Infinite-volume Metastability: on the shape of early droplets

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Abstract. We study a very simple model for nucleation-and-growth in infinite volume in the low-temperature limit. Despite its simplicity, this model exhibits a very rich metastable behavior. Depending on the speed of growth, the system goes trough four different regimes: 1) both the "shape of the critical droplet" and the typical relaxation time are the same as in finite volume. 2) the "shape of the critical droplet" and its "formation rate" are the same as in finite volume but the "relaxation time" is shorter 3) the "shape of the critical droplet" is the same as in finite volume while the nucleation rate is smaller 4) the "shape of the critical droplet" is different from what we have in finite volume and its formation rate is smaller than the finite-volume formation rate of the finite-volume droplet.

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1. Introduction.

Phase transitions and metastability. Metastability is a phenomenon of great importance for the description of phase transitions. The first rigorous approach traces back to [**LP**], where metastability is characterized as a slow evolution of the averages over the process towards the stable equilibrium value. In [CGOV] a new "pathwise approach" to metastability has been introduced where the analysis of the behaviour of typical trajectories has a crucial importance. However, the Curie-Wiess model studied in [**CGOV**] revealed some peculiarities like the fact of being 1-dimensional and the "smoothness" of the "basin of attraction of the metastable state". A long time and the development of new ideas were required to treat the full Ising model in a finite volume and in the low-temperature limit in [**NS**].

The Friedlin-Wentzell regime. This work opened the way to the development of general techniques (see [**S**], [**OS**]) to study metastability in Markov processes with exponentially small transition rates in a finite space (see [**KO**], [**CO**], [**NO**]). Similar results were obtained with a contemporary approach focused on simulated annealing (see

[**C**], [**T**], [**CC**]). The main idea in these papers was to study the exit from a special kind of "basin of attraction", called cycle, a problem analogous to that analyzed in [**FW**] by Friedlin and Wentzell, with the additional complication that the basin is not necessarily smooth but can contain many other basins. Although this feature is a source of technical difficulties, it turns out to influence neither the exit time nor the exit state. Heuristically, this feature can be explained by observing that, starting from any point of the cycle, the process typically visits many times the bottom of the cycle before the exit. Thus, a renewal scheme can be used leading to a quasi-exponential exit law. In this sense, only the bottom and the boundary of the cycle play a role in the exit problem.

Different regimes. While the above mentioned regime is the easiest case to study, it is also the least interesting from a physical point of view. Nevertheless, the methods developed in this framework have been the starting point in the analysis of a large variety of regimes like the low-temperature limit in infinite volume (see [**DS2**]), the low temperature limit in the case of conservative dynamics (see [**HOS**]) and, perhaps the most interesting regime, the case of finite temperature in the phase coexistence limit (see [**SS**]). Recently, a new approach to the problem (carried out in [**BEGK2**]), based on spectral analysis instead of on large deviation methods, allowed to improve considerably the precision of the known results (see [**BM**]) and to deal with cases where the energy landscape is not described in detail like disordered mean field systems (see [**BEGK1**], [**BBG1**], [**BBG2**]).

Zero-temperature limit in infinite volume. In this short paper, we will focus our attention on the regime where the inverse temperature β goes to infinity and the volume is infinite. This regime was treated in two series of works on the Ising model (see [**DS1**], [**DS2**]) and on the Blume-Capel model (see [**MO1**], [**MO2**]). In particular, the papers on the Blume-Capel model are part of a group of works on the competition among different transition mechanisms and show how there is not always a correspondence between finite and infinite-volume transition mechanisms.

Both Ising and Blume-Capel models have a significant peculiarity: the speed of growth of super-critical droplets is relatively low. Due to this characteristic the inner structure of the metastable cycle is irrelevant (such as in a finite volume). Dehghanpour and Schonmann noticed that this is not a general property (see [**DS2**]). In [**MO2**] a condition is given to ensure that the details of the energy landscape do not affect the nucleation pattern .

