SCATTERING AND WAVE OPERATORS FOR ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH SLOWLY DECAYING NONSMOOTH POTENTIALS

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ABSTRACT. We prove existence of modified wave operators for one-dimensional Schrödinger equations with potential in $L^p(\mathbb{R})$, $p < 2$. If in addition the potential is conditionally integrable, then the usual Möller wave operators exist. We also prove asymptotic completeness of these wave operators for some classes of random potentials, and for almost every boundary condition for any given potential.

1. INTRODUCTION

Let H_V be a one-dimensional Schrödinger operator defined by

$$H_V = -\frac{d^2}{dx^2} + V(x). \tag{1.1}$$

Let us discuss the case where the operator is defined on a half-axis, with some self-adjoint boundary condition at zero. We are interested in potentials decaying at infinity, for which we may expect that asymptotically as time tends to infinity, motion of the associated perturbed quantum system resembles the free evolution. What is the critical rate of decay of the potential for which the dynamics remains close to free for large times? The mathematical framework for studying these questions is provided by scattering theory. Recall that the wave operators associated with H_V and H_0 are defined by

$$\Omega_\pm f = \lim_{t \to \mp\infty} e^{itH_V} e^{-itH_0} f, \tag{1.2}$$

where the limit is understood in the strong L^2 sense. In particular, existence of the wave operators implies that the absolutely continuous spectrum of H_V fills all of $\mathbb{R}^+$ (see, e.g. [33]) and moreover provides significant information on large time dynamics e^{-itH_V}.

The wave operators will be called asymptotically complete if the range of $\Omega_\pm$ coincides with the orthogonal complement of the subspace spanned by eigenvectors of the operator H_V. An alternative equivalent characterization is that the range of the wave operators is equal to the absolutely continuous subspace $\mathcal{H}_{\rm ac}(H_V)$ of the operator H_V, and that the singular continuous spectrum $\sigma_{\rm sc}(H_V)$ is empty. We note

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that the intended intuitive meaning of asymptotic completeness is that the dynamics of the perturbed operator can be divided into two well-understood parts: scattering states traveling to infinity in a way similar to the free evolution, and bound states which remain confined in a certain sense for all times. We postpone more detailed discussion of the notion of asymptotic completeness to Section 10.

A well-known result, going back to the very first years of the rigorous scattering theory, says that if $V \in L^1$, then the wave operators exist and are asymptotically complete. Moreover, in this case the spectrum on the positive semi-axis is purely absolutely continuous, and there can be only discrete spectrum below zero, possibly accumulating at zero. There has been much work extending the existence of wave operators to wider classes of potentials, with some additional conditions on derivatives, or for potentials of certain particular forms. Generally, for more slowly decaying potentials one needs to modify the free dynamics in the definition of wave operators in order for the limits to exist. The first work of this type was that of Dollard [19], who studied the Coulomb potential. Further developments used computation of asymptotic classical trajectories by means of a Hamilton-Jacobi equation to build an appropriate phase correction to the free dynamics, which was used to prove existence of modified wave operators (in any dimension). See, for instance, Buslaev and Matveev [7], Ahsholm and Kato [1]; the strongest results are contained in Hörmander [22]. For example, Hörmander [22] gives existence of wave operators if $|V(x)| \le C(1+|x|)^{-1/2-\epsilon}$ and $|D^\alpha V(x)| \le C(1+|x|)^{-3/2-\epsilon}$ for any $|\alpha| = 1$. One needs more conditions on derivatives to compensate for slower decay.

Another series of works (see [3, 4, 18, 42] for further references) studied oscillating potentials of Wigner-von Neumann type, for example $V(x) = \sin(cx^\alpha)/x^\beta$. Interest in this class of potentials was in part fueled by the Wigner-von Neumann example [41], where $V(x) \sim c\sin(2x)/x$ at infinity, which leads to a positive eigenvalue embedded in the absolutely continuous spectrum. For a wide class of potentials of this type (including the original example of [41]), existence and asymptotic completeness of (sometimes modified) wave operators has been shown.

In the opposite direction, Pearson [30] constructed examples of potentials in $\cap_{p>2} L^p(\mathbb{R})$ for which the spectrum is purely singular, and hence wave operators do not exist. Kotani and Ushiroya [26] also provided power decaying examples where the spectrum is purely singular (pure point) for the rate of decay $x^{-\alpha}$, $\alpha < 1/2$.

In the last few years, there has been a series of works studying existence of absolutely continuous spectrum for slowly decaying potentials in full generality, with no additional conditions on derivatives or specific representation. In [8, 34] it was shown that the absolutely continuous spectrum is preserved if $|V(x)| \le C(1+x)^{-\alpha}$, $\alpha > 1/2$, and moreover the generalized eigenfunctions have WKB-type plane wave asymptotics. Recently, we improved the result of [8] to treat potentials in L^p, $p < 2$ [9]. The sharpest result on the stability of the absolutely continuous spectrum is due to Deift and Killip [16], who showed it for $V \in L^2$. This result is optimal in the L^p scale, as Pearson's examples show. The method of [16] is quite different from [8, 34] and yields much less information concerning the nature of the generalized eigenfunctions.

Although the absolute continuity of the spectrum is an important characterization of an operator with direct consequences for the dynamical behavior of the particle, wave operators provide a much finer description of long-time dynamics. The purpose of this paper is to use the additional information contained in the asymptotic behavior of generalized eigenfunctions [9] to prove the existence of (modified) wave operators for potentials $V \in L^p$, $p < 2$. Let

$$W(\lambda, t) = -(2\lambda)^{-1} \int_0^{2\lambda t} V(s)\, ds. \tag{1.3}$$

Define $e^{-iH_0 t \pm iW(H_0, \mp t)}$ to be the Fourier multiplier operator on $L^2(\mathbb{R}^+)$ that maps $\int_0^\infty F(\lambda) \sin(\lambda x)\, d\lambda$ to $\int_0^\infty e^{-i\lambda^2 t \pm iW(\lambda^2, \mp t)} F(\lambda) \sin(\lambda x)\, d\lambda$, for all $F \in L^2(\mathbb{R}^+, d\lambda)$. Define

$$\Omega_\pm^m f = \lim_{t \to \mp\infty} e^{itH_V} e^{-itH_0 \pm iW(H_0, \mp t)} f \tag{1.4}$$

for all $f \in L^2(\mathbb{R}^+)$. Among our main conclusions are the following two theorems.

Theorem 1.1. *Let V be a potential in the class $L^1 + L^p(\mathbb{R}^+)$ for some $1 < p < 2$, and let H_V be the associated Schrödinger operator on $L^2(\mathbb{R}^+)$ with any self-adjoint boundary condition at 0. Then for every $f \in L^2(\mathbb{R}^+)$, the limits in (1.4) exist in $L^2(\mathbb{R}^+)$ norm, as $t \to \mp\infty$. The modified wave operators $\Omega_\pm^m$ thus defined are both unitary bijections from $\mathcal{H} = L^2(\mathbb{R}^+)$ to $\mathcal{H}_{ac}(H_V)$.*

Theorem 1.2. *In addition to the hypotheses of the preceding theorem, suppose that the improper integral $\int_0^\infty V(s)\, ds$ exists. Then for every $f \in L^2(\mathbb{R}^+)$, the limits in (1.2) exist in $L^2(\mathbb{R}^+)$ norm, as $t \to \mp\infty$. The wave operators $\Omega_\pm$ thus defined are both unitary bijections from $\mathcal{H} = L^2(\mathbb{R}^+)$ to $\mathcal{H}_{ac}$.*

By the improper integral we mean of course $\lim_{N\to\infty} \int_0^N V$; we are not assuming that $V \in L^1$. We also prove analogous results on wave operators for the whole line case and for some Dirac operators, see Section 9.

Another way to put the conclusion is that for each $f \in L^2(\mathbb{R}^+)$ there exist functions $F_\pm \in \mathcal{H}_{\mathrm{ac}}$ such that $\|e^{itH_V} F_\pm - e^{itH_0} f\|_{L^2(\mathbb{R}^+)} \to 0$ as $t \to \pm\infty$, and each map $f \mapsto F_\pm$ is unitary and surjective onto $\mathcal{H}_{\mathrm{ac}}$. Thus there exists a full family of scattering states, and any state that is asymptotically free at $t = \mp\infty$ is likewise free at $t = \pm\infty$.

The main idea of the proof is to use generalized eigenfunctions to construct the spectral decomposition of the operator H_V, and in particular to derive an explicit expression for the evolution group of the perturbed operator. Generalized eigenfunctions play a role parallel to the Fourier transform in the free case. The existence of wave operators is then proven by direct analysis, comparing the perturbed evolution with modified free dynamics. While the analysis is generally based on results of [8, 9], we need several essential improvements in the estimates to fulfill this plan. In Sections 2, 3 we extend the analysis of multilinear expressions encountered in the series for generalized eigenfunctions to cover the situations arising in applications to wave operators. In Section 5 we consider solutions of $H_V u = zu$ for complex energies z, and establish certain uniform bounds and asymptotics as $z \to \mathbb{R}^+$. In Section 6 we prove a limiting absorption principle, which allows us to write an explicit formula

for the evolution. Sections 7, 8 and 9 contain long-time asymptotic analysis and the rest of the proof of the existence of wave operators. The key step is Lemma 7.2, where the full strength of the multilinear analysis is required to justify discarding all terms with which it is concerned. In Section 10 we discuss the issue of asymptotic completeness for generic potentials within certain classes.

Although all theorems, and many of the intermediate results, are stated for potentials in $L^1 + L^p$ for some $1 < p < 2$, for simplicity we first give the proofs under the simplifying assumption that $V \in L^p$. In the last section we review the machinery [9] needed to extend the analysis from L^p to $L^1 + L^p$, and to the more general amalgamated class $\ell^p(L^1)$.

We note that the existence of (modified) wave operators for general long-range potentials in higher dimensions remains an open problem. For recent progress in this direction, see [37].

Some of the results of this paper were announced in [12].

2. A numerical bound for iterated multiple integrals

Let $\{f_i\}$ be a collection of integrable functions from $\mathbb{R}$ to $\mathbb{C}$. Consider multilinear expressions

$$M_n(f_1, \dots, f_n) = \iint_{x_1 \le \dots \le x_n} \prod_{i=1}^{n} f_i(x_i) dx_i \tag{2.1}$$

and their maximal variants

$$M_n^*(f_1, \dots, f_n) = \sup_y \Big| \iint_{x_1 \le \dots \le x_n \le y} \prod_{i=1}^{n} f_i(x_i) dx_i \Big|. \tag{2.2}$$

The purpose of this section is to establish upper bounds for M_n, M_n^*, in terms of certain auxiliary functions g_δ of the functions f_i, with particular attention to the dependence on n as $n \to \infty$. These bounds will play an essential part in our analysis of asymptotics of wave groups.

Definition. A martingale structure $\{E_j^m\}$ on an interval $I \subset \mathbb{R}$ is a collection of subintervals $E_j^m \subset I$, indexed by $m \in \{0, 1, 2, \dots\}$ and $j \in \{1, 2, \dots, 2^m\}$, possessing the following two properties. (i) Except for endpoints, $\{E_j^m : 1 \le j \le 2^m\}$ is a partition of I, for each m. (ii) $E_j^m = E_{2j-1}^{m+1} \cup E_{2j}^{m+1}$ for all m, j.

To any $f \in L^1$, any $\delta \in \mathbb{R}$, and any martingale structure, we associate

$$g_\delta(f) = \sum_{m=1}^{\infty} 2^{\delta m} \Big(\sum_{j=1}^{2^m} \big|\textstyle\int_{E_j^m} f\big|^2\Big)^{1/2}. \tag{2.3}$$

More generally, define

$$g_\delta(\{f_k\}) = \sum_{m=1}^{\infty} 2^{\delta m} \Big(\sum_{j=1}^{2^m} \sup_k \big|\textstyle\int_{E_j^m} f_k\big|^2\Big)^{1/2}. \tag{2.4}$$

Proposition 2.1. *There exists $C < \infty$ such that for any martingale structure $\{E_j^m\}$, any $\delta \geq 0$, any $f_1, \ldots, f_n \in L^1(\mathbb{R})$, and any $n \geq 2$,*

$$|M_n(f_1, \ldots, f_n)| \leq \frac{C^{n+1}}{\sqrt{n!}} g_{-\delta}(f_1) \cdot g_\delta(\{f_k : k \geq 2\})^{n-1}. \tag{2.5}$$

Moreover for any $\delta' > \delta \geq 0$, there exists $C < \infty$ such that for all $\{f_i\}$ and all $n \geq 2$,

$$|M_n^*(f_1, \ldots, f_n)| \leq \frac{C^{n+1}}{\sqrt{n!}} g_{-\delta}(f_1) \cdot g_{\delta'}(\{f_k : k \geq 2\})^{n-1}. \tag{2.6}$$

In previous work [10] we proved the simpler analogue with $\delta = 0$ and with the bound (for M_n) $C^{n+1}(n!)^{-1/2} g(\{f_k : k \geq 1\})^n$, and applied it to the analysis of generalized eigenfunctions, which can be expanded as sums over n of such iterated multiple integrals, where f_k is essentially $\exp(\pm 2i\lambda x)V(x)$, V is the potential, and λ^2 is a spectral parameter; $g(f)$ is then a function of λ. In the present work, we need a refinement in which f_1 is essentially $\exp(\pm i\lambda x)h(x)$, and h is an arbitrary L^2 function, unrelated to V. The quantity $g(h)$ is not appropriately bounded for $h \in L^2$, forcing the introduction of the mollifying factors $2^{-\delta m}$ in its definition. This in turn forces compensating factors of $2^{+\delta m}$, leading to the above formulation.

Proof. It is proved in [10] that there exist positive constants b_n satisfying $b_n \leq C^{n+1}/\sqrt{n!}$ and $n^{1/2} b_{n+1}/b_n \to C$ as $n \to \infty$, such that for all nonnegative real numbers x, y,

$$b_n y^n + \sum_{i=2}^{n-2} b_i b_{n-i} x^i y^{n-i} + b_n x^n \leq b_n (x^2 + y^2)^{n/2}. \tag{2.7}$$

It is also shown in [10] that

$$\begin{aligned} |M_n(f, \ldots, f)| &\leq b_n g_0(f)^n \\ M_n^*(f, \ldots, f) &\leq C_\varepsilon^n b_n g_\varepsilon(f)^n \end{aligned} \tag{2.8}$$

for every $\varepsilon > 0$. Moreover, for distinct functions f_i,

$$\begin{aligned} |M_n(f_1, \ldots, f_n)| &\leq b_n g_0(\{f_i\})^n \\ M_n^*(f_1, \ldots, f_n)| &\leq C_\varepsilon^n b_n g_\varepsilon(\{f_i\})^n. \end{aligned} \tag{2.9}$$

Although this bound is not explicitly formulated in [10], it follows directly from exactly the argument given there.

For $n \geq 2$ define $\tilde{b}_n = R^n b_{n-2}$, where R is a sufficiently large positive constant, to be determined later in the proof. In order to simplify notation, we will prove the result in the special case $f_2 = f_3 = \cdots = f_n = f$, and will write $f_1 = \tilde{f}$. The proof will be by induction on n. First we will treat only the case where $n \geq 6$, assuming the result for all $n \leq 5$, and at the end will discuss the modification for small n.

We write $f_j^m = \chi_{E_j^m} \cdot f$ and $\tilde{f}_j^m = \chi_{E_j^m} \cdot \tilde{f}$, where χ_E denotes always the characteristic function of a set E.

Lemma 2.2. *If R is chosen to be sufficiently large then for all $n \geq 2$ and any $\delta \geq 0$,*

$$|M_n(\tilde{f}, f, \ldots, f)| \leq \tilde{b}_n g_{-\delta}(\tilde{f}) \cdot g_\delta(f)^{n-1}. \tag{2.10}$$

Proof. By inequality (4.6) of [10],

$$\begin{aligned}
|M_n(\tilde f, f, \dots, f)| \le & |M_n(\tilde f_1^1, f_1^1, \dots, f_1^1)| \\
& + |M_{n-1}(\tilde f_1^1, f_1^1, \dots, f_1^1)| \cdot |\textstyle\int f_2^1| \\
& + \sum_{i=2}^{n-2} |M_{n-i}(\tilde f_1^1, f_1^1, \dots, f_1^1)| \cdot |M_i(f_2^1, \dots, f_2^1)| \\
& + |\textstyle\int \tilde f_1^1| \cdot M_{n-1}(f_2^1, \dots, f_2^1)| + |M_n(\tilde f_2^1, f_2^1, \dots, f_2^1)| \\
\le & |M_n(2^{-\delta}\tilde f_1^1, 2^\delta f_1^1, \dots, 2^\delta f_1^1)| \\
& + |M_{n-1}(2^{-\delta}\tilde f_1^1, 2^\delta f_1^1, \dots, 2^\delta f_1^1)| \cdot |\textstyle\int 2^\delta f_2^1| \\
& + \sum_{i=2}^{n-2} |M_{n-i}(2^{-\delta}\tilde f_1^1, 2^\delta f_1^1, \dots, 2^\delta f_1^1)| \cdot |M_i(2^\delta f_2^1, \dots, 2^\delta f_2^1)| \\
& + |\textstyle\int 2^{-\delta}\tilde f_1^1| \cdot |M_{n-1}(2^\delta f_2^1, \dots, 2^\delta f_2^1)| \\
& + |M_n(2^{-\delta}\tilde f_2^1, 2^\delta f_2^1, \dots, 2^\delta f_2^1)|.
\end{aligned} \tag{2.11}$$

We have assumed that $n \ge 2$ and $\delta \ge 0$ to ensure that at least one factor of 2^δ offsets the factor of $2^{-\delta}$.

The first and last terms in the preceding bound involve M_n itself, but the former involves only the restrictions of $\tilde f, f$ to E_1^1, while the latter involves only their restrictions to E_2^1; thus these expressions are in a sense simpler than the original expression M_n. We will therefore use as part of our induction hypothesis the desired inequalities for $M_n(\tilde f_1^1, f_1^1, \cdots, f_1^1)$ and for $M_n(\tilde f_2^1, f_2^1, \cdots, f_2^1)$. For the justification of this method of reasoning see the two paragraphs immediately following inequality (4.12) of [10].

The collection of all those sets E_j^m with $m \ge 1$ and $j \le 2^{m-1}$ forms a martingale structure on E_1^1; however, when it is considered as such, the index m should be replaced by $m-1$. Thus the induction hypothesis, for the first term on the right-hand side of the preceding bound, may be stated as

$$\begin{aligned}
&|M_n(2^{-\delta}\tilde f_1^1, 2^\delta f_1^1, \cdots, 2^\delta f_1^1)| \\
&\qquad \le \tilde b_n \sum_{m=2}^{\infty} 2^{-m\delta} \Big(\sum_{j=1}^{2^{m-1}} |\textstyle\int \tilde f_j^m|^2 \Big)^{1/2} \cdot \Big[\sum_{m=2}^{\infty} 2^{m\delta} \Big(\sum_{j=1}^{2^{m-1}} |\textstyle\int f_j^m|^2 \Big)^{1/2} \Big]^{n-1} .
\end{aligned} \tag{2.12}$$

There is a corresponding bound for $|M_n(2^{-\delta}\tilde f_2^1, 2^\delta f_2^1, \cdots, 2^\delta f_2^1)|$, with the inner sum running instead over $2^{m-1} < j \le 2^m$.

To formulate the induction hypothesis for the general term of the preceding expression, introduce

$$\tilde g_t^1 = \sum_{m=2}^{\infty} 2^{-\delta m} \Big(\sum_{j : E_j^m \subset E_t^1} |\textstyle\int \tilde f_j^m|^2 \Big)^{1/2} \quad \text{and} \quad g_t^1 = \sum_{m=2}^{\infty} 2^{\delta m} \Big(\sum_{j : E_j^m \subset E_t^1} |\textstyle\int f_j^m|^2 \Big)^{1/2} \tag{2.13}$$

for $t = 1, 2$ and

$$\tilde{g}^1 = ((\tilde{g}_1^1)^2 + (\tilde{g}_2^1)^2)^{1/2}, \qquad g^1 = ((g_1^1)^2 + (g_2^1)^2)^{1/2}. \tag{2.14}$$

Note that $(g^1)^2 + 2^{2\delta}|\int f_1^1|^2 + 2^{2\delta}|\int f_2^1|^2 \leq g_\delta(f)^2$, with a corresponding relation between $\tilde{g}^1$, $g_{-\delta}(\tilde{f})$.

