Relaxation of Excited States in Nonlinear Schrödinger Equations

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Abstract

We consider a nonlinear Schrödinger equation in $\mathbb{R}^3$ with a bounded local potential. The linear Hamiltonian is assumed to have two bound states with the eigenvalues satisfying some resonance condition. Suppose that the initial data is small and is near some nonlinear *excited* state. We give a sufficient condition on the initial data so that the solution to the nonlinear Schrödinger equation approaches to certain nonlinear *ground* state as the time tends to infinity.

1 Introduction

Consider the nonlinear Schrödinger equation

$$i\partial_t \psi = (-\Delta + V)\psi + \lambda|\psi|^2\psi, \qquad \psi(t=0) = \psi_0 \tag{1.1}$$

where V is a smooth localized potential, λ is an order 1 parameter and $\psi = \psi(t,x) : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{C}$ is a wave function. The goal of this paper is to study the asymptotic dynamics of the solution for initial data ψ_0 near some *nonlinear excited state.*

Recall that for any solution $\psi(t) \in H^1(\mathbb{R}^3)$ the L^2-norm and the Hamiltonian

$$\mathcal{H}[\psi] = \int \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}V|\psi|^2 + \frac{1}{4}\lambda|\psi|^4\, dx \ , \tag{1.2}$$

are constants for all t. The global well-posedness for small solutions in $H^1(\mathbb{R}^3)$ can be proved using these conserved quantities and a continuity argument.

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We assume that the linear Hamiltonian $H_0 := -\Delta + V$ has two simple eigenvalues $e_0 < e_1 < 0$ with normalized eigen-functions ϕ_0, ϕ_1. We further assume that

$$e_0 < 2\, e_1 \ . \tag{1.3}$$

The nonlinear bound states to the Schrödinger equation (1.1) are solutions to the equation

$$(-\Delta + V)Q + \lambda |Q|^2 Q = EQ \ . \tag{1.4}$$

They are critical points to the Hamiltonian $\mathcal{H}[\phi]$ defined in (1.2) subject to the constraint that the L^2-norm of ψ is fixed. For any bound state $Q = Q_E$, $\psi(t) = Qe^{-iEt}$ is a solution to the nonlinear Schrödinger equation.

We may obtain two families of such bound states by standard bifurcation theory, corresponding to the two eigenvalues of the linear Hamiltonian. For any E sufficiently close to e_0 so that $E - e_0$ and λ have the same sign, there is a unique positive solution $Q = Q_E$ to (1.4) which decays exponentially as $x \to \infty$. See Lemma 2.1. We call this family the *nonlinear ground states* and we refer to it as $\{Q_E\}_E$. Similarly, there is a *nonlinear excited state* family $\{Q_{1,E}\}_E$. We will abbreviate them as Q and Q_1. From Lemma 2.1, we also have $\|Q_E\| \sim |E - e_0|^{1/2}$ and $\|Q_{1,E}\| \sim |E - e_1|^{1/2}$.

It is well-known that the family of nonlinear ground states is stable in the sense that if

$$\inf_{\Theta, E} \left\| \psi(t) - Q_E\, e^{i\Theta} \right\|_{L^2}$$

is small for $t = 0$, it remains so for all t, see [6]. Let $\|\cdot\|_{L^2_{\text{loc}}}$ denote a local L^2 norm (a precise choice will be made later on). One expects that this difference actually approaches zero in local L^2 norm, i.e.,

$$\lim_{t \to \infty} \inf_{\Theta, E} \left\| \psi(t) - Q_E\, e^{i\Theta} \right\|_{L^2_{\text{loc}}} = 0 \ . \tag{1.5}$$

If $-\Delta + V$ has only one bound state, it is proved in [10] [5] that the evolution will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E. Suppose now that $-\Delta + V$ has multiple bound states, say, two bound states: a ground state ϕ_0 with eigenvalue e_0 and an excited state ϕ_1 with eigenvalue e_1. It is proved in [12] that the evolution with initial data ψ_0 near some Q_E will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E. See also [1] for the one dimensional case and [11] for nonlinear Klain-Gorden equations.

Denote by L^2_r the weighted L^2 spaces (r may be positive or negative)

$$L^2_r(\mathbb{R}^3) \equiv \left\{ \phi \in L^2(\mathbb{R}^3) \ : \ \langle x \rangle^r \phi \in L^2(\mathbb{R}^3) \right\} . \tag{1.6}$$

The space for initial data in [12] is

$$Y \equiv H^1(\mathbb{R}^3) \cap L^2_{r_0}(\mathbb{R}^3) , \qquad r_0 > 3 . \tag{1.7}$$

We shall use $L^2_{\rm loc}$ to denote $L^2_{-r_0}$. The parameter $r_0 > 3$ is fixed and we can choose, say, $r_0 = 4$ for the rest of this paper. We now state the assumptions in [12] on the potential V.

Assumption A0: $-\Delta + V$ acting on $L^2(\mathbb{R}^3)$ has 2 simple eigenvalues $e_0 < e_1 < 0$, with normalized eigenvectors ϕ_0 and ϕ_1.

Assumption A1: Resonance condition. Let $e_{01} = e_1 - e_0$ be the spectral gap of the ground state. We assume that $2e_{01} > |e_0|$ so that $2e_{01}$ is in the continuum spectrum of H_1. Let

$$\gamma_0 := \lim_{\sigma \to 0+} \operatorname{Im} \left(\phi_0 \phi_1^2, \frac{1}{H_0 + e_0 - 2e_1 - \sigma i} \mathbf{P}_{\rm c}{}^{H_0} \phi_0 \phi_1^2 \right) \tag{1.8}$$

Since the expression is quadratic, we have $\gamma_0 \geq 0$. We assume, for some $s_0 < C n_0^2$ small enough,

$$\inf_{|s| < s_0} \lim_{\sigma \to 0+} \operatorname{Im} \left(\phi_0 \phi_1^2, \frac{1}{H_0 + e_0 - 2e_1 + s - \sigma i} \mathbf{P}_{\rm c}{}^{H_0} \phi_0 \phi_1^2 \right) \geq \frac{3}{4} \gamma_0 > 0 . \tag{1.9}$$

We shall use $0i$ to replace σi and the limit $\lim_{\sigma \to 0+}$ later on.

Assumption A2: For λQ_E^2 sufficiently small, the bottom of the continuous spectrum to $-\Delta + V + \lambda Q_E^2$, 0, is not a generalized eigenvalue, i.e., not a resonance. Also, we assume that V satisfies the assumption in [14] so that the $W^{k,p}$ estimates $k \leq 2$ for the wave operator $W_H = \lim_{t \to \infty} e^{itH} e^{it(\Delta + E)}$ hold for $k \leq 2$, i.e., there is a small $\sigma > 0$ such that,

$$|\nabla^\alpha V(x)| \leq C \langle x \rangle^{-5-\sigma} , \qquad \text{for } |\alpha| \leq 2 .$$

Also, the functions $(x \cdot \nabla)^k V$, for $k = 0, 1, 2, 3$, are $-\Delta$ bounded with a $-\Delta$-bound < 1:

$$\left\| (x \cdot \nabla)^k V \phi \right\|_2 \leq \sigma_0 \left\| -\Delta \phi \right\|_2 + C \left\| \phi \right\|_2 , \qquad \sigma_0 < 1, \quad k = 0, 1, 2, 3 .$$

Assumption A2 contains some standard conditions to assure that most tools in linear Schrödinger operators apply. These conditions are certainly not optimal. The main assumption in A0-A2 is the condition $2e_{01} > |e_0|$ in assumption A1. The rest of assumption A1 are just generic assumptions. This condition states that the excited state energy is closer to the continuum spectrum than to the ground state energy. It guarantees that twice the excited state energy of H_1 (which one obtains from taking the square of the excited state component) becomes a resonance in the continuum spectrum (of H_1). This resonance produces the main relaxation mechanism. If this condition fails, the resonance occurs in higher order terms and a proof of relaxation will be much more complicated. Also, the rate of decay will be different.

The main result in [12] concerning the relaxation of the ground states can be summarized in the following theorem.

Theorem A *Suppose that suitable assumptions on V hold. Then there are small universal constants $\varepsilon_0, n_0 > 0$ such that, if the initial data ψ_0 satisfies $\left\|\psi_0 - ne^{i\Theta_0}\phi_0\right\|_Y \le \varepsilon_0^2 n^2$ for some $n \le n_0$ and some $\Theta_0 \in \mathbb{R}$, then there exists an E_∞ and a function $\Theta(t)$ such that $\|Q_{E_\infty}\|_Y - n = O(\varepsilon_0^2 n)$, $\Theta(t) = -E_\infty t + O(\log t)$ and*

$$\left\|\psi(t) - Q_{E_\infty} e^{i\Theta(t)}\right\|_{L^2_{\mathrm{loc}}} \le C(1+t)^{-1/2}\ . \tag{1.10}$$

This theorem settles the question of asymptotic profile near ground states. Suppose that the initial data ψ_0 is now near some nonlinear excited state. From the physical ground, we expect that ψ_t will eventually decay to some ground state unless the initial data ψ_0 is exactly a nonlinear excited state. We call this the "strong relaxation property". For comparison, we define a weaker property, the "generic relaxation property", as follows. Denote the space of initial data by X. Let X_1 (X_0 resp.) be the subspace of initial data such that the asymptotic profiles are given by some nonlinear excited (ground resp.) states. We shall say that the dynamics satisfy the generic relaxation property if X_1 has "measure zero". This concept depends on a notion of measure which should be specified in each context.

With this definition, the strong relaxation property means that X_1 is exactly the space of excited states. In particular, X_1 is finite dimensional. We first note that the strong relaxation property is false. For any nonlinear excited state Q_1, define X_{1,Q_1} to be the set of initial data converging to Q_1 asymptotically. It is proved in [13] that for any given nonlinear excited state Q_1, X_{1,Q_1} contains a finite co-dimensional set. Thus our goal is to establish some weaker statement such as the generic relaxation property. This is the first step toward a classification of asymptotic dynamics of the nonlinear Schrödinger equation.

In this paper, we shall prove that for any excited state, there is a small neighborhood $\mathcal{N}$ so that

$$|\mathcal{N} \cap X_1| \le C\|\psi_0\|^2 |\mathcal{N} \cap X_0|$$

This estimates states that the ratio between X_1 and X around excited states are bounded by the mass of the initial wave function $\|\psi_0\|^2$. (Since we have not given a measure on the space of initial data, this statement is not well-defined and a precise statement will be given later on.)

In order to state the main result, we first decompose the wave function using the eigenspaces of the Hamiltonian H_0 as

$$\psi = \underline{x}\phi_0 + \underline{y}\phi_1 + \underline{\xi}, \quad \underline{\xi} = \mathbf{P}_c{}^{H_0}\, \psi\ . \tag{1.11}$$

For initial data near excited states, this decomposition contains an error of order y_0^3 and it is difficult to read from (1.11) whether the wave function is exactly an excited state. Thus we shall use the decomposition

$$\psi = x\phi_0 + Q_1(y) + \xi\ . \tag{1.12}$$

where

$$y = \underline{y}, \quad x = \underline{x} - (\phi_0, Q_1(y)), \quad \xi = \underline{\xi} - \mathbf{P}_c Q_1(y)\ . \tag{1.13}$$

Here we have used the convention that

$$Q_1(y) := Q_1(m)e^{i\Theta}, \qquad m = |y|, \ \ me^{i\Theta} = y$$

We shall prove that for ψ with sufficiently small Y norm (1.7), such a decomposition exists and is unique in section 2. Thus we assume that $\psi_0 = x_0\phi_0 + Q_1(y_0) + \xi_0$ is sufficiently small in Y. Let n_0 and ε_0 be the same small constants given in Theorem A. By choosing a smaller n_0, we may assume $n_0 \le \varepsilon_0^2/4$. We assume the initial data satisfies that

$$\begin{aligned} &\|\psi_0\|_Y = n, \ 0 < n \le n_0, \qquad |y_0| \ge \tfrac{1}{2}n, \\ &|x_0| \ge 2n\, e^{-1/n}, \qquad |x_0| \ge \varepsilon_2^{-1} n^2\, \|\xi_0\|_Y\ , \end{aligned} \tag{1.14}$$

where $\varepsilon_2 > 0$ is a small universal constant to be fixed later in the proof. These conditions can be interpreted as follows: The excited state component, y_0, should account for at least half the mass of the initial data. Under this condition, if the ground state component, x_0 is not too small compared with the continuum component

ξ, then the dynamics relaxes to some ground state. The condition $|x_0| \geq 2n\, e^{-1/n}$ is a very mild assumption to make sure that x_0 is not incredibly small. The following constant will be used in many places in this paper. Define

$$\varepsilon := \min\left\{\varepsilon_0/2,\, (\log(2n/x_0))^{-1/2}\right\} . \tag{1.15}$$

Since $n \leq n_0 \leq \varepsilon_0^2/4$ and $|x_0| \geq 2n\, e^{-1/n}$, we have $n \leq \varepsilon^2$.

Theorem 1.1 *Suppose the assumptions on V given above hold. Let $\psi(t,x)$ be a solution of* (1.1) *with the initial data ψ_0 satisfying* (1.14)*. Let*

$$n_1 = \left(|x_0|^2 + \tfrac{1}{2}\,|y_0|^2\right)^{1/2} \sim n . \tag{1.16}$$

Then, there exists an E_∞ and a function $\Theta(t)$ such that $\|Q_{E_\infty}\|_Y - n_1 = O(\varepsilon n)$, $\Theta(t) = -E_\infty t + O(\log t)$ and

$$C_1(1+t)^{-1/2} \leq \left\|\psi(t) - Q_{E_\infty} e^{i\Theta(t)}\right\|_{L^2_{\rm loc}} \leq C_2(1+t)^{-1/2} . \tag{1.17}$$

for some constants C_1 and C_2.

It is instructive to compare our result with the linear stability analysis of [8] [7] [2] [3] and [9]. In our setup the main result in [3] states that the linearized operator around a nonlinear excited state is structural stable if $e_0 < 2\,e_1$ and unstable if $e_0 > 2\,e_1$. Hence the excited states considered in this article is unstable and is expected to decay under generic perturbations. The instability of the excited state contains in Theorem 1.1 is thus consistent with the linear analysis. Notice that Theorem 1.1 tracks the dynamics for all time including time regime when the dynamics are far away from the excited states. Furthermore, *for all initial data considered in Theorem 1.1, the relaxation rate to the asymptotic ground state is exactly of order $t^{-1/2}$*, a rate very different from the standard linear Schrödinger equations.

In view of the linear analysis, the existence [13] of (nonlinear) stable directions for excited states is a more surprising result. For the linear stable case, i.e., $e_0 > 2\,e_1$, the only rigorous result is the existence [13] of (nonlinear) stable directions in this case. Although the linear analysis states that all directions are linearly stable, on physics ground we still expect excited states remains generically unstable.

We now explain the main idea of the proof for Theorem 1.1. The relaxation mechanism can be divided into three time regimes:

1. **The initial layer**: The component of the wave function in the continuum spectrum direction gradually disperses away; the components in the bound states directions do not change much.

2. **The transition regime**: Transition from the excited state to the ground state takes place in this interval. The component along the ground state grows in this regime; that along the excited state is slightly more complicated. We can further divide this time regime into two intervals. In part (i), the component along the excited state does not change much. In part (ii), it decreases steadily and eventually becomes smaller than the component along the ground state.

3. **Stabilization**: The ground state dominates and is stable. Both the excited states and dispersive part gradually decay.

In different time regimes, the dominant terms are different and we have to linearize the dynamics according to the dominant terms. In the first time region, $\psi(t)$ is near an excited state, and it is best to use operator linearized around the excited state. In the third time regimes, $\psi(t)$ is near a ground states, and it is best to use operator linearized around a ground state. For the transition regimes, the dynamics are far away from both excited and ground states and we can only use the linear Hamiltonian H_0.

