}}$, which respectively contribute terms $x_{(1)}x_{(i)}$, $i=2,3,4$, and $x_{(1)}^3x_{(12)}$ in $\Phi_{(1)}$.

Similarly, paths sets

$$\begin{array}{l}(\overline{Ov_1v_{1,1}}, \overline{v_{1,2}(v_3+v_2)(v_3+v_1)v_{1,3}}),\\ (\overline{Ov_1v_{1,1}}, \overline{v_{1,2}(v_3+v_2)(v_3+v_1)(v_3+v_4)v_{1,3}}),\\ (\overline{Ov_1v_{1,1}}, \overline{v_{1,2}(v_3+v_2)(v_3+v_1)(v_3+v_4)v_3v_{1,3}}),\\ (\overline{Ov_1v_{1,1}}, \overline{v_{1,2}v_2v_4v_3v_{1,3}}),\end{array}$$

respectively contribute terms $x_{(1)}^3x_{(i)}$, $i=2,3,4$, and $x_{(1)}^3x_{(12)}$ in $\Phi_{(11)}$.

(v) The first part follows from (74) and similar expressions for $\Phi_{(3)}$, $\Phi_{(4)}$. The second part follows from an observation similar to that used to prove (74). That $\Phi_{(2)}$ contains a term $x_{(1)}^3x_{(11)}x_{(12)}$ follows from an existence of a path

$$\overline{Ov_3(v_1+v_3)(v_1+v_4)v_4v_1v_{1,1}(v_1+v_2)v_{1,2}},$$

and $\Phi_{(12)}$ contains a term $x_{(1)}^4x_{(2)}$ because of a path

$$(\overline{Ov_1v_{1,1}}, \overline{v_{1,2}(v_2+v_3)v_{1,3}(v_3+v_4)v_{1,4}}).$$

□

Theorem 30 *(i)* $\Phi_{(1)}(x,y,0,0,0,0)=x$, *and* $\Phi_{(11)}(x,y,0,0,0,0)=y$ *has a unique solution* $\vec{x}_c=(x_c,y_c,0,0,0,0)$ *in* $\{(x,y,0,0,0,0)\in\mathbb{R}_+^{\mathcal{K}_{res}} \mid (x,y,0,0,0,0)\in\Xi_4\setminus\{\vec{0}\}\}$.

$x_c=0.326490898\cdots$ *and* $y_c=0.027929572\cdots$ *are positive. (In particular,* $\mathcal{K}_{res}=\mathcal{K}_{\vec{x}_c}$ *.)*

(ii) $\vec{x}_c$ *is a self-avoiding fixed point, i.e., satisfies (FP1) – (FP4).*

(iii) There exists a critical point of the restricted model $\beta_{c,res}$ *.*

Proof. (i) Since (73) implies $x>x^2+3x^3+6x^4+6x^5$, it follows that $0<x<\frac{3}{8}$.

Eliminating y from $\Phi_{(1)}(x,y,0,0,0,0)=x$ and $\Phi_{(11)}(x,y,0,0,0,0)=y$ (in a similar spirit with that of the proof in [6, Proposition 3.1], but with lengthy calculations), we obtain an algebraic equation $g(x)=0$, where

$$\begin{aligned}g(x) \;=\; & -3162456+3162456x+27935028x^2+82351390x^3+534340195x^4-22712313853x^5\\ & -22749190609x^6+173488539516x^7+520491536505x^8+159919155293x^9\\ & -1067593750255x^{10}-3355567112768x^{11}-7117707818273x^{12}-8049744033921x^{13}\\ & +3218074725393x^{14}+29132597332920x^{15}+58986824992938x^{16}+74778447132144x^{17}\\ & +70897214418552x^{18}+55063893147408x^{19}+36096140965140x^{20}\\ & +19669482325692x^{21}+7841354208804x^{22}+1771680351168x^{23}+149567809608x^{24},\end{aligned}$$

and y is expressed by a rational function of x, say $h(x)$.

The signs of the coefficients imply (by an argument from elementary calculus similar to those in the proof of [6, Proposition 3.1]) that $g'(x)$ changes sign no more than 4 times on $x\geqq 0$ (i.e., $g'(x)$ has no more than 4 positive roots). Since, by explicit calculation, $g'(0)>0$, $g'(1/5)<0$, $g'(3/8)>0$, $g'(3/7)<0$, and $g'(+\infty)>0$, $g'(x)$ changes sign 4 times on $x\geqq 0$, and 2 times on $0\leqq x\leqq 3/8$.