¿From the induction hypothesis and (2.9) we obtain

(2.15)

$$\begin{aligned}|M_n(\tilde{f}, f, \dots, f)| \leq & \tilde{b}_n \tilde{g}_1^1 (g_1^1)^{n-1} + \tilde{b}_{n-1} \tilde{g}_1^1 (g_1^1)^{n-2} |\int 2^\delta f_2^1| \\ & + \sum_{i=2}^{n-2} \tilde{b}_{n-i} b_i \tilde{g}_1^1 (g_1^1)^{n-1-i} (g_2^1)^i + |\int 2^{-\delta} \tilde{f}_1^1| \cdot b_{n-1} (g_2^1)^{n-1} + \tilde{b}_n \tilde{g}_2^1 (g_2^1)^{n-1}.\end{aligned}$$

Consider now the sum of the first term on the right, together with all terms of the summation for which the index i is either in $[2, n-4]$, or equals $n-2$:

$$\begin{aligned}&\tilde{b}_n \tilde{g}_1^1 (g_1^1)^{n-1} + \sum_{i=2}^{n-4} \tilde{b}_{n-i} b_i \tilde{g}_1^1 (g_1^1)^{n-1-i} (g_2^1)^i + \tilde{b}_2 b_{n-2} \tilde{g}_1^1 g_1^1 (g_2^1)^{n-2} \\ &\quad = R^n \tilde{g}_1^1 g_1^1 \Big(b_{n-2} (g_1^1)^{n-2} + \sum_{i=2}^{n-4} R^{-i} b_{n-i-2} b_i (g_1^1)^{n-2-i} (g_2^1)^i + R^{-n+2} b_0 b_{n-2} (g_2^1)^{n-2} \Big) \\ &\quad \leq R^n \tilde{g}_1^1 g_1^1 \Big(b_{n-2} (g_1^1)^{n-2} + \sum_{i=2}^{n-4} b_{n-i-2} b_i (g_1^1)^{n-2-i} (g_2^1)^i + b_0 b_{n-2} (g_2^1)^{n-2} \Big) \\ &\quad \leq R^n \tilde{g}_1^1 g_1^1 b_{n-2} \big((g_1^1)^{2(n-2)} + (g_2^1)^{2(n-2)} \big)^{1/2} \\ &\quad = \tilde{b}_n \tilde{g}_1^1 g_1^1 (g^1)^{n-2}.\end{aligned}$$

To pass from the third line to the fourth we have invoked (2.7).

We now have

$$\begin{aligned}|M_n(\tilde{f}, f, \dots, f)| \leq & \tilde{b}_n \tilde{g}_1^1 g_1^1 (g^1)^{n-2} + \tilde{b}_3 b_{n-3} \tilde{g}_1^1 (g_1^1)^2 (g_2^1)^{n-3} + \tilde{b}_n \tilde{g}_2^1 (g_2^1)^{n-1} \\ & + \tilde{b}_{n-1} \tilde{g}_1^1 (g_1^1)^{n-2} |\int 2^\delta f_2^1| + |\int 2^{-\delta} \tilde{f}_1^1| b_{n-1} (g_2^1)^{n-1}.\end{aligned} \tag{2.16}$$

Set $\beta = \tilde{b}_3 b_{n-3} / \tilde{b}_n = R^{3-n} b_1 b_{n-3} / b_{n-2}$ and note that $\beta \leq C R^{3-n} n^{1/2} \leq 1$, if R is chosen to be sufficiently large, since we are assuming $n \geq 6$. Applying Cauchy-Schwarz to the sum of the first three terms on the right-hand side of the preceding inequality, we obtain

$$\begin{aligned}&\tilde{b}_n \tilde{g}_1^1 g_1^1 (g^1)^{n-2} + \tilde{b}_3 b_{n-3} \tilde{g}_1^1 (g_1^1)^2 (g_2^1)^{n-3} + \tilde{b}_n \tilde{g}_2^1 (g_2^1)^{n-1} \\ &\leq \tilde{b}_n \tilde{g}^1 \cdot \Big((g_1^1)^2 (g^1)^{2n-4} + (g_2^1)^{2n-2} + 2\beta (g_1^1)^3 (g_2^1)^{n-3} (g^1)^{n-2} + \beta^2 (g_1^1)^4 (g_2^1)^{2n-6} \Big)^{1/2}.\end{aligned} \tag{2.17}$$

We claim that

$$(g_1^1)^2 (g^1)^{2n-4} + (g_2^1)^{2n-2} + 2\beta (g_1^1)^3 (g_2^1)^{n-3} (g^1)^{n-2} + \beta^2 (g_1^1)^4 (g_2^1)^{2n-6} \leq (g^1)^{2n-2}, \tag{2.18}$$

provided that $n \geq 6$ and β is sufficiently small. Writing $x = g_1^1, y = g_2^1$, this is simply

$$x^2(x^2+y^2)^{n-2} + 2\beta x^3 y^{n-3}(x^2+y^2)^{(n-2)/2} + \beta^2 x^4 y^{2n-6} + y^{2n-2} \leq (x^2+y^2)^{n-1},$$

which is equivalent to

$$2\beta x^3 y^{n-3}(x^2+y^2)^{(n-2)/2} + \beta^2 x^4 y^{2n-6} + y^{2n-2} \leq y^2(x^2+y^2)^{n-2}.$$

By homogeneity we may assume that $x^2+y^2=1$, so we wish to have

$$2\beta x^3 y^{n-3} + \beta^2 x^4 y^{2n-6} + y^{2n-2} \leq y^2.$$

This holds whenever $x^2+y^2=1$ and $x, y \geq 0$, provided β is sufficiently small, provided that each of the exponents on y on the left-hand side is strictly larger than 2; this holds provided $n \geq 6$.

We have thus established that for $n \geq 6$,

$$|M_n(\tilde f, f, \dots, f)| \leq \tilde b_n \tilde g^1 (g^1)^{n-1} + \tilde b_{n-1} \tilde g_1^1 (g_1^1)^{n-2} |\textstyle\int 2^\delta f_2^1| + |\int 2^{-\delta} \tilde f_1^1| b_{n-1} (g_2^1)^{n-1}. \tag{2.19}$$

Representing the right-hand side as $\tilde b_n \tilde x x^{n-1} + \tilde b_{n-1} \tilde x x^{n-2} y + b_{n-1} \tilde y x^{n-1}$ where $x = g^1, y = |\int 2^\delta f_2^1|$ and $\tilde x = \tilde g^1, \tilde y = |\int 2^{-\delta} \tilde f_1^1|$, we seek to bound this right-hand side by

$$\tilde b_n (\tilde x + \tilde y)(x+y)^{n-1} \geq \tilde b_n \tilde x x^{n-1} + \tilde b_n \tilde x (n-1) x^{n-2} y + \tilde b_n \tilde y x^{n-1}. \tag{2.20}$$

Now[1] $\tilde b_{n-1} \lesssim R n^{1/2} \tilde b_n \leq (n-1)\tilde b_n$ for all sufficiently large n. Likewise, $b_{n-1} \leq C n^{1/2} R^{-1} \tilde b_n \leq (n-1)\tilde b_n$. Hence a term-by-term comparison completes the proof of the bound for M_n, for $n \geq 6$. □

Observe that the restriction $n \geq 6$ was only used above to control the first line of the right-hand side of (2.16); once that term is majorized by $\tilde b_n \tilde g^1 (g^1)^{n-1}$, the reasoning of the final paragraph allows us to absorb the two remaining special terms.

The cases $n = 2, 3$ are quite simple; beginning again with (2.11), we obtain $n+1$ terms, all of which are rather simple for $n = 2, 3$. The details are left to the reader.

Consider $n = 5$, the most complicated case remaining. From (2.11) we obtain an upper bound of

$$C_5 b_5 \tilde g_1^1 (g_1^1)^4 + C_3 b_2 \tilde g_1^1 (g_1^1)^2 (g_2^1)^2 + C_2 b_3 \tilde g_1^1 g_1^1 (g_2^1)^3 + b_5 C_5 \tilde g_2^1 (g_2^1)^4 \tag{2.21}$$

plus the two special terms involving $\int_{E_1^1} \tilde f$, $\int_{E_2^1} f$. By Cauchy-Schwarz we may bound the square of (2.21) by

$$b_5 (C_5 \tilde g^1)^2 \big[(g_1^1)^4 + \beta (g_1^1)^2 (g_2^1)^2 + \beta g_1^1 (g_2^1)^3\big]^2 + (g_2^1)^4,$$

where β may be made as small as desired by choosing C_5 to be sufficiently large relative to C_2, C_3. Thus the desired inequality reduces to

$$(x^4 + \beta x^2 y^2 + \beta x y^3)^2 + y^8 \leq (x^2+y^2)^4,$$

[1] It is in order to be able to assert this last inequality that we incorporate an ℓ^1 with respect to m, rather than an ℓ^2 norm, in the definitions of the functionals g_δ.

which holds for small β. The case $n = 4$ is similar but simpler since one fewer term appears in the analogue of (2.21).

We next discuss M_n^*. Let χ_y denote the characteristic function of $(-\infty, y]$ and apply the result just proved to $M_n(\tilde{f}, f\cdot\chi_y, \cdots, f\cdot\chi_y)$ to obtain

$$M_n^*(\tilde{f}, f, \ldots, f) \le \frac{C^{n+1}}{\sqrt{n!}} g_{-\delta}(\tilde{f}) \cdot [\sup_y g_\delta(f\cdot\chi_y)]^{n-1}. \tag{2.22}$$

It was shown in the proof of Proposition 4.2 of [10] that for any function F,

$$\sup_y \sum_{m=1}^{\infty} (\sum_j |\textstyle\int_{E_j^m} F\chi_y|^2)^{1/2} \le C \sum_{m=1}^{\infty} m (\sum_j |\textstyle\int_{E_j^m} F|^2)^{1/2}, \tag{2.23}$$

and we may dominate the multiplicative coefficient m by $C_\epsilon 2^{m\epsilon}$ on the right-hand side. Exactly the same proof allows the insertion of a factor of $2^{\delta m}$ after the first sum on both sides of the inequality, provided that $\delta \ge 0$, yielding the bound asserted.

We have so far discussed only the special case where $f_2 = f_3 = \cdots = f_n = f$, but the general case is treated by exactly the same argument, simply bounding $|\int_{E_j^m} f_i|$ by $\max_k |\int_{E_j^m} f_k|$ wherever the former arises in the proof. □

3. Multilinear operators

The bounds derived in the preceding section for multiple integrals will be used to obtain Lebesgue space norm bounds for certain multilinear operators. Consider a family of integral operators

$$T_i f(\lambda) = \int_{\mathbb{R}} K_i(\lambda, x) f(x)\, dx. \tag{3.1}$$

Denote by $\|T\|_{p,q}$ the norm of T as an operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$. Consider associated multilinear operators

$$\mathcal{T}_n(f_1, f_2, \ldots, f_n)(\lambda) = \iint_{x_1 \le x_2 \le \cdots \le x_n} \prod_{i=1}^n K_i(\lambda, x_i) f_i(x_i) dx_i \tag{3.2}$$

and their maximal variants

$$\mathcal{T}_n^*(f_1, f_2, \ldots, f_n)(\lambda) = \sup_y \Big| \iint_{x_1 \le x_2 \le \cdots \le x_n \le y} \prod_{i=1}^n K_i(\lambda, x_i) f_i(x_i) dx_i \Big| \,. \tag{3.3}$$

Theorem 3.1. *For any $N < \infty$, $p < q$, and $2 \le q$, there exists $C < \infty$ such that for any $n \ge 2$ and any collection of operators $\{T_i : 1 \le i \le n\}$ of cardinality $\le N$, for any collection of functions $\{f_1, \ldots, f_n\}$ of cardinality $\le N$,*

$$\|\mathcal{T}_n^*(f_1, \ldots, f_n)\|_{L^r} \le \frac{C^{n+1}}{\sqrt{n!}} \|T_1\|_{2,2} \|f_1\|_{L^2} \prod_{i=2}^n \big(\|T_i\|_{p,q} \|f_i\|_{L^p}\big) \tag{3.4}$$

where

$$\frac{1}{r} = \frac{1}{2} + \frac{n-1}{q} \,. \tag{3.5}$$

Proof. It suffices to prove this under the assumption that $\|f_1\|_{L^2} = 1 = \|f_i\|_{L^p}$ for every $i \geq 2$. Construct a martingale structure $\{E_j^m\}$ for $\mathbb{R}$ so that

$$\int_{E_j^m} |f_1|^2 + \sum_{i\geq 2}^{*} \int_{E_j^m} |f_i|^p = 2^{-m} \int_{\mathbb{R}} \big(|f_1|^2 + \sum_{i\geq 2}^{*} |f_i|^p\big) \tag{3.6}$$

where the notation $\sum^*$ indicates that the sum is to be taken over a maximal set of indices i for which the functions f_i are distinct. Denote by χ_j^m the characteristic function of E_j^m.

Let $\delta > 0$ be a constant to be specified. By Proposition 2.1,

$$\mathcal{T}_n^*(f_1, \dots, f_n)(\lambda) \leq \frac{C^{n+1}}{\sqrt{n!}} \tilde{G}(\lambda) G(\lambda)^{n-1} \tag{3.7}$$

where

$$\tilde{G}(\lambda) = \sum_{m=1}^{\infty} 2^{-\delta m} \Big(\sum_j |T_1(f_1 \cdot \chi_j^m)(\lambda)|^2\Big)^{1/2} \tag{3.8}$$

$$G(\lambda) = \sum_{m=1}^{\infty} 2^{\delta m} \Big(\sum_j \max_i |T_i(f_i \cdot \chi_j^m)(\lambda)|^2\Big)^{1/2}. \tag{3.9}$$

Now

$$\begin{aligned} \|\tilde{G}\|_{L^2} &\leq \sum_m 2^{-\delta m} \Big(\sum_j \|T_1(f_1 \cdot \chi_j^m)\|_{L^2}^2\Big)^{1/2} \\ &\leq \sum_m 2^{-\delta m} \|T_1\|_{2,2} \Big(\sum_j \|f_1 \cdot \chi_j^m)\|_{L^2}^2\Big)^{1/2} \leq C_\delta \|T_1\|_{2,2} \|f_1\|_{L^2} \end{aligned} \tag{3.10}$$

provided that $\delta > 0$. As for G, we may majorize

$$G(\lambda) \leq \sum_i^{*} \sum_{m=1}^{\infty} 2^{\delta m} \Big(\sum_j |T_i(f_i \cdot \chi_j^m)(\lambda)|^2\Big)^{1/2}. \tag{3.11}$$

It is shown on page 413 of [10] (see also the proof of Corollary 5.4 below) that since $p < 2 \leq q$, there exists $\varepsilon > 0$ such that

$$\|\Big(\sum_j |T_i(f_i \cdot \chi_j^m)|^2\Big)^{1/2}\|_{L^q} \leq C 2^{-\varepsilon m}, \tag{3.12}$$

under the condition that $\|f_i \chi_j^m\|_{L^p}^p \leq C' 2^{-m}$ for all j, m. Choosing $\delta < \varepsilon$ and summing over m gives

$$\|G\|_{L^q} \leq C < \infty, \tag{3.13}$$

of course under the hypothesis that $\|f_i\|_{L^p} = 1$ for all $i \geq 2$. An application of Hölder's inequality concludes the proof. □

Our main theorems are based on estimates for such multilinear operators; however, we require not the conclusion of Theorem 3.1, but rather the more detailed information contained in (3.7) together with the norm bounds for $\tilde{G}, G$.

4. Nontangential convergence and the maximal function

Throughout the paper we write $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. Consider the cones

$$\Gamma_\alpha(w) = \{z \in \mathbb{C}^+ : |\operatorname{Re}(z) - w| < \alpha \operatorname{Im}(z)\},$$
$$\Gamma_{\alpha,\delta}(w) = \{z \in \mathbb{C}^+ \cap \Gamma_\alpha(w) : \operatorname{Im}(z) < \delta\}.$$

A function $f(z)$ defined on $\mathbb{C}^+$ is said to converge to a limit a as $z \to w$ nontangentially if for every $\alpha < \infty$, $f(z) \to a$ as $z \to w$ within the cone $\Gamma_\alpha(w)$. The nontangential maximal function of f is defined by

$$Nf(w) = N_{\alpha,\delta} f(w) = \sup_{z \in \Gamma_{\alpha,\delta}(w)} |f(z)|. \tag{4.1}$$

We will need the following local variant of a standard property of functions holomorphic in the whole half plane $\mathbb{C}^+$.

Lemma 4.1. *Let $\delta > 0$. Let Λ be an open subinterval of $\mathbb{R}$, and let $\Lambda' \Subset \Lambda$ be a relatively compact subinterval. Let $0 \le q \le \infty$. Let B be a Banach space. Suppose that F is a holomorphic B-valued function in $\{\lambda + i\varepsilon : 0 < \varepsilon < \delta,\ \lambda \in \Lambda\}$. Then for any $\alpha < \infty$, there exist $C < \infty$ and $\delta' > 0$ depending on $\alpha, \Lambda, \Lambda'$, but not on F, such that*

$$\|N_{\alpha,\delta'} F(\cdot)\|_{L^q(\Lambda')} \le C \sup_{0<\varepsilon<\delta} \|F(\cdot + i\varepsilon)\|_{L^q(\Lambda)}. \tag{4.2}$$

By a B-valued holomorphic function we mean a continuous function f from an open subset of $\mathbb{C}$ to B, such that $z \mapsto \ell(f)(z)$ is holomorphic for every bounded linear functional ℓ on B.

Proof. Suppose first that f is continuous on the closed rectangle $\Lambda + i[0,\delta]$, and that $1 < q < \infty$. Let $0 < \delta'' < \delta$ be a small number to be chosen. Choose a third interval Λ'', so that $\Lambda' \subset \Lambda'' \subset \Lambda$ and the distance from each interval to the boundary of the next larger interval is strictly positive. Consider the rectangle $R = \Lambda'' + i[0,\delta''] = \{w : \operatorname{Re}(w) \in R \text{ and } \operatorname{Im}(w) \in [0,\delta'']\}$.

For any $z \in R$ with $\operatorname{Re}(z) \in \Lambda'$ and $0 < \operatorname{Im}(z) < \delta''/2$, we may write $f(z) = \int_{\partial R} f(\zeta) d\omega_z(\zeta)$ where ω_z is harmonic measure on ∂R. Fix a conformal map φ from R to the unit disk; this map is smooth everywhere except at the corners of R, where it behaves locally like $z \mapsto z^2$ in appropriate local coordinates. Harmonic measure for R may be computed by pulling back harmonic measure from the unit disk under φ. From this we readily deduce that for any $z \in R$ with $\operatorname{Re}(z) \in \Lambda$,

$$f(z) = \int_{\Lambda''} k(z,\zeta) f(\zeta)\, d\sigma(\zeta) + O\Big(\int_{\partial R \setminus \Lambda''} |f(\zeta)|\, d\sigma(\zeta) \Big) \tag{4.3}$$

where σ denotes arc length measure on ∂R, and where k satisfies upper bounds of Poisson kernel type: $|k(x+it,y)| \le Ct[|x-y|^2 + t^2]^{-1}$.

Denote by F the restriction of f to Λ'', and by M the Hardy-Littlewood maximal function. We combine these bounds with the usual majorization of the nontangential maximal function of a Poisson integral by M to conclude that for $z \in \Gamma_{\alpha,\delta'}(y)$ with $0 < \operatorname{Im}(z) < \delta''$ and $y \in \Lambda'$, the first term on the right-hand side of (4.3) is bounded by $C'MF(y)$.

The second term on the right is not suitably bounded, in general. Therefore we consider all translates $R_s = \{z + s : z \in R\}$ where $s \in \mathbb{R}$ ranges over a small interval I centered at 0. For each R_s we have a variant of (4.3), obtained by conjugating with the translation $z \mapsto z + s$; the integral in the first term now extends over $\Lambda'' + s$. By averaging all these variants over $s \in I$ with respect to normalized Lebesgue measure, we obtain

$$f(z) = \int_{\tilde\Lambda} \tilde k(z,w) d\sigma(w) + O\Big(\int_Q |f(w)|\, dw\Big) + O\Big(\int_{\tilde\Lambda} |f(w+i\delta'')|\, d\sigma(w)\Big)$$

where $\tilde\Lambda = \cup_{s\in I}\Lambda'' + s$ is slightly larger than Λ'', $\tilde k$ satisfies the same bounds as k, dw denotes Lebesgue measure in $\mathbb{C}$, and Q is a certain compact subset of the closed rectangle $\Lambda + i[0,\delta]$. The nontangential maximal function of the first term is majorized by CMF, just as before. The second is majorized, uniformly in z, by $\sup_\varepsilon \|f(\cdot + i\varepsilon)\|_{L^q(\Lambda)}$. The third is already under control, by hypothesis, since $q \geq 1$. Because M is bounded on L^q, we obtain the desired conclusion, under the supplementary hypothesis that f extends continuously to $\mathrm{Im}\,(z) = 0$.