Besides technical problems associated with changing coordinate systems in different time intervals, there is an intrinsic difficulty related to the time reversibility of the Schrodinger equation. Imagine that we are now ready to show that our dynamics is in the third time regime and will stabilize around some nonlinear ground state. If we take the wave function ψ_t at this time and time reverse the dynamics, then the dynamics will drive this wave function back to the initial state near some excited state. The time reversed state ψ_t and the wave function ψ_t itself will satisfy the same estimates in the usual Sobolev or L_p senses. However, their dynamics are completely different: one stabilizes to a ground state; the other back to near an excited state. This suggests that ψ_t carries information concerning the time direction and this information will not show if we measure it by the usual estimates.

This time reversal difficulty manifest itself in the technical proofs as follows. We shall see that, when the third time regime begin, the dispersive part is not well-localized and its L^2-norm can be larger than that of the bound states—both violate conditions for approaching to ground states in [12]. To resolve this issue, we need to extract information which are time-direction sensitive so that even though the disperive part may be large, it is irrelevant since it is "out-going". Though the concept of "out-going" wave is known for linear Schrödinger equations, it is difficult to implement it for nonlinear Schrödinger equations. Our strategy is to identify the main terms of the dispersive part and calculate them explicitly. These terms carry sufficient information concerning the time direction. The rest are error terms and we can use various Sobolev or L_p estimates.

Resonance induced decay and growth

To illustrate the mechanism of resonance induced decay and growth, we consider the problem in the coordinates with respect to the linear Hamiltonian $H_0 = -\Delta + V$,

$$\psi(t) = x(t)\phi_0 + y(t)\phi_1 + \xi(t) \ , \quad , \xi(t) \in \mathbf{P}_c^{H_0}\, \psi(t)$$

The nonlinear term $\psi^2\bar{\psi}$, (assume $\lambda = 1$) can be split into a sum of many terms using this decomposition. However, we claim that there is only one important nonlinear term in the equation for each component:

$$\begin{aligned} i\dot{x} &= e_0 x + (\phi_0, (y\phi_1)^2\bar{\xi}) + \cdots && (1.18)\\ i\dot{y} &= e_1 y + (\phi_1, 2(x\phi_0)(\bar{y}\phi_1)\xi) + \cdots && (1.19)\\ i\partial_t\xi &= H_0\xi + \mathbf{P}_c^{H_0}\, \bar{x}y^2\phi_0\phi_1^2 + \cdots && (1.20) \end{aligned}$$

From (1.18), we know $u(t) = e^{ie_0 t}x(t)$ has less oscillation of lower order than $x(t)$. Hence we say $x(t)$ has a phase factor $-e_0$. Similarly, $y(t)$ has a phase factor $-e_1$. The nonlinear term $\bar{x}y^2\phi_0\phi_1^2$ has a phase factor $e_0 - 2e_1$, which, due to the assumption (1.3), is the only term in $\psi^2\bar{\psi}$ with a negative phase factor. It gives a term in ξ:

$$\xi(t) = \bar{x}y^2(t)\Phi + \cdots , \qquad \Phi = \frac{1}{H + e_0 - 2e_1 - 0i}\,\phi_0\phi_1^2$$

Notice that Φ is complex. Substituting this term into (1.18) and (1.19), we have

$$\begin{aligned} i\dot{x} &= i\gamma_0|y|^4 x + \cdots , \\ i\dot{y} &= -2i\gamma_0|x|^2|y|^2 y + \cdots , \end{aligned} \qquad (1.21)$$

with γ_0 given in (1.8). In (1.21) we have omitted two types of irrelevant terms:

1. Terms with same phase factors as x or y: for example, $e_0 x$ and $|y|^2 x$ in (1.18). Since their coefficients are real, they disappear when we consider the equations for $|x|$ and $|y|$.

2. Terms with different phase factors: for example, $\bar{x}y^2$ in (1.18). Since these terms have different phases, their contribution averaging over time will be small. This can be made precise by the Poincaré normal form.

From (1.21) we obtain the decay of y and the growth of x as well as the three time regimes mentioned previously. However, it should be warned that this set-up is only suitable when both x and y are of similar sizes.

2 The initial layer and the transition regimes: The set up

We now outline the basic strategy for the initial layer and the transition regimes. We first review the construction of the bound state families.

2.1 Nonlinear bound states

The basic properties of the ground state families can be summarized in the following lemma from [12].

Lemma 2.1 *Suppose that $-\Delta + V$ satisfies the assumptions (A0) and (A2). Then there is an n_0 sufficiently small such that for E between e_0 and $e_0 + \lambda n_0^2$ there is a nonlinear ground states $\{Q_E\}_E$ solving* (1.4)*. The nonlinear ground state Q_E is real, local, smooth, $\lambda^{-1}(E - e_0) > 0$, and*

$$Q_E = n\,\phi_0 + O(n^3)\ , \qquad n \approx C\,[\lambda^{-1}(E - e_0)]^{1/2}\ , \quad C = (\int \phi_0^4\, dx)^{-1/2}.$$

Moreover, we have $R_E = \partial_E Q_E = O(n^{-2})\,Q_E + O(n) = O(n^{-1})$ and $\partial_E^2 Q_E = O(n^{-3})$. If we define $c_1 \equiv (Q, R)^{-1}$, then $c_1 = O(1)$ and $\lambda c_1 > 0$.

This lemma can be proved using standard perturbation argument and similar conclusions hold for excited states as well. For the purpose of this paper, we prefer to use the value $m = (\phi_1, Q_1)$ as the parameter and refer to the family of excited states as $Q_1(m)$. It is straightforward to compute the leading corrections of $Q_1(m)$ via standard perturbation argument used in proving Lemma 2.1. Thus we can write Q_1 as

$$Q_1(m) = m\phi_1 + \left(m^3 q_3 + q^{(5)}(m)\right) := m\phi_1 + q(m)\ , \quad q^{(5)}(m) = O(m^5), \quad q(m) \perp \phi_1\ , \tag{2.1}$$

where $q_3 = -\lambda(H_0 - e_1)^{-1}\pi\phi_1^3$ and π is the projection

$$\pi h = h - (\phi_1, h)\phi_1. \tag{2.2}$$

Similarly, we can also expand $E_1(m)$ in m as

$$E_1(m) = e_1 + E_{1,2}m^2 + E_{1,4}m^4 + E_1^{(6)}(m), \qquad E_1^{(6)}(m) = O(m^6) \ . \tag{2.3}$$

Moreover, we can differentiate the relation of $Q_1(m)$ w.r.t. m to have

$$Q_1'(m) = \frac{d}{dm}Q_1 = \phi_1 + q'(m) \ , \qquad q'(m) = \frac{d}{dm}q(m) = O(m^2), \quad q'(m) \perp \phi_1, \tag{2.4}$$

2.2 Equations

Thus in the first and second time regimes, we write

$$\psi(t) = x(t)\phi_0 + Q_1(m(t))e^{i\Theta(t)} + \xi(t) \ , \tag{2.5}$$

where $\xi \in \mathtt{H}_c(H_0)$, see (1.12). If we write $\Theta(t) = \theta(t) - \int_0^t E_1(m(s))\,ds$, we can write $y(t)$ as

$$y = me^{i\Theta} = m\exp\left\{i\theta(t) - i\int_0^t E_1(m(s))\,ds\right\} \ . \tag{2.6}$$

Denote the part orthogonal to ϕ_1 by $h = x\phi_0 + \xi$. From the Schrödinger equation, we have h satisfies the equation

$$\begin{aligned} i\partial_t h &= H_0 h + G + \Lambda, \\ G &= \lambda|\psi|^2\psi - \lambda Q_1^3 e^{i\Theta} \\ &= \lambda Q_1^2(e^{i2\Theta}\bar{h} + 2h) + \lambda Q_1(e^{i\Theta}2h\bar{h} + e^{-i\Theta}h^2) + \lambda|h|^2 h \qquad (2.7) \\ \Lambda &= \left(\dot{\theta}Q_1 - i\dot{m}Q_1'\right)e^{i\Theta} \ . \qquad (2.8) \end{aligned}$$

Since $m(t)$ and $\theta(t)$ are chosen so that (2.5) holds, we have $0 = (\phi_1, i\partial_t h(t)) = \left(\phi_1,\, G + (\dot{\theta}Q_1 - i\dot{m}Q_1')e^{i\Theta}\right)$. Hence $m(t)$ and $\theta(t)$ satisfy

$$\dot{m} = \left(\phi_1, \operatorname{Im} Ge^{-i\Theta}\right) \ , \qquad \dot{\theta} = -\frac{1}{m}\left(\phi_1, \operatorname{Re} Ge^{-i\Theta}\right) \ . \tag{2.9}$$

We also have the equation for y:

$$i\dot{y} = i\dot{m}e^{i\Theta} - (\dot{\theta} - E_1(m))me^{i\Theta} = E_1(m)y + e^{i\Theta}(i\dot{m} - m\dot{\theta}) = E_1(m)y + (\phi_1, G) \ .$$

Here we have used (2.9). Denote $\Lambda_\pi = \pi\Lambda$. We can decompose the equation for h into equations for x and ξ. Summarizing, we have the original Schrödinger equation is equivalent to

$$\begin{cases} i\dot{x} = e_0\,x + (\phi_0,\, G + \Lambda_\pi) \\ i\dot{y} = E_1(m)y + (\phi_1, G) \\ i\partial_t\xi = H_0\,\xi + \mathbf{P}_c\,(G + \Lambda_\pi) \end{cases} \tag{2.10}$$

Clearly, x has an oscillation factor $e^{-ie_0 t}$, and y has a factor $e^{-ie_1 t}$. Hence we define

$$x = e^{-ie_0 t} u, \qquad y = e^{-ie_1 t} v \; . \tag{2.11}$$

Together with the integral form of the equation for ξ, we have

$$\dot{u} = -ie^{ie_0 t} \left(\phi_0,\, G + \Lambda_\pi\right), \tag{2.12}$$

$$\dot{v} = -ie^{ie_1 t} \left[(E_1(m) - e_1)y + (\phi_1, G)\right], \tag{2.13}$$

$$\xi(t) = e^{-iH_0 t}\xi_0 + \int_0^t e^{-iH_0(t-s)}\, \mathbf{P}_c^{H_0}\, G_\xi(s)\, ds \;, \qquad G_\xi = i^{-1}(G + \Lambda_\pi) \; . \tag{2.14}$$

This is the system we shall study.

2.3 Basic estimates and decompositions

It is useful to decompose various terms according to orders in n so that we can identify their contributions. We now proceed to so this for G, Λ_π, $E(|y|)$ and $\xi(t)$. We expect that $x, y = O(n)$ and $\xi = O(n^3)$ locally.

1. G Recall that G is given by

$$G = \lambda Q_1^2 (e^{i2\Theta}\bar{h} + 2h) + \lambda Q_1 (e^{i\Theta} 2h\bar{h} + e^{-i\Theta}h^2) + \lambda |h|^2 h$$

with $h = x\phi_0 + \xi$ and $Q_1 = Q_1(|y|)$. From the the decomposition (2.1) of $Q_1 = |y|\phi_1 + |y|^3 q_3 + q^{(5)}(|y|)$, we decompose G as

$$\begin{aligned} G = {} & \lambda(y^2\phi_1^2 + 2y^3\bar{y}\phi_1 q_3)\, \bar{h} + \lambda(|y|^2\phi_1^2 + 2|y|^4\phi_1 q_3)\, 2h \\ & + \lambda(y\phi_1 + y^2\bar{y}q_3)\, 2|h|^2 + \lambda(\bar{y}\phi_1 + y\bar{y}^2 q_3)\, h^2 + \lambda|h|^2 h + (*) \end{aligned}$$

where $(*) = \lambda\left[2|y|\phi_1 q^{(5)} + (|y|^3 q_3 + q^{(5)})^2\right]\,(e^{i2\Theta}\bar{h} + 2h) + \lambda q^{(5)}\,(e^{i\Theta}2h\bar{h} + e^{-i\Theta}h^2)$ with $q^{(5)} = q^{(5)}(|y|)$. We then substitute $h = x\phi_0 + \xi$ to obtain

$$G = G_3 + G_5 + G_7, \tag{2.15}$$

where

$$G_3 = \lambda(y^2\bar{x} + 2|y|^2 x)\phi_0\phi_1^2 + \lambda(2|x|^2 y + x^2\bar{y})\phi_0^2\phi_1 + \lambda|x|^2 x\phi_0^3 \tag{2.16}$$

$$\begin{aligned} G_5 = {} & \lambda(2y^3\bar{y}\bar{x} + 4|y|^4 x)\, \phi_0\phi_1 q_3 + \lambda(2|x|^2 y^2\bar{y} + x^2 y\bar{y}^2)\, \phi_0^2 q_3 \\ & + \lambda(x\phi_0 + y\phi_1)^2\overline{\xi} + 2\lambda|(x\phi_0 + y\phi_1)|^2\xi \end{aligned} \tag{2.17}$$

and

$$G_7 = \lambda \left[2|y|\phi_1 q^{(5)}(|y|) + (|y|^3 q_3 + q^{(5)}(|y|))^2\right] \, (e^{i2\Theta}\bar{h} + 2h) \tag{2.18}$$
$$+ \lambda q^{(5)}(|y|)\,(e^{i\Theta} 2h\bar{h} + e^{-i\Theta}h^2)$$
$$+ \lambda \left(2y^3\bar{y}\phi_1 q_3\,\bar{\xi} + 4|y|^4\phi_1 q_3\,\xi\right) + \lambda(y\phi_1 2|\xi|^2 + \bar{y}\phi_1\xi^2)$$
$$+ \lambda y^2\bar{y}q_3\, 2(x\phi_0\bar{\xi} + \bar{x}\phi_0\xi + |\xi|^2) + \lambda y\bar{y}^2 q_3\,(2x\phi_0\xi + \xi^2)$$
$$+ \lambda\phi_0(\bar{x}\xi^2 + 2x|\xi|^2) + \lambda|\xi|^2\xi\ .$$

Note that $G_3 = O(n^3)$, $G_5 = O(n^5)$ and $G_7 = O(n^7)$. If we use the convention that

$$f \lesssim g_1 + g_2 + \cdots$$

for $\|f\| \le C\,\|g_1\| + \|g_2\| + \cdots$ for some suitable norms, we have

$$G \lesssim n^2 x + n^2\xi + \xi^3 \tag{2.19}$$
$$G_5 \lesssim n^4 x + n^2\xi \tag{2.20}$$
$$G_7 \lesssim n^6 x + n^4\xi + n\xi^2 + \xi^3 \tag{2.21}$$

It is crucial to observe that *no term in G_3 is of order y^3*. This is due to our setup emphasizing the role of nonlinear excited states. The price we pay is the introduction of terms involving q_3 and $q^{(5)}$.

We now identify the main oscillation factors of various terms. For example, $y^2\bar{x} = e^{i(-2e_1+e_0)t}v^2\bar{u}$, and its factor is $-2e_1 + e_0$. For terms in G_3 we have

$$\begin{array}{ccccc} y^2\bar{x} & |y|^2 x & |x|^2 y & x^2\bar{y} & |x|^2 x \\ -2e_1 + e_0 & -e_0 & -e_1 & -2e_0 + e_1 & -e_0 \end{array} \tag{2.22}$$

From the spectral assumption $|e_0| > 2|e_1|$, $-2e_1 + e_0$ is the only negative phase factor. Hence it is the only term of order (n^3) that have resonance effect when we compute the main part of ξ. Also, since $|x|^2 y$ has the same phase as y, it will be resonant in the y-equation. Similarly, $|y|^2 x$ and $|x|^2 x$ have same phase as x and will be resonant in x-equation.