Also we have $g(0) < 0$, $g(1/5) < 0$, and $g(3/8) > 0$. In particular, we have one and only one solution $x = x_c$ of $g(x) = 0$ in $x \in (1/5, 3/8)$.

Again by explicit calculation, we have $g'(21/200) > 0$ and $g'(11/100) < 0$, therefore, the other point that $g'(x)$ changes sign in $0 \leqq x \leqq 3/8$ is in the interval $(21/200, 11/100)$. By explicit calculation $g'(x) \leqq g'(21/200) \leqq 9 \times 10^5$, $x \in [21/200, 11/100]$, and $g(21/200) \leqq -2 \times 10^6$. Therefore, if $x \in [21/200, 11/100]$, then $g(x) \leqq -2 \times 10^6 + 4.5 \times 10^3 < 0$. This implies $g(x) < 0$, $0 \leqq x \leqq 1/5$, hence x_c obtained above is the unique solution to $g(x) = 0$ on $x \geqq 0$. With $y_c = h(x_c)$, (x_c, y_c) is therefore the unique solution to $\Phi_{(1)}(x, y, 0, 0, 0, 0) = x$, and $\Phi_{(11)}(x, y, 0, 0, 0, 0) = y$ on $\mathbb{R}^2_+$. It is easy to see that $x_c = 0.326490898\cdots$ and $y_c = 0.027929572\cdots$, which prove that $(x_c, y_c, 0, 0, 0, 0) \in \Xi_4 \setminus \{\vec{0}\}$.

(ii) We prove each of (FP1) – (FP4) in turn.

(a) Proposition 29 (v) implies that if $I \notin \mathcal{K}_{res} = \{(1), (11)\}$ then $\Phi_I(\vec{x}_c) = 0$. Therefore $\vec{\Phi}(\vec{x}_c) = \vec{x}_c$.

(b) Proposition 29 (v) implies

$$\mathcal{J}_{(3)J}(\vec{x}_c) = \mathcal{J}_{(4)J}(\vec{x}_c) = 0, \quad J \in \mathcal{I}_d\,,$$

and

$$\mathcal{J}_{I(1)}(\vec{x}_c) = \mathcal{J}_{I(11)}(\vec{x}_c) = 0, \quad I \notin \mathcal{K}_{res}\,.$$

Therefore, $B = \mathcal{J}(\vec{x}_c)$ has a form

$$B = \begin{pmatrix} p & 3q & B_{13} & B_{14} & B_{15} & B_{16} \\ q & r & B_{23} & B_{24} & B_{25} & B_{26} \\ 0 & 0 & B_{33} & B_{34} & B_{35} & B_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{63} & B_{64} & B_{65} & B_{66} \end{pmatrix},$$

where, by (73),

$$\begin{aligned} p &= 2x_c + 9x_c^2 + 24x_c^3 + 30x_c^4 + 36x_c^2y_c + 120x_c^3y_c + 36x_cy_c^2 + 234x_c^2y_c^2 + 192x_cy_c^3 + 132y_c^4\,, \\ q &= 4x_c^3 + 10x_c^4 + 12x_c^2y_c + 52x_c^3y_c + 96x_c^2y_c^2 + 176x_cy_c^3 + 220y_c^4\,, \\ r &= 4x_c^3 + 13x_c^4 + 64x_c^3y_c + 264x_c^2y_c^2 + 88y_c^3 + 880x_cy_c^3 + 930y_c^4\,, \end{aligned}$$

and B_{ij}'s are non-negative, and (74) implies

$$\begin{aligned} B_{33} &= \mathcal{J}_{(2)(2)}(\vec{x}_c) = \Phi_{(1),1}(\vec{x}_c), \\ B_{36} &= \mathcal{J}_{(2)(12)}(\vec{x}_c) = \Phi_{(1),4}(\vec{x}_c), \\ B_{63} &= \mathcal{J}_{(12)(2)}(\vec{x}_c) = \Phi_{(11),1}(\vec{x}_c), \\ B_{66} &= \mathcal{J}_{(12)(12)}(\vec{x}_c) = \Phi_{(11),4}(\vec{x}_c). \end{aligned}$$

A 2×2 matrix $\begin{pmatrix} p & 3q \\ q & r \end{pmatrix}$ has eigenvalues $\lambda = 3.17282866849\cdots > 1$ and $\lambda_2 = 0.249393708\cdots$. In fact, they are the solutions of $x^2 - (p + r)x + pr - 3q^2 = 0$.