Before introducing the model and showing how the growth speed can influence nucleation, it is useful to recall the heuristics of [**DS1**]: we consider a lattice model on $\mathbb{Z}^d$ in which the critical nuclei originate independently of each other with rate $e^{-\beta\Gamma}$ and grow with speed[1] $e^{-\beta\nu}$. Let us consider the "space-time cone" with vertex in (o, t) and slope $e^{-\beta\nu}$. By definition, if a critical droplet is generated in the cone, it

[1]In the Ising model $e^{-\beta\Gamma}$ is the inverse of the typical nucleation time at finite volume and the growth speed is determined by a $(d-1)$-dimensional metastability problem in infinite volume.

reaches the origin within t. The value t_c of the typical time required by the droplet to reach the origin is given by

$$t_c \left(e^{-\beta\nu} t_c\right)^d e^{-\beta\Gamma} \sim 1, \tag{1.1}$$

which gives

$$t_c \sim e^{\beta \frac{\Gamma + d\nu}{d+1}}. \tag{1.2}$$

This value has to be compared with the typical nucleation time at finite volume, that is $e^{\beta\Gamma}$. However, in realistic models, the finite-volume exit law is not strictly exponential and the typical exit configuration conditioned to a given exit time may be different from the unconditioned exit configuration. This is due to the fact that, for small times, the process typically does not reach the bottom of the cycle. Therefore the renewal scheme cannot be applied and the results à la Friedlin and Wentzell are not valid.[2]

Shape of the droplets. From the results in [**MO2**] it is clear that all the sub-cycles with depth smaller than the logarithm divided by β of the given exit time do not influence the exit and can be ignored safely. With the help of a very simple model, we will show that metastability in the infinite volume and $\beta \to \infty$ regime cannot be characterized only by the depth and the boundary of the critical cycle but a more detailed description of the energy landscape is needed. We give a rather dramatic example of the fact that the "formation rate" and the "shape" of the "critical droplet" may depend on the parameter ν even with a fixed "finite-volume" Hamiltonian. While we do not give a general recipe, we show that the results of Lemma 4.3 in [**MO2**] are sufficient to analyze the problem in a model-dependent way.

2. Notation and results.

We denote by $[x]_+ := \max\{x, 0\}$ the *positive part* of of the real number x.

The model. We consider a one-dimensional spin system on the lattice $\mathbb{Z}$ where the spin variable $\sigma(i)$ can take values in $\Omega := \{-1, 0, 1, 2, 3\}$. The values 1 and 2 of the spin variable should be thought of as "inner degrees of freedom" (sub-critical droplets) of the system, while $-1, 0$, and 3 as observable states.

[2]This phenomenon could be observed in a model with the same finite-volume energy landscape of the Ising model and a very small growth exponent ν. For instance, a similar system could be obtained by adding to the Ising Hamiltonian a non-local term proportional to the squared number of pluses or a many-body interaction.

The *single site Hamiltonian* is

$$\begin{aligned} H(-1) &= \frac{40}{36} \\ H(0) &= 0 \\ H(1) &= \frac{33}{36} \\ H(2) &= \frac{3}{36} \\ H(3) &= 1 \end{aligned} \tag{2.1}$$

and is shown in Fig. 1.

FIGURE 1. Single-site Hamiltonian.

The *single site dynamics* is given by the following transition rates:
For $a \notin \{-1, 3\}$,

$$c_\beta(b, a) := \begin{cases} e^{-\beta[H(b)-H(a)]_+} & \text{if } a = b \pm 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

If $a = -1$ or $a = 3$, then $c_\beta(a, b) \equiv 0$ and no transition is allowed. This dynamics is Metropolis reversible in $\{0, 1, 2\}$ with absorbing states in -1 and in 3. Therefore,

we will be allowed to use the results in [**OS**] and [**MO2**] about reversible Markov chains up to the hitting time to $\{-1, 3\}$.
The parameter β has the meaning of the *inverse temperature*.
We denote by σ_t^* the *single-site process* on Ω distributed according with the above defined dynamics and by

$$\tau_Q^* := \min\{t; \sigma_t^* \in Q\}, \tag{2.3}$$

$$\tau^* := \tau_{-1,3}^*. \tag{2.4}$$

The *infinite volume dynamics* is defined as follows: at time $t = 0$ the initial configuration is $\underline{0}$ (all zeroes). Afterwards, the sites evolve according to their single-site dynamics (namely, with the same law of σ^*) until they have one nearest neighbor with spin -1 or 3. Then, they assume the value of the spin of their nearest neighbor with spin -1 or 3 with rate