2. Λ_π and $E(m)$ Recall $\Lambda_\pi = \pi(\dot{\theta}Q_1 - i\dot{m}Q_1')e^{i\Theta}$. Since $\dot{\theta} = O(n^{-1}\,\|G\|_{\rm loc})$ and $\dot{m} = O(\|G\|_{\rm loc})$,

$$\|\Lambda_\pi(s)\| = O(\dot{\theta})\,O(\pi Q_1) + O(\dot{m})\,O(\pi Q_1') \le Cn^2\,\|G\|_{\rm loc}\ , \tag{2.23}$$

To find out the main part of Λ_π, we substitute equation (2.9) for $\dot m$ and $\dot\theta$ to obtain ($m = |y|$),

$$\begin{aligned}\Lambda_\pi &= \pi(\dot\theta Q_1 - i\dot m Q_1')\, e^{i\Theta}\\ &= -\left\{(\phi_1,\, G/2)m^{-1}\pi Q_1 + (\phi_1,\, \bar G/2)m^{-1}\pi Q_1 e^{2i\Theta}\right\}\\ &\quad - i\left\{(\phi_1,\, G/2i)\pi Q_1' + (\phi_1,\, \bar G/2i)\pi Q_1' e^{2i\Theta}\right\}\end{aligned}$$

Since $G = G_3 + (G_5 + G_7)$ and $\pi Q_1(m) = m^3 q_3 + q^{(5)}(m)$ by (2.1), we have $\pi Q_1'(m) = 3m^2 q_3 + O(m^4)$, and the main part of Λ_π is (also recall $y = me^{i\Theta}$)

$$\begin{aligned}\Lambda_{\pi,5} &= -\tfrac12\left\{(\phi_1,\, G_3)\,|y|^2 q_3 + (\phi_1,\, \bar G_3)\, y^2 q_3\right\}\\ &\quad - \tfrac12\left\{(\phi_1,\, G_3)\, 3|y|^2 q_3 + (\phi_1,\, \bar G_3)\, 3y^2 q_3\right\}\\ &= -2q_3\, (\phi_1,\, G_3|y|^2 + \bar G_3 y^2)\end{aligned} \tag{2.24}$$

Let $\Lambda_{\pi,7} = \Lambda_\pi - \Lambda_{\pi,5}$. We have

$$\Lambda_\pi = \Lambda_{\pi,5} + \Lambda_{\pi,7} \tag{2.25}$$

$$\Lambda_{\pi,5} \lesssim \|G_3\|_{\mathrm{loc}}\, |y|^2 \lesssim n^4 x \tag{2.26}$$

$$\Lambda_{\pi,7} \lesssim \|G_5 + G_7\|_{\mathrm{loc}}\, |y|^2 + \|G\|_{\mathrm{loc}}\, |y|^4\ . \tag{2.27}$$

3. ξ Recall the equation for ξ in (2.14),

$$\xi(t) = e^{-iH_0 t}\xi_0 + \int_0^t e^{-iH_0(t-s)}\ \mathbf{P}_c^{H_0}\, G_\xi(s)\, ds\ , \qquad G_\xi = i^{-1}(G + \Lambda_\pi)\ .$$

Since $\|\Lambda_\pi\| \le Cn^2\, \|G\|_{\mathrm{loc}}$, the main terms in $G_\xi = i^{-1}(G + \Lambda_\pi)$ is $i^{-1}G_3$. We now compute the first term $\lambda y^2 \bar x \phi_0 \phi_1^2$ in G_3 using integration by parts:

$$\begin{aligned}&- i\lambda \int_0^t e^{-iH_0(t-s)}\ \mathbf{P}_c y^2 \bar x \phi_0\phi_1^2\, ds\\ &= -i\lambda e^{-iH_0 t}\int_0^t e^{i(H_0 - 0i)s}\, e^{i(e_0 - 2e_1)s}\, v^2 \bar u\, \mathbf{P}_c \phi_0\phi_1^2\, ds\\ &= -i\lambda e^{-iH_0 t}\left\{\left[\frac{1}{i(H_0 - 0i + e_0 - 2e_1)} e^{iH_0 s}\, e^{i(e_0-2e_1)s}\, v^2\bar u\, \mathbf{P}_c\phi_0\phi_1^2\right]_0^t\right.\\ &\qquad\left. - \int_0^t \frac{1}{i(H_0 - 0i + e_0 - 2e_1)} e^{iH_0 s}\, e^{i(e_0-2e_1)s}\frac{d}{ds}\left(v^2\bar u\right)\, \mathbf{P}_c\phi_0\phi_1^2\, ds\right\}\\ &= y^2\bar x\Phi_1 - e^{-iH_0 t} y^2 \bar x(0)\Phi_1 - \int_0^t e^{-iH_0(t-s)}\, e^{i(e_0-2e_1)s}\frac{d}{ds}\left(v^2\bar u\right)\Phi_1\, ds\ ,\end{aligned}$$

where

$$\Phi_1 = \frac{-\lambda}{H_0 - 0i + e_0 - 2e_1}\ \mathbf{P}_c\phi_0\phi_1^2\ . \tag{2.28}$$

This term, with the phase factor e_0-2e_1, is the only one in G_3 having a negative phase factor (see (2.5)). Since $-(e_0-2e_1)$ is in the continuous spectrum of H_0, $H_0+e_0-2e_1$ is not invertible, and needs a regularization $-0i$. We choose $-0i$, not $+0i$, so that the term $e^{-iH_0t}y^2\bar{x}(0)\Phi_1$ decays as $t\to\infty$. See Lemma 2.2.

We can integrate all terms in G_3 and obtain the main terms of $\xi(t)$ as

$$\xi^{(2)}(t)=y^2\bar{x}\Phi_1+|y|^2x\Phi_2+|x|^2y\Phi_3+x^2\bar{y}\Phi_4+|x|^2x\Phi_5 \tag{2.29}$$

where

$$\begin{aligned}\Phi_2&=\frac{-2\lambda}{H_0-e_0}\,\mathbf{P}_\mathrm{c}\phi_0\phi_1^2\ , &\qquad \Phi_3&=\frac{-2\lambda}{H_0-e_1}\,\mathbf{P}_\mathrm{c}\phi_0^2\phi_1\ ,\\ \Phi_4&=\frac{-\lambda}{H_0-2e_0+e_1}\,\mathbf{P}_\mathrm{c}\phi_0^2\phi_1\ , &\qquad \Phi_5&=\frac{-\lambda}{H_0-e_0}\,\mathbf{P}_\mathrm{c}\phi_0^3\ .\end{aligned} \tag{2.30}$$

The rest of $\xi(t)$ is

$$\begin{aligned}\xi^{(3)}(t)&=e^{-iH_0t}\xi_0-e^{-iH_0t}\xi^{(2)}(0)-\int_0^t e^{-iH_0(t-s)}\,\mathbf{P}_\mathrm{c}\,G_4\,ds\\ &\quad+\int_0^t e^{-iH_0(t-s)}\,\mathbf{P}_\mathrm{c}\left(G_\xi-i^{-1}G_3-i^{-1}\lambda|\xi|^2\xi\right)\,ds\\ &\quad+\int_0^t e^{-iH_0(t-s)}\,\mathbf{P}_\mathrm{c}\left(i^{-1}\lambda|\xi|^2\xi\right)\,ds\\ &\equiv\xi_1^{(3)}(t)+\xi_2^{(3)}(t)+\xi_3^{(3)}(t)+\xi_4^{(3)}(t)+\xi_5^{(3)}(t)\end{aligned}$$

where $\xi_4^{(3)}(t)$ and $\xi_5^{(3)}(t)$ are higher order terms in G_ξ which we did not integrate and the integrand G_4 in $\xi_3^{(3)}(t)$ consists of the remainders from the integration by parts:

$$\begin{aligned}G_4=&\,e^{i(e_0-2e_1)s}\frac{d}{ds}\left(v^2\bar{u}\right)\Phi_1+e^{i(-e_0)s}\frac{d}{ds}\left(|v|^2u\right)\Phi_2\\ &+e^{i(-e_1)s}\frac{d}{ds}\left(|u|^2v\right)\Phi_3+e^{i(-2e_0+e_1)s}\frac{d}{ds}\left(u^2\bar{v}\right)\Phi_4+e^{i(-e_0)s}\frac{d}{ds}\left(u^2\bar{u}\right)\Phi_5\ ,\end{aligned} \tag{2.31}$$

Here we single out $\xi_5^{(3)}(t)$ since $|\xi|^2\xi$ is a non-local term. Thus we have following decomposition for ξ:

$$\xi(t)=\xi^{(2)}(t)+\xi^{(3)}(t)=\xi^{(2)}+\left(\xi_1^{(3)}+\cdots+\xi_5^{(3)}\right)\ , \tag{2.32}$$

2.4 Linear estimates

We now summarize known results concerning the linear analysis. The decay estimate was contained in [4] and [14]; the estimate (2.34) was taken from [11] and [12];

Lemma 2.2 (decay estimates for e^{-itH_0}) *For $q \in [2,\infty]$ and $q' = q/(q-1)$,*

$$\left\| e^{-itH_0}\ \mathbf{P}_{\mathrm{c}}{}^{H_0}\phi \right\|_{L^q} \le C\,|t|^{-3\left(\frac{1}{2}-\frac{1}{q}\right)}\,\|\phi\|_{L^{q'}}\ . \tag{2.33}$$

For smooth local functions ϕ and sufficiently large r_0, we have

$$\lim_{\sigma\to 0+}\left\| \langle x\rangle^{-r_0}\ e^{-itH_0}\,\frac{1}{(H_0+e_0-2e_1-\sigma i)^k}\,\mathbf{P}_{\mathrm{c}}{}^{H_0}\,\langle x\rangle^{-r_0}\,\phi \right\|_{L^2} \le C\,\langle t\rangle^{-9/8} \tag{2.34}$$

where $k=1,2$.

Intuitively, we can write

$$e^{-iH_0 t}\Phi_1 = \lim_{\varepsilon\to 0+} e^{-iH_0 t}\int_0^\infty e^{-i(H_0-\varepsilon i+e_0-2e_1)s}\,\mathbf{P}_{\mathrm{c}}\phi_0\phi_1^2\,ds\ .$$

3 The initial layer and the transition regimes: The estimates

In this section we wish to show the following picture for the solution $\psi(t)$: In the initial layer regime, the dispersive part gradually disperses away, while the sizes of the bound states do not change much. In the transition regime, the *original* dispersive part becomes negligible, while the ϕ_0-components of $\psi(t)$ increases and the ϕ_1-component decreases.

Recall the orthogonal decomposition $\psi(t) = \underline{x}\phi_0 + \underline{y}\phi_1 + \underline{\xi}$ (1.11). We have $|\underline{x}(t)|^2 + |\underline{y}(t)|^2 + \left\|\underline{\xi}(t)\right\|_{L^2}^2 = \|\psi(t)\|_{L^2}^2 \le n^2$. If we decompose ψ via (1.12), i.e.,

$$\psi(t) = x\phi_0 + Q_1(y) + \xi\ , \qquad \xi = \xi^{(2)} + \xi^{(3)}\ , \tag{3.1}$$

we have $y = \underline{y}$, $x = \underline{x} + O(y^3)$ and $\xi = \underline{\xi} + O(y^3)$. Thus

$$|x(t)|,\ |y(t)|,\ \|\xi(t)\|_{L^2} \le \tfrac{5}{4}n\ , \qquad \|\xi_0\|_Y \le 4n\ . \tag{3.2}$$

For $p = 1,2,4$, define the space

$$L^p_{\mathrm{loc}} \equiv \left\{ f:\ \langle x\rangle^{-r_0} f(x) \in L^p(\mathbb{R}^3) \right\}\ , \tag{3.3}$$

where $r_0 > 3$ is the exponent appeared in the linear estimate (2.34) in Lemma 2.2. Since $r_0 > 3$, we have $\|f\|_{L^1_{\mathrm{loc}}} \le C\,\|f\|_{L^p}$ for any p,

The following proposition is the main result for the dynamics in the initial layer and the transition regimes.

Proposition 3.1 *Suppose that V satisfies the assumptions given in §1. Let $\psi(t,x)$ be a solution of* (1.1) *with the initial data ψ_0 satisfying* (1.14). *Let $\varepsilon_3 > 0$ be a sufficiently small constant to be fixed later. Let $t_0 = \varepsilon_3 n^{-4}$. Then there exists t_1 and t_2 such that for some constant $C \le 10000$ we have*

$$t_0 \le t_1 \le \frac{1.01}{\gamma_0 n^4} \log\left(\frac{n}{|x_0|}\right) , \qquad t_1 + C\,(n^4\varepsilon^2)^{-1} \le t_2 \le t_1 + 10100\,(\gamma_0 n^4 \varepsilon^2)^{-1} , \tag{3.4}$$

and the following estimates hold:

(i) For $0 \le t \le 2\,t_2$,

$$|x(t)| \ge \tfrac{3}{4} \sup_{0\le s\le t} |x(s)| , \tag{3.5}$$

$$\begin{aligned} \|\xi(t)\|_{L^4} &\le C_2\, n^2 t^{1/4}\, |x(t)| + C_2\, \|\xi_0\|_Y \langle t\rangle^{-3/4} , \\ \|\xi(t)\|_{L^4_{\rm loc}} &\le C_2 n^2\, |x(t)| \quad + C_2\, \|\xi_0\|_Y \langle t\rangle^{-9/8} , \\ \left\|\xi^{(3)}(t)\right\|_{L^2_{\rm loc}} &\le\; C_2\, n^{15/4} |x(t)| + C_2\, \|\xi_0\|_Y \langle t\rangle^{-9/8} . \end{aligned} \tag{3.6}$$

where the constant C_2 will be specified in (3.19) *of next subsection.*

(ii) (Initial layer) for $0 \le t \le t_0$,

$$\begin{aligned} &\tfrac{1}{2}|x_0| \le |x(t)| \le \tfrac{3}{2}\,|x_0|, \\ &0.99\,|y_0| \le |y(t)| \le 1.01\,|y_0| \end{aligned} \tag{3.7}$$

(iii) For $n_1 = \left(|x_0|^2 + \frac{1}{2}\,|y_0|^2\right)^{1/2}$ defined in (1.16),

$$|x(t_1)| \ge 0.01 n , \quad |x(t_2)| \ge 0.99 n_1 , \quad \tfrac{1}{2}\varepsilon n \le |y(t_2)| \le 2\varepsilon n . \tag{3.8}$$

Notice that (3.4) implies

$$t_2 \le C_3 n^{-4} \tag{3.9}$$

for some constant C_3.

We will prove these estimates using (1.14), (3.2) and a continuity argument. Hence we can assume the following weaker estimates: For $0 \le t \le 2t_2$:

$$\begin{aligned} |x(t)| &\ge \tfrac{1}{2} \sup_{0\le s\le t} |x(s)| , \\ |x(t)| &\le 2|x_0| \qquad \text{for } t < t_0 , \\ \|\xi(t)\|_{L^4} &\le 2C_2\, n^2 t^{1/4}\, |x(t)| + 2C_2\, \|\xi_0\|_Y \langle t\rangle^{-3/4} , \\ \|\xi(t)\|_{L^4_{\rm loc}} &\le 2C_2 n^2\, |x(t)| \quad + 2C_2\, \|\xi_0\|_Y \langle t\rangle^{-9/8} , \\ \left\|\xi^{(3)}(t)\right\|_{L^2_{\rm loc}} &\le\; 2C_2\, n^{15/4} |x(t)| + 2C_2\, \|\xi_0\|_Y \langle t\rangle^{-9/8} . \end{aligned}$$

By continuity, if we prove Proposition 3.1 assuming these weaker estimates, we have proved the proposition itself. We shall see also estimates (3) will be used only in estimating higher order terms.