For general f not necessarily continuous up to $\mathrm{Im}\,(z) = 0$, consider $f_\eta(z) = f(z + i\eta)$, for small $\eta \in \mathbb{R}^+$. Apply the result just proved to f_η, and pass to the limit $\eta \to 0^+$ using Fatou's lemma.

In order to extend this argument to $0 < q \leq 1$, it suffices to recall that if F is a $\mathbb{C}$-valued holomorphic function, then $|F|^s$ is subharmonic for any $s > 0$. Fixing any $0 < s < q$ any bounded linear functional ℓ on B, and setting $F(z) = \ell(f(z))$, we may therefore write instead of (4.3)

$$|F(z)|^s \leq \int_{\Lambda''} k(z,\zeta)|F(\zeta)|^s\, d\sigma(\zeta) + O\Big(\int_{\partial R\setminus\Lambda''} |F(\zeta)|^s\, d\sigma(\zeta)\Big).$$

The proof then proceeds as above, using the fact that $f \mapsto [M(|f|^s)]^{1/s}$ is bounded on L^q for all $q > s$. □

5. Generalized eigenfunctions associated to complex spectral parameters

Consider the generalized eigenfunction equation

$$-u'' + V(x)u = zu, \tag{5.1}$$

where z is permitted to be complex. In earlier work we have analyzed the solutions to this equation for z real and positive, and have shown that for almost every such z there exists a solution with certain asymptotic behavior as $x \to +\infty$. Our present purpose is to analyze solutions for complex z and to show that they tend almost everywhere to the solutions for real z, as z approaches the positive real axis.

We restrict attention to parameters $z \in \mathbb{C}^+ \cup \mathbb{R}^+$, and choose a branch of $\sqrt{z}$ which has nonnegative imaginary part for such z. Define the phases

$$\xi(x,z) = \sqrt{z}x - (2\sqrt{z})^{-1}\int_0^x V. \tag{5.2}$$

Theorem 5.1. *Let $1 \le p < 2$ and assume that $V \in L^1 + L^p$. For each $z \in \mathbb{C}^+$ there exists a (unique) solution $u(x,z)$ of the generalized eigenfunction equation* (5.1) *satisfying*

$$u(x,z) - e^{i\xi(x,z)} \to 0 \text{ and } \partial u(x,z)/\partial x - i\sqrt{z}e^{i\xi(x,z)} \to 0 \qquad \text{as } x \to +\infty. \tag{5.3}$$

$u(x,z)$ is continuous as a function on $\mathbb{C}^+ \times \mathbb{R}$, and is holomorphic with respect to z for each fixed x.

Likewise, there exists such a (unique) solution for almost every $z \in \mathbb{R}^+$. For almost every $E \in \mathbb{R}$, $u(x,z)$ converges to $u(x,E)$ uniformly for all x in any interval bounded below, as $z \to E$ nontangentially.

Here "almost every" means with respect to Lebesgue measure. The existence of such solutions for almost every $z \in \mathbb{R}^+$ is proved in [9]. For $z \in \mathbb{C}^+$ it is well known under weaker hypotheses on V. The new point here is the convergence as $z \to \mathbb{R}^+$, and in particular, the fact that it is globally uniform in x.

It suffices to prove the theorem for $x \in [0,\infty)$, since the conclusion for $x \in [\rho,\infty)$, for any $\rho > -\infty$, then follows via the eigenfunction equation (5.1).

By rewriting (5.1) as a first-order system, performing a couple of algebraic transformations, reducing to an integral equation, and solving it by iteration, one arrives [9] at a formal series representation for solutions of (5.1):

$$\begin{pmatrix} u(x,z) \\ u'(x,z) \end{pmatrix} = \begin{pmatrix} e^{i\xi(x,z)} & e^{-i\xi(x,z)} \\ i\sqrt{z}e^{i\xi(x,z)} & -i\sqrt{z}e^{-i\xi(x,z)} \end{pmatrix} \cdot \begin{pmatrix} \sum_{n=0}^{\infty} T_{2n}(V,\dots,V)(x,z) \\ -\sum_{n=0}^{\infty} T_{2n+1}(V,\dots,V)(x,z) \end{pmatrix} \tag{5.4}$$

where

$$T_n(V_1,\dots,V_n)(x,z) = (2\sqrt{z})^{-n} \int_{x \le t_1 \le \cdots \le t_n} \prod_{j=1}^{n} e^{2i(-1)^{n-j}\xi(t_j,z)} V_j(t_j)\, dt_j, \tag{5.5}$$

with the convention $T_0(\cdot) \equiv 1$. To prove Theorem 5.1 we will show that each multilinear expression T_n is well-defined for all $z \in \mathbb{C}^+$, that $T_n(\cdot)(x,z) \to 0$ as $x \to +\infty$ for all $n \ge 1$, that they have the natural limits as $z \to \mathbb{R}^+$ nontangentially, and that these expressions satisfy bounds sufficiently strong to enable us to sum the infinite series to obtain the desired conclusions.

Substitute $\zeta = \sqrt{z}$ and write $\zeta = \lambda + i\varepsilon$, noting that $\varepsilon > 0$. Also write $\phi(x,\zeta) = \xi(x,z)$, $S_n(V_1,\dots)(x,\zeta) = T_n(V_1,\dots)(x,z)$. Let $\{E_j^m\}$ be a martingale structure on $\mathbb{R}^+$. Denote by $t_{m,j}^{\pm}$ respectively the right $(+)$ and left $(-)$ endpoints of the interval E_j^m.

The real part of the exponent $2i\sum_{j=1}^{n}(-1)^{n-j}\xi(t_j,z)$ is to leading order

$$-2\,\mathrm{Im}\,(\sqrt{z}[(t_n - t_{n-1}) + (t_{n-2} - t_{n-3}) + \cdots]) = -2\varepsilon \cdot [(t_n - t_{n-1}) + (t_{n-2} - t_{n-3}) + \cdots],$$

which is nonnegative (for $x \ge 0$) for all $z \in \mathbb{C}^+$ since $t_1 \le t_2 \cdots$; the exponential factor decays rapidly as $[(t_n - t_{n-1}) + (t_{n-2} - t_{n-3}) + \cdots] \to \infty$. This accounts for the difference between $z \in \mathbb{C}^+$ and $z \in \mathbb{R}^+$.

Define

$$G_m(V)(\zeta) = \Big(\sum_{j=1}^{2^m} |s_j^{m,-}(V,\zeta)|^2 + |s_j^{m,+}(V,\zeta)|^2 \Big)^{1/2}$$
$$G(V) = \sum_{m=1}^{\infty} m G_m(V) \tag{5.6}$$

where

$$s_j^{m,-}(V,\zeta) = \int_{E_j^m} e^{2i[\phi(t,\zeta)-\phi(t_{m,j}^-,\zeta)]} V(t)\, dt$$
$$s_j^{m,+}(V,\zeta) = \int_{E_j^m} e^{2i[\phi(t_{m,j}^+,\zeta)-\phi(t,\zeta)]} V(t)\, dt. \tag{5.7}$$

When $j = 2^m$, and only then, the right endpoint of E_j^m is infinite. To simplify notation we make the convention that for $j = 2^m$, the second term $|s_j^{m,+}(V,\zeta)|^2$ is always to be omitted, in the definition of G and anywhere else that the quantities $\phi(t_{m,j}^+,\zeta)$ arise.

The following definitions will allow us to regard G as a linear operator, and hence to exploit properties of holomorphic functions.

Definition. $\mathcal{B}$ denotes the Banach space consisting of all sequences $\{\mathbb{C}^2 \ni s_j^m : m \geq 0,\ 1 \leq j \leq 2^m\}$, with the norm $\|s\|_{\mathcal{B}} = \sum_m m\big(\sum_{j=1}^{2^m} |s_j^m|^2 \big)^{1/2}$.

$\mathfrak{G} : L^p \mapsto \mathcal{B}$ denotes the operator

$$\mathfrak{G}(V)(\zeta) = \{(s_j^{m,+}(V,\zeta), s_j^{m,-})(V,\zeta) : 1 \leq m < \infty, 1 \leq j \leq 2^m\}. \tag{5.8}$$

Thus

$$G(V)(\zeta) = \|\mathfrak{G}(V)(\zeta)\|_{\mathcal{B}}.$$

Likewise we may write $G_M(V) = \|\mathfrak{G}_M(V)\|_{\mathcal{B}}$ with the analogous definition of $\mathfrak{G}_M$.

The real parts of $i\phi(t,\zeta) - i\phi(t_{m,j}^-,\zeta)$ and $i\phi(t_{m,j}^+,\zeta) - i\phi(t,\zeta)$ are bounded above uniformly for $0 < \operatorname{Im}(\zeta) \leq 1$, and are $\leq -c \operatorname{Im}(\zeta)(t - t_{m,j}^-)$ and $\leq -c \operatorname{Im}(\zeta)(t_{m,j}^+ - t)$, respectively; see (5.18). Therefore for $V \in L^1 + L^\infty$ and $\zeta \in \mathbb{C}^+$, each of these integrals converges absolutely, and each defines a holomorphic scalar-valued function of $\zeta \in \mathbb{C}^+$. Thus $\mathfrak{G}(V)$ may be regarded as a $\mathcal{B}$–valued holomorphic function[2] in any open set where it can be established that the series defining $\|\mathfrak{G}(V)\|_{\mathcal{B}}$ converges uniformly.

We will also use the following variant. Given a collection of functions $(V_1, \ldots, V_n)$, we define

$$G(\{V_i\})(\zeta) = \sum_{m=1}^{\infty} m \cdot \Big(\sum_{j=1}^{2^m} \sum_{i}^{*} |s_j^{m,+}(V_i,\zeta)|^2 + |s_j^{m,-}(V_i,\zeta)|^2 \Big)^{1/2}, \tag{5.9}$$

[2]By this we mean simply that it is a continuous mapping into $\mathcal{B}$ with respect to the norm topology, and that each $s_j^{m,\pm}$ is a holomorphic scalar-valued function.

where $\sum^*$ indicates that the sum is taken over a maximal set of indices i for which the functions V_i are all distinct.

Lemma 5.2. *For all V, n and all $\zeta \in \mathbb{C}^+ \cup \mathbb{R}^+$,*

$$\sup_{x\in\mathbb{R}^+} |S_n(V,V,\dots,V)(x,\zeta)| \le \frac{C^{n+1}}{\sqrt{n!}} G(V)(\zeta)^n. \tag{5.10}$$

More generally, for all n and all $\{V_1,\dots,V_n\}$,

$$\sup_{x\in\mathbb{R}^+} |S_n(V_1,V_2,\dots,V_n)(x,\zeta)| \le \frac{C_k^{n+1}}{\sqrt{n!}} G(\{V_i\})(\zeta)^n, \tag{5.11}$$

provided that $\{V_i\}_{i=1}^n$ has cardinality $\le k$.

Write $p' = p/(p-1)$.

Lemma 5.3. *For any compact interval $\Lambda \Subset (0,\infty)$, there exists $C < \infty$ such that for any $1 \le p \le 2$, for all $t' \in \mathbb{R}$ and $f \in L^p(\mathbb{R})$, for every $\varepsilon \ge 0$,*

$$\| \int_{t\ge t'} e^{2i[\phi(t,\lambda+i\varepsilon)-\phi(t',\lambda+i\varepsilon)]} f(t)\,dt\|_{L^{p'}(\Lambda,d\lambda)} \le C\|f\|_{L^p} \tag{5.12}$$

$$\| \int_{t\le t'} e^{2i[\phi(t',\lambda+i\varepsilon)-\phi(t,\lambda+i\varepsilon)]} f(t)\,dt\|_{L^{p'}(\Lambda,d\lambda)} \le C\|f\|_{L^p}. \tag{5.13}$$

For $\varepsilon = 0$ these integrals need not converge absolutely, hence require interpretation. They are initially well-defined for compactly supported f, and the lemma asserts an *a priori* bound for such functions. Then they are defined for general $f \in L^p$ by approximating in L^p norm by compactly supported functions, and passing to the limit in $L^{p'}$ norm.

Let an exponent $p < \infty$ be specified. Recall that a martingale structure is said to be adapted to f in L^p if $\int_{E_j^m} |f|^p = 2^{-m}\int |f|^p$ for all m, j. Recall from the preceding section the definitions of the cones $\Gamma_{\alpha,\delta}$ and associated nontangential maximal functions $N_{\alpha,\delta}$.

Corollary 5.4. *Let $\alpha < \infty$, let $1 \le p \le 2$, and let $\Lambda \Subset (0,\infty)$ be any compact subinterval. Then there exist $C < \infty, \delta > 0$ such that for any $f \in L^p(\mathbb{R})$ and for any martingale structure $\{E_j^m\}$ on $\mathbb{R}^+$*

$$\|N_{\alpha,\delta} G_m(f)(\lambda)\|_{L^{p'}(\Lambda,d\lambda)} \le C\|f\|_{L^p}. \tag{5.14}$$

Moreover for each $1 \le p < 2$ there exists $\rho > 0$ such that for any $f \in L^p$ and for any martingale structure adapted to f in L^p,

$$\|N_{\alpha,\delta} G_m(f)(\lambda)\|_{L^{p'}(\Lambda,d\lambda)} \le C2^{-\rho m}\|f\|_{L^p}. \tag{5.15}$$

Consequently under these additional hypotheses,

$$\|N_{\alpha,\delta} G(f)(\lambda)\|_{L^{p'}(\Lambda,d\lambda)} \le C\|f\|_{L^p}. \tag{5.16}$$

Moreover, for almost every $\lambda \in \Lambda$,

$$\|\mathfrak{G}(f)(\zeta) - \mathfrak{G}(f)(\lambda)\|_{\mathcal{B}} \to 0 \tag{5.17}$$

as $\zeta \to \lambda$ nontangentially.

We postpone the proofs of Lemmas 5.2, 5.3 and Corollary 5.4 until the end of the section.

For any $t \geq t'$ and any $\zeta = \lambda + i\varepsilon$ with $\varepsilon \geq 0$,

$$\operatorname{Re}\left(i\phi(t,\lambda+i\varepsilon) - i\phi(t',\lambda+i\varepsilon)\right) = -\varepsilon(t-t') - \varepsilon(\lambda^2+\varepsilon^2)\int_{t'}^{t} V, \tag{5.18}$$

and $|\int_{t'}^{t} V| \leq |t-t'|^{1/p'}\|V\|_{L^p}$. Therefore

$$|e^{i[\phi(t,\lambda+i\varepsilon)-\phi(t',\lambda+i\varepsilon)]}| \leq Ce^{-c\varepsilon|t-t'|} \tag{5.19}$$

where $C, c \in \mathbb{R}^+$ are constants which depend only on the L^p norm of V. Hence for all $n \geq 1$,

$$\begin{aligned} |S_{2n}(V,V,\dots,V)(x,\lambda+i\varepsilon)| &\leq \iint_{x\leq t_1\leq\cdots\leq t_{2n}} e^{-c\varepsilon(t_{2n}-t_{2n-1}+t_{2n-2}-\cdots)}\prod_j V(t_j)\,dt_j \\ &\leq \frac{C^n}{n!}\Big(\iint_{x\leq t'\leq t} e^{-c\varepsilon(t-t')}|V(t')V(t)|\,dt'\,dt\Big)^n \\ &\leq \frac{C^n}{n!}\varepsilon^{-2n(1-p^{-1})}\|V\|^{2n}_{L^p([x,\infty)}. \end{aligned} \tag{5.20}$$

In the same way, for $x \geq 0$, one obtains for $n \geq 0$

$$|S_{2n+1}(V,V,\dots,V)(x,\lambda+i\varepsilon)| \leq \frac{C^n}{n!}\varepsilon^{-(2n+1)(1-p^{-1})}\|V\|^{2n+1}_{L^p([x,\infty)}. \tag{5.21}$$

Let z belong to any compact subset of $\mathbb{C}^+$. Then $\varepsilon = \operatorname{Im}(\sqrt{z})$ has a strictly positive lower bound. Therefore the individual terms of the series (5.4) defining a formal solution of (5.1) do define uniformly bounded functions of (x,ζ) for $x \geq 0$ and ζ in any compact subset of $\mathbb{C}^+$, and moreover, the series are uniformly absolutely convergent. As in Lemma 4.2 of [9], it follows that the sums of these series define solutions of the generalized eigenfunction equation (5.1) for all such z. Because the L^p norm of V over $[x,\infty)$ tends to zero as $x \to +\infty$, only $T_0(x)$ contributes in that limit, so these solutions do have the desired WKB asymptotics $\exp(i\xi(x,z))$. Clearly each summand depends holomorphically on ζ, hence so do the sums.

Existence of a (unique) solution for almost every $z \in \mathbb{R}^+$ is proved in [9]. We come now to the main step, where we relate $z \in \mathbb{C}^+$ to $z \in \mathbb{R}^+$. For compactly supported $f \in L^1$, the quantities $s_j^{m,\pm}(f,\zeta)$ are clearly holomorphic functions of ζ where $\operatorname{Im}(\zeta) > 0$, and are continuous at $\varepsilon = 0$ for each $0 \neq \lambda \in \mathbb{R}$. The same holds for $S_n(f,f,\dots,f)(x,\zeta)$, for each $x \in \mathbb{R}$.

$\mathfrak{G}(f)$ is likewise a $\mathcal{B}$–valued holomorphic function of ζ, continuous on $\mathbb{C}^+ \cup \mathbb{R}^+$, for compactly supported $f \in L^1$. This follows by combining the holomorphy of the individual terms (5.7) with the rapid convergence bound (5.15).

Lemma 5.5. *Let $1 \leq p < 2$ and $V \in L^p(\mathbb{R})$. For almost every $E \in \mathbb{R}$, for every $n \geq 1$, $T_n(V,V,\dots,V)(x,z) \to T_n(V,V,\dots,V)(x,E)$ uniformly for all $x \geq 0$ as $\mathbb{C}^+ \ni z \to E$ nontangentially.*

Proof. It is equivalent to show that

$$\sup_{x\geq 0}\left|S_n(V,V,\dots,V)(x,\zeta)-S_n(V,V,\dots,V)(x,\lambda)\right|\to 0$$

as $\zeta\to\lambda$ nontangentially, for almost all $\lambda\in\Lambda$, for any fixed compact interval $\Lambda\subset\mathbb{R}^+$. For $V=W\in L^1$ with compact support, we have already established convergence uniformly in x,λ, as $\mathbb{C}^+\ni\zeta\to\lambda$ unrestrictedly (rather than merely nontangentially), since the phases $\phi(t,\zeta)$ converge uniformly to $\phi(t,\lambda)$ for t,ζ,λ in any compact set.

Let V be given, and remain fixed for the remainder of this proof. Set $\phi_V(t,\zeta)=\zeta t-(2\zeta)^{-1}\int_0^t V$. Whenever we write $S_n(f_1,f_2,\dots,f_n)$, it is defined in terms of the phases $\phi(t_i,\zeta)=\phi_V(t_i,\zeta)$, independent of $f_1,f_2,\dots$. Thus the S_n are here genuine multilinear operators.