Consider a 2×2 matrix $B' = \begin{pmatrix} B_{33} & B_{36} \\ B_{63} & B_{66} \end{pmatrix}$. Its characteristic polynomial $f(x) = \det(x\,I - B)$ is

$$f(x) = x^2 - (B_{33} + B_{66})x + (B_{33}B_{66} - B_{36}B_{63}) = (x - \frac{1}{2}(B_{33} + B_{66}))^2 - \frac{1}{4}(B_{33} - B_{66})^2 - B_{36}B_{63}\,. \tag{75}$$

Note that (74) implies

$$\begin{aligned} x_c &= \Phi_{(1)}(\vec{x}_c) = B_{33}\,x_c + B_{36}\,y_c + \Phi_{(1),0}(\vec{x}_c) \geqq B_{33}\,x_c + B_{36}\,y_c + x_c^2, \\ y_c &= \Phi_{(11)}(\vec{x}_c) = B_{63}\,x_c + B_{66}\,y_c + \Phi_{(11),0}(\vec{x}_c) \geqq B_{63}\,x_c + B_{66}\,y_c + x_c^4, \end{aligned}$$

which further imply

$$\begin{aligned} &0 \leqq x_a := \frac{1}{2}(B_{33} + B_{66}) \leqq \frac{1}{2}((1 - x_c) + (1 - \frac{x_c^4}{y_c})) < 1, \\ &1 - B_{33} - B_{66} + B_{33}B_{66} = (1 - B_{33})(1 - B_{66}) \geqq (\frac{y_c}{x_c}B_{36} + x_c)(\frac{x_c}{y_c}B_{63} + \frac{x_c^4}{y_c}) > B_{36}B_{63}\,. \end{aligned}$$

Note also that Proposition 29 (v) implies that

$$B_{36} \geqq x_c^3 y_c \quad \text{and} \quad B_{63} \geqq x_c^4.$$

Using these estimates in (75) we find that $f(x)$ is increasing on $x \geqq x_a$, symmetric with respect to $x = x_a$ (hence in particular, $f(-1) > f(-1+2x_a) = f(1)$), $f(x_a) \leqq -B_{36}B_{63} \leqq -x_c^7 y_c < 0$, and $f(1) > 0$. Therefore the two eigenvalues λ_3 and λ_4 of B' are real, distinct, and have absolute values less than 1.

The other 2 eigenvalues are 0, for which we have 2 independent left eigen vectors $(0,0,0,1,0,0)$ and $(0,0,0,0,1,0)$.

Therefore, $B = \mathcal{J}(\vec{x}_c)$ is diagonable by an invertible matrix, whose eigenvalues are λ, λ_2, λ_3, λ_4, 0, 0, and the eigenvalue λ which is largest in absolute value satisfies $\lambda > 1$ and all the other eigenvalues have absolute values strictly less than 1.

Denote the left eigenvector of B for λ by $\vec{v}_L = (v_{L,(1)}, v_{L,(11)}, v_{L,(2)}, v_{L,(3)}, v_{L,(4)}, v_{L,(12)})$. It is defined by

$$\begin{aligned}
&\frac{v_{L,(11)}}{v_{L,(1)}} = \frac{\lambda - p}{q} = \frac{3q}{\lambda - r}\,,\\
&v_{L,(2)} = \frac{1}{\lambda - B_{33}}(B_{13}v_{L,(1)} + B_{23}v_{L,(11)} + B_{63}v_{L,(12)})\\
&v_{L,(3)} = \frac{1}{\lambda}(B_{14}v_{L,(1)} + B_{24}v_{L,(11)} + B_{34}v_{L,(2)} + B_{64}v_{L,(12)}),\\
&v_{L,(4)} = \frac{1}{\lambda}(B_{15}v_{L,(1)} + B_{25}v_{L,(11)} + B_{35}v_{L,(2)} + B_{65}v_{L,(12)}),\\
&v_{L,(12)} = \frac{1}{\lambda - B_{66}}(B_{16}v_{L,(1)} + B_{26}v_{L,(11)} + B_{36}v_{L,(2)}).
\end{aligned}$$

Obviously, we can take $v_{L,(1)} = 1$. λ is the larger solution of $x^2 - (p+r)x + pr - 3q^2 = 0$, hence $\lambda > p$. Therefore, $v_{L,(11)} > 0$. Note that Proposition 29 (iv) implies

$$B_{i,j} > 0, \quad i = 1,2,\ j = 3,4,5,6\,.$$

Note also that $\lambda > 1 > \max\{B_{33}, B_{66}\}$. Therefore $v_{L,I} > 0$, $I = (3), (4), (5), (12)$,

Finally, note that the right eigenvector of B for λ can be chosen to be $\begin{pmatrix} 3q \\ \lambda - p \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, hence has positive (non-zero) (1)-component.