Recall from (2.23) that the local term Λ_π satisfies $\|\Lambda_\pi\|_r \le Cn^2 \|G\|_{L^1_{\mathrm{loc}}}$ for any r. Thus we have

$$|\dot{u}(t)| \lesssim \|G\|_{L^1_{\mathrm{loc}}}, \qquad |\dot{v}(t)| \lesssim \|G\|_{L^1_{\mathrm{loc}}} + |y|^3. \tag{3.10}$$

The following lemma provides estimates for G assuming the estimate (3).

Lemma 3.2 *Let G be given by* (2.15)–(2.18). *Suppose n is sufficiently small and the estimate* (3) *holds for $t \le C_3 n^{-4}$. Then we have the following estimates for G:*

$$\|G(t)\|_{L^{4/3}\cap L^{8/7}} \le C_4 n^2 |x(t)| + C(C_2) n^2 \|\xi_0\|_Y \langle t\rangle^{-9/8} . \tag{3.11}$$

$$\|(G-G_3)(t)\|_{L^{4/3}\cap L^{8/7}} \le C_4 n^{15/4} |x(t)| + C(C_2) n^2 \|\xi_0\|_Y \langle t\rangle^{-9/8} . \tag{3.12}$$

$$\|(G-G_3)(t)\|_{L^1_{\mathrm{loc}}} \le C_4 n^4 |x(t)| + C(C_2) n^2 \|\xi_0\|_Y \langle t\rangle^{-9/8} . \tag{3.13}$$

where C_4 is a constant independent of C_2 and $C(C_2)$ denotes constants depending on C_2.

Moreover, (3.11) *and* (3.12) *remain true if we replace G by G_ξ, and $(G-G_3)$ by $(G_\xi - i^{-1}G_3)$. Furthermore, we have*

$$\|G_\xi(t)\|_{L^1} \le C_5 n^3 . \tag{3.14}$$

By the assumption (1.14), when $t > t_0$ the last term $Cn^2 \|\xi_0\|_Y \langle t\rangle^{-9/8}$ is smaller and can be removed. The proof of this lemma is a straightforward application of the Holder and Schwarz inequalities.

We will use (3.11) and (3.12) for ξ, and (3.13) for x and y.

Proof. Recall (2.15) that $G = G_3 + G_5 + G_7$. We first consider only the non-local term $\lambda|\xi|^2\xi$ in G, which belongs to G_7. Since $n \le \varepsilon^2$ and $t_2 \le C_3 \varepsilon^{-2} n^{-4}$, by (3) we have $\|\xi(s)\|_{L^4} \le Cn^{3/4}|x(s)| + C\|\xi_0\|_Y \langle s\rangle^{-3/4}$. Also using (3.2) and the H older inequality we have

$$\left\||\xi|^2\xi(s)\right\|_{L^{4/3}} \le C\|\xi(s)\|_{L^4}^3 \le C\left(n^{3/4}|x(s)|\right)^3 + C\|\xi_0\|_Y^3 \langle s\rangle^{-9/4} . \tag{3.15}$$

$$
\begin{aligned}
\left\| |\xi|^2 \xi(s) \right\|_{L^{8/7}} &\le C \, \|\xi(s)\|_{L^2}^{1/2} \, \|\xi(s)\|_{L^4}^{5/2} \\
&\le C n^{1/2} \left\{ \left(n^{3/4} |x(s)| \right)^{5/2} + \|\xi_0\|_Y^{5/2} \langle s \rangle^{-15/8} \right\} \\
&\le C n^{4-1/8} |x(s)| + C n^2 \, \|\xi_0\|_Y \langle s \rangle^{-15/8} \ .
\end{aligned} \tag{3.16}
$$

Hence this non-local term satisfies (3.11)–(3.12). Moreover, to prove (3.13), we can bound $\| |\xi|^2 \xi(s) \|_{L^1_{\mathrm{loc}}}$ by $\|\xi(s)\|_{L^4}^3$.

For the local term $G - \lambda |\xi|^2 \xi = G_3 + G_5 + (G_7 - \lambda |\xi|^2 \xi)$, all L^p-norms are equivalent. We can read from the explicit expressions of G the following estimates:

$$
\begin{aligned}
G_3 &\lesssim n^2 x \\
G_5 &\lesssim n^4 x + n^2 \, \xi \\
G_7 - \lambda |\xi|^2 \xi &\lesssim n^6 \, |x| + n^4 \, \xi + n \, \xi^2
\end{aligned}
$$

To estimate ξ in $G_\xi - \lambda |\xi|^2 \xi$, we can use $\|\xi\|_{L^4_{\mathrm{loc}}}$. For example,

$$
\left\| \bar{y} \phi_1 \xi^2 \right\|_{L^{4/3}} \le C |y| \left\| \phi_1 \langle x \rangle^{2r_0} \right\|_{L^4} \|\xi\|_{L^4_{\mathrm{loc}}}^2 \le C n \left((n^2 |x|)^2 + \|\xi_0\|_Y^2 \langle s \rangle^{-9/4} \right) \ .
$$

Together with the explicit expressions of G and G_3, similar arguments show (3.11)–(3.13).

Since $G_\xi = i^{-1}(G + \Lambda_\pi)$ and $\|\Lambda_\pi\| \le C n^2 \, \|G\|_{\mathrm{loc}}$ by (2.23), (3.11) and (3.13) hold if we replace G and $G - G_3$ by G_ξ and $G_\xi - i^{-1} G_3$. obtain (3.14), we only need to check the non-local term $\lambda |\xi|^2 \xi$. Since $\|\xi(s)\|_{L^4} \le (2 C_2 C_3^{1/4} + 8 C_2) n$, we have

$$
\left\| |\xi|^2 \xi \right\|_{L^1} \le C_{4,1} \, \|\xi(s)\|_{L^2} \, \|\xi(s)\|_{L^4}^2 \le \frac{1}{10} C_5 n \, n^2 \ , \tag{3.17}
$$

provided we choose $C_5 \ge 10 (2 C_2 C_3^{1/4} + 8 C_2) C_{4,1}$. Thus the lemma is proved. **Q.E.D.**

3.1 Estimates of the dispersive part

We now prove the estimates for ξ in Proposition 3.1 by using (3.2), (3), Lemma 2.2 and Lemma 3.2.

Step 1. L^4 and L^4_{loc} norms, $0 \le t \le 2 t_2$

Recall the equation (2.14) for ξ:

$$
\xi(t) = e^{-i H_0 t} \xi_0 + \int_0^t e^{-i H_0 (t-s)} \, \mathbf{P}_{\mathrm{c}}^{H_0} \, G_\xi(s) \, ds \ , \qquad G_\xi = i^{-1} (G + \Lambda_\pi) \ . \tag{3.18}
$$

By (3), Lemma 2.2 and Lemma 3.2, we have

$$\begin{aligned}
\|\xi(t)\|_{L^4} &\le \left\|e^{-iH_0t}\xi_0\right\|_{L^4} + \int_0^t C\,|t-s|^{-3/4}\,\|G_\xi(s)\|_{4/3}\,ds \\
&\le C\,\|\xi_0\|_Y\,\langle t\rangle^{-3/4} + \int_0^t C\,|t-s|^{-3/4}\left(C_4 n^2|x|(s) + C(C_2)n^2\,\|\xi_0\|_Y\,\langle s\rangle^{-9/8}\right)ds \\
&\le C_{2,1}\,\|\xi_0\|_Y\,\langle t\rangle^{-3/4} + C_{2,1}n^2\,t^{1/4}\,|x(t)| \;+\; C(C_2)n^2\,\|\xi_0\|_Y\,\langle t\rangle^{-4/3},
\end{aligned}$$

where $C_{2,1}$ is some explicit constant.

We now estimate $\|\xi(t)\|_{L^4_{\mathrm{loc}}}$. If $t \le 1$, we can bound L^4_{loc}-norm by L^4-norm. Hence we may assume $t > 1$. For , We divide the time integral in (3.18) into $s \in [0, t-1]$ and $s \in [t-1, t]$. In the first interval, we estimate L^4_{loc} by L^8 norm; in the second we estimate L^4_{loc} simply by L^4. Using similar arguments in the previous estimate of the L_4 norm, we have

$$\begin{aligned}
\|\xi(t)\|_{L^4_{\mathrm{loc}}} &\le C\,\|\xi_0\|_Y\,\langle t\rangle^{-9/8} + \int_0^{t-1}\frac{C}{|t-s|^{9/8}}\,\|G_\xi\|_{L^{8/7}}\,ds + \int_{t-1}^t \frac{C}{|t-s|^{3/4}}\,\|G_\xi\|_{L^{4/3}}\,ds \\
&\le C\,\|\xi_0\|_Y\,\langle t\rangle^{-9/8} + \int_0^t \frac{C}{\langle t-s\rangle^{9/8}}\left(C_4 n^2|x|(s) + C(C_2)n^2\,\|\xi_0\|_Y\,\langle s\rangle^{-9/8}\right)ds \\
&\qquad + \sup_{t-1\le s\le t}\|G_\xi(s)\|_{L^{4/3}} \\
&\le C_{2,2}\,\|\xi_0\|_Y\,\langle t\rangle^{-9/8} + C_{2,2}n^2|x(t)| \;+\; C(C_2)n^2\,\|\xi_0\|_Y\,\langle t\rangle^{-9/8}
\end{aligned}$$

where $C_{2,2}$ is some explicit constant.

Step 2. L^2_{loc}-norm, $0 \le t \le 2t_2$

Recall the decomposition (2.32): $\xi = \xi^{(2)} + \xi^{(3)} = \xi^{(2)} + (\xi_1^{(3)} + \cdots + \xi_5^{(3)})$. We will estimate the L^2_{loc}-norm of each term.

0. $\xi^{(2)}$. Since $\Phi_1 \in L^2_{\mathrm{loc}}$, and $\Phi_j \in L^2$, $(j > 1)$, we have

$$\left\|\xi^{(2)}(t)\right\|_{L^2_{\mathrm{loc}}} \le C_{2,3}Cn^2\,|x(t)|\,,$$

for some explicit constant $C_{2,3}$.

1. $\xi_1^{(3)}$. We have

$$\left\|\xi_1^{(3)}(t)\right\|_{L^2_{\mathrm{loc}}} \le C_{2,4}\,\|\xi_0\|_Y\,\langle t\rangle^{-9/8}\;,$$

for some explicit constant $C_{2,4}$ by the $L^{p',p}$ estimate of e^{-itH_0} in Lemma 2.2.

2. $\xi_2^{(3)}$. By the linear estimate (2.34) in Lemma 2.2 we have, for some constant $C_{2,5}$,

$$\left\|\xi_2^{(3)}(t)\right\|_{L^2_{\mathrm{loc}}} \le C_{2,5}n^2|x_0|\,\langle t\rangle^{-9/8}\;.$$

3. $\xi_3^{(3)}$. To estimate $\xi_3^{(3)}(t) = -\int_0^t e^{-iH_0(t-s)}\,\mathbf{P}_{\rm c} G_4\,ds$ with G_4 defined in (2.31), we need estimates (3.10) for $\dot u$ and $\dot v$ and the linear estimate (2.34) in Lemma 3.2. Hence

$$\begin{aligned}
\left\|\xi_3^{(3)}(t)\right\|_{L^2_{\rm loc}} &\le \int_0^t \left\|e^{-iH_0(t-s)}\,\mathbf{P}_{\rm c} G_4\right\|_{L^2_{\rm loc}} ds\\
&\overset{(2.34)}{\le} C\int_0^t \langle t-s\rangle^{-9/8}\left(n^2|\dot u| + n|x\dot v|\right) ds\\
&\overset{(3.10)}{\le} C\int_0^t \langle t-s\rangle^{-9/8}\left(n^2\|G\|_{L^{4/3}} + n^4|x|\right) ds\\
&\overset{(3.11)}{\le} C\int_0^t \langle t-s\rangle^{-9/8}\left(n^4|x| + C(C_2)n^4\|\xi_0\|_Y\langle s\rangle^{-9/8}\right) ds\\
&\le C_{2,6}n^4|x(t)| + C(C_2)n^4\|\xi_0\|_Y\langle t\rangle^{-9/8}\ ,
\end{aligned}$$

4. $\xi_4^{(3)}+\xi_5^{(3)}$. We write $\xi_4^{(3)}+\xi_5^{(3)} = \int_0^t e^{-iH_0(t-s)}\,\mathbf{P}_{\rm c} G_{\xi,5}(s)\,ds$, where $G_{\xi,5}(s) := (G_\xi - i^{-1}G_3)(s)$. By Lemma 3.2 and Lemma 2.2, we have for $t>1$,

$$\begin{aligned}
&\left\|(\xi_4^{(3)}+\xi_5^{(3)})(t)\right\|_{L^2_{\rm loc}}\\
&\le \int_0^{t-1}\left\|e^{-iH_0(t-s)}\,\mathbf{P}_{\rm c} G_{\xi,5}(s)\right\|_{L^8} ds + \int_{t-1}^t \left\|e^{-iH_0(t-s)}\,\mathbf{P}_{\rm c} G_{\xi,5}(s)\right\|_{L^4} ds\\
&\le C\int_0^{t-1} C\,|t-s|^{-9/8}\|G_{\xi,5}(s)\|_{8/7}\,ds + \int_{t-1}^t C\,|t-s|^{-3/4}\|G_{\xi,5}(s)\|_{4/3}\,ds\\
&\le C\left(\int_0^{t-1}|t-s|^{-9/8} + \int_{t-1}^t |t-s|^{-3/4}\right)\left((C_4 n^{15/4}|x(s)| + C(C_2)n^2\|\xi_0\|\langle s\rangle^{-9/8}\right) ds\\
&\le C_{2,7}n^{15/4}|x(t)| + C(C_2)n^2\|\xi_0\|_Y\langle t\rangle^{-9/8}\ ,
\end{aligned}$$

for some explicit constant $C_{2,7}$. If $t<1$, we can bound the $L^2_{\rm loc}$-norm by the L^4-norm. Hence the last estimate for $t<1$ follows from the estimate in Step 1.

We have obtained estimates on ξ involving explicit constants $C_{2,1},\dots,C_{2,7}$ and $C(C_2)$. We now define the constant C_2 in (3.6) to be:

$$C_2 \equiv C_{2,1} + \cdots + C_{2,7}\ . \tag{3.19}$$

Since all terms involving $C(C_2)$ have some extra n factor, ξ satisfies the estimates in (3.6) provided that n is sufficiently small.

Summarizing, we have proved the following lemma:

Lemma 3.3 *If n is sufficiently small, there is an explicit constant C_2 such that, if* (3) *holds in* $[0,t]$ *for some* $t\le C_3 n^{-4}$, *then the estimates* (3.6) *in Proposition 3.1 also hold in* $[0,t]$.

Remark (3.20) In the proof, we only used (3.2), (3), Lemma 2.2 and Lemma 3.2. The information we need on the size of bound states is in (3.2) and the first estimate of (3). Since (3.2) is always true, we only need to ensure that the first estimate in (3) holds.

3.2 Normal form for equations of bound states

We now compute the Poincaré normal form for the bound states. This normal form will be used to estimate the bound states components x and y in next subsection.