Let $\varepsilon>0$ be arbitrary, and fix a martingale structure $\{E_j^m\}$ adapted to V in L^p on $\mathbb{R}^+$. Decompose $V=W+(V-W)$ where $W(x)=V(x)\chi_{(0,R]}(x)$, with R chosen so that $\|V-W\|_{L^p}<\varepsilon$ and moreover so that $\|NG(V-W)\|_{L^{p'}(\Lambda)}<\varepsilon$. Such a choice is possible, since

$$\|NG_M(V\chi_{[R,\infty)})\|_{L^{p'}(\Lambda)}\leq C\min(2^{-M\delta}\|V\|_{L^p},\,\|V\chi_R\|_{L^p}).$$

Then

$$\begin{aligned}(5.22)\quad &|S_n(V,V,\dots,V)(x,\zeta)-S_n(V,V,\dots,V)(x,\lambda)|\\ &\quad\leq|S_n(W,W,\dots,W)(x,\zeta)-S_n(W,W,\dots,W)(x,\lambda)|\\ &\qquad+|S_n(V,V,\dots,V)(x,\zeta)-S_n(W,W,\dots,W)(x,\zeta)|\\ &\qquad\qquad+|S_n(V,V,\dots,V)(x,\lambda)-S_n(W,W,\dots,W)(x,\lambda)|.\end{aligned}$$

The first term on the right tends to zero, in the sense desired. Majorize the second by

$$\begin{aligned}(5.23)\quad &|S_n(V,V,\dots,V)(x,\zeta)-S_n(W,W,\dots,W)(x,\zeta)|\\ &\qquad\leq\sum_{i=1}^n|S_n(V,\dots,V,V-W,W,\dots,W)(z,\zeta)|\end{aligned}$$

where in the i-th summand, the argument of S_n has $i-1$ copies of V and $n-i$ copies of W. Fix any aperture $\alpha\in\mathbb{R}^+$. Thus as established in the proof of Proposition 4.1 of [9],

$$\begin{aligned}(5.24)\quad &\sup_{x\geq 0}\sup_{\zeta\in\Gamma_{\alpha,\delta}(\lambda)}|S_n(V,V,\dots,V)(x,\zeta)-S_n(W,W,\dots,W)(x,\zeta)|\\ &\qquad\leq C_n\sum_{i=1}^n\sup_{\zeta\in\Gamma_{\alpha,\delta}(\lambda)}G(V)^{i-1}(\zeta)G(W)^{n-i}(\zeta)G(V-W)(\zeta)\\ &\qquad\qquad\leq C_n\sum_{i=1}^n NG(V)^{i-1}(\lambda)NG(W)^{n-i}(\lambda)NG(V-W)(\lambda).\end{aligned}$$

Let $q = p'$. By Chebyshev's inequality, for any $\beta > 0$,

(5.25)
$$\begin{aligned}
&|\{\lambda \in \Lambda : \sup_{\zeta \in \Gamma_{\alpha,\delta}(\lambda)} \sup_{x \geq 0} |S_n(V,V,\dots,V)(x,\zeta) - S_n(W,W,\dots,W)(x,\zeta)| > \beta\}| \\
&\qquad \leq C_n \beta^{-q/n} \sum_{i=1}^{n} \|NG(V)^{i-1} NG(W)^{n-i} NG(V-W)\|_{L^{q/n}(\Lambda)}^{q/n} \\
&\qquad \leq C_n \beta^{-q/n} \sum_{i=1}^{n} (\|NG(V)\|_{L^q(\Lambda)}^{i-1} \|NG(W)\|_{L^q(\Lambda)}^{n-i} \|NG(V-W)\|_{L^q(\Lambda)})^{q/n} \\
&\qquad \leq C_n \beta^{-q/n} \sum_{i=1}^{n} (\|V\|_{L^p}^{n-1} \|V-W\|_{L^p})^{q/n} \\
&\qquad \leq C_n \beta^{-q/n} \|V\|_{L^p}^{q(n-1)/n} \varepsilon^{q/n}.
\end{aligned}$$

The same reasoning gives

$$|\{\lambda \in \Lambda : \sup_{x \geq 0} \big|S_n(V,V,\dots,V)(x,\lambda) - S_n(W,W,\dots,W)(x,\lambda)| > \beta\}\big| \leq C_n \beta^{-q/n} \|V\|_{L^p}^{q(n-1)/n} \varepsilon^{q/n}. \tag{5.26}$$

Consequently

$$|\{\lambda \in \Lambda : \limsup_{\Gamma_\alpha(\lambda) \ni \zeta \to \lambda} \sup_{x \geq 0} \big|S_n(V,V,\dots,V)(x,\zeta) - S_n(V,V,\dots,V)(x,\lambda)\big| > \beta\} \leq C_n \beta^{-q/n} \|V\|_{L^p}^{q(n-1)/n} \varepsilon^{q/n}, \tag{5.27}$$

for all $\beta, \varepsilon \in \mathbb{R}^+$. Letting $\varepsilon \to 0$, we conclude that the lim sup vanishes for almost every λ. □

It is now straightforward to sum the series to obtain the same conclusion regarding convergence of $u(x,z) = u(x,\zeta^2)$ to $u(x,E) = u(x,\lambda^2)$:

$$|u(x,\zeta^2) - u(x,\lambda^2)| \leq \sum_{n=0}^{\infty} |S_n(V,V,\dots,V)(x,\zeta) - S_n(V,V,\dots,V)(x,\lambda)| \tag{5.28}$$

and

(5.29)

$$\begin{aligned}
&\sup_{\zeta\in\Gamma_{\alpha,\delta}(\lambda)}\sup_{x\geq 0}\sum_{n=M}^{\infty}|S_n(V,V,\dots,V)(x,\zeta)-S_n(V,V,\dots,V)(x,\lambda)|\\
&\qquad\leq \sup_{\zeta\in\Gamma_{\alpha,\delta}(\lambda)}\sup_{x\geq 0}\sum_{n=M}^{\infty}(|S_n(V,V,\dots,V)(x,\zeta)|+|S_n(V,V,\dots,V)(x,\lambda)|)\\
&\qquad\leq \sum_{n=M}^{\infty}\frac{C^{n+1}}{\sqrt{n!}}\big(NG(V)(\lambda)+G(V)(\lambda)\big)^n\\
&\qquad\leq \frac{C^{M+1}}{\sqrt{M!}}\big(NG(V)(\lambda)+G(V)(\lambda)\big)^M\sum_{k=0}^{\infty}\frac{C^{k+1}}{\sqrt{k!}}\big(NG(V)(\lambda)+G(V)(\lambda)\big)^k\\
&\qquad\leq \frac{C^{M+1}}{\sqrt{M!}}\big(NG(V)(\lambda)+G(V)(\lambda)\big)^M\exp(C\big(NG(V)(\lambda)+G(V)(\lambda)\big)^2).
\end{aligned}$$

For almost every $\lambda\in\mathbb{R}$, $\big(NG(V)(\lambda)+G(V)(\lambda)\big)<\infty$, and hence this expression tends to zero as $M\to\infty$. Coupled with the convergence established for the individual terms S_n in Lemma 5.5, this completes the proof of Theorem 5.1, modulo the proofs of Lemmas 5.3 and 5.2, and Corollary 5.4. □

Proof of Lemma 5.3. The proofs of the two inequalities are essentially the same, so we discuss only the first. The exponent here is $i\Phi(t,\lambda+i\varepsilon)=i\phi(t,\lambda+i\varepsilon)-i\phi(t',\lambda+i\varepsilon)=(i\lambda-\varepsilon)(t-t')-(i\lambda+\varepsilon)(\lambda^2+\varepsilon^2)^{-1}(\int_{t'}^t V)$. Since $t\geq t'$ and $|\int_{t'}^t V|\leq C+C|t-t'|^{1/2}$, the real part of Φ is bounded above, uniformly for all λ in any compact subinterval $\Lambda\Subset\mathbb{R}\backslash\{0\}$ and $\varepsilon\geq 0$. Thus L^1 is mapped boundedly to $L^\infty(\Lambda)$, uniformly in ε; by interpolation it suffices to prove the L^2 estimate.

Fix any cutoff function $\eta\in C^\infty(\mathbb{R}\backslash\{0\})$ and consider

$$\int\Big|\int_{t>t'}e^{i\Phi(t,\lambda+i\varepsilon)}f(t)\,dt\Big|^2\eta(\lambda)\,d\lambda=\iint_{s,t\geq t'}f(t)\bar f(s)K(t,s)\,dt\,ds \tag{5.30}$$

where

$$K(t,s)=\int e^{\Psi(t,s,\lambda+i\varepsilon)}\eta(\lambda)\,d\lambda \tag{5.31}$$

with

$$\begin{aligned}
\Psi(t,s,\lambda+i\varepsilon)&=2i\Phi(t,\lambda+i\varepsilon)-2i\bar\Phi(s,\lambda+i\varepsilon)\\
&=2i\big[\lambda(t-s)-\lambda(\lambda^2+\varepsilon^2)^{-1}\textstyle\int_s^t V\big]\\
&\quad-2\varepsilon\big[(t-t')+(s-t')+(\lambda^2+\varepsilon^2)^{-1}(\textstyle\int_{t'}^t V)+(\lambda^2+\varepsilon^2)^{-1}(\int_{t'}^s V)\big].
\end{aligned} \tag{5.32}$$

We claim that $|K(t,s)|\leq C(1+|s-t|)^{-2}$, uniformly in $\varepsilon\geq 0$; this would suffice to imply the L^2 bound. The integrand itself is bounded, uniformly in all parameters, so it suffices to restrict attention to the case where $|s-t|\geq C_0$, where C_0 is a sufficiently large constant. In that case we integrate by parts, integrating $\exp(2i[\lambda(t-s)-\lambda(\lambda^2+\varepsilon^2)^{-1}\int_s^t V])$, and differentiating $\eta(\lambda)\cdot\exp(-2\varepsilon[(t-t')+(s-t')+(\lambda^2+\varepsilon^2)^{-1}(\int_{t'}^t V)+$

$(\lambda^2+\varepsilon^2)^{-1}(\int_{t'}^{s} V))$, noting that $\partial[\lambda(t-s)-\lambda(\lambda^2+\varepsilon^2)^{-1}\int_s^t V]/\partial\lambda \geq |s-t|/2$ provided C_0 is chosen to be sufficiently large. Thus we gain a factor of $(s-t)^{-1}$. On the other hand, differentiating the other exponential with respect to λ brings in an unfavorable term $O(\varepsilon\int_{t'}^{\max(s,t)}|V|)$. After two integrations by parts, the integrand is

$$O(|s-t|^{-2})\cdot e^{-2\varepsilon(s-t')-2\varepsilon(t-t')}\cdot O(1+\varepsilon^2(\int_{t'}^{\max(s,t)}|V|)^2) = O(|s-t|^{-2}), \tag{5.33}$$

uniformly in ε. □

Proof of Lemma 5.2. The new feature here is the introduction of the modifying factors $\exp(\pm 2i\phi(t^{\pm}_{m,j},\zeta))$; without these, this is proved in [10]. We will merely indicate the modification needed in the argument, referring to [10] for the rest. Consider

$$\iint_{x\leq t_1\leq\cdots\leq t_n} e^{2i[\phi(t_n,\zeta)-\phi(t_{n-1},\zeta)+\phi(t_{n-2},\zeta)-\cdots]} f(t_1)f(t_2)\cdots f(t_n)\,dt_1\cdots dt_n. \tag{5.34}$$

Decompose the region of integration $\{t=(t_1,\cdots,t_n): x\leq t_1\leq\cdots\leq t_n\}$ as $\cup_{k=0}^n\Omega_k$ where $\Omega_k=\{t: x\leq t_1\leq\cdots\leq t_k\leq t^+_{1,1}=t^-_{1,2}\leq t_{k+1}\leq\cdots\leq t_n\}$. The total integral is

(5.35)

$$\begin{aligned}
&\sum_k \iint_{\Omega_k} e^{2i[\phi(t_n,\zeta)-\phi(t_{n-1},\zeta)+\cdots\pm\phi(t_{k+1},\zeta)\mp\phi(t^-_{1,2},\zeta)]}\cdot e^{2i[\pm\phi(t^+_{1,1}\mp\phi(t_k,\zeta)\pm\phi(t_{k-1},\zeta)\mp\cdots]}\prod_{j=1}^n f(t_j)\,dt_j\\
&\quad=\sum_k\Big(\iint_{t^-_{1,2}\leq t_{k+1}\leq\cdots\leq t_n} e^{2i[\phi(t_n,\zeta)-\phi(t_{n-1},\zeta)+\cdots+(-1)^{n-k-1}\phi(t_{k+1},\zeta)]}\prod_{j=k+1}^n f(t_j)\,dt_j\Big)\\
&\qquad\cdot\Big(\iint_{x\leq t_1\leq\cdots\leq t_k\leq t^+_{1,1}} e^{2i(-1)^{n-k}[\phi(t_k,\zeta)-\phi(t_{k-1},\zeta)+\cdots+(-1)^{n-1}\phi(t_1,\zeta)]}\prod_{j=1}^k f(t_j)\,dt_j\Big).
\end{aligned}$$

For each k, each of the two factors on the right-hand side has the same form as the multiple integral with which we began, except that when $n-k$ is odd, an extra factor of -1 appears in the exponent in the integral with respect to $dt_k\cdots dt_1$; this minus sign destroys the bounds we seek, as is clear from (5.18). Therefore when $n-k$ is odd, we rewrite the corresponding term as the modified product

$$\begin{aligned}
&\Big(\iint_{t^-_{1,2}\leq t_{k+1}\leq\cdots\leq t_n} e^{2i[\phi(t_n,\zeta)-\phi(t_{n-1},\zeta)+\cdots+\phi(t_{k+1},\zeta)-\phi(t^-_{1,2},\zeta)]}\prod_{j=k+1}^n f(t_j)\,dt_j\Big)\\
&\qquad\cdot\Big(\iint_{x\leq t_1\leq\cdots\leq t_k\leq t^+_{1,1}} e^{2i[\phi(t^+_{1,1},\zeta)-\phi(t_k,\zeta)+\phi(t_{k-1},\zeta)+\cdots+(-1)^{n-1}\phi(t_1,\zeta)]}\prod_{j=1}^k f(t_j)\,dt_j\Big).
\end{aligned}$$

Suppose now that n is even. The proof in [10] is an induction based on a repeated application of this decomposition; each step of that recursion involves a "cut point" $t^+_{m,j}=t^-_{m,j+1}$ playing the same role as $t^+_{1,1}=t^-_{1,2}$ in the above formula. At each step, the region of integration is decomposed into subregions as above, and corresponding to each subregion there is a splitting of the terms in the phase into two subsets. At any

step which results in an odd number of terms appearing in one (hence both) subsets, we modify the resulting phases by introducing a factor $1 = \exp(\pm 2i[\phi(t^+_{m,j},\zeta) - \phi(t^-_{m,j+1},\zeta)])$, factoring it as a product of one exponentials, and splitting those two exponential factors as above. This, together with the argument in [10], yields the assertion of the lemma for even n.

For odd n we introduce a factor of $1 = \exp(2i[\phi(x,\zeta) - \phi(x,\zeta)])$, incorporate $\exp(-2i\phi(x,\zeta))$ into the phase, thus reducing matters again to the case where there an even number of terms. The remaining factor of $\exp(2i\phi(x,\zeta))$ is bounded above, uniformly for $0 \le \mathrm{Im}\,(\zeta) \le 1$ and $x \ge 0$, so is harmless. □

Proof of Corollary 5.4. Let $\Lambda \Subset (0,\infty)$ be a compact interval, and let $1 < p < 2$, the case $p = 1$ being trivial. Set $q = p' = p/(p-1)$. We discuss only the contributions of terms involving $\phi(t^-_{m,j},\zeta)$ to G and G_M; those involving $t^+_{m,j}$ are treated in exactly the same way. For any $m \ge 1$ and any $f \in L^p$ we have, since $q/2 \ge 1$,

$$\|\Big(\sum_{j=1}^{2^m}\Big|\int_{E^m_j} e^{2i[\phi(t,\lambda+i\varepsilon)-\phi(t^-_{m,j},\lambda+i\varepsilon)]}f(t)\,dt\Big|^2\Big)^{1/2}\|^q_{L^q(\Lambda,d\lambda)} \le \Big(\sum_j\Big[\int_\Lambda\Big|\int_{E^m_j} e^{2i[\phi(t,\lambda+i\varepsilon)-\phi(t^-_{m,j},\lambda+i\varepsilon)]}f(t)\,dt\Big|^q d\lambda\Big]^{2/q}\Big)^{q/2}, \tag{5.36}$$

by Minkowski's integral inequality. By Lemma 5.3, the right-hand side is

$$\le C\Big(\sum_j \|f\cdot\chi_{E^m_j}\|^2_{L^p}\Big)^{q/2}.$$

Since $p \le 2$, this is

$$\le C\Big(\sum_j \|f\|^{2-p}_{L^p}\|f\cdot\chi_{E^m_j}\|^p_{L^p}\Big)^{q/2} = C\|f\|^q_{L^p}.$$

If we assume that the martingale structure is adapted to f in L^p, then for $p < 2$ we have the improved majorization

$$\|f\cdot\chi_{E^m_j}\|^2_{L^p} \le 2^{-m(2-p)/p}\|f\|^{2-p}_{L^p}\|f\cdot\chi_{E^m_j}\|^p_{L^p},$$

whence the final bound is $2^{-\rho m}\|f\|^q_{L^p}$ for some $\rho(p) > 0$.

Therefore

$$\int_\Lambda \|\mathfrak{G}_M(f)(\lambda+i\varepsilon)\|^q_{\mathcal{B}}\,d\lambda \le C2^{-\rho M}\|f\|^q_{L^p}, \tag{5.37}$$

uniformly for all $\varepsilon > 0$. The first two conclusions of the Corollary now follow from Lemma 4.1, since Λ is an arbitrary compact interval.

That $\mathfrak{G}(f)(\zeta)$ converges almost everywhere to $\mathfrak{G}(f)(\lambda)$ in the $\mathcal{B}$ norm as $\zeta \to \lambda$ nontangentially, is an immediate consequence of the bound $\|NG_M(f)\|_{L^q} \le C2^{-\varepsilon M}\|f\|_{L^p}$, since

$$\int_{t\ge t'} e^{2i[\phi(t,\zeta)-\phi(t',\zeta)]}f(t)\,dt \to \int_{t\ge t'} e^{2i[\phi(t,\lambda)-\phi(t',\lambda)]}f(t)\,dt \tag{5.38}$$

almost everwhere as $\zeta \to \lambda$ nontangentially, for all $f \in L^p$. This holds for all f in the dense subspace $L^1 \cap L^p$, and then follows for general f by Lemma 4.1, Lemma 5.3, and standard reasoning. □

6. Resolvents and spectral projections

6.1. **The half-line case.** Consider the operator

$$H_V = -\frac{d^2}{dx^2} + V(x) \tag{6.1}$$

on $\mathbb{R}^+$, with Dirichlet boundary condition at the origin. Any other selfadjoint boundary condition can be treated in a similar way. For each $z \in \mathbb{C}$, let $u_1(x,z), u_2(x,z)$ be the unique solutions of

$$-u'' + V(x)u = zu \tag{6.2}$$

satisfying the boundary conditions $u_1(0,z) = (0,1)^t, u_2(0,z) = (1,0)^t$; the superscripts t denote transposes. The classical theory of second order differential operators (see e.g. [13, 40]) tells us that if (6.1) is in the limit point case, then for any $z \in \mathbb{C}\setminus\mathbb{R}$ there exists a unique complex number $m(z)$, called the Weyl m-function, such that

$$f(x,z) = u_1(x,z)m(z) + u_2(x,z) \in L^2(\mathbb{R}^+).$$

We will always consider potentials which lead to the limit point situation, as is the case if $V \in L^1 + L^p$ for some $1 \le p < \infty$; see for example [32]. The Weyl m function is a Herglotz function, that is, it is analytic in the upper half-plane and has positive imaginary part there.

By direct computation, the resolvent $(H_V - z)^{-1}g$ for $z \in \mathbb{C}^+$ is given by

$$(H_V - z)^{-1}g(x) = u_1(x,z)\int_x^\infty f(y,z)g(y)\,dy + f(x,z)\int_0^x u_1(y,z)g(y)\,dy. \tag{6.3}$$

Denote by $P_{(a,b)}$ the spectral projection associated to H_V and to the interval (a,b). ¿From the resolvent formula (6.3) we may derive a formula for the projections $P_{(a,b)}$. In doing so, we will use the following three routine facts.

1. The functions $u_1(x,z), u_2(x,z)$ are continuous in x for each z, and are entire holomorphic functions of z for each x.

2. The m function (in fact, any Herglotz function) has a representation

$$m(z) = C_1 + C_2 z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t) \tag{6.4}$$

for some positive Borel measure μ satisfying $\int (1+|t|^2)^{-1} d\mu(t) < \infty$, see e.g. [2]. In the m function context, μ is often called the spectral measure. For Dirichlet boundary conditions, as considered here, it is the spectral measure corresponding to the generalized vector δ_0', defined by $\delta_0'(u) = u'(0)$ for any function u in the domain of H_V. The moment condition above corresponds to the fact that the derivative δ_0' belongs to the Sobolev-like space $H_{-2}(H_V)$ associated to H_V (see e.g. [5] for details on families of Sobolev-like spaces associated with any selfadjoint operator A).

3. Im $m(E + i\epsilon)$ converges weakly to $\pi\mu$ as $\epsilon \to 0^+$. Moreover, Im $m(E + i\epsilon)$

has limiting boundary values for Lebesgue-almost every E, and the density of the absolutely continuous part of μ satisfies

$$d\mu_{\mathrm{ac}}(E) = \frac{1}{\pi} \operatorname{Im} m(E+i0)\, dE, \tag{6.5}$$

where $m(E+i0) = \lim_{\varepsilon\to 0^+} m(E+i\varepsilon)$. Since the imaginary part of m is simply a Poisson integral of μ, this is straightforward.