(c) The explicit form (73) shows that $\Phi_{(1)}$ and $\Phi_{(11)}$ contain terms $x_{(1)}^2$ and $x_{(1)}^4$, respectively.

(d) We have already shown $\vec{x}_c \in \Xi_4 \cap \mathbb{R}_+^{\mathcal{K}_{res}}$ and since $x_c > 0$, $\vec{x}_c \neq \vec{0}$.

(iii) Theorem 13 implies that there exists one and only one $\beta_0 \in \mathbb{R}$ such that $\vec{x}_{can,\mathcal{K}_{res}}(\beta_0) \in \partial D$. To prove that β_0 is the critical point of the restricted model, it is sufficient to prove, from (CS2) and (DA1) – (DA2), that

$$\lim_{n\to\infty} \vec{X}_n(\vec{x}_{can,\mathcal{K}_{res}}(\beta_0)) = \vec{x}_c\,, \tag{76}$$

and

$$x_{can,\mathcal{K}_{res},I}(\beta_0) \neq 0,\ I \in \mathcal{K}_{res}\,.$$

The latter obviously holds, because

$$\vec{x}_{can,\mathcal{K}_{res}}(\beta_0) = (e^{-\beta_0}, e^{-2\beta_0}, 0, 0, 0, 0).$$

Hence it suffices to prove (76).

Since $\mathbb{R}_+^{\mathcal{K}_{res}} \subset \mathbb{R}_+^{\mathcal{I}_4}$ is an invariant subset (Proposition 4 or Proposition 6), we may restrict (76) to $\mathbb{R}_+^{\mathcal{K}_{res}}$. Define

$$\begin{aligned}
&x_n(x,y) = X_{n,(1)}(x,y,0,0,0,0), \quad y_n(x,y) = X_{n,(11)}(x,y,0,0,0,0),\\
&\vec{\phi} = (\phi_1, \phi_2); \quad \phi_1(x,y) = \Phi_{(1)}(x,y,0,0,0,0), \quad \phi_2(x,y) = \Phi_{(11)}(x,y,0,0,0,0).
\end{aligned}$$

Then $(x_0(x,y), y_0(x,y)) = (x,y)$ and $(x_{n+1}, y_{n+1}) = \vec{\phi} \circ (x_n, y_n)$, $n \in \mathbb{Z}_+$.

Next define

$$\Xi_4^{(2)} = \{(x,y) \in \mathbb{R}_+^2 \mid (x,y,0,0,0,0) \in \Xi_4\} = \{(x,y) \in \mathbb{R}_+^2 \mid x^2 \geqq y\},$$

and

$$D^{(2)} = \{(x,y) \in \Xi_4^{(2)} \mid (x,y,0,0,0,0) \in D\},$$

where $D \subset \mathbb{R}_+^6$ is as in (31). Denote by $D^{(2)o}$, $D^{(2)c}$, $\partial D^{(2)}$, the interior, exterior, and boundary of $D^{(2)}$ in $\Xi_4^{(2)}$.

Note that

$$(e^{-\beta_0}, e^{-2\beta_0}) \in \partial D^{(2)}.$$

To prove (76), it then suffices to prove

$$\lim_{n\to\infty} (x_n(x,y), y_n(x,y)) = (x_c, y_c), \quad (x,y) \in \partial D^{(2)}. \tag{77}$$

The definition and Theorem 13 implies that these sets are invariant sets of (ϕ_1, ϕ_2) and that

$$\begin{aligned} D^{(2)} &= \{(x,y) \in \Xi_4^{(2)} \mid \sup_{n\in\mathbb{Z}_+} \max\{x_n(x,y), y_n(x,y)\} < \infty\} \\ &= \{(x,y) \in \Xi_4^{(2)} \mid \sup_{n\in\mathbb{Z}_+} \max\{x_n(x,y), y_n(x,y)\} \leqq 1\}, \end{aligned}$$

and

$$D^{(2)o} = \{(x,y) \in \Xi_4^{(2)} \mid \lim_{n\to\infty} \max\{x_n(x,y), y_n(x,y)\} = 0\}.$$