Recall that we write

$$x(t) = e^{-ie_0 t}u(t), \qquad y(t) = e^{-ie_1 t}v(t)$$

and the equations (2.12) and (2.13) for u and v,

$$\dot{u} = -ie^{ie_0 t}\left(\phi_0, G + \Lambda_\pi\right), \qquad \dot{v} = -ie^{ie_1 t}\left[(E_1(m) - e_1)y + (\phi_1, G)\right].$$

Using the decompositions (2.25) for Λ_π and (2.3) for $E_1(m)$, we can decompose the equations for u and v according to orders in n:

$$\begin{aligned} \dot{u} &= -ie^{ie_0 t}(\phi_0, G_3) - ie^{ie_0 t}(\phi_0, G_5 + \Lambda_{\pi,5}) - ie^{ie_0 t}(\phi_0, G_7 + \Lambda_{\pi,7}) \\ &\equiv R_{u,3} + R_{u,5} + R_{u,7} \ , \end{aligned} \tag{3.21}$$

$$\begin{aligned} \dot{v} &= -ie^{ie_1 t}[(\phi_1, G_3) + E_{1,2}|y|^2 y] - ie^{ie_1 t}[(\phi_1, G_5) + E_{1,4}|y|^4 y] \\ &\quad - ie^{ie_1 t}[(\phi_1, G_7) + E_1^{(6)}(|y|)\, y] \\ &\equiv R_{v,3} + R_{v,5} + R_{v,7} \ . \end{aligned} \tag{3.22}$$

Using (2.26), (2.27) and Lemma 3.2, which assume (3), we have

$$|R_{u,5}| \lesssim \|G_5\|_{L^1_{\text{loc}}} + \|\Lambda_{\pi,5}\|_{L^1} \lesssim n^4|x| + n^2 \|\xi_0\|_Y \langle t\rangle^{-9/8} \tag{3.23}$$

$$|R_{u,7}| \lesssim \|G_7\|_{L^1_{\text{loc}}} + \|\Lambda_{\pi,7}\|_{L^1} \lesssim n^6|x| + n^2 \|\xi_0\|_Y \langle t\rangle^{-9/8} \tag{3.24}$$

$$|R_{v,5}| \lesssim \|G_5\|_{L^1_{\text{loc}}} + |y|^5 \lesssim n^5 + n^2 \|\xi_0\|_Y \langle t\rangle^{-9/8} \tag{3.25}$$

$$|R_{v,7}| \lesssim \|G_7\|_{L^1_{\text{loc}}} + |y|^7 \lesssim n^7 + n^2 \|\xi_0\|_Y \langle t\rangle^{-9/8} \tag{3.26}$$

We shall first integrate $R_{u,3}$ and $R_{v,3}$ in step 1, and then integrate $R_{u,5}$ and $R_{v,5}$ in step 2.

Step 1 Integration of terms of order n^3

In the equation of u, (3.21), the terms of order n^3 are contained in $R_{u,3} = -ie^{ie_0 t}(\phi_0, G_3)$. The resonant terms from G_3 are $|y|^2 x$ and $|x|^2 x$, whose phase cancels the factor $e^{ie_0 t}$. The other four terms of order n^3 in G_3 have different frequencies and can be exploited using integration by parts. By (2.16) we have

$$\begin{aligned}\dot{u} &= -ie^{ie_0 t}(\phi_0, G_3) + R_{u,5} + R_{u,7} \\ &= c_1\,|u|^2 u + c_2\,|v|^2 u + \frac{d}{dt}(u_1^-) + g_{u,1} + R_{u,5} + R_{u,7}\end{aligned}$$

where

$$c_1 = -i\lambda(\phi_0^2, \phi_0^2), \qquad c_2 = -i2\lambda(\phi_0^2, \phi_1^2)$$

$$u_1^- = -\left(\lambda\phi_0, \frac{e^{i(2e_0-2e_1)t}v^2\bar{u}}{2e_0-2e_1}\phi_0\phi_1^2 + \frac{e^{i(e_0-e_1)t}2|u|^2 v}{e_0-e_1}\phi_0^2\phi_1 + \frac{e^{i(-e_0+e_1)t}u^2\bar{v}}{-e_0+e_1}\phi_0^2\phi_1\right)$$

and

$$g_{u,1} = \left(\lambda\phi_0, \frac{e^{i(2e_0-2e_1)t}\frac{d}{dt}(v^2\bar{u})}{2e_0-2e_1}\phi_0\phi_1^2 + \frac{e^{i(e_0-e_1)t}\frac{d}{dt}(2|u|^2 v)}{e_0-e_1}\phi_0^2\phi_1 + \frac{e^{i(-e_0+e_1)t}\frac{d}{dt}(u^2\bar{v})}{-e_0+e_1}\phi_0^2\phi_1\right).$$

In the equation of v, (3.22), the terms of order n^3 are in $R_{v,3} = -ie^{ie_1 t}[(\phi_1, G_3) + E_{1,2}|y|^2 y]$. There is only one resonant term in G_3, namely, $|x|^2 y$. Another resonant term of order n^3 is from the term $E_{1,2}|y|^2 y$. The other four terms of order n^3 in G_3 have different frequencies and can be integrated. We thus have

$$\begin{aligned}\dot{v} &= -ie^{ie_1 t}[(\phi_1, G_3) + E_{1,2}|y|^2 y] + R_{v,5} + R_{v,7} \\ &= c_6|u|^2 v + c_7|v|^2 v + \frac{d}{dt}(v_1^-) + g_{v,1} + R_{v,5} + R_{v,7}\end{aligned}$$

where

$$c_6 = -i2\lambda(\phi_0^2, \phi_1^2), \qquad c_7 = -iE_{1,2}$$

$$\begin{aligned}v_1^- = -\Bigg(\lambda\phi_1, &\frac{e^{i(-e_1+e_0)t}v^2\bar{u}}{-e_1+e_0}\phi_0\phi_1^2 + \frac{e^{i(e_1-e_0)t}2|v|^2 u}{e_1-e_0}\phi_0\phi_1^2 \\ &+ \frac{e^{i(2e_1-2e_0)t}u^2\bar{v}}{2e_1-2e_0}\phi_0^2\phi_1 + \frac{e^{i(e_1-e_0)t}|u|^2 u}{e_1-e_0}\phi_0^3\Bigg)\end{aligned}$$

and

$$\begin{aligned}g_{v,1} = \Bigg(\lambda\phi_1, &\frac{e^{i(-e_1+e_0)t}\frac{d}{dt}(v^2\bar{u})}{-e_1+e_0}\phi_0\phi_1^2 + \frac{e^{i(e_1-e_0)t}\frac{d}{dt}(2|v|^2 u)}{e_1-e_0}\phi_0\phi_1^2 \\ &+ \frac{e^{i(2e_1-2e_0)t}\frac{d}{dt}(u^2\bar{v})}{2e_1-2e_0}\phi_0^2\phi_1 + \frac{e^{i(e_1-e_0)t}\frac{d}{dt}(|u|^2 u)}{e_1-e_0}\phi_0^3\Bigg)\end{aligned}$$

We now define

$$u_1 = u - u_1^- \ , \qquad v_1 = v - v_1^- \ . \tag{3.27}$$

The equations for u_1 and v_1 are

$$\begin{aligned}\dot u_1 &= c_1|u|^2u + c_2|v|^2u + g_{u,1} + R_{u,5} + R_{u,7} \\ &= c_1|u_1|^2u_1 + c_2|v_1|^2u_1 + g_{u,2} + g_{u,1} + R_{u,5} + R_{u,7}\end{aligned}$$

$$g_{u,2} = c_1(|u|^2u - |u_1|^2u_1) + c_2(|v|^2u - |v_1|^2u_1)$$

and

$$\begin{aligned}\dot v_1 &= c_6|u|^2v + c_7|v|^2v + g_{v,1} + R_{v,5} + R_{v,7} \\ &= c_6|u_1|^2v_1 + c_7|v_1|^2v_1 + g_{v,2} + g_{v,1} + R_{v,5} + R_{v,7}\end{aligned}$$

$$g_{v,2} = c_6(|u|^2v - |u_1|^2v_1) + c_7(|v|^2v - |v_1|^2v_1) \ .$$

We have finished the integration of order n^3 terms. Note that both u_1^- and v_1^- enter the equations of u_1 and v_1. This is the reason we compute their normal form together.

Observe that

$$|u_1^-| \lesssim n^2|u| \ , \qquad |v_1^-| \lesssim n^2|v| \ . \tag{3.28}$$

We now decompose $g_{u,1}$, $g_{v,1}$, $g_{u,2}$ and $g_{v,2}$ according to their orders in n. We want to write them as sum of order n^5 and order n^7 terms. We first claim that $g_{u,1}$ and $g_{v,1}$ are of the forms

$$g_{u,1} = e^{ie_0t}g_{u,1,5} + g_{u,1,7} \ , \qquad g_{v,1} = e^{ie_1t}g_{v,1,5} + g_{v,1,7} \ ,$$

where $g_{u,1,7}$ and $g_{v,1,7}$ are order n^7 terms, and $g_{u,1,5}$ and $g_{v,1,5}$ are explicit homogeneous polynomials of degree 5 in $x, \bar x, y, \bar y$ with purely imaginary coefficients. Moreover, every term in $g_{u,1,5}$ has a factor x or $\bar x$. For example, the first term in $g_{u,1}$ is

$$\begin{aligned}&Ce^{i(2e_0-2e_1)t}\frac{d}{dt}(v^2\bar u) \\ &= Ce^{i(2e_0-2e_1)t}(2\bar u v\dot v + v^2\dot{\bar u}) \\ &= Ce^{ie_0t}\left(2\bar x y e^{-ie_1t}\dot v + y^2e^{ie_0t}\overline{\dot u}\right) \\ &= Ce^{ie_0t}\left(2\bar x y e^{-ie_1t}[R_{v,3} + R_{v,5} + R_{v,7}] + y^2e^{ie_0t}[\overline{R_{u,3}} + \overline{R_{u,5}} + \overline{R_{u,7}}]\right)\end{aligned}$$

where $C = (\lambda\phi_0, (2e_0 - 2e_1)^{-1}\phi_0\phi_1^2)$ is real. Repeating this calculations for all terms in $g_{u,1}$ and collecting terms of order n^5, we obtain $g_{u,1,5}$. The rest belongs to $g_{u,1,7}$. There are two terms of order n^5 in the last expression: the terms of the form $2\bar{x}yR_{v,3}$ and $y^2\overline{R_{u,3}}$. By definitions of $R_{u,3}$ and $R_{v,3}$, they are explicit polynomials of degree 5 in $x, \bar{x}, y, \bar{y}$ with purely imaginary coefficients.

From the estimates of (3.23)–(3.26), we can bound $g_{u,1,7}$ by

$$\begin{aligned} |g_{u,1,7}(t)| &\lesssim n^6|u| + n^4 \|\xi_0\|_Y \langle t\rangle^{-9/8} , \\ |g_{v,1,7}(t)| &\lesssim n^6|v| + n^4 \|\xi_0\|_Y \langle t\rangle^{-9/8} . \end{aligned} \tag{3.29}$$

Similarly, we can write $g_{u,2}$ and $g_{v,2}$ as

$$g_{u,2} = e^{ie_0 t}g_{u,2,5} + g_{u,2,7} , \qquad g_{v,2} = e^{ie_1 t}g_{v,2,5} + g_{v,2,7} ,$$

where $g_{u,1,5}$ and $g_{v,1,5}$ are explicit homogeneous polynomials of degree 5 in $x, \bar{x}, y, \bar{y}$ with purely imaginary coefficients and $g_{u,2,7}$ and $g_{v,2,7}$ are order n^7 terms satisfying

$$|g_{u,2,7}(t)| \lesssim n^6|u| , \qquad |g_{v,2,7}(t)| \lesssim n^6|v| . \tag{3.30}$$

Here we have used (3.28) in last estimate. Moreover, every term in $g_{u,2,5}$ has a factor x or $\bar{x}$. We shall not perform calculations and estimates in details as they are similar to the previous step. To gain some idea, we shall do one example and show it is of the right form. By using $u - u_1 = u_1^-$, the first term in $g_{u,2}$ can be written as

$$\begin{aligned} |u|^2u - |u_1|^2u_1 &= u^2\bar{u} - (u - u_1^-)^2(\bar{u} - \overline{u_1^-}) = u^2\overline{(u_1^-)} + 2|u|^2u_1^- + O(u|u_1^-|^2) \\ &= x^2\overline{(u_1^-)} + 2|x|^2u_1^- + O(u|u_1^-|^2) \end{aligned}$$

The first two terms, $x^2\overline{(u_1^-)} + 2|x|^2u_1^-$, contributes to $g_{u,2,5}$. Since u_1^- equals to $e^{ie_0 t}$ times a polynomial of degree 3 in $x, \bar{x}, y, \bar{y}$ with real coefficients and c_1 in $g_{u,2}$ is purely imaginary, they are of the desired form.

Summarizing, we can write

$$g_{u,1} + g_{u,2} = e^{ie_0 t}\widetilde{R}_{u,5} + g_{u,3}$$

$$g_{v,1} + g_{v,2} = e^{ie_1 t}\widetilde{R}_{v,5} + g_{v,3}$$

where $\widetilde{R}_{u,5} = g_{u,1,5}+g_{u,2,5}$ and $\widetilde{R}_{v,5} = g_{u,1,5}+g_{u,2,5}$ are explicit homogeneous polynomials of degree 5 in $x, \bar{x}, y$ and $\bar{y}$ with **purely imaginary** coefficients. Moreover, every term in $\widetilde{R}_{u,5}$ has a factor x or $\bar{x}$. Also, $g_{u,3} = g_{u,1,7} + g_{u,2,7}$ and $g_{v,3} = g_{v,1,5} + g_{v,2,5}$. From the assumption (3), we have

$$|\widetilde{R}_{u,5}| \lesssim n^4|x|, \quad |\widetilde{R}_{v,5}| \lesssim n^5. \tag{3.31}$$

$$|g_{u,3}| \lesssim n^6|x| + n^4 \|\xi_0\|_Y \langle t\rangle^{-9/8}, \quad |g_{v,3}| \lesssim n^7 + n^4 \|\xi_0\|_Y \langle t\rangle^{-9/8} . \tag{3.32}$$

The final equations for u_1 and v_1 are

$$\dot{u}_1 = c_1|u_1|^2u_1 + c_2|v_1|^2u_1 + (R_{u,5} + e^{ie_0t}\widetilde{R}_{u,5}) + (R_{u,7} + g_{u,3}) \tag{3.33}$$

$$\dot{v}_1 = c_6|u_1|^2v_1 + c_7|v_1|^2v_1 + (R_{v,5} + e^{ie_1t}\widetilde{R}_{v,5}) + (R_{v,7} + g_{v,3}) \tag{3.34}$$

Step 2 Integration of terms of order n^5

We now integrate terms of order n^5. In u_1-equation (3.33) we have $R_{u,5}+e^{ie_0t}\widetilde{R}_{u,5}$, where $R_{u,5}$ is from the decomposition of original equation (3.21) and $\widetilde{R}_{u,5}$ is from the error terms $g_{u,1} + g_{u,2}$. Similarly, terms of order n^5 in v_1-equation (3.34) is $R_{v,5} + e^{ie_1t}\widetilde{R}_{v,5}$. Observe that they are either of the form $x^\alpha y^\beta$ with $|\alpha| + |\beta| = 5$, or of the form $xy\xi$. Also note that there are two sources in $R_{u,5}$: G_5 and $\Lambda_{\pi,5}$. Among all these terms the main term is G_5.

We have already studied $\widetilde{R}_{u,5}$ and $\widetilde{R}_{v,5}$. They are explicit homogeneous polynomials of degree 5 in $x, \bar{x}, y$ and $\bar{y}$ with **purely imaginary** coefficients. Moreover, every term in $\widetilde{R}_{u,5}$ has a factor x or $\bar{x}$.

We next look at Λ_π. Recall (2.25) that $\Lambda_\pi = \Lambda_{\pi,5}+\Lambda_{\pi,7}$ and $\Lambda_{\pi,5} = -2q_3\,(\phi_1,\, G_3|y|^2 + \bar{G}_3y^2)$ (2.24). Thus $\Lambda_{\pi,5}$ is a homogeneous polynomial in $x, \bar{x}, y$ and y of degree 5 with purely real functions as coefficients. Therefore the term $-ie^{ie_0t}(\phi_0, \Lambda_{\pi,5})$ in $\dot{u}$ equation (3.21) gives only polynomials with purely imaginary coefficients and a phase e^{ie_0t}.