$$e^{-\beta\nu}. \tag{2.5}$$

If a site has both nearest neighbors with spin in $\{-1, 3\}$, it will assume one of the two values with uniform probability and rate $e^{-\beta\nu}$. ν is the only parameter in our model and $e^{-\beta\nu}$ has the meaning of growth speed of super-critical configurations.
This model can be considered as the counterpart of the nucleation-and-growth model introduced in [**DS1**]. While in that case Dehghanpour and Schonmann focused their attention on the supercritical growth, here we are interested on how the speed of growth influence the nucleation pattern.
Given a volume $\Phi \subset \mathbb{Z}$ and a configuration ρ (boundary condition), we define the *restriction* $\sigma_{\Phi;t}^{\rho}$ of the process to Φ by freezing the spins outside Φ to $\rho(i)$.
We focus our attention on the following hitting time

$$\tau := \min\{t;\ \sigma_t(0) \in \{-1, 3\}\}. \tag{2.6}$$

For $\Phi \subset \mathbb{Z}$, we consider the auxiliary hitting time:

$$\hat{\tau}^{\rho}(\Phi) := \min\left\{t;\ \exists\, i \in \Phi\ \ \sigma_{\Phi;t}^{\rho}(i) \in \{-1, 3\}\right\}. \tag{2.7}$$

We omit the volume from notation if $\Phi = \mathbb{Z}$ and the boundary condition if $\rho = \underline{0}$.
Obviously, if $\Phi' \subseteq \Phi''$ and for all i, $\rho'(i) \le \rho''(i)$, then

$$\hat{\tau}^{\rho'}(\Phi') \le \hat{\tau}^{\rho''}(\Phi'') \tag{2.8}$$

We call the first appearance of a -1 or a 3 in a given volume *nucleation*, the site where we see this -1 or 3 *critical droplet* and the value of this spin *shape of the critical droplet*. If $\nu < 1/3$, we say that the -1-droplets are of the *right kind* whereas the 3-droplet are of the *wrong kind*; if $\nu > 1/3$, it is vice versa.

Let us introduce some useful notation before stating our main result.
Given a set $B \subset \Omega$, we define $F(B)$ as the set of the minima of the Hamiltonian H in B. We denote by ∂B its *outer boundary* and the energy of the points in this set by $H(F(B))$.

A *cycle* $A \subset \Omega$ is a connected set such that $H(F(\partial A)) > \max_{a\in A} H(a)$ (in our case, $\{0\}$, $\{2\}$, and $\{0,1,2\}$ are cycles). Given a cycle A, we define its *depth* as $\Gamma(A) := H(F(\partial A)) - H(F(A))$ and its *largest inner resistance* $\Theta(A)$ as the maximal depth of a sub-cycle $A' \subset A$ that does not contain the whole $F(A)$:

$$\Theta(A) := \max_{F(A)\not\subset A' \subset A} \Gamma(A') \tag{2.9}$$

if such a sub-cycle does not exist, we set $\Theta(A) := 0$. We will use results about the exit from cycles from [**OS**] and [**MO2**].
Let us introduce some useful functions of ν:

$$k^a(\nu) := \frac{20}{36} + \frac{\nu}{2} \tag{2.10}$$

$$k^b(\nu) := \frac{22}{36} + \frac{\nu}{3} \tag{2.11}$$

$$k^c(\nu) := \frac{1}{2} + \frac{\nu}{2} \tag{2.12}$$

$$k^d(\nu) := 1 \tag{2.13}$$

The time of the first appearance of a stable phase in the origin is characterized by the following exponent:

$$k(\nu) := \min\left\{k^a, k^d, \max\left\{k^b, k^c\right\}\right\} \tag{2.14}$$

In Fig. 2, k is plotted v.s. ν.