The explicit form (73) and Theorem 13 also imply, by an elementary argument (as in [6, Proposition 3.2 (2)]), that there exist a constant c, satisfying $0 < c < \frac{1}{2}$, and a continuous sitrictly decreasing function $p:\ [0,c] \to \mathbb{R}_+$, satisfying $p(0) < \frac{1}{2}$ and $p(c) = 0$, such that

$$\partial D^{(2)} = \{(x, p(x)) \mid x \in [0,c]\}. \tag{78}$$

The Jacobi matrix of the map $(x,y) \to \vec{\phi}(x,y)$ is, by (73),

$$\begin{aligned} \mathcal{J}^{(2)}(x,y) &= \frac{\partial(\phi_1,\phi_2)}{\partial(x,y)}(x,y) \\ &= 8x^4 + 62x^5 + 165x^6 + 192x^7 + 90x^8 + 128x^4y + 432x^5y + 516x^6y + 360x^7y \\ &\quad + 528x^3y^2 + 2088x^4y^2 + 3996x^5y^2 + 4770x^6y^2 + 176xy^3 + 2552x^2y^3 \\ &\quad + 10032x^3y^3 + 25200x^4y^3 + 35040x^5y^3 + 1860xy^4 + 11538x^2y^4 + 56640x^3y^4 \\ &\quad + 113520x^4y^4 + 3168xy^5 + 69912x^2y^5 + 206640x^3y^5 + 50376xy^6 + 201780x^2y^6 \\ &\quad + 11616y^7 + 62400xy^7 - 22440y^8. \end{aligned}$$

We note the following fact proved by an elementary argument in [6, Proposition 3.3].

Lemma 31 *If there exists $\epsilon > 0$ such that*

$$\mathcal{J}^{(2)}(x,y) > \epsilon, \ (x,y) \in \partial D^{(2)}, \tag{79}$$

then, $\phi_1(x_1,y_1) < \phi_1(x_2,y_2)$ holds for any $(x_1,y_1) \in \partial D^{(2)}$ and $(x_2,y_2) \in \partial D^{(2)}$ satisfying $x_1 < x_2$.

Remark. The proof of [6, Proposition 3.3] applies to the present case. The proof there (after [6, (3.7)]) uses only (78) and the fact that ϕ_1 and ϕ_2 are polynomials of positive coefficients. ◇

To prove (79), note that the only negative term in $\mathcal{J}^{(2)}(x,y)$ is $-22440y^8$. If $(x,y) \in \partial D^{(2)}$, then $y = p(x) < p(0) < 1/2$, hence $22440y^8 < 11220y^7 < 11616y^7$. With other positive terms in $\mathcal{J}^{(2)}(x,y)$, (79) follows easily.

We are ready to prove (77). Let $(x,y) \in \partial D^{(2)}$, and assume that $\{(x_n(x,y), y_n(x,y)) \mid n \in \mathbb{Z}_+\}$ accumulates at $(x_a, y_a) \neq (x_c, y_c)$. Let $(x_b, y_b) = \vec{\phi}(x_a, y_a)$. Since Theorem 30 implies that (x_c, y_c) is the only fixed point in $\Xi_4^{(2)} \setminus \{\vec{0}\}$, it follows that $x_b \neq x_a$.

Assume that $x_b > x_a$. For $\epsilon > 0$ denote the ϵ neighborhood of (x,y) by $U_\epsilon(x,y)$. Since $\vec{\phi}$ is continuous, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\vec{\phi}(U_\delta(x_a, y_a)) \subset U_\epsilon(x_b, y_b)$. Taking ϵ small enough so that $(x', y') \in U_\epsilon(x_b, y_b)$ implies $x' > x_m := (x_a + x_b)/2$, we therefore see that there exists $\delta > 0$ such that $(x,y) \in U_\delta(x_a, y_a) \cap \partial D^{(2)}$ implies $\phi_1(x,y) > x_m$ $(> x_a)$. Then Lemma 31 further implies, by induction,

$$\phi_1(\vec{\phi}^n(x,y)) > x_m\,, \quad n \in \mathbb{N},\ (x,y) \in U_\epsilon(x_a, y_a) \cap \partial D^{(2)}.$$

By assumption that $\{(x_n(x,y), y_n(x,y)) \mid n \in \mathbb{Z}_+\}$ accumulates at (x_a, y_a), there exists a positive integer N such that $(x_N(x,y), y_N(x,y)) \in U_\epsilon(x_a, y_a) \cap \partial D^{(2)}$. Therefore $x_n(x,y) > x_m$, $n = N+1, N+2, \cdots$, which contradicts the assumption that $\{x_n(x,y) \mid n \in \mathbb{Z}_+\}$ accumulates at $x_a < x_m$.