Recall G_5 is given by

$$\begin{aligned} G_5 &= \lambda(2y^3\bar{y}\bar{x} + 4|y|^4x)\,\phi_0\phi_1q_3 + \lambda(2|x|^2y^2\bar{y} + x^2y\bar{y}^2)\,\phi_0^2q_3 \\ &\quad + \lambda(x\phi_0 + y\phi_1)^2\overline{\xi} + 2\lambda|(x\phi_0 + y\phi_1)|^2\xi \end{aligned}$$

Recall the decomposition $\xi = \xi^{(2)} + \xi^{(3)}$, where

$$\xi^{(2)}(t) = y^2\bar{x}\Phi_1 + |y|^2x\Phi_2 + |x|^2y\Phi_3 + x^2\bar{y}\Phi_4 + |x|^2x\Phi_5$$

with Φ_1 the only function with nonzero imaginary part (2.28), (2.30). Let

$$\Phi_1 = \Phi_{1,R} + i\Phi_{1,I} \tag{3.35}$$

with both $\Phi_{1,R}$ and $\Phi_{1,I}$ real. Denote the real part of $\xi^{(2)}(t)$ by

$$\xi_R^{(2)}(t) = y^2\bar{x}\Phi_{1,R} + |y|^2 x\Phi_2 + |x|^2 y\Phi_3 + x^2\bar{y}\Phi_4 + |x|^2 x\Phi_5. \tag{3.36}$$

We can write $\xi = y^2\bar{x}i\Phi_I + \xi_R^{(2)} + \xi^{(3)}$. Thus we can further decompose G_5 as

$$G_5 = G_{5,1} + G_{5,2} + G_{5,3} \tag{3.37}$$

where

$$\begin{aligned}
G_{5,1} &= (x\phi_0 + y\phi_1)^2\bar{y}^2 x(-i)\Phi_{1,I} + 2|(x\phi_0 + y\phi_1)|^2 y^2\bar{x}i\Phi_{1,I} \\
G_{5,2} &= \lambda(2y^3\bar{y}\bar{x} + 4|y|^4 x)\,\phi_0\phi_1 q_3 + \lambda(2|x|^2 y^2\bar{y} + x^2 y\bar{y}^2)\,\phi_0^2 q_3 \\
&\quad + \lambda(x\phi_0 + y\phi_1)^2\overline{\xi_R^{(2)}} + 2\lambda|(x\phi_0 + y\phi_1)|^2\xi_R^{(2)} \\
G_{5,3} &= \lambda(x\phi_0 + y\phi_1)^2\overline{\xi^{(3)}} + 2\lambda|(x\phi_0 + y\phi_1)|^2\xi^{(3)}
\end{aligned}$$

The term $G_{5,3}$ will be shown to be smaller than $G_{5,1}$ and $G_{5,2}$. Although $G_{5,1}$ and $G_{5,2}$ are of the same size, $G_{5,2}$ consists of monomials in x, $\bar{x}$, y and $\bar{y}$ with *real* functions as coefficients, while $G_{5,1}$ with purely imaginary coefficients. The reason that $G_{5,1}$ has purely imaginary coefficients is due to the resonance of some linear combination of eigenvalues with the continuum spectrum of H_0 appearing in the form $(H_0 - 0i - 2e_1 + e_0)^{-1}$.

The only resonant term in u-equation from $G_{5,1}$ is $|y|^4 x$ (from $y^2\bar{\xi}$):

$$-ie^{ie_0 t}(\phi_0, (y\phi_1)^2\bar{y}^2 x(-i)\Phi_{1,i}) = -(\phi_0\phi_1^2, \Phi_{1,i})|v|^4 u\ ,$$

and the only resonant term in v-equation from $G_{5,1}$ is $|x|^2|y|^2 y$ (from $x\bar{y}\xi$):

$$-ie^{ie_1 t}(\phi_1, 2(x\phi_0)(\bar{y}\phi_1)y^2\bar{x}i\Phi_{1,i}) = 2(\phi_0\phi_1^2, \Phi_{1,i})|u|^2|v|^2 v\ .$$

Note their coefficients only differ by a factor -2. We recall

$$\gamma_0 = -(\phi_0\phi_1^2, \Phi_{1,i}) = -\operatorname{Im}\left(\lambda\phi_0\phi_1^2\,,\ \frac{-\lambda}{H_0 - 0i - 2e_1 + e_0}\ \mathbf{P}_c\phi_0\phi_1^2\right) > 0 \tag{3.38}$$

Together with the definitions of $R_{u,5}$ and $R_{v,5}$ in (3.21), we can rewrite

$$\begin{aligned}
R_{u,5} + e^{ie_0 t}\widetilde{R}_{u,5} &= e^{ie_0 t}\left[\widetilde{R}_{u,5} - i(\phi_0,\, G_{5,1} + G_{5,2} + \Lambda_{\pi,5})\right] - ie^{ie_0 t}(\phi_0,\, G_{5,3}) \\
R_{v,5} + e^{ie_1 t}\widetilde{R}_{v,5} &= e^{ie_1 t}\left[\widetilde{R}_{v,5} - i(\phi_1,\, G_{5,1} + G_{5,2} + E_{1,4}|y|^4 y)\right] - ie^{ie_1 t}(\phi_1,\, G_{5,3})
\end{aligned}$$

As in Step 1, we now integrate by parts the non-resonant terms inside the square brackets. The resonant terms can't be integrated and we shall only collect them.

This procedure is the same as in Step 1 and we only summarize the conclusion: there exists constants $c_3, c_4, c_5, c_8, c_9, c_{10}$, $u_2^- = O(u^5 + uv^4)$ and $v_2^- = O(u^5 + uv^4)$ two homogeneous polynomials in u and v of degree 5, and $g_{u,4}$ and $g_{v,4}$ the integration remainders such that

$$
\begin{aligned}
R_{u,5} + e^{ie_0 t}\widetilde{R}_{u,5} &= \left(c_3|u|^4 + c_4|u|^2|v|^2 + c_5|v|^4\right) u \\
&\quad + \frac{d}{dt}\left(u_2^-\right) + g_{u,4} - ie^{ie_0 t}(\phi_0,\, G_{5,3}) \ , \\
R_{v,5} + e^{ie_1 t}\widetilde{R}_{v,5} &= \left(c_8|u|^4 + c_9|u|^2|v|^2 + c_{10}|v|^4\right) v \\
&\quad + \frac{d}{dt}\left(v_2^-\right) + g_{v,4} - ie^{ie_1 t}(\phi_1,\, G_{5,3})
\end{aligned}
$$

Furthermore, except c_3 and c_9, all other constants are purely imaginary. The real parts of c_3 and c_9 are from $G_{5,1}$ and they are given explicitly by

$$
\operatorname{Re} c_3 = \gamma_0, \qquad \operatorname{Re} c_9 = -2\gamma_0 \ . \tag{3.39}
$$

The explicit forms of u_2 or v_2 are not important. We only need to know their sizes.

We can now write the equations for u and v as

$$
\begin{aligned}
\dot{u}_1 &= c_1\,|u_1|^2 u_1 + c_2\,|v_1|^2 u_1 + \left(c_3|u|^4 + c_4|u|^2|v|^2 + c_5|v|^4\right) u \\
&\quad + \frac{d}{dt}\left(u_2^-\right) + g_{u,4} - ie^{ie_0 t}(\phi_0,\, G_{5,3}) + g_{u,3} + R_{u,7} \ , \\
\dot{v}_1 &= c_6\,|u_1|^2 v_1 + c_7\,|v_1|^2 v_1 + \left(c_8|u|^4 + c_9|u|^2|v|^2 + c_{10}|v|^4\right) v \\
&\quad + \frac{d}{dt}\left(v_2^-\right) + g_{v,4} - ie^{ie_1 t}(\phi_1,\, G_{5,3}) + g_{v,3} + R_{v,7} \ ,
\end{aligned}
$$

We now define

$$
\mu \equiv u_1 - u_2^- = u - u_1^- - u_2^- \tag{3.40}
$$

$$
\nu \equiv v_1 - v_2^- = v - v_1^- - v_2^- \tag{3.41}
$$

We have

$$
\begin{aligned}
\dot{\mu} &= c_1\,|u_1|^2 u_1 + c_2\,|v_1|^2 u_1 + \left(c_3|u|^4 + c_4|u|^2|v|^2 + c_5|v|^4\right) u \\
&\quad + g_{u,4} - ie^{ie_0 t}(\phi_0,\, G_{5,3}) + g_{u,3} + R_{u,7} \\
&= c_1\,|\mu|^2\mu + c_2\,|\nu|^2\mu + \left(c_3|\mu|^4 + c_4|\mu|^2|\nu|^2 + c_5|\nu|^4\right)\mu + g_u
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\nu} &= c_6\,|u_1|^2 v_1 + c_7\,|v_1|^2 v_1 + \left(c_8|u|^4 + c_9|u|^2|v|^2 + c_{10}|v|^4\right) v \\
&\quad + g_{v,4} - ie^{ie_1 t}(\phi_1,\, G_{5,3}) + g_{v,3} + R_{v,7} \\
&= c_6\,|\mu|^2\nu + c_7\,|\nu|^2\nu + \left(c_8|\mu|^4 + c_9|\mu|^2|\nu|^2 + c_{10}|\nu|^4\right)\nu + g_v
\end{aligned}
$$

with

$$g_u = g_{u,4} + g_{u,5} + g_{u,3} + R_{u,7} - ie^{ie_0 t}(\phi_0,\, G_{5,3}) \tag{3.42}$$

$$g_v = g_{v,4} + g_{v,5} + g_{v,3} + R_{v,7} - ie^{ie_1 t}(\phi_1,\, G_{5,3}) \tag{3.43}$$

and

$$\begin{aligned} g_{u,5} &= c_1\left(|u_1|^2 u_1 - |\mu|^2\mu\right) + c_2\left(|v_1|^2 u_1 - |\nu|^2\mu\right) \\ &\qquad + c_3(|u|^4 u - |\mu|^4\mu) + c_4(|u|^2|v|^2 u - |\mu|^2|\nu|^2\mu) + c_5(|v|^4 u - |\nu|^4\mu) \\ g_{v,5} &= c_6\left(|u_1|^2 v_1 - |\mu|^2\nu\right) + c_7\left(|v_1|^2 v_1 - |\nu|^2\nu\right) \\ &\qquad + c_8(|u|^4 v - |\mu|^4\nu) + c_9(|u|^2|v|^2 v - |\mu|^2|\nu|^2\nu) + c_{10}(|v|^4 v - |\nu|^4\nu)\ . \end{aligned}$$

Observe that the error terms g_u and g_v are of the form

$$g_u, g_v \sim (x^7 + xy^6) + (x^2+y^2)\,\xi^{(3)} + (x^4+y^4)\xi + \left(\phi, \xi^3\right) + \cdots\ ,$$

where ϕ denotes some local function. For g_v we should add $|y|^7$: Since g_v has a term $-ie^{ie_1 t}\,(\phi_1, E^{(6)}(|y|)y)$ from $R_{v,7}$. This term has no factor in x and is of order $|y|^7$.

Under the assumption of (3), the error terms $g_{u,4}$ and $g_{v,4}$ can be estimated similarly as for $g_{u,1}$ and $g_{v,1}$. Also, $g_{u,5}$ and $g_{v,5}$ can be estimated similarly as for $g_{u,2}$ and $g_{v,2}$. We also have, for $j=1,2$,

$$|e^{ie_j t}(\phi_j,\, G_{5,3})| \le Cn^2 \left\|\xi^{(3)}\right\|_{L^2_{\rm loc}} \overset{(3)}{\le} Cn^{6-1/4}|x| + Cn^2\left\|\xi_0\right\|_Y \langle t\rangle^{-9/8}\ .$$

Together with the estimates (3.24), (3.26) and (3.32), we conclude

$$\begin{aligned} |g_u(t)| &\le C_6 n^{6-1/4}|x(t)| + C_6 n^2 \left\|\xi_0\right\|_Y \langle t\rangle^{-9/8} \\ |g_v(t)| &\le C_6 n^{7-1/4} \qquad\qquad + C_6 n^2 \left\|\xi_0\right\|_Y \langle t\rangle^{-9/8} \end{aligned}$$

Summarizing our effort, we have obtained the following lemma.

Lemma 3.4 *Let μ and ν be defined as in* (3.40)–(3.41)*. They satisfy*

$$\begin{aligned} \dot\mu &= (c_1|\mu|^2 + c_2|\nu|^2)\mu + (c_3|\mu|^4 + c_4|\mu|^2|\nu|^2 + c_5|\nu|^4)\mu + g_u \\ \dot\nu &= (c_6|\mu|^2 + c_7|\nu|^2)\nu + (c_8|\mu|^4 + c_9|\mu|^2|\nu|^2 + c_{10}|\nu|^4)\nu + g_v \end{aligned} \tag{3.44}$$

All coefficients $c_1, \cdots, c_{10}$ except c_3 and c_9 are purely imaginary and we have (3.39)*, i.e.,*

$$\operatorname{Re} c_3 = \gamma_0\ , \qquad \operatorname{Re} c_9 = -2\gamma_0\ , \tag{3.45}$$

where $\gamma_0 > 0$ is defined in (1.8).

Moreover, assuming (3) *and using the estimates* (3.2) *and Lemma 3.2, we have*

$$|u(t) - \mu(t)| \le C_6\, n^2 |x(t)| \ , \quad |v(t) - \nu(t)| \le C_6\, n^3 \tag{3.46}$$

$$|g_u(t)| \le C_6 n^{6-1/4} |x(t)| + C_6 n^2 \left\|\xi_0\right\|_Y \langle t\rangle^{-9/8} \tag{3.47}$$

$$|g_v(t)| \le C_6 n^{7-1/4} \qquad + C_6 n^2 \left\|\xi_0\right\|_Y \langle t\rangle^{-9/8} \tag{3.48}$$

for some explicit constant C_6.

3.3 Estimates for bound states

In this subsection we will conclude the estimates for x and y stated in Proposition 3.1. Recall tha, t under the assumption of (3) and (3.2), we have proved Lemmas 3.2–3.4 which contains estimates for ξ, g_u and g_v.

We now derive some preliminary estimates. Let $f = 2|\mu|^2$ and $g = |\nu|^2$. We have, by (3.46),

$$\left| f(t) - 2|x|^2 \right| \le 5C_6 n^2 |x|^2, \qquad \left| g(t) - |y|^2 \right| \le 5C_6 n^4. \tag{3.49}$$

We also have from (3.44) that

$$\dot f = \mathrm{Re}\, 4\bar\mu\dot\mu = 2\gamma_0 g^2 f + \mathrm{Re}\, 4\bar\mu g_u \ , \tag{3.50}$$

$$\dot g = \mathrm{Re}\, 2\bar\nu\dot\nu = -2\gamma_0 f g^2 + \mathrm{Re}\, 2\bar\nu g_v \ . \tag{3.51}$$

By (3.47) and (3.48), we have

$$\left| \dot f + \dot g \right| = |\mathrm{Re}\, 4\bar\mu g_u + \mathrm{Re}\, 4\bar\nu g_v| \le 4C_6 n^{8-1/4} + 4C_6 n^3 \left\|\xi_0\right\|_Y \langle t\rangle^{-9/8} \ .$$

Recall $n_1^2 = |x(0)|^2 + \frac12 |y(0)|^2$. By (3.49) we have $|(f+g)(0) - 2n_1^2| \le 10 C_6 n^4$. Thus, for $t < C_3 n^{-4}$,

$$\begin{aligned} |(f+g)(t) - 2n_1^2| \ &\le \ 10C_6 n^4 + \int_0^t 4C_6 n^{8-1/4} + 4C_6 n^3 \left\|\xi_0\right\|_Y \langle s\rangle^{-9/8} ds \\ &\le \ 5C_3 C_6 n^{4-1/4} \ . \end{aligned} \tag{3.52}$$

We now prove Proposition 3.1 in three steps.