$\partial_\nu k(\nu)$ has three discontinuity points for the values $\nu = 1/3,\ 2/3$, and 1. These points correspond to "dynamical phase transitions", namely to changes in the nucleation patterns. We remark once more that the single-point energy landscape does not depend on ν and thus these transitions have no finite-volume counterpart.
Notice that $k(\nu)$ is strictly monotonic for $\nu \le 1$. We denote the inverse function of $k(\nu)$ by

$$\nu(\kappa) : [\tfrac{5}{9}, 1] \to [0,1]\,. \tag{2.15}$$

A particular role will be played by the slab

$$\Lambda := \left\{-\lfloor e^{\beta(k(\nu)-\nu)}\rfloor, ..., \lfloor e^{\beta(k(\nu)-\nu)}\rfloor\right\} \tag{2.16}$$

that corresponds to the heuristic notion of base of the "critical space-time cone" described in the introduction (see (1.1)).
Our main result is contained in the following Theorem:

Theorem 2.1. *$\forall \nu > 0$, $\forall\, \varepsilon > 0$ in the limit $\beta \uparrow \infty$,*

$$\mathbb{P}_{\underline{0}}\left(\tau > e^{\beta(k(\nu)-\varepsilon)}\right) \to 1 \tag{2.17}$$

$$\mathbb{P}_{\underline{0}}\left(\tau < e^{\beta(k(\nu)+\varepsilon)}\right) \to 1. \tag{2.18}$$

FIGURE 2. k v.s. ν.

Moreover,

$$\mathbb{P}\left(\sigma_\tau(0) = -1\right) \quad \to 1 \textit{ if } \nu < \frac{1}{3} \tag{2.19}$$

$$\mathbb{P}\left(\sigma_\tau(0) = 3\right) \quad \to 1 \textit{ if } \nu > \frac{1}{3} \tag{2.20}$$

The Theorem above shows that both the exit time and the exit state may depend on the inner structure of the critical cycle. This dependence shows up only at very high speed of growth and it is hidden in the Ising and Blume-Capel model studied in [**DS1**], [**DS2**], [**MO1**], and [**MO2**] where the speed of growth is not independent of the energy of the critical droplet. Depending on the parameter, we detect in our model four different nucleation behaviors:

i) for $\nu > 1$ (where $k(\nu) = k^d(\nu)$), the system behaves like in finite volume, and both the exit time and state are "the same" as in finite volume;
ii) for $2/3 < \nu < 1$ (where $k(\nu) = k^c(\nu)$), the system is in the Dehghanpour and Schonmann regime: the typical exit time is $\exp\left(\beta\left(\frac{1+\nu}{2}\right)\right)$ and the exit state is the same as in finite volume; both this case and case i were described in [**DS1**] (see the introduction for the heuristic discussion of case ii).
iii) for $1/3 < \nu < 2/3$ (where $k(\nu) = k^b(\nu)$), the inner structure of the cycle $\{0, 1, 2\}$ in Fig. 1 becomes relevant. Indeed, since it is very unlikely that the process once reached 2 goes back to 0, both the exit from $\{0\}$ and the exit

from $\{2\}$ are rare event to take into consideration. The exit rate is therefore lower than the one we get from the heuristics in [**DS1**] while the exit state is the same as in previous cases;

iv) for $\nu < 1/3$ (where $k(\nu) = k^a(\nu)$), like in case iii, the exit trough 3 entails two rare events and its probability is so low that it is more likely to exit from -1; the exit rate is consistently affected. In this case the system reaches the "state" where value of the spin is a.s. -1 despite of the fact that the Gibbs measure gives a.s. the value 3 and the fact that the energy barrier between 0 and -1 is higher than that between 0 and 3.[3]

3. Basic tools.

In this Section, we review the basic results about finite-volume metastability in the context of [**OS**] and [**MO2**]. The setting is that of Markov chains with exponentially small transition rates (e.g. Metropolis dynamics in the $\beta \to \infty$ limit) with finite state-space.

The extension to the continuous-time case is immediate (via large-deviation estimates) as far as exponential times are concerned.

We will use these results to bound the probability of exit through a given state at a given time from above and from below.