Similar argument also holds for if we assume $x_b < x_a$. Therefore (x_c, y_c) is the only point that $\{(x_n(x,y), y_n(x,y)) \mid n \in \mathbb{Z}_+\}$ accumulates, which proves (77).

□

Theorem 30 and the results in Section 3 imply the following results on the asymptotic behaviors of restricted self-avoiding paths on the 4 dimensional pre-Sierpiński gasket.

Theorem 32 *Let $\vec{x}_c = (x_c, y_c, 0, 0, 0, 0) \in \mathbb{R}_+^6$ and $\beta_{res} = \beta_{c,\mathcal{K}_{res}} \in \mathbb{R}$ be the constants defined in Theorem 30. Then the following holds for the restricted self-avoiding paths on the 4 dimensional pre-Sierpiński gasket.*

(i) If $\vec{x} \in \mathcal{D}om(\vec{x}_c)$, then the following hold.

For $I \in \mathcal{K}_{res} = \{(1), (11)\}$, the joint distribution of scaled generalized path length $(\lambda^{-n} s_{(1)}\,,\ \lambda^{-n} s_{(11)})$ under $\mu_{\vec{x},n,I}$ converges weakly to a Borel probability measure $p^*_{\vec{x},I}$ on $\mathbb{R}^6$ as $n \to \infty$.

The generating function $\varphi^*_{\vec{x},I}$, defined by

$$\varphi^*_{\vec{x},I}(\vec{t}) = \int_0^\infty e^{\vec{t}\cdot\vec{\xi}} p^*_{\vec{x},I}[d\vec{\xi}], \quad \vec{t} \in \mathbb{C}^6\,,$$

is an entire function in $\vec{t}$, and the set of functions $(\varphi^*_{\vec{x},(1)}\,,\ \varphi^*_{\vec{x},(11)})$ are uniquely determined by (29) for $d = 4$.

If $\vec{x} \in \mathcal{D}om(\vec{x}_c) \cap \Xi_4$ and $I \in \mathcal{K}_{res}$, then the distribution of $\lambda^{-n} L(w)$, the scaled length of $w \in W_I^{(n)}$, under $\mu_{\vec{x},n,I}$ converges weakly to a Borel probability measure $\bar{p}^*_{\vec{x},I}$, which has a C^∞ density $\bar{\rho}^*_{\vec{x},I}$.

In particular, $\bar{\rho}^*_{\vec{x},(1)}(\xi) > 0$, $\xi > 0$.

(ii) For $I \in \mathcal{I}_4 = \{(1), (11), (2), (3), (4), (12)\}$,

$$\lim_{n\to\infty} Z_{\mathcal{K}_{res},n,I}(\beta) = \begin{cases} 0, & \beta > \beta_{res}\,, \\ x_{c,I}\,, & \beta = \beta_{res}\,, \end{cases}$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{4} Z_{\mathcal{K}_{res},n,(i)}(\beta) = \infty, \quad \beta < \beta_{res}\,.$$

(iii) The number $N_{res}(k) = N_{\mathcal{K}_{res}}(k)$ of restricted self-avoiding paths of length k starting from 0 satisfies

$$\lim_{k\to\infty} \frac{1}{k} \log N_{res}(k) = \beta_{res}\,.$$

(iv) The exponent for mean square displacement for the restricted model is $d_w = \dfrac{\log\lambda}{\log w} = 1.6657696\cdots$, in the sense that

$$\lim_{k\to\infty} \frac{1}{\log k} \log E_{res,k}[|w(k)|^{s\,d_w}] = s, \quad s \geqq 0,$$

where $E_{res,k}$ is the expectation with respect to the probability measure with equal weight on length k restricted self-avoiding paths starting at O.

Remark. The convergence of $\mu_{\vec{x},n,I}$ and the properties of the limit measure holds both for the full model and the restricted model (if $\vec{x} \in \mathcal{D}om(\vec{x}_c)$), because they hold independently of (CS2).

$\diamond$

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