Fix functions h and g with compact support. We integrate the resolvent element $\langle (H_V - z)^{-1} g, h\rangle$ over the contour γ_ϵ in complex plane consisting of two horizontal intervals $(a \pm i\epsilon, b \pm i\epsilon)$, and two vertical intervals at the ends connecting them. In the limit $\epsilon \to 0^+$, the contributions of the vertical intervals disappear unless a or b is an eigenvalue (point mass of μ); we will assume this is not the case. By the spectral theory (see, e.g. [31], Stone formula) we get then the following expression for $\langle P_{(a,b)} g, h\rangle$:

$$\begin{aligned} \langle P_{(a,b)} g, h\rangle &= \frac{1}{2\pi i} \lim_{\epsilon\to 0} \int_{\gamma_\epsilon} \langle (H_V - z)^{-1} g, h\rangle \, dz \\ &= \int_a^b \int_{\mathbb{R}} \int_{\mathbb{R}} u_1(x,E) u_1(y,E) g(x) \overline{h}(y)\, dx\, dy\, d\mu(E). \end{aligned} \tag{6.6}$$

In passing to the last line, we have taken into account the resolvent formula (6.3), the properties of u_1, u_2 (in particular, u_2 drops out because of analyticity), and the fact that $\pi^{-1} \operatorname{Im} m(E+i\epsilon)$ converges weakly to μ. The compact supports of h, g ensure that the integral is well-defined. Similar formulas can be found in [13, 40].

Since $P_{(a,b)}$ is by its definition an orthogonal projection, an immediate consequence of (6.6) is that the mapping $g \mapsto \int_{\mathbb{R}} u_1(x,E) g(x)\, dx$, initially defined for continuous g having compact support, extends to an orthogonal projection from $L^2(\mathbb{R}^+, dx)$ to $L^2((a,b), \mu)$. Dually, from (6.6) we see that for each $g \in L^2(\mathbb{R}, d\mu)$,

$$U_0 g(x) = \lim_{N\to\infty} \int_{-N}^{N} u_1(x,E) g(E) d\mu(E) \tag{6.7}$$

exists in $L^2(\mathbb{R}^+, dx)$ norm, and that the linear operator U_0 thus defined is a unitary bijection from $L^2(\mathbb{R}, d\mu)$ to $L^2(\mathbb{R}, dx)$ with inverse

$$U_0^{-1} g(E) = \lim_{N\to\infty} \int_{-N}^{N} u_1(x,E) g(x) dx, \tag{6.8}$$

where the limit is again taken in L^2 norm.

With the formula (6.6) for the spectral projections in hand, we can invoke general spectral theory to find expressions for the spectral representation and other functions of H_V (see, e.g. [6]). To H_V and any interval (a,b) are associated a maximal closed subspace of $L^2(\mathbb{R}^+)$ on which H_V has purely absolutely continuous spectrum, and spectrum contained in (a,b). By (6.6) and (6.5) as well as by definition of the absolutely continuous part of the spectral projection, the projection $P^{\mathrm{ac}}_{(a,b)}$ of $L^2(\mathbb{R}^+)$

onto this subspace can be written as

$$P^{\rm ac}_{(a,b)}g(x) = \frac{1}{\pi}\int_a^b u_1(x,E)\Big(\int_{\mathbb{R}} u_1(y,E)g(y)\,dy\Big)\,\mathrm{Im}\ m(E+i0)\,dE. \tag{6.9}$$

The integral over $\mathbb{R}$ here is generally understood in the L^2-limiting sense; we will omit such explanatory remarks in the future. Consequently the operator U mapping continuous functions with compact support to $L^2(\mathbb{R}, dx)$, defined by

$$Uh(x) = \frac{1}{\pi}\int_{\mathbb{R}} u_1(x,E)h(E)\,\mathrm{Im}\ m(E+i0)\,dE \tag{6.10}$$

extends to an isometry of $L^2(\mathbb{R}, \mathrm{Im}\ m(E+i0)\,dE)$ onto the absolutely continuous subspace associated to H_V.

The unitary evolution operator on the absolutely continuous subspace is given by

$$e^{-iH_V t}g(x) = \frac{1}{\pi}\int_{\mathbb{R}} e^{-iEt}u_1(x,E)\tilde{g}(E)\,\mathrm{Im}\ m(E+i0)\,dE, \tag{6.11}$$

where

$$\tilde{g}(E) = \frac{1}{\pi}\int_{\mathbb{R}} u_1(y,E)g(y)\,dy.$$

Finally, in the case $V = 0$ we can compute explicitly $u_1(x,E) = \sqrt{E}^{-1}\sin\sqrt{E}x$, $m(z) = \sqrt{z}$, and so the evolution operator can be written as

$$e^{-iH_0 t}g(x) = \frac{1}{\pi}\int_{\mathbb{R}} e^{-iEt}\sin(\sqrt{E}x)\hat{g}(E)\,\frac{dE}{\sqrt{E}}, \tag{6.12}$$

where

$$\hat{g}(E) = \int \sin(\sqrt{E}x)g(x)\,dx.$$

For almost every $E > 0$, define the scattering coefficient $\gamma(E) \in \mathbb{C}$ by

$$\gamma(E) = 1/u(0,E) \tag{6.13}$$

where $u(x,E)$ is the unique generalized eigenfunction asymptotic to $\exp(i\xi(x,E))$ as $x \to +\infty$, whose existence was established in Theorem 5.1.

The following proposition connects the formulae of this section with the generalized eigenfunctions analyzed in §5.

Proposition 6.1 (Limiting absorption principle)**.** *Assume that $V \in L^1 + L^p(\mathbb{R})$ for some $1 < p < 2$. For almost every $E \in \mathbb{R}^+$, the generalized eigenfunction $f(x, E+i0) = u_1(x,E)m(E+i0) + u_2(x,E)$ satisfies*

$$f(x, E+i0) = \gamma(E)e^{i\xi(x,E)}(1+o(1)), \tag{6.14}$$

and

$$|\gamma(E)|^2 = \mathrm{Im}\ m(E+i0)/\sqrt{E}. \tag{6.15}$$

Proof of Proposition 6.1. Denote by $u(x,z)$ the generalized eigenfunctions, with spectral parameter z, whose existence was established in Theorem 5.1. By the uniqueness of L^2 solutions of (5.1) for $z \in \mathbb{C}^+$, $f(x,z) = u(x,z)/u(0,z)$. For a.e. E, $f(x,z)$ converges uniformly as a function of $x \in \mathbb{R}^+$ to $u_1(x,E)m(E+i0) + u_2(x,E)$, as z converges to E nontangentially. The uniform convergence ensures that $f(x,E+i0)$ is indeed a generalized eigenfunction associated to the spectral parameter E. At the same time,

$$\frac{u(x,E+i\epsilon)}{u(0,E+i\epsilon)} \to \frac{u(x,E)}{u(0,E)}$$

for almost every E by Theorem 5.1, and therefore for $E > 0$

$$u_1(x,E)m(E+i0) + u_2(x,E) = \frac{u(x,E)}{u(0,E)} = \gamma(E)e^{i\xi(x,E)}(1+o(1))$$

as $x \to \infty$ (there is no absolutely continuous spectrum for $E < 0$). The relation between γ and $m(E+i0)$ follows by comparing the Wronskians of the left and right hand sides (taken with their complex conjugates). □

Now we are going to rewrite the spectral representation in a manner convenient for the scattering theory. Notice that

$$u_1(x,E) = \frac{1}{2i\,\mathrm{Im}\; m(E+i0)}\left(\gamma(E)u(x,E) - \overline{\gamma}(E)\overline{u}(x,E)\right).$$

Introduce

$$\psi(x,\lambda) = \lambda\overline{\gamma}(\lambda^2)u_1(x,\lambda^2) = \frac{1}{2i}\Big(u(x,\lambda^2) - \frac{\overline{\gamma}(\lambda^2)}{\gamma(\lambda^2)}\cdot\overline{u}(x,\lambda^2)\Big). \tag{6.16}$$

where $u(x,E)$ continue to denote the generalized eigenfunctions whose existence was established in Theorem 5.1, now for $E \in \mathbb{R}^+$. Then the results of this subsection may be summarized as follows.

Proposition 6.2. *Suppose that $V \in L^1 + L^p(\mathbb{R}^+)$ for some $1 < p < 2$. Then for the associated selfadjoint Schrödinger operator H_V on $L^2(\mathbb{R}^+)$ with Dirichlet boundary conditions, the spectral projection $P^{\mathrm{ac}}_{(a,b)}$ can be expressed as*

$$P^{\mathrm{ac}}_{(a^2,b^2)}g(x) = \frac{2}{\pi}\int_{[a,b]\cap\mathbb{R}^+} \psi(x,\lambda)\tilde{g}(\lambda)\,d\lambda \tag{6.17}$$

for any $g \in L^2(\mathbb{R}^+)$, where the modified Fourier transform $\tilde{g}$ is defined by

$$\tilde{g}(\lambda) = \int_{\mathbb{R}} \overline{\psi}(x,\lambda)g(x)\,dx. \tag{6.18}$$

Similarly, the associated wave group is

$$e^{-itH_V}g(x) = \frac{2}{\pi}\int_0^\infty e^{-i\lambda t}\psi(x,\lambda)\tilde{g}(\lambda)\,d\lambda. \tag{6.19}$$

The operator

$$U_V f(x) = \sqrt{2/\pi} \int_0^\infty f(\lambda)\psi(x,\lambda)\, d\lambda \tag{6.20}$$

is a unitary surjection from $L^2(\mathbb{R}^+, d\lambda)$ *onto* $\mathcal{H}_{ac}$.

6.2. **Formulae for the case of the whole real line.** Consider $-d^2/dx^2 + V(x)$ as an essentially selfadjoint operator on $L^2(\mathbb{R})$. The asymptotic analysis of the preceding section, in which $x \to +\infty$, works equally well as $x \to -\infty$. Whereas it was necessary in the half-line case to relate asymptotics at $+\infty$ to boundary conditions at $x = 0$, now we must relate asymptotics at $+\infty$ to those at $-\infty$. The absolutely continuous spectrum now has multiplicity two, which complicates some of the formulae.

Let $z \in \mathbb{C}^+$. Solutions $u_1(x,z), u_2(x,z)$ are defined precisely as before, with the same initial conditions at $x = 0$; now they are considered as global solutions on $\mathbb{R}$ rather than merely on $[0,\infty)$. Introduce two solutions $f_\pm(x,z) = u_1(x,z)m_\pm(z) + u_2(x,z)$, so that $f_+ \in L^2(\mathbb{R}^+, dx)$ and $f_- \in L^2(\mathbb{R}^-, dx)$ for each $z \in \mathbb{C}^+$. In terms of these, the resolvent is given by

$$(H_V - z)^{-1} g(x) = \frac{-1}{W[f_+, f_-]} \Big(f_+(x,z) \int_{-\infty}^{x} f_-(y,z) g(y)\, dy + f_-(x,z) \int_x^\infty f_+(y,z) g(y)\, dy \Big). \tag{6.21}$$

Notice that $W[f_+, f_-] = m_+ - m_-$ (since Im $m_-(z) < 0$ in $\mathbb{C}^+$).

The formula for the spectral projection associated to the absolutely continuous spectrum can be computed from the resolvent in a way similar to the half-line case.

In the free case $V = 0$, it simplifies to

$$P^{\rm ac}_{(a^2,b^2)} g(y) = \frac{1}{2\pi} \int_a^b \chi_{\mathbb{R}^+}(\lambda) \left(e^{i\lambda x} \int_{\mathbb{R}} e^{-i\lambda y} g(y)\, dy + e^{-i\lambda x} \int_{\mathbb{R}} e^{i\lambda y} g(y)\, dy \right) d\lambda. \tag{6.22}$$

The evolution operator is given by

$$e^{-iH_0 t} g(x) = \frac{1}{2\pi} \int_0^\infty e^{-i\lambda^2 t} \left(e^{i\lambda x} g_+(\lambda) + e^{-i\lambda x} g_-(\lambda) \right) d\lambda, \tag{6.23}$$

where

$$g_\pm(\lambda) = \int_{\mathbb{R}} e^{\mp i\lambda y} g(y) dy.$$

In the general case, we introduce scattered waves, defining $\psi_\pm(x,\lambda)$ for almost every $\lambda \in \mathbb{R}^+$ to be the unique generalized eigenfunctions with the asymptotic behavior

$$\psi_+(x,\lambda) = \begin{cases} t_1(\lambda) e^{i\phi(x,\lambda)}(1 + o(1)), & x \to +\infty, \\ \left(e^{i\phi(x,\lambda)} + r_1(\lambda) e^{-i\phi(x,\lambda)} \right) + o(1), & x \to -\infty \end{cases} \tag{6.24}$$

$$\psi_-(x,\lambda) = \begin{cases} t_2(\lambda) e^{-i\phi(x,\lambda)}(1 + o(1)), & x \to -\infty, \\ \left(e^{-i\phi(x,\lambda)} + r_2(\lambda) e^{i\phi(x,\lambda)} \right) + o(1), & x \to +\infty, \end{cases} \tag{6.25}$$

where we recall that $\phi(x,\lambda) = \lambda x - (2\lambda)^{-1}\int_0^x V(s)\,ds$. That is, $\psi_\pm$ are defined by the stated asymptotics as $x \to \pm\infty$, respectively; then writing them as linear combinations of the generalized eigenfunctions $\exp(\pm i\xi(x,\lambda^2)) + o(1)$ as $x \to \mp\infty$, respectively, we define r_i, t_i to be the scalar coefficients in those linear combinations. The asymptotics in (6.24), (6.25) can be differentiated, in the sense that $\psi_+(x,\lambda) = i\lambda t_1(\lambda)\exp(i\xi(x,\lambda^2)) + o(1)$ as $x \to +\infty$, with analogous formulae as $x \to -\infty$ and for ψ_-.

As in the half-line case, there exist coefficients $\gamma_\pm(\lambda)$ such that

$$\psi_\pm(x,\lambda) = \gamma_\pm^{-1}(\lambda)(u_1(x,\lambda^2)m_\pm(\lambda^2+i0) + u_2(x,\lambda^2)) \tag{6.26}$$

for almost every λ. Computations of Wronskians, examination of transfer matrices between $\psi_+, \overline{\psi}_+$ and $\psi_-, \overline{\psi}_-$, and exploitation of complex conjugation leads to the following formulae relating the transmission and reflection coefficients t_i, r_i with one another, and with the coefficients $\gamma_\pm, m_\pm$:

$$|r_i|^2 + |t_i|^2 = 1 \text{ for } i = 1,2, \tag{6.27}$$

$$t_1 = t_2, \tag{6.28}$$

$$r_2 = -\frac{t_1}{\overline{t}_1}\overline{r}_1, \tag{6.29}$$

$$|\gamma_\pm|^2|t_1|^2\lambda = \pm\,\mathrm{Im}\; m_\pm(\lambda^2+i0). \tag{6.30}$$

After a computation, we obtain

(6.31)

$$P^{\mathrm{ac}}_{(a^2,b^2)}g(y) = \frac{1}{2\pi}\int_a^b \chi_{\mathbb{R}^+}(\lambda)\left(\psi_+(x,\lambda)\int_{\mathbb{R}}\overline{\psi}_+(y,\lambda)g(y)\,dy + \psi_-(x,\lambda)\int_{\mathbb{R}}\overline{\psi}_-(y,\lambda)g(y)\,dy\right)d\lambda.$$

Therefore, the evolution operator is given by

$$e^{-iH_V t}g(x) = \frac{1}{2\pi}\int_0^\infty e^{-i\lambda^2 t}\Big(\psi_+(x,\lambda)\tilde{g}_+(\lambda) + \psi_-(x,\lambda)\tilde{g}_-(\lambda)\Big)\,d\lambda, \tag{6.32}$$

where the transforms $\tilde{g}_\pm$ are defined by

$$\tilde{g}_\pm(\lambda) = \int_{\mathbb{R}}\overline{\psi}_\pm(y,\lambda)g(y)dy. \tag{6.33}$$

6.3. **Dirac-type operators on the whole real line.** Here we consider a system of differential equations of second order which plays an important role in analysis of the defocusing nonlinear Schrödinger equation (NLS) by the inverse scattering method. This system is given by [29]

$$y' = iz\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}y + \begin{pmatrix}0 & q\\ \overline{q} & 0\end{pmatrix}y. \tag{6.34}$$

We are going to use two alternative equivalent representations of this system. The first is

$$\begin{pmatrix} -i\partial_x & V \\ \overline{V} & i\partial_x \end{pmatrix} y = zy, \tag{6.35}$$

where $V = iq$. System (6.35) is obtained from (6.34) by multiplying both sides by a matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The second is obtained by setting $y = Q^{-1}\phi$ in (6.35), where

$$Q = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

We obtain

$$\begin{pmatrix} 0 & -\partial_x \\ \partial_x & 0 \end{pmatrix} \phi + \begin{pmatrix} \operatorname{Re} V & \operatorname{Im} V \\ \operatorname{Im} V & -\operatorname{Re} V \end{pmatrix} \phi = z\phi. \tag{6.36}$$

The operator on the left hand side of (6.36) is a particular case of a second order Dirac-type operator (the most general case does not require the second diagonal entry to be minus the first; in the most general case there will be a phase shift in the main term of the solution asymptotics, unlike the situation here).

We digress briefly to comment on the possibility of embedded point spectrum for this operator. Let $z = E \in \mathbb{R}$. The equation (6.35) preserves the analogue of the Wronskian for two solutions $f = (f_1, f_2)$ and $g = (g_1, g_2)$, $W[f, g] = i(f_2 g_1 - f_1 g_2)$. Notice that if (g_1, g_2) is a solution, so is $(\overline{g}_2, \overline{g}_1)$. The constancy of the Wronskian for these two solutions implies that $|g_2|^2 - |g_1|^2$ is constant. Consider for simplicity the case where $|g_1| = |g_2|$. Representing $g_1 = R(x)e^{i\theta_1(x)}$, $g_2 = R(x)e^{i\theta_2(x)}$ and using (6.35), we find that $\theta_1 + \theta_2 = c$ is constant, and

$$\begin{aligned} (\log R)' &= -\operatorname{Re} V \sin(2\theta_1 - c) + \operatorname{Im} V \cos(2\theta_1 - c) \\ \theta_1' &= E - \operatorname{Re} V \cos(2\theta_1 - c) - \operatorname{Im} V \sin(2\theta_1 - c). \end{aligned} \tag{6.37}$$

From these Prüfer-like equations one can easily recover certain constructions of embedded eigenvalues available in the Schrödinger case [28, 24, 25]; see [28] for an earlier alternative approach to the Dirac case. In particular, there exist potentials $V = O(|x|^{-1})$ with isolated eigenvalues embedded in the continuous spectrum, and for any function $g(|x|)$ tending to infinity, there exists $|V(x)| \le g(|x|)/|x|$ for which there is a dense set of eigenvalues in $\mathbb{R}^+$.

The form (6.35) will prove useful for studying solutions; the form (6.36) will allow us to set up scattering in a way completely parallel to the setup for Schrödinger operators on the whole axis. First, setting

$$y_1 = \begin{pmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{pmatrix} y$$

we arrive at

$$y_1' = \begin{pmatrix} 0 & -iVe^{-2izx} \\ i\overline{V}e^{2izx} & 0 \end{pmatrix} y_1.$$

Iterating as in the Schrödinger case, we get solutions $\psi_\pm(x,z)$ for any $z \in \mathbb{C}^+$ and almost every $z \in \mathbb{R}$ with the asymptotic behavior

$$\psi_+(x,z) = e^{izx}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1)\right), \quad x \to +\infty \tag{6.38}$$

$$\psi_-(x,z) = e^{-izx}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)\right), \quad x \to -\infty.$$

An analogue of Proposition 6.1 also holds.

Before proceeding, we record a corollary of the existence of such generalized eigenfunctions for real z.

Corollary 6.3. *For any $1 < p < 2$ and any $V \in L^1 + L^p(\mathbb{R})$, the Dirac operator D_V on $L^2(\mathbb{R})$ has nonempty absolutely continuous spectrum. More precisely, an essential support for the ac spectrum is the entire real line $\mathbb{R}$.*

As remarked above, this hypothesis does not preclude the presence of a dense set of eigenvalues embedded in the continuous spectrum.