The following Lemma gives the desired upper bound:

Lemma 3.1 (Lemma 3.1 in [**OS**])**.** *For all a, b such that $H(b) > H(a)$, for all $\kappa > 0$ and $\varepsilon > 0$*

$$\mathbb{P}_a\left(\tau_b^* \leq e^{\beta\kappa}\right) \leq e^{-\beta(H(b)-H(a)-\varepsilon)} \tag{3.1}$$

If $\kappa < \Gamma(A)$, we immediately get the bound

$$\mathbb{P}_a\left(\tau_{\partial A}^* = \tau_b^*, \tau_b^* \leq e^{\beta\kappa}\right) \leq e^{-\beta(H(b)-H(a)-\varepsilon)}. \tag{3.2}$$

In particular, when $\kappa < \Gamma(A)$, the exit probability goes to zero. The counterpart of this fact is the content of the following Lemma from [**OS**]:

Lemma 3.2 (Proposition 3.7 in [**OS**] i and iii)**.** *For all $a \in A$, for all $\varepsilon > 0$*

$$\mathbb{P}_a\left(\tau_{\partial A}^* < e^{\beta(\Gamma(A)+\varepsilon)}\right) \geq 1 - e^{-\beta c} \tag{3.3}$$

for some positive constant c and sufficiently large β.
Moreover, for all $a \in A$, $b \in \partial A$, for all $\varepsilon > 0$,

$$\mathbb{P}_a\left(\tau_{\partial A}^* = \tau_b^*\right) \geq e^{-\beta(H(b)-H(F(\partial A))-\varepsilon)} \tag{3.4}$$

While the two previous Lemmata give sharp bounds on the exit time at finite volume, their results are not sufficient to deal with the infinite-volume case.

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[3] In a reversible situation, the system would go to the intermediate state and then reach the Gibbs state at a later time.

The following Lemma from [**MO2**] shows that if the exit time is not too small, the inner details of the cycle do not influence the exit state. In this case, the bound on the exit probability is "exponentially equivalent" to the bound (3.2).

Lemma 3.3. (Lemma 4.3 in [**MO2**].) *Given a non-trivial cycle A and a positive number κ such that*

$$\Theta(A) < \kappa \leq \Gamma(A),$$

we have $\forall\, a \in A$, $\forall\, b \in \partial A$, $\forall\, \varepsilon > 0$ and β sufficiently large

$$\mathbb{P}_a\left(\tau^*_{\partial A} < e^{\beta\kappa}\ ,\ \tau^*_{\partial A} = \tau^*_b\right) \geq e^{-\beta(H(b)-H(F(A))-\kappa+\varepsilon)}. \tag{3.5}$$

4. Proof of Theorem 2.1.

We now focus on the model described in §2. In the following key Lemma we estimate the exit probability at a given time T, showing that the most likely exit state depends on T.

Lemma 4.1. *For all $0 < \kappa < 1$, $\forall \varepsilon > 0$ and sufficiently large β,*

$$e^{-\beta(\kappa-\nu(\kappa)+\varepsilon)} \leq \mathbb{P}_0\left(\tau^* \leq e^{\beta\kappa}\right) \leq e^{-\beta(\kappa-\nu(\kappa)-\varepsilon)}. \tag{4.1}$$

Moreover,

$$\mathbb{P}_0\left(\tau^* = \tau^*_3 \mid \tau^* \leq e^{\beta\kappa}\right) \tag{4.2}$$

tends to 0 if $\kappa < 26/36$ (i.e. $\nu(\kappa) < 1/3$) and to 1 if $\kappa > 26/36$.

Proof. We split the proof into three parts:[4]

a) $\kappa < 26/36$ (i.e. $\nu(\kappa) < 1/3$).

By applying Lemma 3.3 on the cycle $\{0\}$:

$$\mathbb{P}_0\left(\left\{\tau^* = \tau^*_{-1}\right\} \cap \left\{\tau^* \leq e^{\beta\kappa}\right\}\right) \geq e^{-\beta(40/36-\kappa+\varepsilon)} \tag{4.3}$$

while, from 3.2

$$\mathbb{P}_0\left(\left\{\tau^* = \tau^*_{-1}\right\} \cap \left\{\tau^* \leq e^{\beta\kappa}\right\}\right) \leq e^{-\beta(40/36-\kappa)} \tag{4.4}$$

On the other hand, the exit passing through 3 entails a transition from 0 to 1 and a transition from 2 to 3; therefore, by using Lemma 3.1 and the Markov property, we get

$$\mathbb{P}_0\left(\left\{\tau^* = \tau^*_3\right\} \cap \left\{\tau^* \leq e^{\beta\kappa}\right\}\right) \leq e^{-2\beta(33/36-\kappa)} \tag{4.5}$$

[4] In the general case, the analogue of this proof would be to pass to a "renormalized Markov chain" (see [**S**]) where the state space is partitioned into the subsets of states that are "equivalent" at time T (meaning that starting from any state in a subset all the other states in the same subset are visited within T with large probability). The probability of a transition is "exponentially equivalent" to the probability of the best path in this renormalized Markov chain (the product of the transition probabilities in the path).