1. Initial layer regime

In this period the dispersive part disperses away so much that it becomes negligible locally. The times it takes for this to happen is of order $t_0 = \varepsilon_3 n^{-4}$. We first prove that $\frac{1}{2}|x_0| \le |x(t)| \le \frac{3}{2}|x_0|$ for $t \in [0, t_0]$. The main ingredients of the proof are the norm form equation of x from the last section and the following observation. The ξ dependent term is of the form $n^2\xi$ or of higher orders. Because of our assumption $\|\xi_0\|_Y \le \varepsilon_2|x_0|n^{-2}$ and the decay of $\xi(t)$, this term will not change $x(t)$ very much. More precisely, for $t \in [0, t_0]$, $t_0 = \varepsilon_3 n^{-4}$, we have by (3.50), (3.47), (3.49), the assumptions $\|\xi_0\|_Y \le \varepsilon_2|x_0|n^{-2}$ and (3),

$$\begin{aligned}
|f(t) - f(0)| &\le \int_0^t 4\gamma_0 g^2 f(s) + 5|x_0||g_u(s)|\, ds \\
&\le \left(8\gamma_0 n^4 + Cn^{6-1/4}\right)|x_0|^2 t_0 + 10|x_0|n^2\, \|\xi_0\|_Y \\
&\le (10\gamma_0 + 1)\varepsilon_2\, |x_0|^2 + 10\varepsilon_3|x_0|^2 \le \tfrac{1}{8} f(0),
\end{aligned}$$

provided n, ε_2 and ε_3 are sufficiently small. By (3.49), we have $||x(t)|^2 - |x_0|^2| \le \frac{1}{4}|x_0|^2$. Hence we have $\frac{1}{2}|x_0| \le |x(t)| \le \frac{3}{2}|x_0|$ for $t \in [0, t_0]$.

Similarly, we can show $|g(t) - g(0)| \le ((10\gamma_0 + 1)\varepsilon_2 + 10\varepsilon_3)n^2$. Hence we have $||y(t)| - |y_0|| \le 0.01|y_0|$ for $t \in [0, t_0]$ if ε_2 and ε_3 are small. The smallness of ε_2 and ε_3 can be guaranteed if we define

$$\varepsilon_2 = \frac{1}{2000(\gamma_0 + 1)}, \qquad \varepsilon_3 = \frac{1}{2000} \ . \tag{3.53}$$

2. Transition regime (i)

In this period most mass of the disperse wave is far away and has no effect on the local dynamics; the ground state begins to grow exponentially until it has the order $n/100$. The time it takes is of order n^{-4}.

Define

$$t_1 \equiv \inf_{t \ge t_0} \{t : |x(t)| \ge 0.01n\} := t_1' \ . \tag{3.54}$$

We want to show that

$$0 \le t_1 \le t_0 + \frac{1.01}{\gamma_0 n^4} \log\left(\frac{2n}{|x_0|}\right) , \tag{3.55}$$

Suppose (3.55) fails, that is, $|x(t)| < 0.01n$ for all $t \le t_1'$. By (3.46) and (3.49), we have $f(t) \le 0.0004\, n^2$ and $g(t) \ge 0.9995\, n^2$ for $t \le t_1'$. Hence

$$\dot f(t) \ge 2\gamma_0(0.9995\, n^2)^2 f + O(n^6) f \ge \frac{2}{1.01}\gamma_0 n^4 f \ ,$$

if n is sufficiently small. Hence

$$f(t) \ge f(0) \exp\left\{\frac{2}{1.01}\gamma_0 n^4 t\right\} \ , \tag{3.56}$$

for $t \le t_1'$. We have

$$f(t_1') \ge f(0) \exp \left\{ \frac{2}{1.01} \gamma_0 n^4 \, \frac{1.01}{\gamma_0 n^4} \, \log \left(\frac{n}{|x_0|} \right) \right\} = f(0) n^2 |x_0|^{-2} \ge 0.99 n^2$$

which is a contradiction to the assumption that $|x|(t_1') < 0.01n$. This shows that t_1 satisfies (3.55). We also have that (3.56) holds for all $t \le t_1$, and that $f(t_1) \ge 5 \cdot 10^{-5}\, n^2$.

3. Transition regime (ii)

Recall the definition of ε (1.15). Define

$$t_2 \equiv \inf \left\{ t : g(t) \le (\varepsilon n)^2 \right\} \ . \tag{3.57}$$

We want to show that

$$t_1 \le t_2 \le t_1 + 10100 \, (\gamma_0 n^4 \varepsilon^2)^{-1} := t_2' \ . \tag{3.58}$$

Suppose the contrary, then $g(t) \ge (\varepsilon n)^2$ for all $t \le t_2'$. Then $\dot f > 0$ and we have that $f(t) \ge 5 \cdot 10^{-5}\, n^2$ for $t_1 \le t \le t_2'$. Hence

$$\dot g \le -(1.99)\, \gamma_0 f g^2 \le -9.95 \cdot 10^{-5}\, \gamma_0 n^2 g^2 \ .$$

Hence

$$g(t) \le [g(t_1)^{-1} + 9.95 \cdot 10^{-5}\, \gamma_0 n^2 (t - t_1)]^{-1} \ , \quad (t_1 \le t \le t_2').$$

and $g(t_2) < (\varepsilon n)^2$. This contradiction shows the existence of t_2 satisfying (3.58).

Since $\dot g \ge -(2.01) f g^2$, similar argument shows $t_2 \ge C \varepsilon^{-2} n^{-4}$ if $|y(0)| > 2\varepsilon n$. Combining with estimate (3.52) for $f + g$, we have estimates for $f(t_2)$. From (3.49), these estimates of f and g can be translated into estimates of $x(t_2)$ and $y(t_2)$ stated in Proposition 3.1.

We have proved (3.4), (3.5), (3.7) and (3.8) in Proposition 3.1, using the estimates (1.14), (3.2) and the assumption (3). Since (3) holds for $t = 0$, by continuity it holds for all $t \le t_2$. From Lemmas 3.2–3.4 and the above estimates, Proposition 3.1 is proved.

4 Stabilization regime

In this section we study the solution $\psi(t)$ in the third time regime, after the solution has become near some nonlinear ground state. In this regime, it is natural to use the decomposition (4.3) for the solution $\psi(t)$ which emphasizes nonlinear ground states. A key issue here is to pass the information from the coordinates system (1.12) to the (4.3). As emphasized in the introduction, it is not sufficient to use only the estimates of ψ at $t = t_2$. We will also use the explicit form of the main terms in the dispersive part of $\psi(t)$ to ensure that they do not come back to affect the local dynamics, i.e., the part of the wave represented by these terms is "out-going'. The set-up and proof here are similar to those in [12] except the big terms of the dispersive part just mentioned. We shall first show that certain local estimates used in [12] are still small (see Lemma 4.2).

4.1 Preliminaries

In the time regime $t \geq t_2$, we will use the set-up in [12], which we now briefly recall. For the proofs we refer the reader to [12].

For all nonlinear ground state Q_E with frequency E, Let $\mathcal{L}_E$ be the linearized operator around Q_E:

$$\mathcal{L}h = -i\left\{(-\Delta + V - E + 2\lambda Q_E^2)\, h + \lambda Q_E^2\, \overline{h}\,\right\} \tag{4.1}$$

With respect to $\mathcal{L}_E$, we can decompose $L^2(\mathbb{R}^3, \mathbb{C})$, as a real vector space, as the direct sum of three invariant subspaces:

$$L^2(\mathbb{R}^3, \mathbb{C}) = S(\mathcal{L}_E) \oplus \mathtt{E}_1(\mathcal{L}_E) \oplus \mathtt{H}_c(\mathcal{L}_E) \tag{4.2}$$

where $S(\mathcal{L}_E)$ and $\mathtt{E}_1(\mathcal{L}_E)$ are generalized eigenspaces, obtained from perturbation of ϕ_0 and ϕ_1 respectively and $\mathtt{H}_c(\mathcal{L}_E)$ corresponds to the continuous spectrum of $\mathcal{L}_E$. Notice that this decomposition is not orthogonal. Also, $S(\mathcal{L}_E) = \mathrm{span}_{\mathbb{R}}(iQ_E, R_E)$, where $R_E = \partial_E Q_E$.

For each ψ sufficiently close to Q_E, we can decompose ψ as

$$\psi = [Q_E + a_E R_E + \zeta_E + \eta_E]\; e^{i\Theta_E}\ . \tag{4.3}$$

Here $a_E, \Theta_E \in \mathbb{R}$ and $\zeta_E \in \mathtt{E}_1(\mathcal{L})$ and $\eta_E \in \mathtt{H}_c(\mathcal{L})$. The direction iQ_E is implicitly given in $Q_E(e^{i\Theta} - 1)$. Moreover, for this ψ there is a unique frequency E' such that in the decomposition (4.3) the coefficient a vanishes. In some sense it means that $Q_{E'}$ is the closest nonlinear ground states to ψ.

4.2 Estimates

Our aim is to show that $\psi(t_2)$ satisfies the conditions of Theorem 3 (resonance dominated solutions) in [12]. By Proposition 3.1 one can show that $\psi(t_2)$ is close to a nonlinear ground state $Q_{E_0}e^{i\Theta_0}$ in L^2_{loc}-norm, i.e., $\|Q_{E_0}\|_{L^2} = n_{t_2} \le n_0$, $\left\|\psi(t_2) - Q_{E_0}e^{i\Theta_0}\right\|_{L^2_{\text{loc}}} \le \varepsilon_0 n_{t_2}$. From the same Proposition, we have $n_{t_2} \sim n_1 \sim n$ where n_1 is defined in (1.16) and n is defined in (1.14). Thus for the purpose of order of magnitude, we are free to interchange n_{t_2} with n. We now state the conditions of Theorem 3 (resonance dominated solutions) in [12]: Suppose that the initial data $\psi(t_2)$ is decomposed as in (4.3) with the frequency $E = E_0$ chosen so that the coefficient a vanishes. Then the excited state component ζ satisfies

$$0 < \|\zeta\| \le \varepsilon_0 n \tag{4.4}$$

and the dispersive part satisfies

$$\|\eta_E(t_2)\|_Y \le C\,\|\zeta\|^2 \tag{4.5}$$

for all E close to E_0 with $|E - E_0| \le \|\zeta\|^2$. Here the Y norm is defined in (1.7). We shall see that the condition (4.4) is easy to verify by Proposition 3.1. The dispersive part, however, is no longer localized and there is no hope to satisfy (4.5). In fact, even its L^2-norm is not small enough. Recall we know from the previous sections that $|x(t_2)|^2 - |x_0|^2$ is roughly one half of $|y_0|^2 - |y(t_2)|^2$. Thus, by conservation of L^2-norm, the dispersive part gains the other half of $|y_0|^2 - |y(t_2)|^2$ and $\|\xi(t_2)\| \approx n$.

One believes on physics ground that most mass of the dispersive part is far away and it has little influence to the local dynamics. The local part of the dispersive component, on the other hand, is generated by changes of the bound states and is small. Thus the results in [12] should still hold. This idea, however, requires to clarify the concept of "out-going waves" for nonlinear equations. Instead of directly approach this problem, we examine the condition (4.5) in the proof in [12]. It is used only to guarantee that for all $s \ge 0$

$$\begin{aligned} \left\|e^{s\mathcal{L}}\eta_E(t_2)\right\|_{L^4} &\le \tfrac{1}{20}\left((\varepsilon n)^{-2} + \gamma_0 n^2 s\right)^{-3/4+1/100}, \\ \left\|e^{s\mathcal{L}}\eta_E(t_2)\right\|_{L^2_{\text{loc}}} &\le C\left((\varepsilon n)^{-2} + \gamma_0 n^2 s\right)^{-1}, \end{aligned} \tag{4.6}$$

These estimates are used to estimate the local contributions of $e^{s\mathcal{L}}\eta_E(t_2)$ in (3.19) (see the following remark) and in the proofs for the L^4 and L^2_{loc} estimates for $\eta_E(t)$ in Lemmas 5.2 and 5.3. of [12].

Remark: In [12], (4.6) is used to estimate $\tilde{\eta}_1^{(3)}(t) = e^{-iA(t-t_2)}\tilde{\eta}(t_2)$, with $\tilde{\eta}(t_2) = e^{i\Theta(t_2)}U\eta_E(t_2)$, and A being a self-adjoint perturbation of $-\Delta + V$, $\mathcal{L} = U^{-1}(-iA)U$, U a bounded operator. Since $e^{s\mathcal{L}} = U^{-1}e^{-isA}U$,

$$\tilde{\eta}_1^{(3)}(t) = e^{-iA(t-t_2)}\, e^{i\Theta(t_2)}U\eta_E(t_2) \ = e^{i\Theta(t_2)}Ue^{(t-t_2)\mathcal{L}}\eta_E(t_2)$$

Thus we can choose freely to estimate either A or $\mathcal{L}$. Since $e^{(t-t_2)\mathcal{L}}\eta_E(t_2)$ is easier to estimate than $e^{-iA(t-t_2)}\tilde{\eta}(t_2)$ by using the Duhamel's expansion, we state all conditions in terms of $\mathcal{L}$.

Because we only need (4.6), the same proof in [12] actually gives the following stronger result.

Theorem 4.1 *Suppose that $\psi(t_2)$ is close to a nonlinear ground state $Q_{E_0}e^{i\Theta_0}$ in L^2_{loc}-norm, and suppose that in the decomposition* (4.3) *with $E = E_0$ one has $\|Q_{E_0}\|_{L^2} = n_{t_2} \sim n \le n_0$, $\|\zeta_{E_0}\| \le \varepsilon_0 n$, $|a| + \|\eta_{E_0}\|_{L^2_{\mathrm{loc}}} \le \varepsilon_0^2 n^2$.*

If for all E close to E_0 with $|E - E_0| \le \varepsilon_0^2 n^2$ the dispersive part $\eta_E(t_2)$ in the decomposition (4.3) *satisfies the estimates* (4.6)*, then the conclusion and the proof of Theorem 1 in [12] remain valid. In particular, there is a frequency E_∞ with $|E_\infty - E_0| \le \varepsilon_0^2 n^2$ and a function $\Theta(t) = -E_\infty t + O(\log(t))$ for $t \in [t_2, \infty)$ such that*

$$\left\|\psi(t) - Q_{E_\infty}e^{i\Theta(t)}\right\|_{L^2_{\mathrm{loc}}} \le C_2\left((\varepsilon n)^{-2} + \gamma_0 n^2(t-t_2)\right)^{-1/2}$$

for some constant C_2.

Suppose that the initial data $\psi(t_2)$ is decomposed as in (4.3) *with the frequency E chosen so that the coefficient a vanishes. Suppose that, in addition to the previous assumption that* (4.6) *holds for all frequency E with $|E - E_0| \le \varepsilon_0^2 n^2$, the excited state component ζ satisfies* (4.4)*–*(4.5)*. Then the lower bound*

$$C_1\left((\varepsilon n)^{-2} + \gamma_0 n^2(t-t_2)\right)^{-1/2} \le \left\|\psi(t) - Q_{E_\infty}e^{i\Theta(t)}\right\|_{L^2_{\mathrm{loc}}}$$

holds as well.