Now let us return to (6.36). The existence of solution (6.38) and the relation between (6.36) and (6.35) imply existence of solutions $\eta_\pm(x,z)$ of (6.36), for $z \in \mathbb{C}^+ \cup \mathbb{R}$, of the form

$$\eta_+(x,z) = e^{izx}\left(\begin{pmatrix} 1 \\ i \end{pmatrix} + o(1)\right), \quad x \to +\infty \tag{6.39}$$

$$\eta_-(x,z) = e^{-izx}\left(\begin{pmatrix} 1 \\ -i \end{pmatrix} + o(1)\right), \quad x \to -\infty.$$

We proceed very much as in the Schrödinger case. The role of the Wronskian is played by $W[f,g] = f_2 g_1 - f_1 g_2$. As before, we denote by $u_{1,2}(x,z)$ solutions satisfying $u_1(0,z) = (0,1)^T$, $u_2(0,z) = (1,0)^T$. Let $f_\pm(x,z) = u_1(x,z)m_\pm(z) + u_2(x,z)$ be L^2 solutions on $\pm\infty$. Denote the operator on the left hand side of (6.36) by D_V. Its resolvent is given by

(6.40)

$$(D_V - z)^{-1}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \frac{1}{W[f_-, f_+]}\left(f_+ \int_{-\infty}^{x} (f_{-,1}g_1 + f_{-,2}g_2)\,dy + f_- \int_{x}^{\infty} (f_{+,1}g_1 + f_{+,2}g_2)\,dy\right).$$

Let us denote in this section

$$\langle f(x), g(x)\rangle = \int_{\mathbb{R}} (f_1(x)g_1(x) + f_2(x)g_2(x))\,dx;$$

when necessary, the integral is understood in a limiting sense (in L^2 if f is a solution which also depends on E). An analog of (6.9) can be derived using contour integration. The conclusion is that (one version of) the spectral representation of the absolutely continuous part of the spectral measure is given by the map

$$U : L^2(\mathbb{C}^2, dx) \mapsto L^2(\mathbb{C}^2, MdE), \;\; U\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{pmatrix},$$

where

$$\tilde{g}_i(E) = \langle u_i(x,E), g\rangle$$

and

$$M(E) = \operatorname{Im} \begin{pmatrix} \frac{m_+ m_-}{m_- - m_+} & \frac{m_-}{m_- - m_+} \\ \frac{m_-}{m_- - m_+} & \frac{1}{m_- - m_+} \end{pmatrix} (E + i0).$$

Then U is an isometry, $UD_V U^{-1} = E$ (with U^{-1} given by

$$U^{-1}\tilde{g}(x) = \int\limits_{\mathbb{R}} \begin{pmatrix} u_1(x,E) \\ u_2(x,E) \end{pmatrix} M\tilde{g}\, dE,$$

where (u_1, u_2) is a 2×2 matrix with columns u_1, u_2. The unitary evolution of the absolutely continuous part is then given by

$$e^{-iD_V t} g(x) = \int\limits_{\mathbb{R}} e^{-iEt} \begin{pmatrix} u_1(x,E) \\ u_2(x,E) \end{pmatrix} M\tilde{g}(E)\, dE. \tag{6.41}$$

In the free case, the spectral representation can be simplified. In particular, we get

$$e^{-iD_0 t} g(x) = \frac{1}{4} \int e^{-iEt} \left(\eta_{+,0}(x,E) \langle \overline{\eta}_{+,0}, g\rangle + \eta_{-,0}(x,E) \langle \overline{\eta}_{-,0}, g\rangle \right), \tag{6.42}$$

where

$$\eta_{\pm,0}(x,E) = e^{\pm iEx} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}.$$

In the perturbed case, we introduce the scattered waves

$$\eta_+(x,E) = \begin{cases} t_1(E) e^{iEx} \begin{pmatrix} 1 \\ i \end{pmatrix} + o(1), & x \to +\infty, \\ \left(e^{iEx} \begin{pmatrix} 1 \\ i \end{pmatrix} + r_1(E) e^{-iEx} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right) + o(1), & x \to -\infty \end{cases} \tag{6.43}$$

$$\eta_-(x,E) = \begin{cases} t_2(E) e^{-iEx} \begin{pmatrix} 1 \\ -i \end{pmatrix} + o(1), & x \to -\infty, \\ \left(e^{-iEx} \begin{pmatrix} 1 \\ -i \end{pmatrix} + r_2(E) e^{iEx} \begin{pmatrix} 1 \\ i \end{pmatrix} \right) + o(1), & x \to +\infty \end{cases} \tag{6.44}$$

Existence of such solutions follows from (6.38). A computation parallel to the whole axis Schrödinger operator case allows us to rewrite the dynamics (6.41) as

$$e^{-iD_V t} g(x) = \frac{1}{4} \int\limits_{\mathbb{R}} e^{-iEt} \left(\eta_+(x,E) \langle \overline{\eta}_+, g\rangle + \eta_-(x,E) \langle \overline{\eta}_-, g\rangle \right). \tag{6.45}$$

7. Long-time asymptotics

Let $V \in L^1 + L^p(\mathbb{R}^+)$ for some $1 < p < 2$, and consider the selfadjoint Schrödinger operator $H_V = -d^2/dx^2 + V(x)$ on $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition. Let $\mathcal{H} = L^2(\mathbb{R}^+)$ and let $\mathcal{H}_{\rm ac}$ denote the maximal closed subspace of $\mathcal{H}$ on which H_V has purely absolutely continuous spectrum.

Let $\phi(x,\lambda) = \lambda x - (2\lambda)^{-1}\int_0^x V$ be as before, and for almost every $\lambda \in \mathbb{R}^+$ let $u(x,\lambda)$ be the unique generalized eigenfunction associated to the spectral parameter λ^2, satisfying $\Psi(x,\lambda) = \exp(i\phi(x,\lambda)) + o(1)$ as $x \to +\infty$. ¿From the preceding section recall the generalized eigenfunctions $\psi(x,\lambda) = (2i)^{-1}(u - e^{i\omega(\lambda)}\overline{u})$, where $\exp(i\omega(\lambda)) = \overline{\gamma}(\lambda^2)/\gamma(\lambda^2)$. Recall the unitary bijection $U_V : L^2(\mathbb{R}^+, d\lambda) \mapsto \mathcal{H}_{\rm ac}$ defined by $U_V f(x) = \sqrt{2/\pi}\int_0^\infty f(\lambda)\psi(x,\lambda)\,d\lambda$. Finally, recall that $e^{-iH_V t}(U_V f)(x) = \sqrt{2/\pi}\int_0^\infty e^{-i\lambda^2 t} f(\lambda)\psi(x,\lambda)\,d\lambda$.

Fix a martingale structure $\{E_j^m\}$ on $\mathbb{R}^+$ that is adapted to V, in the sense that $\int_{E_j^m}|V|^p = 2^{-m}\int_{\mathbb{R}^+}|V|^p$ for all $m \geq 0$ and all j. For any sufficiently small $\delta > 0$, recall the functional

$$g_\delta(f)(\lambda) = \sum_{m=1}^\infty 2^{m\delta}\Big(\sum_{j=1}^{2^m}\Big|\int_{E_j^m} e^{2i\phi(x,\lambda)} f(x)\,dx\Big|^2 + \Big|\int_{E_j^m} e^{-2i\phi(x,\lambda)} f(x)\,dx\Big|^2\Big)^{1/2}.$$

We have shown that for all sufficiently small δ_0, $g_{\delta_0}(V)(\lambda) < \infty$ for almost every $\lambda \in \mathbb{R}^+$. Fix some $\delta < \delta_0$.

Definition. A compact set $\Lambda \Subset (0,\infty)$ is said to be a *set of uniformity* if $g_\delta(V)$ is a bounded function of $\lambda \in \Lambda$, and if $u(x,\lambda) - e^{i\phi(x,\lambda)} \to 0$ as $x \to +\infty$, uniformly for all $\lambda \in \Lambda$.

Lemma 7.1. *For any $f \in L^2(\mathbb{R}^+)$, for any $R < \infty$,*

$$\|e^{-itH_V} U_V f\|_{L^2([0,R])} \to 0 \quad \text{as } t \to \infty.$$

Proof. Since $\exp(-itH_V)$ is unitary, it suffices to prove this for all f in some dense subspace of $L^2(\mathbb{R}^+)$; we choose the subspace consisting of all f supported on some set of uniformity Λ (which depends on f). The functions $\psi(x,\lambda)$ are uniformly bounded on $[0,\infty)\times\Lambda$, as are their derivatives with respect to x, from which it follows that U_V is a compact mapping from $L^2(\Lambda)$ to $L^2([0,R])$. Thus it suffices to establish weak convergence to zero. Now for any $h \in L^2([0,R])$,

$$\langle e^{-itH_V} U_V f, h\rangle = \int_\Lambda e^{-i\lambda^2 t} f(\lambda) U_V^* h(\lambda)\,d\lambda, \tag{7.1}$$

which converges to zero by the Riemann-Lebesgue lemma, since $U_V^* h \in L^2(\mathbb{R}^+)$. □

Define

$$\psi_0(x,\lambda) = (2i)^{-1}\big(e^{i\phi(x,\lambda)} - e^{i\omega(\lambda)}e^{-i\phi(x,\lambda)}\big) \tag{7.2}$$

$$U_V^\dagger f(x) = \sqrt{2/\pi}\int_0^\infty f(\lambda)\psi_0(x,\lambda)\,d\lambda; \tag{7.3}$$

these are approximations to ψ, U_V, respectively. The next lemma is the key step in showing that only the leading-order approximation ψ_0 to ψ contributes to the long-time asymptotics of the wave group.

Lemma 7.2. *For any set of uniformity Λ and any $R>0$, there exists $C(R,\Lambda)<\infty$ such that for all $f\in L^\infty(\Lambda)$,*

$$\|(U_V - U_V^\dagger)f\|_{L^2([R,\infty))} \le C(R,\Lambda)\|f\|_{L^\infty}, \quad \text{where } C(R,\Lambda)\to 0 \text{ as } R\to\infty. \tag{7.4}$$

Proof. It suffices to prove this with $\psi(x,\lambda)-\psi_0(x,\lambda)$ replaced by

$$\begin{aligned}\Psi(x,\lambda) - e^{i\phi(x,\lambda)} \\ = e^{i\phi(x,\lambda)}\sum_{n=1}^\infty S_{2n}(V,V,\dots,V)(x,\lambda) - e^{-i\phi(x,\lambda)}\sum_{n=0}^\infty S_{2n+1}(V,V,\dots,V)(x,\lambda),\end{aligned} \tag{7.5}$$

for the conclusion for the other terms follows from this by complex conjugation. We argue by duality; let h be an arbitrary function in $L^2([R,\infty))$ of norm 1, and consider

$$\begin{aligned}\int_R^\infty e^{i\phi(x,\lambda)}h(x)S_{2n}(V,V,\dots,V)(x,\lambda) \\ = \iint_{R\le t_0\le t_1\le\cdots\le t_{2n}} e^{i\phi(t_0,\lambda)}h(t_0)\,dt_0\prod_{k=1}^{2n} e^{\pm_k 2i\phi(t_k,\lambda)}V(t_k)\,dt_k.\end{aligned} \tag{7.6}$$

Here $\pm_k$ denotes a plus or minus sign depending on k in any manner; in fact these signs alternate in our expansion, but that is of no importance here. Introduce

$$g_{-\delta}(h)(\lambda) = \sum_{m=1}^\infty 2^{-m\delta}\Big(\sum_{j=1}^{2^m}\Big|\int_{E_j^m} e^{i\phi(x,\lambda)}h(x)\,dx\Big|^2 + \Big|\int_{E_j^m} e^{-i\phi(x,\lambda)}h(x)\,dx\Big|^2\Big)^{1/2}. \tag{7.7}$$

By Proposition 2.1,

$$\Big|\iint_{R\le t_0\le t_1\le\cdots\le t_{2n}} e^{i\phi(t_0,\lambda)}h(t_0)\,dt_0\prod_{k=1}^{2n} e^{\pm_k 2i\phi(t_k,\lambda)}V(t_k)\,dt_k\Big| \le \frac{C^{n+1}}{\sqrt{2n!}}g_{-\delta}(h)(\lambda)g_\delta(V_R)^{2n}(\lambda) \tag{7.8}$$

where $V_R(x) = V(x)\chi_{[R,\infty)}(x)$; there is a corresponding bound for the terms arising from the multilinear expressions S_{2n+1}. We may dominate $\sup_R g_\delta(V_R)(\lambda)^2$ by $C'g_{\delta'}(V)(\lambda)^2$ for any $\delta'>\delta$; to simplify notation we replace δ' again by δ. Consequently by summing over n and comparing with the Taylor expansion for the exponential function,

$$\begin{aligned}\Big|\Big\langle\int_\Lambda\int_R^\infty f(\lambda)\big(u(x,\lambda)-e^{i\phi(x,\lambda)}\big)h(x)\,dx\,d\lambda\Big\rangle\Big| \\ \le \int_\Lambda C g_{-\delta}(h)(\lambda)g_\delta(V_R)(\lambda)e^{Cg_\delta(V)(\lambda)^2}|f(\lambda)|\,d\lambda.\end{aligned} \tag{7.9}$$

Recall that for any $\delta>0$, $\|g_{-\delta}(h)\|_{L^2(\Lambda)}\le C_{\Lambda,\delta}\|h\|_{L^2}\le C_{\Lambda,\delta}<\infty$, for any compact $\Lambda\Subset(0,\infty)$, uniformly over all martingale structures, not necessarily adapted to h.

The factor $\exp(Cg_\delta(V)(\lambda)^2)$ is bounded uniformly for $\lambda \in \Lambda$, by definition of a set of uniformity. Thus it suffices to show that $\|g_\delta(V_R)(\lambda)\|_{L^2(\Lambda,d\lambda)} \to 0$ as $R \to \infty$. Now in the sum (2.3) defining $g_\delta(V)$, the ℓ^2 sum over j for fixed m is $\le C2^{-\varepsilon m}\|V_R\|_{L^2}$ in $L^2(\Lambda)$ for some $\varepsilon > 0$, so it suffices to show that $\int_{E_j^m} e^{2i\phi(x,\lambda)} V_R(\lambda)\,dx \to 0$ in $L^2(\Lambda)$. This holds by Lemma 5.3, since $V_R \to 0$ in $L^1 + L^p$ norm. □

Corollary 7.3. *Let $\rho > 0$. For any $f \in L^2(\mathbb{R}^+, d\lambda)$ supported on $[\rho, \infty)$,*

$$\|e^{-itH_V} U_V f - \sqrt{2/\pi} \int_0^\infty e^{-i\lambda^2 t} f(\lambda)\psi(x,\lambda)\,d\lambda\|_{L^2(\mathbb{R}^+)} \to 0 \text{ as } |t| \to \infty. \tag{7.10}$$

The restriction on the support of f comes about because of the factor of λ^{-1} in the definition of ϕ, which causes difficulties as $\lambda \to 0$.

Proof. It is straightforward to show that $\int_0^\infty e^{-i\lambda^2 t} f(\lambda)\psi(x,\lambda)\,d\lambda \to 0$ in $L^2([0,R])$ for any finite R.

Let $\varepsilon > 0$. Choose a set of uniformity $\Lambda \subset [\rho,\infty)$ and a function $F \in L^\infty(\Lambda)$ such that $\|f - F\|_{L^2(\mathbb{R}^+)} < \varepsilon$. Then $\|e^{-itH_V}(U_V f - U_V F)\|_{L^2(\mathbb{R}^+)} < \varepsilon$ for every t, as well. By the preceding lemma, there exists $R < \infty$ such that $\|e^{-itH_V} U_V F - \sqrt{2/\pi}\int_0^\infty e^{-i\lambda^2 t} F(\lambda)\psi_0(x,\lambda)\,d\lambda\|_{L^2([R,\infty))} < \varepsilon$ for all $t \in \mathbb{R}$. Fixing such an R, there exists $T < \infty$ such that both $e^{-itH_V} U_V F$ and $\sqrt{2/\pi}\int_0^\infty e^{-i\lambda^2 t} F(E)\psi_0(x,\lambda)\,d\lambda$ have $L^2([0,R])$ norms $< \varepsilon$, for all $|t| \ge T$. Thus

$$\|e^{-itH_V} U_V f - \sqrt{2/\pi} \int_0^\infty e^{-i\lambda^2 t} F(\lambda)\psi_0(x,\lambda)\,d\lambda\|_{L^2(\mathbb{R}^+)} < 4\varepsilon \text{ for all } |t| \ge T. \tag{7.11}$$

Finally, note that $\|\int_\rho^\infty g(\lambda)u_0(x,\lambda)\,d\lambda\|_{L^2(\mathbb{R}^+)} \le C_\rho \|g\|_{L^2([\rho,\infty))}$ for all g, where $C_\rho < \infty$ for all $\rho > 0$; this follows from the proof of Lemma 5.3. Applying this to $g = f - F$ allows us to replace F again by f in the preceding inequality, at the expense of replacing 4ε by $C_\rho\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this completes the proof. □

8. A time-dependent phase correction

The goal of this section is to convert the leading-order asymptotics $\exp(\pm i\lambda x \mp i(2\lambda)^{-1}\int_0^x V)$ to a more standard form, $\exp(\pm i\lambda x \mp i(2\lambda)^{-1}\int_0^{2\lambda|t|} V)$.

Lemma 8.1. *Let $V \in L^1 + L^2$. For any $f \in L^2$ supported in a compact subinterval of $(0,\infty)$,*

$$\Big\| \int_0^\infty e^{-i\lambda^2 t + i\lambda x - i(2\lambda)^{-1}\int_0^x V} f(\lambda)\,d\lambda \Big\|_{L^2(\mathbb{R}^+)} + \Big\| \int_0^\infty e^{-i\lambda^2 t + i\lambda x - i(2\lambda)^{-1}\int_0^{2\lambda|t|} V} f(\lambda)\,d\lambda \Big\|_{L^2(\mathbb{R}^+)} \to 0 \text{ as } t \to -\infty. \tag{8.1}$$

An analogous statement holds as $t \to +\infty$, and follows by taking complex conjugates.

Proof. Since the mapping $f \mapsto \int_0^\infty e^{-i\lambda^2 t + i\lambda x} e^{-i(2\lambda)^{-1}\int_0^x V} f(\lambda)\,d\lambda$ is bounded from $L^2([\rho,\infty))$ to $L^2(\mathbb{R}^+)$ for every $\rho > 0$, and since the variant defined by replacing $\int_0^x V$ by $\int_0^{2\lambda t} V$ is unitary, it suffices to prove this merely for $f \in C_0^\infty$, which we assume henceforth.