Since $40/36-\kappa=\kappa-\nu(\kappa)<66/36-2\kappa$, by (4.3), (4.4) and (4.5), we get (4.1) in case a). By (4.3) and (4.5) we get (4.2):

$$\mathbb{P}_0\left(\{\tau^*=\tau_3^*\}\ \mid\ \left\{\tau^*\le e^{\beta\kappa}\right\}\right)\le\frac{\mathbb{P}_0\left(\{\tau^*=\tau_3^*\}\cap\left\{\tau^*\le e^{\beta\kappa}\right\}\right)}{\mathbb{P}_0\left(\{\tau^*=\tau_1^*\}\cap\{\tau^*\le e^{\beta\kappa}\}\right)}\to 0 \tag{4.6}$$

b) if $26/36\le\kappa<30/36$ (i.e. $1/3\le\nu(\kappa)<2/3$) then we still have the bounds in (4.4) and (4.5) but now $\kappa-\nu(\kappa)=66/36-2\kappa$ and the leading term is (4.5). To get a lower bound on the probability to exit through 3, we observe that

$$\begin{gathered}\mathbb{P}_0\left(\{\tau^*=\tau_3^*\}\cap\left\{\tau^*\le e^{\beta\kappa}\right\}\right)\ge\\ \mathbb{P}_0\left(\tau_1^*\le\tfrac{1}{2}e^{\beta\kappa}\right)\mathbb{P}_1\left(\tau_2^*\le 1\right)\mathbb{P}_2\left(\tau_3^*\le\tfrac{1}{2}e^{\beta\kappa}-1\right)\ge\\ e^{-2\beta(33/36-\kappa+\varepsilon)},\end{gathered} \tag{4.7}$$

where, to get the last inequality, we used Lemma 3.3 on the cycles $\{0\}$ and $\{2\}$. By (4.3), (4.4) and (4.7), we get (4.1) in case b). By (4.4) and (4.7) we get (4.2):

$$\mathbb{P}_0\left(\{\tau^*=\tau_1^*\}\ \mid\ \left\{\tau^*\le e^{\beta\kappa}\right\}\right)\le\frac{\mathbb{P}_0\left(\left\{\tau^*=\tau_{-1}^*\right\}\cap\left\{\tau^*\le e^{\beta\kappa}\right\}\right)}{\mathbb{P}_0\left(\{\tau^*=\tau_3^*\}\cap\{\tau^*\le e^{\beta\kappa}\}\right)}\to 0 \tag{4.8}$$

c) $30/36<\kappa<1$ (i.e. $2/3<\nu(\kappa)$). In this case, we can deal directly with the cycle $\{0,1,2\}$. By (3.2) and Lemma 3.3, respectively, we get the following bounds on the probability to exit through 3:

$$\mathbb{P}_0\left(\{\tau^*=\tau_3^*\}\cap\left\{\tau^*\le e^{\beta\kappa}\right\}\right)\le e^{-\beta(1-\kappa)} \tag{4.9}$$

and

$$\mathbb{P}_0\left(\{\tau^*=\tau_3^*\}\cap\left\{\tau^*\le e^{\beta\kappa}\right\}\right)\ge e^{-\beta(1-\kappa+\varepsilon)}. \tag{4.10}$$

By (4.3), (4.4) and (4.10), we get (4.1) in case c). By the same procedure leading to (4.8) we get (4.2).

□

The lower bound on τ (proof of (2.17)).

We use Lemma 4.1 when $\kappa=k(\nu)-\varepsilon$ (notice that if $\nu>1$ this choice gives $\kappa-\nu(\kappa)=\varepsilon$).