The merit of this modification is that we do not need the initial data to be localized. We only need to know that its dispersive part is "outgoing" in certain sense. Notice that the condition on the size of the excited component ζ is a simple consequence of the estimates (3.6), (3.8) and (3.4). To see this, we first pretend that the size of ζ is given by $y(t_2)$ and the size of the ground state component is given by $x(t_2)$. Then the condition (4.4) is just a simple consequence of (3.8). Since the difference between the decompositions (1.12) and (4.3) are higher order terms, the condition (4.4) is easy to check. Therefore, Theorem 1.1 follows from the following Lemma:

Lemma 4.2 *Let $\psi(t)$ be the solution of* (1.1) *in Theorem 1.1 and t_2 be the time in Proposition 3.1. Let $E_0 = E(t_2)$ be the unique energy such that in the decomposition* (4.3) *the coefficient a vanishes. Then for all E close to E_0, i.e., $|E - E_0| \le C\varepsilon_0^2 n^2$, we have for all $t \ge t_2$*

$$\left\| e^{(t-t_2)\mathcal{L}} \eta_E(t_2) \right\|_{L^4} \le n(n^2 t)^{-3/4+1/100} \ , \quad \left\| e^{(t-t_2)\mathcal{L}} \eta_E(t_2) \right\|_{L^2_{\text{loc}}} \le n(n^2 t)^{-1} \ . \tag{4.7}$$

Notice that, since $t_2 \sim \varepsilon^{-2} n^{-4}$ by Proposition 3.1, we have $(\varepsilon n)^{-2} + \gamma_0 n^2 (t - t_2) \sim n^2 t$ for all $t \ge t_2$, no matter $t > 2t_2$ or $t < 2t_2$. Hence (4.7) implies (4.6) with a big margin

Proof of Lemma 4.2

We have the two decompositions at $t = t_2$

$$\begin{aligned} \psi(t) &= x(t)\phi_0 + Q_1(y(t)) + \xi(t) \\ &= \left[Q_E + a_E(t) R_E + \zeta_E(t) + \eta_E(t) \right] e^{i\Theta_E(t)} \end{aligned} \tag{4.8}$$

Since E will be fixed for the rest of this proof, we shall drop all subscripts E. Hence

$$\zeta(t) + \eta(t) = \left[x(t)\phi_0\, e^{-i\Theta(t)} - Q_T \right] + \left[Q_1(y(t)) + \xi(t) \right] e^{-i\Theta(t)} - a_E(t) R_E \ . \tag{4.9}$$

Thus we have

$$\eta(t_2) = \mathbf{P}_{\mathrm{c}}^{\mathcal{L}} \left\{ \left[x(t_2)\phi_0\, e^{-i\Theta(t_2)} - Q_T \right] + \left[Q_1(y(t_2)) + \xi(t_2) \right] e^{-i\Theta(t_2)} \right\} = \eta_{0,1} + \eta_{0,2} \ ,$$

where

$$\begin{aligned} \eta_{0,1} &= \mathbf{P}_{\mathrm{c}}^{\mathcal{L}} \left\{ \xi(t_2) e^{-i\Theta(t_2)} \right\} \ , \\ \eta_{0,2} &= \mathbf{P}_{\mathrm{c}}^{\mathcal{L}} \left\{ \left(x(t_2)\phi_0\, e^{-i\Theta(t_2)} - Q_E \right) + Q_1(y(t_2))\, e^{-i\Theta(t_2)} \right\} \ . \end{aligned}$$

Note that $\eta_{0,2}$ is a local H^1 function and is bounded by $O(n^3)$, i.e., $\|\eta_{0,2}\|_Y \le Cn^3$. Therefore we have

$$\left\| e^{(t-t_2)\mathcal{L}} \eta_{0,2} \right\|_{L^4} \le Cn^3 \ \langle t - t_2 \rangle^{-3/4} \ , \qquad \left\| e^{(t-t_2)\mathcal{L}} \eta_{0,2} \right\|_{L^2_{\text{loc}}} \le Cn^3 \ \langle t - t_2 \rangle^{-3/2} \ . \tag{4.10}$$

Hence $e^{t\mathcal{L}} \eta_{0,2}$ satisfies the desired estimates with a big margin.

We now focus on the non-local term $\eta_{0,1} = \mathbf{P}_{\mathrm{c}}^{\mathcal{L}} \left\{ \xi(t_2) e^{-i\Theta(t_2)} \right\}$. Recall $\xi(t_2)$ is bounded by $2n$ in L^2 (3.2) and by $4C_2 n^{5-1/4}$ in L^2_{loc} from Proposition 3.1. In particular, we have $\|\eta_{0,1}\|_{L^2} \le C \|\xi(t_2)\|_{L^2} \le Cn$. Thus, by Lemmas 2.6 and 2.9 of [12], we have $\left\| e^{(t-t_2)\mathcal{L}} \eta_{0,1} \right\|_{L^2} \le Cn$.

For convenience of notation, we write

$$\mathcal{L} = -iH_0 + iV_1 + iV_2\mathtt{C}\ ,$$

where $V_1 = 2\lambda Q_E^2$, $V_2 = \lambda Q_E^2$ and $\mathtt{C}$ denotes the conjugation operator. By Duhamel's principle,

$$\begin{aligned} e^{(t-t_2)\mathcal{L}}\eta_{0,1} &= \mathbf{P}_{\rm c}^{\mathcal{L}}\, e^{(t-t_2)\mathcal{L}} \left\{\xi(t_2)e^{-i\Theta(t_2)}\right\} \\ &= \mathbf{P}_{\rm c}^{\mathcal{L}}\, e^{-i(t-t_2)H_0}\xi(t_2)e^{-i\Theta(t_2)} \\ &\quad + \int_{t_2}^{t} e^{(t-s)\mathcal{L}}\,\mathbf{P}_{\rm c}^{\mathcal{L}}\, i(V_1+V_2\mathtt{C})e^{-i(s-t_2)H_0}\xi(t_2)e^{-i\Theta(t_2)}\, ds\ . \end{aligned}$$

We now substitute (2.14) with $t = t_2$, i.e.,

$$\xi(t_2) = e^{-it_2H_0}\xi_0 + \int_0^{t_2} e^{-i(t_2-\tau)H_0}\,\mathbf{P}_{\rm c}^{H_0}\, G_\xi(\tau)\, d\tau, \tag{4.11}$$

into the above equation. We have

$$e^{(t-t_2)\mathcal{L}}\eta_{0,1} = E_1 + E_2 + E_3 + E_4$$

where

$$\begin{aligned} \Omega_1 &= \mathbf{P}_{\rm c}^{\mathcal{L}}\, e^{-itH_0}\, e^{-i\Theta(t_2)}\xi_0 \\ \Omega_2 &= \int_{t_2}^{t} e^{(t-s)\mathcal{L}}\,\mathbf{P}_{\rm c}^{\mathcal{L}}\, i(V_1+V_2\mathtt{C})e^{-isH_0}\, e^{-i\Theta(t_2)}\xi_0\, ds \\ \Omega_3 &= \int_0^{t_2} \mathbf{P}_{\rm c}^{\mathcal{L}}\, e^{-i(t-\tau)H_0}\,\mathbf{P}_{\rm c}^{H_0}\ e^{-i\Theta(t_2)}\, G_\xi(\tau)\, d\tau \\ \Omega_4 &= \int_{t_2}^{t}\int_0^{t_2} e^{(t-s)\mathcal{L}}\,\mathbf{P}_{\rm c}^{\mathcal{L}}\, i(V_1+V_2\mathtt{C})e^{-i(s-\tau)H_0}\,\mathbf{P}_{\rm c}^{H_0}\ e^{-i\Theta(t_2)}\, G_\xi(\tau)\, d\tau\, ds \end{aligned}$$

The only estimates we need here from §3 are: (cf.(3.14))

$$\|\xi_0\|_Y \le 4n, \qquad \|G_\xi(\tau)\|_{L^1} \le Cn^3, \quad (0\le\tau\le t_2), \qquad t_2 \sim \varepsilon^{-2}n^4\ . \tag{4.12}$$

From the linear estimate Lemma 2.2 we can estimate Ω_1 by

$$\begin{aligned} &\|\Omega_1\|_{L^4} \le Ct^{-3/4}\,\|\xi_0\|_{L^{4/3}} \\ &\|\Omega_1\|_{L^2_{\rm loc}} \le C\,\|\Omega_1\|_{L^8} \le Ct^{-9/8}\,\|\xi_0\|_{L^{8/7}} \end{aligned}$$

For Ω_2, since $t_2 > 1$, we have

$$\|\Omega_2\|_{L^4} \le C\int_{t_2}^{t} |t-s|^{-3/4}\,|s|^{-9/8}\,\|\xi_0\|_{L^{8/7}}\ ds \le Ct^{-3/4}\,\|\xi_0\|_{L^{8/7}}\ .$$

If $t > t_2+1$, we bound its $L^2_{\rm loc}$-norm by L^8 for $s \in [t_2, t-1]$ and by L^4 for $s \in [t-1, t]$. Thus we have

$$\begin{aligned}\|\Omega_2\|_{L^2_{\rm loc}} &\le C\left\{\int_{t_2}^{t-1} |t-s|^{-9/8} + \int_{t-1}^{t} |t-s|^{-3/4}\right\} |s|^{-9/8} \|\xi_0\|_{L^{8/7}}\, ds \\ &\le Ct^{-9/8}\|\xi_0\|_{L^{8/7}} \ .\end{aligned}$$

If $t \le t_2+1$, we use only L^4 norm for the whole interval $[t_2, t]$ and the same estimate holds.

For Ω_3, notice that $\Omega_3 = \mathbf{P}_{\rm c}^{\mathcal{L}}\, e^{-i(t-t_2)H_0} J$ where J is the integral in (4.11),

$$J = \int_0^{t_2} e^{-i(t_2-\tau)H_0}\, \mathbf{P}_{\rm c}^{H_0}\, G_\xi(\tau)\, d\tau = \xi(t_2) - e^{-it_2H_0}\xi_0 \ .$$

Since it is the difference of two L^2 functions of order n, $\|J\|_{L^2} \le Cn$ and $\|\Omega_3(t)\|_{L^2} \le Cn$ for all t.

Suppose that $t \ge 2t_2$. Since $\|G_\xi\|_{L^1} \le Cn^3$ by (3.14), we have

$$\|\Omega_3\|_{L^\infty} \le C\int_0^{t_2} |t-\tau|^{-3/2}\|G_\xi(\tau)\|_{L^1}\, d\tau \le C\int_0^{t_2} t^{-3/2}n^3\, d\tau \le Cn^3t_2t^{-3/2}$$

Interpolating, we have

$$\|\Omega_3\|_{L^4} \le C\|\Omega_3\|_{L^2}^{1/2}\ \|\Omega_3\|_{L^\infty}^{1/2} \le Cn^2t_2^{1/2}t^{-3/4} \ .$$

Since $L^2_{\rm loc}$-norm is bounded by L^∞-norm, we conclude that

$$\|\Omega_3\|_{L^2_{\rm loc}} \le Cn^3t_2t^{-3/2} \le Cn^3t_2^{5/8}t^{-9/8} \ .$$

Suppose now that $t < 2t_2$. From similar arguments we have

$$\|\Omega_3\|_{L^4} \le C\int_0^{t_2} |t-\tau|^{-3/4}n^3\, d\tau \le C\int_0^{t_2} |t_2-\tau|^{-3/4}n^3\, d\tau = Ct_2^{1/4}n^3$$

$$\begin{aligned}\|\Omega_3\|_{L^2_{\rm loc}} &\le C\int_0^{t_2-1} \|\cdot\|_{L^8} + C\int_{t_2-1}^{t_2} \|\cdot\|_{L^4} \\ &\le C\int_0^{t_2-1} |t_2-\tau|^{-9/8}n^3\, d\tau + C\int_{t_2-1}^{t_2} |t_2-\tau|^{-3/4}n^3\, d\tau \le Cn^3 + Cn^3\end{aligned}$$

We now estimate Ω_4. Since $\|G_\xi\|_{L^1} \le Cn^3$ by (3.14), we have,

$$\begin{aligned}\|\Omega_4\|_{L^4} &\le C\int_{t_2}^{t} |t-s|^{-3/4}\int_0^{t_2} |s-\tau|^{-3/2}\|G_\xi(\tau)\|_{L^1}\, d\tau\, ds \\ &\le C\int_{t_2}^{t} |t-s|^{-3/4}t_2s^{-3/2}n^3\, ds \\ &\le Ct_2n^3\int_{t_2}^{t} |t-s|^{-3/4}s^{-9/8}t_2^{-3/8}\, ds \\ &\le Cn^3t_2^{5/8}\int_0^{t} |t-s|^{-3/4}\langle s\rangle^{-9/8}\, ds \\ &\le Cn^3t_2^{5/8}t^{-3/4}\end{aligned}$$

We can bound the $L^2_{\rm loc}$-norm by L^8 for $s \in [t_2, t-1]$, and by L^4 for $s \in [t-1, t]$ (if $t \le t_2 + 1$, we use only L^4 norm for the whole interval $[t_2, t]$). Thus we have the bound

$$
\begin{aligned}
\|\Omega_4\|_{L^2_{\rm loc}} &\le \left(C \int_{t_2}^{t-1} |t-s|^{-9/8} + C \int_{t-1}^{t} |t-s|^{-3/4} \right) \left(\int_0^{t_2} |s-\tau|^{-3/2} \|G_\xi(\tau)\|_{L^1}\, d\tau \right) ds \\
&\le C \int_{t_2}^{t-1} |t-s|^{-9/8} (t_2 s^{-3/2} n^3)\, ds + C \int_{t-1}^{t} |t-s|^{-3/4} (t_2 s^{-3/2} n^3)\, ds \qquad (4.13)
\end{aligned}
$$

The second integral in (4.13) is bounded by $Cn^3 t_2 t^{-3/2}$. The first integral, when $t \ge 2t_2$, is bounded by

$$
\begin{aligned}
&\le Cn^3 t_2 \int_{t_2}^{t/2} |t-s|^{-9/8} s^{-3/2}\, ds + Cn^3 t_2 \int_{t/2}^{t-1} |t-s|^{-9/8} s^{-3/2}\, ds \qquad (4.14) \\
&\le Cn^3 t_2 t^{-9/8} \int_{t_2}^{t/2} s^{-3/2}\, ds + Cn^3 t_2 t^{-3/2} \int_{t/2}^{t-1} |t-s|^{-9/8}\, ds \\
&\le Cn^3 t_2 t^{-9/8} t_2^{-1/2} + Cn^3 t_2 t^{-3/2}
\end{aligned}
$$

On the other hand, if $t_2 + 1 \le t \le 2t_2$, then the first integral in (4.13) is bounded by

$$
Cn^3 t_2 \int_{t/2}^{t-1} |t-s|^{-9/8} s^{-3/2}\, ds
$$

which is the second integral in (4.14) and is bounded by $Cn^3 t_2 t^{-3/2}$. Thus, for all $t \in [t_2, T]$,

$$
\|\Omega_4\|_{L^2_{\rm loc}} \le Cn^3 t_2^{1/2} t^{-9/8} + Cn^3 t_2 t^{-3/2}
$$

Summarizing, we have

$$
\left\| e^{(t-t_2)\mathcal{L}} \eta_{0,1} \right\|_{L^4} \le \sum_{j=1}^{4} \|\Omega_j\|_{L^4} \le C \left\{ \|\xi_0\|_{L^{4/3} \cap L^{8/7}} + n^2 t_2^{1/2} + n^3 t_2^{5/8} \right\} t^{-3/4}
$$

$$
\left\| e^{(t-t_2)\mathcal{L}} \eta_{0,1} \right\|_{L^2_{\rm loc}} \le \sum_{j=1}^{4} \|\Omega_j\|_{L^2_{\rm loc}} \le C \left\{ \|\xi_0\|_{L^{4/3} \cap L^{8/7}} + n^3 t_2^{5/8} \right\} t^{-9/8}
$$

Since $t_2 \le \varepsilon^{-2} n^{-4}$, equation (4.7) holds if n is sufficiently small. We have proved Lemma 4.2.

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