Let $\Lambda \Subset \mathbb{R}^+$ be the support of f. By integrating by parts once with respect to λ, integrating $\exp(i[-\lambda^2 t + \lambda x])$ and differentiating the rest, and by moving absolute value signs inside the integral, we obtain a pointwise in (x,t) bound

$$|(8.1)| \le C \int_\Lambda |x-2\lambda t|^{-1} \big(1 + |\textstyle\int_0^x V| + |\int_0^{2\lambda|t|} V| + |tV(2\lambda t)|\big)\, d\lambda \tag{8.2}$$
$$\le C(|x|+|t|)^{-1}\big(\big(1 + |\textstyle\int_0^x V| + |\int_0^{C|t|} V|\big)$$

for the integrands in (8.1). For $t \le -1$, $|x-2\lambda t|^{-1} \sim (x+|t|)^{-1}$, which tends to zero in $L^2(\mathbb{R}^+, dx)$. Thus the first term behaves as desired. The second and third terms do also, if $V \in L^1$. For $V \in L^2$, $x^{-1}\int_0^x V \in L^2$ as well, so the term $(x+|t|)^{-1}\int_0^x V$ is an L^2 function times $x/(x+|t|)$, hence $\to 0$ in L^2 norm. Lastly, $\int_0^{C|t|} V = o(|t|^{1/2})$, while $|t|^{1/2}/(x+|t|)$ is $O(1)$ in $L^2(\mathbb{R}^+)$. □

Lemma 8.2. *Let $V \in L^1 + L^2$. For any $f \in L^2$ supported in a compact subinterval of $(0,\infty)$,*

$$\Big\| \int_0^\infty e^{-i\lambda^2 t + i\lambda x} \Big[e^{-i(2\lambda)^{-1}\int_0^x V} - e^{-i(2\lambda)^{-1}\int_0^{2\lambda|t|} V} \Big] f(\lambda)\, d\lambda \Big\|_{L^2(\mathbb{R}^+)} \to 0 \text{ as } t \to +\infty. \tag{8.3}$$

Proof. As in the preceding lemma, it suffices to prove this for $f \in C_0^\infty(\mathbb{R}^+)$. Fix a cutoff function $\eta \in C_0^\infty(\mathbb{R})$ supported on $(-2,2)$ and $\equiv 1$ on $[-1,1]$, with $0 \le \eta \le 1$. Let ϵ be a strictly positive function, such that $\epsilon(t) \to 0$ as $t \to +\infty$, at a rate to be specified below. Set

$$\eta(\lambda, t, x) = \eta(\epsilon(t) t^{-1/2}(x - 2\lambda t)). \tag{8.4}$$

Consider the function of (x,t)

$$\int_0^\infty e^{-i\lambda^2 t + i\lambda x} \Big[e^{-i(2\lambda)^{-1}\int_0^x V} - e^{-i(2\lambda)^{-1}\int_0^{2\lambda t} V} \Big] f(\lambda)\, \eta(\lambda,t,x)\, d\lambda\ . \tag{8.5}$$

We have

$$\begin{aligned} |(8.5)| &\le C \int \int_{|y-2\lambda t| \le 2\epsilon(t)^{-1} t^{1/2}} |V(y)|\, dy \chi_{|x-2\lambda t| \le 2\epsilon(t)^{-1}t^{1/2}}\, d\lambda \\ &\le C \int \int_{|y-x| \le 4\epsilon(t)^{-1} t^{1/2}} |V(y)|\, dy \chi_{|x-2\lambda t| \le 2\epsilon(t)^{-1}t^{1/2}}\, d\lambda \\ &\le C\epsilon(t)^{-1} t^{-1/2} \int_{|y-x| \le 4\epsilon(t)^{-1}t^{1/2}} |V(y)|\, dy \\ &\le C\epsilon(t)^{-2} MV(x), \end{aligned} \tag{8.6}$$

where M is the maximal function of Hardy and Littlewood. Observe that the restriction $\eta(\lambda,t,x) \ne 0$ for some $\lambda \in \Lambda$ implies that $ct \le x \le Ct$ for some $c, C \in \mathbb{R}^+$ depending only on Λ, provided $\epsilon(t) \ll t^{1/2}$, and moreover that V may be replaced by its restriction to such an interval. If $V \in L^2$ then the L^2 norm of M applied to the restriction of V to such an interval tends to zero as $t \to \infty$, and by choosing $\epsilon(t)$ to tend to zero sufficiently slowly we find that (8.5) $\to 0$ in L^2. For $V \in L^1$ we have the

pointwise bound $C\epsilon(t)^{-1}t^{-1/2}\int_{ct}^{Ct}|V|$, which is $o(1)\cdot\epsilon(t)^{-1}$ in $L^2(x\sim t)$; this tends to zero provided $\epsilon(t)$ does so sufficiently slowly. In the general case $V\in L^1+L^2$, we decompose (8.5) into two parts, and estimate them separately.

There remains the contribution of $1-\eta(\lambda,t,x)$:

$$\int_0^\infty e^{-i\lambda^2 t+i\lambda x}f(\lambda)\Big(e^{-i(2\lambda)^{-1}\int_0^x V}-e^{-i(2\lambda)^{-1}\int_0^{2\lambda t}V}\Big)(1-\eta)(\lambda,t,x)\,d\lambda. \tag{8.7}$$

We would like to apply to it the method of stationary phase. However, the phase function $-t\lambda^2+\lambda x-(2\lambda)^{-1}\int_0^{2\lambda t}V$ is not well behaved; its partial derivative with respect to λ is $x-2\lambda t+(2\lambda)^{-2}(\int_0^{2\lambda t}V)-(2\lambda)^{-1}2tV(2\lambda t)$, and the final term is not well under control unless one assumes $V(x)=O(|x|^{-1/2})$. Instead, we integrate by parts, integrating $\exp(i[\lambda x-\lambda^2 t])$ and differentiating the rest, to obtain

(8.8)
$$i\int_0^\infty e^{-i\lambda^2 t+i\lambda x}f(\lambda)\frac{\partial}{\partial\lambda}\Big[(x-2\lambda t)^{-1}\Big(e^{-i(2\lambda)^{-1}\int_0^x V}-e^{-i(2\lambda)^{-1}\int_0^{2\lambda t}V}\Big)(1-\eta)(\lambda,t,x)\Big]\,d\lambda.$$

When the derivative is expanded according to Leibniz's rule, various terms result. The main terms are those in which $\partial/\partial\lambda$ acts on either of the two exponentials $\exp(i(2\lambda)^{-1}(\int_0^* V))$, and we discuss these first. One such term is a constant multiple of

$$\begin{aligned}\int_0^\infty e^{-i\lambda^2 t+i\lambda x-i(2\lambda)^{-1}\int_0^{2\lambda t}V}f(\lambda)(x-2\lambda t)^{-1}(1-\eta)(\lambda,t,x)\lambda^{-1}tV(2\lambda t)\,d\lambda\\ =c\int_0^\infty e^{-i(y^2/4t)+i(xy/2t)-ity^{-1}\int_0^y V}\tilde f(y/2t)(x-y)^{-1}(1-\eta)(\lambda,t,x)v(y)\,dy\end{aligned} \tag{8.9}$$

where we have substituted $y=2\lambda t$ and written $\tilde f(\lambda)=\lambda^{-1}f(\lambda)\in C_0^\infty$, and where $v=V$ (for the present moment only). The integral operators with kernels $\exp(iAxy)(x-y)^{-1}\eta(\delta(x-y))$ are bounded on $L^2(\mathbb{R})$, uniformly in $A,\delta\in\mathbb{R}^+$. Thus the $L^2(dx)$ norm of this last expression is $O(\|v\|_{L^2})$. Moreover, since f is supported where $\lambda\geq\rho>0$, only the restriction of v to $[2\rho t,\infty)$ comes into play, so we obtain a bound of $C\|v\|_{L^2([2\rho t,\infty))}$. This holds uniformly in all real-valued functions V appearing in the exponent.

There is also an easy alternative bound $O(t^{-1/4}\|v\|_{L^1})$, obtained directly by inserting absolute values inside the integral. Thus the general case $V\in L^1+L^2$ may be treated by decomposing v as a sum, and estimating the two terms separately.

Another term arising from (8.8) differs only in that $tV(2\lambda t)$ is replaced by $c\lambda^{-2}\int_0^{2\lambda t}V$; in terms of the new variable y, $v(y)$ is replaced by $t^{-1}\int_0^y v$. Since λ ranges over a compact interval Λ, y ranges over an interval $[ct,Ct]$ where $0<c<C<\infty$. Therefore $t^{-1}\int_0^y v$ is majorized in $L^q(dy)$ by $\|v\|_{L^q(d\lambda)}$, for every $1\leq q\leq\infty$. Thus the same reasoning applies to this term.

In the last of the main terms, the derivative falls on $\exp(-i(2\lambda)^{-1}\int_0^x V)$, and we obtain

$$c\big(\textstyle\int_0^x V\big)\int e^{-i\lambda^2 t+i\lambda x-i(2\lambda)^{-1}\int_0^x V}(x-2\lambda t)^{-1}f(\lambda)(1-\eta)(\lambda,t,x)\,d\lambda. \tag{8.10}$$

If $x \geq Ct$ where C is a sufficiently large constant, depending on Λ, this is majorized by $Cx^{-1}\big|\int_0^x V\big|$, which is $o(\|V\|_{L^2})$ in $L^2(|x| \geq Ct)$ as $t \to +\infty$. However, for $x \lesssim t$ and for $V \in L^2$, simple estimates on this quantity seem to miss the desired conclusion by a factor of $\log(t)$, so we integrate by parts again and move absolute values inside the integral to obtain an upper bound

$$(8.11)\quad |(8.10)| \leq C\big|\textstyle\int_0^x V\big| \cdot \displaystyle\int_\Lambda |x-2\lambda t|^{-2} \Big(\frac{t|1-\eta|}{|x-2\lambda t|} + |1-\eta| + C\epsilon t^{1/2}|\eta'(\epsilon t^{-1/2}(x-2\lambda t))| + \big|(1-\eta)\textstyle\int_0^x V\big|\Big)\, d\lambda,$$

where $(1-\eta) \equiv (1-\eta)(\lambda,t,x)$ and $\epsilon \equiv \epsilon(t)$. Bearing in mind that now $x \leq Ct$, one term is

$$(8.12)\quad \leq C\big|\textstyle\int_0^x V\big|^2 \displaystyle\int_{|\lambda-(x/2t)|\geq \epsilon^{-1}t^{-1/2}} t^{-2}|\lambda-(x/2t)|^{-2}\, d\lambda \leq C\big|\textstyle\int_0^x V\big|^2 t^{-3/2}\epsilon \leq Ct^{-3/2}\epsilon x^{1/2}\big|\textstyle\int_0^x V\big| \leq C\epsilon(t)x^{-1}\big|\textstyle\int_0^x V\big|,$$

which is $o(\|V\|_{L^2})$ in L^2 norm. Another term is

$$(8.13)\qquad \leq C\big|\textstyle\int_0^x V\big| \displaystyle\int_{|\lambda-(x/2t)|\geq \epsilon^{-1}t^{-1/2}} t^{-2}|\lambda-(x/2t)|^{-3}\, d\lambda \leq C\Big|\int_0^x V\Big|\epsilon^2 t^{-1},$$

which again is $o(\|V\|_{L^2})$ in L^2 norm. The two remaining terms are both majorized by the discussion of this last term, thus completing the analysis of the case where $\partial/\partial\lambda$ acts on $\exp(-i(2\lambda)^{-1}\int_0^x V)$, if $V \in L^2$. In the general case $V \in L^1 + L^2$, the contribution of the L^1 part is better behaved and the details are left to the reader.

Another type of term arises when $\partial/\partial\lambda$ acts in (8.8) on $(1-\eta)(\lambda,t,x)$, producing $c\epsilon(t)t^{1/2}\cdot\eta'(\epsilon(t)t^{-1/2}(x-2\lambda t)$. The factor $\eta'(\epsilon(t)t^{-1/2}(x-2\lambda t)$ is supported where $|x-2\lambda t| \leq 2\epsilon(t)^{-1}t^{1/2}$, and $\epsilon(t)t^{1/2}/(x-2\lambda t) = O(1)$ in this region. Therefore the analysis given above for the contribution of $\eta(\lambda,t,x)$ applies equally well to this term.

When the derivative falls on $f(\lambda)$, we have gained a factor of $(x-2\lambda t)^{-1}$, and have an upper bound $C\int_\Lambda |x-2\lambda t|^{-1}\chi_{|x-2\lambda t|\geq \epsilon(t)^{-1}t^{1/2}}\, d\lambda$. For each $\lambda \in \Lambda$, the integrand is $O(t^{-1/4}\epsilon(t)^{1/2})$ in $L^2(\mathbb{R}^+)$.

There remains the term in which the derivative $\partial/\partial\lambda$ falls on $(x-2\lambda t)^{-1}$; this term is a constant multiple of

$$(8.14)\qquad \int_0^\infty e^{-i\lambda^2 t + i\lambda x}\Big(e^{-i(2\lambda)^{-1}\int_0^{2\lambda t} V} - e^{-i(2\lambda)^{-1}\int_0^x V}\Big) f(\lambda)\frac{t}{(x-2\lambda t)^2}(1-\eta)(\lambda,t,x)\, d\lambda$$

and hence is majorized by $C\int_\Lambda t|x-2\lambda t|^{-2}\chi_{|\lambda-(x/2t)|\geq\epsilon(t)^{-1}t^{-1/2}}\, d\lambda \leq C\epsilon(t)t^{-1/2}$, which gives the desired L^2 bound for $x \leq C_0 t$, for any fixed C_0. Choosing C_0 to be sufficiently large, we can more simply majorize $|x-2\lambda t|^{-2}$ for $x \geq C_0 t$ by Cx^{-2} in the integrand, obtaining the pointwise bound Ct/x^2 for the integral, giving a contribution $\leq Ct^{-1/2}$ to the $L^2(dx)$ norm. □

9. Modified wave operators

9.1. **The half-line case.** We discuss modified wave operators for Schrödinger operators on the half line.

Recall from the introduction the modified wave operators (1.4)

$$\Omega_{\pm}^{m} f = \lim_{t \to \mp\infty} e^{itH_V} e^{-itH_0 \pm iW(H_0, \mp t)} f$$

for all $f \in L^2(\mathbb{R}^+)$, where existence of the limit remains to be established. Here W is given by (1.3):

$$W(\lambda, t) = -(2\lambda)^{-1} \int_0^{2\lambda t} V(s)\, dx.$$

Motivating this definition is the heuristic that a wave packet with frequency $\approx \lambda$ should propagate with velocity $\approx 2\lambda$, so that for large t, $\int_0^x V$ should behave like $\int_0^{2\lambda t} V$.

Proof of Theorem 1.1. We write $A(t) \sim B(t)$ to mean that $\|A(t) - B(t)\|_{L^2(\mathbb{R}^+)} \to 0$ as $t \to +\infty$, or as $t \to -\infty$, depending on which limit is presently under discussion. Given $g \in L^2(\mathbb{R}^+)$, write

$$g(x) = \pi^{-1} \int_0^\infty \hat{g}(\lambda)[e^{i\lambda x} - e^{-i\lambda x}]\, d\lambda, \tag{9.1}$$

where this defines the modified Fourier transform $\hat{g}$, so that $\|\hat{g}\|_{L^2(\mathbb{R}^+)} = \sqrt{\pi}\|g\|_{L^2(\mathbb{R}^+)}$. Then

$$e^{-itH_0} g(x) = \pi^{-1} \int_0^\infty \hat{g}(\lambda) e^{-i\lambda^2 t}[e^{i\lambda x} - e^{-i\lambda x}]\, d\lambda, \tag{9.2}$$

and the modified evolution is

(9.3)

$$e^{-itH_0 - iW(H_0,t)} g(x) = \pi^{-1} \int_0^\infty \hat{g}(\lambda) e^{-i\lambda^2 t} \left[e^{i\lambda x - i(2\lambda)^{-1} \int_0^{2\lambda t} V} - e^{-i\lambda x - i(2\lambda)^{-1} \int_0^{2\lambda t} V} \right] d\lambda.$$

Define $\tilde{g} \in \mathcal{H}_{\mathrm{ac}}$ by

$$\tilde{g}(x) = \pi^{-1} \int_0^\infty \hat{g}(\lambda) \psi(x, \lambda)\, d\lambda. \tag{9.4}$$

The map $g \mapsto \tilde{g}$ is a unitary bijection from $L^2(\mathbb{R}^+)$ to $\mathcal{H}_{\mathrm{ac}}$. We aim to prove that $e^{-itH_0 - W(H_0,t)} g \sim e^{-itH_V} \tilde{g}$, as $t \to +\infty$, for any $g \in L^2(\mathbb{R}^+)$. By unitarity of the evolutions, this implies the existence of the modified wave operator Ω_-^m, and likewise that it is an isometric bijection from $L^2(\mathbb{R}^+)$ to $\mathcal{H}_{\mathrm{ac}}$. Moreover, by unitarity, it suffices to prove this under the assumption that $\hat{g} \in C_0^\infty(\mathbb{R}^+)$, which we assume henceforth.

Now

$$\begin{aligned} e^{-itH_0 - iW(H_0,t)} g(x) &= \pi^{-1} \int_0^\infty \hat{g}(\lambda) e^{-i\lambda^2 t} \left[e^{i\lambda x - i(2\lambda)^{-1} \int_0^{2\lambda t} V} - e^{-i\lambda x - i(2\lambda)^{-1} \int_0^{2\lambda t} V} \right] d\lambda \\ &\sim \pi^{-1} \int_0^\infty \hat{g}(\lambda) e^{-i\lambda^2 t} e^{i\lambda x - i(2\lambda)^{-1} \int_0^{2\lambda t} V}\, d\lambda \end{aligned}$$

by taking complex conjugates in Lemma 8.1. On the other hand,

$$\begin{aligned} e^{-itH_V}\tilde g &= \pi^{-1}\int_0^\infty \hat g(\lambda)e^{-i\lambda^2 t}\psi(x,\lambda)\,d\lambda \\ &= \pi^{-1}\int_0^\infty \hat g(\lambda)e^{-i\lambda^2 t}\big(u(x,\lambda^2)-\frac{\bar\gamma(\lambda^2)}{\gamma(\lambda^2)}\overline{u(x,\lambda^2)}\big)\,d\lambda \\ &\sim \pi^{-1}\int_0^\infty \hat g(\lambda)e^{-i\lambda^2 t}\big(e^{i\lambda x - i(2\lambda)^{-1}\int_0^x V}-\frac{\bar\gamma(\lambda^2)}{\gamma(\lambda^2)}e^{-i\lambda x+i(2\lambda)^{-1}\int_0^x V}\big)\,d\lambda \\ &\sim \pi^{-1}\int_0^\infty \hat g(\lambda)e^{-i\lambda^2 t}e^{i\lambda x - i(2\lambda)^{-1}\int_0^x V}\,d\lambda. \\ &\sim \pi^{-1}\int_0^\infty \hat g(\lambda)e^{-i\lambda^2 t}e^{i\lambda x - i(2\lambda)^{-1}\int_0^{2\lambda t} V}\,d\lambda. \end{aligned}$$

by first Corollary 7.3, then Lemma 8.1, and then Lemma 8.2. Thus we have the asymptotic relation $e^{-itH_0-iW(H_0,t)}g\sim e^{-itH_V}\tilde g$, as $t\to+\infty$, so $\Omega_-^m=U_V^{-1}\circ U_0$.

The analysis as $t\to-\infty$ is the same, except for the appearance of the unimodular factor $\overline{\gamma(\lambda^2)}/\gamma(\lambda^2)$. The conclusion is therefore that $\Omega_+^m=U_V^{-1}\circ[\overline{\gamma(\lambda^2)}/\gamma(\lambda^2)]\circ U_0$, where the inner factor denotes the operator defined on $L^2(\mathbb{R}^+,d\lambda)$ by multiplication by this function of λ.

Composing, we find that $S^m=(\Omega_-^m)^{-1}\Omega_+^m=U_0^{-1}\circ[\overline{\gamma(\lambda^2)}/\gamma(\lambda^2)]\circ U_0$. □

Proof of Theorem 1.2. Suppose now that the improper integral $\int_0^\infty V(y)\,dy$ exists. The asymptotic behavior of V enters both into the definition of the modified evolution $e^{-itH_0-W(H_0,t)}$, and into the definition of the scattering coefficients $\gamma(E)$. To sort this out, we change the definition of the phase $\xi(x,\lambda^2)$ to λx. We correspondingly modify the normalization of the generalized eigenfunctions $u(x,\lambda^2)$ to $u(x,\lambda^2)\sim e^{i\lambda x}$. The scattering coefficient $\gamma(\lambda^2)$ is now defined, for almost every $\lambda>0$, by the relation (6.13) $\gamma(\lambda^2)=1/\psi(0,\lambda)$. Writing $\gamma(\lambda^2)=|\gamma(\lambda^2)|e^{i\arg\gamma(\lambda^2)}$, we change the definition of $\psi(x,\lambda)$ to

$$\psi(x,\lambda)=e^{i\arg\gamma(\lambda^2)}u(x,\lambda^2)-e^{-i\arg\gamma(\lambda^2)}\overline{u(x,\lambda^2)}. \tag{9.5}$$

In the formal expressions for the spectral projectors and wave group, ψ is now replaced by this new ψ. Retracing the above analysis, we find that the wave operators exist, and

$$\begin{aligned} \Omega_- &= U_V^{-1}\circ e^{-i\arg\gamma(\lambda^2)}\circ U_0, \\ \Omega_+ &= U_V^{-1}\circ e^{i\arg\gamma(\lambda^2)}\circ U_0, \\ S &= {U_0}^{-1}\circ e^{2i\arg\gamma(\lambda^2)}\circ U_0. \end{aligned} \tag{9.6}$$

□

Remark. The exact expressions for the modified wave operators $\Omega_\pm^m$ depend on a choice of normalization of the solutions ψ; see below. These solutions can be modified by factors $e^{i\kappa(\lambda)}$, leading to different U_V and hence different looking expressions for the wave operators, like in (9.6) and in the proof of Theorem 1.1. However, the scattering matrix $S(\lambda)$ is invariant under choice of such normalization.