Since the spins are independent from each other until time $\hat\tau(\Lambda)$,

$$\mathbb{P}_0\left(\hat\tau(\Lambda)<t\right)=1-\left(1-\mathbb{P}_0\left(\tau^*<t\right)\right)^{\Lambda} \tag{4.11}$$

by using the definition of Λ (see (2.16)), we get from (4.11)

$$\begin{gathered}\mathbb{P}_0\left(\hat\tau(\Lambda)<e^{\beta(k(\nu)-\varepsilon)}\right)\le\\ \lfloor e^{\beta(k(\nu)-\nu)}\rfloor e^{-\beta([k(\nu)-\nu]_+)}e^{-\beta\varepsilon'}\to 0;\end{gathered} \tag{4.12}$$

namely, it is very unlikely that the nucleation occurs into Λ within time $e^{\beta(k(\nu)+\varepsilon)}$.

Next, we prove that L is too large to be crossed within the allotted time:

$$\mathbb{P}_{\underline{0}}\left(\tau < e^{\beta(k(\nu)-\varepsilon)} \mid \hat{\tau}(\Lambda) > e^{\beta(k(\nu)-\varepsilon)}\right) \leq 2\,\mathbb{P}\left(\sum_{n=0}^{\lfloor e^{\beta(k(\nu)-\nu)}\rfloor} \zeta(n) < e^{\beta(k(\nu)-\varepsilon)}\right), \quad (4.13)$$

where the $\zeta(n)$'s are i.i.d. exponential variables with rate $e^{-\beta\nu}$. Let Z be a Poisson variable with mean $e^{\beta(k(\nu)-\nu-\varepsilon)}$ r.h.s. of (4.13) is equal to

$$2\mathbb{P}\left(Z \geq \lfloor e^{\beta(k(\nu)-\nu)}\rfloor\right) \leq \frac{e^{\beta(k(\nu)-\nu-\varepsilon)}}{\lfloor e^{\beta(k(\nu)-\nu)}\rfloor} \to 0, \quad (4.14)$$

where in the last inequality we used Chebychev inequality.
This concludes the proof of the lower bound (2.17). □

Upper bound on τ (Proof of (2.18)). Let us start by considering the case $\nu < 1$ (so that $k(\nu) + \varepsilon$ can be taken smaller than 1). By using Lemma 4.1 and (4.11), we see that Λ is so large that (with large probability) nucleation in it occurs within $e^{\beta(k(\nu)+\varepsilon)}$.
Now we show that Λ is small enough to be crossed in the allotted time: by the same procedure of (4.13), we get

$$\mathbb{P}_{\underline{0}}\left(\tau > e^{\beta(k(\nu)+\varepsilon)} \mid \hat{\tau}(\Lambda) > e^{\beta(k(\nu)+\varepsilon)}\right) \leq \mathbb{P}_{\underline{0}}\left(\sum_{n=0}^{\lfloor e^{\beta(k(\nu)-\nu)}\rfloor} \zeta(n) > e^{\beta(k(\nu)+\varepsilon)}\right), \quad (4.15)$$

where the $\zeta(n)$'s are i.i.d. exponential variables with mean $e^{\beta\nu}$. By using Chebychev inequality, we bound r.h.s. of (4.15) by

$$\frac{\lfloor e^{\beta(k(\nu)-\nu)}\rfloor e^{\beta\nu}}{e^{\beta(k(\nu)+\varepsilon)}} \to 0 \quad (4.16)$$

In the case $\nu \geq 1$, Lemma 3.2 applied on the cycle $\{0, 1, 2\}$ gives

$$\mathbb{P}_{\underline{0}}\left(\tau^* = \tau_3^*\ ,\ \tau^* < e^{\beta(1+\varepsilon)}\right) \geq 1 - e^{-\beta c}. \quad (4.17)$$

Since with large probability nucleation occurs in the origin within the allotted time $e^{\beta(1+\varepsilon)}$ and since, by (2.8), nucleation in other sites can only help, we conclude the proof. □

The shape of the droplet (proof of (2.19) and (2.20)). By the same procedure leading to (4.12), we immediately show that with overwhelming probability all the droplets of the "wrong" kind formed within $e^{\beta(k(\nu)+\varepsilon)}$ are very far away from the origin (more than $\lfloor e^{\beta(k(\nu)-\nu)}\rfloor e^{\beta\delta}$ for some $\delta > 0$). Since the presence of droplets of the "right" kind does not increase the speed of growth of the the droplets of the "wrong kind" (indeed, they prevent the growth), we can proceed as for (4.14) and conclude the proof. □

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