9.2. **The whole-line case.** In the full-axis Schrödinger case, the results are similar. One defines modified wave operators by

$$\Omega_{\pm}^{m} f = \lim_{t \to \mp\infty} e^{itH_V} e^{-iW_a(-i\partial_x, t)} f, \tag{9.7}$$

where ∂_x is the operator of one differentiation in x, and

$$W_a(\lambda, t) = \lambda^2 + \frac{1}{2\lambda} \int_0^{2\lambda t} V(s)\, ds. \tag{9.8}$$

The modified free evolution operator can be written as (compare with (6.23))

$$e^{-iW_a(-i\partial_x, t)} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda^2 t + i\lambda x - \frac{i}{2\lambda}\int_0^{2\lambda t} V(s)\, ds} \hat{f}(\lambda)\, d\lambda, \tag{9.9}$$

where $\hat{f}(\lambda) = \int \exp(-i\lambda x) f(x)\, dx$ is the Fourier transform of f. Recall the representation (6.32) for the perturbed evolution (we set here $E = \lambda^2$)

$$e^{-iH_V t} f(x) = \frac{1}{2\pi} \int_0^{\infty} e^{-i\lambda^2 t} (\psi_+(x,\lambda)\tilde{f}_+(\lambda) + \psi_-(x,\lambda)\tilde{f}_-(\lambda))\, d\lambda. \tag{9.10}$$

Denote by U_V the operator $f \mapsto (f_+, f_-)$, where $f_\pm(\lambda) = \int_{\mathbb{R}} \overline{\psi}_\pm(x,\lambda) f(x)\, dx$. Denote by $\mathcal{H}_{\mathrm{ac}}$ the maximal closed subspace of $\mathcal{H} = L^2(\mathbb{R})$ on which H_V has purely absolutely continuous spectrum.

Theorem 9.1. *Let $V \in L^1 + L^p(\mathbb{R})$ for some $1 < p < 2$. Then for every $f \in L^2$, the limits in (9.7) exist in $L^2(\mathbb{R})$ norm as $t \to \mp\infty$. The modified wave operators $\Omega_\pm^m$ thus defined are surjective and unitary from $L^2(\mathbb{R})$ to $\mathcal{H}_{ac}$. One has*

$$\begin{aligned} \Omega_+^m &= U_V^{-1} U_0, \\ \Omega_-^m &= U_V^{-1} S(\lambda)^{-1} U_0, \\ S^m &= {U_0}^{-1} S(\lambda) U_0 \end{aligned} \tag{9.11}$$

where $S(\lambda)$ denotes multiplication by the scattering matrix

$$S(\lambda) = \begin{pmatrix} t_1(\lambda) & -\overline{r}_1(\lambda)\frac{t_1(\lambda)}{\overline{t_1(\lambda)}} \\ r_1(\lambda) & t_1(\lambda) \end{pmatrix}. \tag{9.12}$$

As in the half-line case, we could make the wave operators look more symmetric by modifying the definition of U_V. Let $A = \sqrt{S}$ be any matrix square root of the unitary operator S. Then if we were to define $\tilde{U}_V = AU_V$, we would obtain $\Omega_+^m = {\tilde{U}_V}^{-1} A^* U_0$, and $\Omega_-^m = {\tilde{U}_V}^{-1} A U_0$.

We sketch the asymptotic analysis of $e^{-itH_V} g$, as $|t| \to \infty$, for arbitrary $g \in \mathcal{H}_{\mathrm{ac}}$. We proceed formally; all steps are justified as in the preceding subsection. Write $g = (2\pi)^{-1} \int_0^\infty \big[\hat{g}_+(\lambda)\psi_+(x,\lambda) + \hat{g}_-(\lambda)\psi_-(x,\lambda)\big] e^{-i\lambda^2 t}\, d\lambda$ with $2\pi\|g\|_{L^2}^2 = \|\hat{g}_+\|_{L^2(\mathbb{R}^+)}^2 + \|\hat{g}_-\|_{L^2(\mathbb{R}^+)}^2$. Define

$$\begin{aligned} \Phi_+(\lambda, x, t) &= \quad \lambda x - (2\lambda)^{-1} \textstyle\int_0^{2\lambda t} V, \\ \Phi_-(\lambda, x, t) &= -\lambda x + (2\lambda)^{-1} \textstyle\int_0^{-2\lambda t} V. \end{aligned} \tag{9.13}$$

Formally,

$$e^{-itH_V}\psi_+(x,t) \sim \begin{cases} \big[t_1(\lambda)e^{i\Phi_+(\lambda,x,t)} + r_1(\lambda)e^{i\Phi_-(\lambda,x,t)}\big]e^{-i\lambda^2 t} & \text{as } t\to+\infty,\\ e^{i\Phi_+(\lambda,x,t)}e^{-i\lambda^2 t} & \text{as } t\to-\infty, \end{cases} \tag{9.14}$$

and

$$e^{-itH_V}\psi_-(x,t) \sim \begin{cases} \big[t_2(\lambda)e^{i\Phi_-(\lambda,x,t)} + r_2(\lambda)e^{i\Phi_+(\lambda,x,t)}\big]e^{-i\lambda^2 t} & \text{as } t\to+\infty,\\ e^{i\Phi_-(\lambda,x,t)}e^{-i\lambda^2 t} & \text{as } t\to-\infty, \end{cases} \tag{9.15}$$

Therefore as $t\to+\infty$,

$$\begin{aligned} e^{-itH_V}(2\pi)^{-1}\int_0^\infty \big[\hat g_+(\lambda)\psi_+(x,\lambda) + \hat g_-(\lambda)\psi_-(x,\lambda)\big]e^{-i\lambda^2 t}\,d\lambda \\ \sim (2\pi)^{-1}\int_0^\infty \Big(\hat g_+(\lambda)\big[t_1(\lambda)e^{i\Phi_+(\lambda,x,t)} + r_1(\lambda)e^{i\Phi_-(\lambda,x,t)}\big] \\ + \hat g_-(\lambda)\big[t_2(\lambda)e^{i\Phi_-(\lambda,x,t)} + r_2(\lambda)e^{i\Phi_+(\lambda,x,t)}\big]\Big)e^{-i\lambda^2 t}\,d\lambda. \end{aligned} \tag{9.16}$$

As $t\to-\infty$, we have instead the asymptotics

$$\sim (2\pi)^{-1}\int_0^\infty \big(\hat g_+(\lambda)e^{i\Phi_+(\lambda,x,t)} + \hat g_-(\lambda)e^{i\Phi_-(\lambda,x,t)}\big)e^{-i\lambda^2 t}\,d\lambda. \tag{9.17}$$

Here "$\sim$" signifies asymptotic equality in $L^2(\mathbb{R},dx)$. Thus for any function $f(\lambda) = (f_+(\lambda), f_-(\lambda))^T$ taking values in $L^2(\mathbb{R}^+,\mathbb{C}^2)$,

(9.18)

$$e^{-itH_V}{U_V}^{-1}\begin{pmatrix} f_+\\ f_-\end{pmatrix} \sim \begin{cases} e^{-iW_a(-i\partial_x,t)}{U_0}^{-1}\begin{pmatrix} t_1(\lambda) & r_2(\lambda)\\ r_1(\lambda) & t_2(\lambda)\end{pmatrix}\begin{pmatrix} f_+\\ f_-\end{pmatrix} & \text{as } t\to+\infty,\\ e^{-iW_a(-i\partial_x,t)}{U_0}^{-1}\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} f_+\\ f_-\end{pmatrix} & \text{as } t\to-\infty. \end{cases}$$

Inverting this and exploiting the identities relating the various scattering coefficients to simplify the resulting formula, we obtain

(9.19)

$$e^{-iW_a(-i\partial_x,t)}{U_0}^{-1}\begin{pmatrix} f_+\\ f_-\end{pmatrix} \sim \begin{cases} e^{-itH_V}{U_V}^{-1}\begin{pmatrix} \bar t_1 & \bar r_1\\ -(r_1\bar t_1/t_1) & \bar t_1\end{pmatrix}\begin{pmatrix} f_+\\ f_-\end{pmatrix} & \text{as } t\to+\infty,\\ e^{-itH_V}{U_V}^{-1}\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} f_+\\ f_-\end{pmatrix} & \text{as } t\to-\infty. \end{cases}$$

with asymptotic equality holding in $L^2(\mathbb{R}^+,\mathbb{C}^2,d\lambda)$ norm. The conclusions of the theorem follow directly.

As in the half-line case, we obtain a cleaner conclusion under the additional hypothesis that the improper integrals $\int_0^{\pm\infty} V(x)\,dx$ exist. Modify the definitions of the

scattering coefficients by redefining

$$\psi_+(x,\lambda) = \begin{cases} t_1(\lambda)e^{i\lambda x} + o(1), & x \to +\infty, \\ \left(e^{i\lambda x} + r_1(\lambda)e^{-i\lambda x}\right) + o(1), & x \to -\infty \end{cases} \tag{9.20}$$

$$\psi_-(x,\lambda) = \begin{cases} t_2(\lambda)e^{-i\lambda x} + o(1), & x \to -\infty, \\ \left(e^{-i\lambda x} + r_2(\lambda)e^{i\lambda x}\right) + o(1), & x \to +\infty. \end{cases} \tag{9.21}$$

Theorem 9.2. *Let $V \in L^1 + L^p(\mathbb{R})$ for some $1 < p < 2$, and suppose that both the improper integrals $\int_0^{\pm\infty} V(x)\,dx$ exist. Then for every $f \in L^2(\mathbb{R})$, the two limits*

$$\Omega_\pm f = \lim_{t \to \mp\infty} e^{itH_V} e^{-itH_0} f \tag{9.22}$$

exist in $L^2(\mathbb{R})$ norm, and the operators thus defined are surjective and unitary from $L^2(\mathbb{R})$ to $\mathcal{H}_{ac}$. One has

$$\begin{aligned} \Omega_+ &= U_V^{-1} U_0, \\ \Omega_- &= U_V^{-1} S(\lambda) U_0, \\ S &= {U_0}^{-1} S(\lambda) U_0 \end{aligned} \tag{9.23}$$

where $S(\lambda)$ is defined in terms of the modified scattering coefficients $t_j(\lambda), r_j(\lambda)$ defined in (9.20),(9.21)*, and U_V is defined as before but in terms of $\psi_\pm$ as in* (9.20), (9.21).

9.3. **The Dirac case.** Consider next the operator D_V on $\mathcal{H} = L^2(\mathbb{R})$. Because no phase correction $\int_0^x V$ appears in the generalized eigenfunction asymptotics, there is no need for modification of the standard wave operators in this case.

We calculate

$$\begin{aligned} e^{-itD_V} {U_V}^{-1} G \sim 4^{-1} \int_{\mathbb{R}} e^{-iEt} \Big[& G_+(E) \Big(t_1(E) e^{iEx} \begin{pmatrix} 1 \\ i \end{pmatrix} + r_1(E) e^{-iEx} \begin{pmatrix} 1 \\ -i \end{pmatrix} \Big) \\ & + G_-(E) \Big(t_2(E) e^{-iEx} \begin{pmatrix} 1 \\ -i \end{pmatrix} + r_2(E) e^{-iEx} \begin{pmatrix} 1 \\ i \end{pmatrix} \Big) \Big] \, dE \end{aligned} \tag{9.24}$$

as $t \to +\infty$, and

$$\sim 4^{-1} \int_{\mathbb{R}} e^{-iEt} \Big(G_+(E) e^{iEx} \begin{pmatrix} 1 \\ i \end{pmatrix} + G_-(E) e^{-iEx} \begin{pmatrix} 1 \\ -i \end{pmatrix} \Big) \, dE \tag{9.25}$$

as $t \to -\infty$.

Theorem 9.3. *Let $V \in L^1 + L^p(\mathbb{R})$ for some $1 < p < 2$. Then for each $f \in L^2(\mathbb{R})$, both of the limits*

$$\Omega_\pm f = \lim_{t \to \mp\infty} e^{-itD_V} e^{itD_0} f \tag{9.26}$$

exist in $L^2(\mathbb{R})$ norm, and define surjective and unitary operators to $\mathcal{H}_{ac}$. Moreover $\Omega_+ = U_V^{-1} U_0$ and $\Omega_- = U_V^{-1} S(E)^{-1} U_0$, where $S(E)$ is defined as before by (9.12). *The scattering operator S is equal to $U_0^{-1} S(E) U_0$.*

10. Asymptotic completeness

Let us denote by $\mathcal{H}_{\rm pp}(A)$ the pure point subspace of the self-adjoint operator A. Recall from [33]

Definition. The (modified) wave operators $\Omega_\pm$ are called asymptotically complete if their ranges coincide with the orthogonal complement of $\mathcal{H}_{\rm pp}(H_V)$.

In our context, since the absolutely continuous spectrum is under control, we have

Corollary 10.1. *The wave operators of Theorems 1.1 and 1.2 are asymptotically complete if and only if the singular continuous spectrum of the operator H_V is empty.*

In general, the question of asymptotic completeness of the wave operators for potentials in L^p remains open for $1 < p < 2$. There exist examples with embedded dense point spectrum on $\mathbb{R}^+$ [28, 38]. A very recent preprint [17] states that singular continuous spectrum can arise for L^2 potentials. There also exist estimates on the size of the set where the singular spectrum may be supported [35, 11] and examples of operators which have decaying solutions on a half axis for a set of spectral parameters having exactly the right dimension [36, 27]. However, no examples have yet been constructed of Schrödinger operators, with potential $V \in L^p$ for some $p < 2$, possessing nonempty singular continuous spectrum.

Nevertheless, there are some settings where generic (in some sense) asymptotic completeness can be inferred. We discuss two such cases: almost surely for certain random models, and for almost every boundary condition with any fixed potential. In both of these cases, the spectrum is purely absolutely continuous where $E > 0$, so asymptotic completeness in the relatively weak sense in which we have defined it implies a stronger form.

Let us say that the half-axis operator H_V^β has boundary condition β at the origin if $u(0) + \beta u'(0) = 0$ for any function u in the domain of H_V^β.

Theorem 10.2. *Let H_V^β be a Schrödinger operator defined on a half-axis with the boundary condition β. Assume that $V \in L^1 + L^p$ for some $1 < p < 2$. Then for almost every β, the modified wave operators defined in* (1.4) *are asymptotically complete. Moreover if $\int_0^\infty V(x)\,dx$ exists, then the usual wave operators defined by* (1.2) *are asymptotically complete.*

Proof. The result is a simple corollary of Theorems 1.1, 1.2, and general rank one perturbation theory. Theorem 1.1 and well-known results of scattering theory imply that the absolutely continuous part of the spectral measure of H_V corresponding to some boundary condition β_0, $\mu_{\rm ac}^{\beta_0}$, fills all of R^+; that is, $D\mu_{\rm ac}^{\beta_0}(E) > 0$ for a.e. E. The variation of the boundary condition can be regarded as rank one perturbation [39], Section I.6. The standard rank one perturbation theory then implies that for any β, the singular part of the spectral measure $\mu_{\rm s}^\beta$ can only be supported on a fixed set of energies $S \subset \mathbb{R}^+$ of zero Lebesgue measure [39], Theorem II.2. But then again by rank one theory for almost every β the singular spectrum on $\mathbb{R}^+$ is empty [39], Theorem I.8. Since the potential belongs to $L^1 + L^p$, the spectrum below zero is discrete (with only 0 as a possible accumulation point). □

Next, let us consider the following random model:

$$V(x) = \sum_{n=1}^{\infty} a_n(\omega) g(n) f(x-n), \tag{10.1}$$

where $f(x) \in C_0^\infty(0,1)$, $g \in l^p$ for some $p < 2$, and $a_n(\omega)$ are independent identically distributed bounded random variables with zero expectation. We have

Theorem 10.3. *Let H_V be a one dimensional Schödinger operator with random potential* (10.1)*. Then with probability one, the wave operators $\Omega_\pm$ exist and are asymptotically complete.*

Indeed, Theorem 9.1 of [24] shows that almost surely, the spectrum of the Schrödinger operator with potential (10.1) is purely absolutely continuous on $\mathbb{R}^+$. Notice that our assumptions on the potential easily imply that the improper integral $\int_0^\infty V$ exists almost surely. Theorem 1.1 and decay of the potential then imply asymptotic completeness.

This illustrates another type of situation where asymptotic completeness holds. We remark that the result holds in a variety of more general random models for which (10.1) is just an illustration. For a more general setting, see [24].

Discussion. The above is only one of various possible definitions of asymptotic completeness. The notion of asymptotic completeness is intended to describe a situation where the Hilbert space is split into two orthogonal subspaces [33]: $\mathcal{H}_{\rm pp}(H_V)$ and the range of wave operators, $\mathcal{H}_{\rm ac}(H_V)$. On $\mathcal{H}_{\rm ac}(H_V)$ the perturbed dynamics is close to the modified free evolution at large times, and corresponds to the scattering states. On $\mathcal{H}_{\rm pp}(H_V)$ the dynamics is supposed to be bounded in some sense. However, the intuitive physical assumption that pure point spectrum leads to dynamics which is bounded needs to be clarified, and in recent years there have appeared examples with very non-trivial transport on the pure point component. A widely accepted way to calibrate transport properties is to consider evolution of the averaged moments of coordinate operator:

$$\langle \langle |X|^m \rangle_\phi \rangle_T = \frac{1}{T} \int_0^T |\langle e^{-iHt}\phi, |X|^m e^{-iHt}\phi \rangle| \, dt, \tag{10.2}$$

where ϕ is the initial state, $\langle \phi_1, \phi_2 \rangle$ is the inner product and $|X|^m$ the operator of multiplication by $(|x|+1)^m$ in coordinate representation. The paper [15] contains an example of a (discrete) Schrödinger operator h with pure point spectrum and exponentially decaying eigenfunctions, such that

$$\limsup_{t\to\infty} \langle \langle |X|^2 \rangle_{\delta_0} \rangle_T / T^\alpha = \infty \tag{10.3}$$

for any $\alpha < 2$. Here the initial state is δ_0, the vector localized at the origin. Given that the rate of growth T^2 for the second moment corresponds to ballistic motion, as for the free Laplace operator, this example shows that in some sense the transport associated with point spectrum can be very fast.

However, there is still an important difference between transport associated with point and singular continuous spectrum. Namely, let B_R denote the ball of radius R centered at the origin and B_R^c its complement. Let $P_{\rm pp}$ and $P_{\rm c}$ be the orthogonal projections on $\mathcal{H}_{\rm pp}(H_V)$ and the continuous subspace $\mathcal{H}_{\rm c}(H_V)$ respectively. Then for any $\epsilon > 0$ there exists R_ϵ such that

$$\|e^{-iH_V t} P_{\rm pp}\phi\|^2_{L^2(B^c_{R_\epsilon})} < \epsilon \tag{10.4}$$

for all t. The growth of the moments in (10.3) is achieved not because of the motion of the whole wavepacket, but because of thin tails escaping to infinity. On the other hand, we have

$$\frac{1}{T}\int_0^T \|e^{-iH_V t} P_{\rm c}\phi\|^2_{L^2(B_R)}\, dt \stackrel{T\to\infty}{\longrightarrow} 0 \tag{10.5}$$

for any finite R. Equation (10.5) is one of the statements of the RAGE theorem (see, e.g. [14]) and is basically a corollary of Wiener's theorem on Fourier transforms of measures. Morever, there exist examples of Schrödinger operators [23] in which the dynamics corresponding to the singular continuous subspace is almost ballistic in a sense that the whole wavepacket is moving to infinity at a fast rate: for any $\rho > 0$ there exists C_ρ such that

$$\frac{1}{T}\int_0^T \|e^{-iH_V t}\phi\|^2_{L^2(B_{C_\rho T^{1-\epsilon}})}\, dt < \rho$$

for all T and ϕ lying in the singular continuous subspace of H_V.

11. Potentials in $\ell^p(L^1)$

Following [9], we sketch here the small modifications needed to extend the analysis of potentials in L^p to those in $L^1 + L^p$, and indeed those in the larger class $\ell^p(L^1)$. A locally integrable function f is said to belong to the amalgamated space $\ell^p(L^1)$ if

$$\sum_{n\in\mathbb{Z}} \Big(\int_n^{n+1} |f(x)|dx\Big)^p < \infty.$$

The norm $\|f\|_{\ell^p(L^1)}$ is the p-th root of this expression. For any $1 \le p < \infty$, this defines the Banach space $\ell^p(L^1)$, which contains $L^1 + L^p$.

A martingale structure $\{E_j^m\}$ is said to be adapted to f in $\ell^p(L^1)$ if

$$\|f\cdot\chi_{E_j^m}\|^p_{\ell^p(L^1)} \le 2^{-m}\|f\|^p_{\ell^p(L^1)} \tag{11.1}$$

for all m, j. For any $f \in \ell^p(L^1)$, there does exist an adapted martingale structure [9].

Lemma 5.3 extends to $\ell^p(L^1)$: this Banach space is mapped boundedly to $L^{p'}(\Lambda)$ for any compact interval $\Lambda \Subset (0,\infty)$, for any $1 \le p \le 2$. The proof is essentially unchanged; see the analogous proof of Proposition 3.5 of [9].

Corollary 5.4 may now be refined by replacing $\|f\|_{L^p}$ by $\|f\|_{\ell^p(L^1)}$ on the right-hand side of each inequality. With these bounds for the operator G in hand, the remainder of the proof is unchanged.

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