THE LAGRANGE INVERSION FORMULA ON NON–ARCHIMEDEAN FIELDS. NON–ANALYTICAL FORM OF DIFFERENTIAL AND FINITE DIFFERENCE EQUATIONS.

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Abstract. The classical formula of Lagrange for the inversion of analytic functions is extended to analytic and non–analytic inversion problems on non–Archimedean fields. We give some applications to the field of formal Laurent series in n variables, where the non–analytic inversion formula gives explicit formal solutions of general semilinear differential and q–difference equations. In particular we will be interested in studying the Siegel center problem (Linearization of germs of functions near a fixed point) and the problem of conjugation of a vector fields with its linear part near a critical point.

In addition to the analytic Siegel Center problem, we give sufficient condition for the linearization to belong to some Classes of ultradifferentiable germs, closed under composition and derivation, including Gevrey Classes. We prove that the Bruno condition is sufficient for the linearization to belong to the same Class of the germ. If one allows the linearization to be less regular than the germ new conditions are introduced, weaker than the Bruno condition. This generalizes to dimension $n \geq 1$ some results of [CM]. A similar statement holds in the case of the linearization of ultradifferentiable vector fields. Moreover our formulation of the non–analytic Lagrange inversion formula by mean of the tree formalism, allows us to point out the strong similarities existing between the two linearization problems, formulated (essentially) with the same functional equation. In the case of analytic vector fields of $\mathbb{C}^2$ we prove a quantitative estimate of a previous qualitative result of [PMY].

1. INTRODUCTION

Let k be a field of characteristic zero complete with respect to a non–trivial absolute value $|\ |$ and let k' denote its residue field. When $k = \mathbb{R}$ or $\mathbb{C}$, the classical Lagrange inversion formula (see [La][1], [Di] chapter VIII, section 7 or [San] p. 286, for the 1–dimensional case, and to [Go] for the multidimensional case) says that if G is an analytic function in a neighborhood of $w \in k$ then there exists a unique

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[1] J. H. Lambert was the first interested in determining the roots x of the equation $x^m + px = q$ developing it in an infinite series [Caj]. His results stimulated J. Lagrange, who first generalized the method to solve the equation $a - x + \phi(x) = 0$ for an analytic function ϕ, and then he applied the idea to the Kepler problem: solving the elliptic motion of a point mass planet about a fixed point, according to the law of the inverse square. To do this Lagrange studied the possibility of inverting the fundamental relation between the mean anomaly, M, and the eccentric anomaly, E: $E = e \sin E + M$, being e the eccentricity of the orbit.

solution $h = H(u,w)$ of

$$h = uG(h) + w\,, \tag{1.1}$$

provided that $|u|$ is sufficiently small. The solution $h = H(u,w)$ depends analytically on u and w and its Taylor series with respect to u is explicitly given by the formula:

$$H(u,w) = w + \sum_{n\geq 1} \frac{u^n}{n!}\frac{d^{n-1}}{dw^{n-1}}(G(w))^n\,. \tag{1.2}$$

In sections 2, after recalling some elementary notions of theory of analytic functions on non–Archimedean fields, we give two generalizations of (1.1): in the n–dimensional vector space k^n, k *non–Archimedean* when G is an analytic function (Corollary 2.5), and for non–analytic G (Theorem 2.3). To deal with this second case we rewrite the Lagrange inversion formula by means of the tree formalism. We refer to [Ge] and references therein for a combinatorial proof of the Lagrange inversion formula using the tree formalism.

In sections 4 and 5 we will give some applications of the previous results in the setting of the formal Laurent series with applications to some dynamical systems problems. The idea of using trees in non–linear small divisors problems (in particular Hamiltonian) is due to H. Eliasson [E] who introduced trees in his study of the absolute convergence of Lindstedt series. The idea has been further developed by many authors (see, for example, [CF, G1, G2] always in the context of Hamiltonian KAM theory, see also [BG1] which we take as reference for many definitions concerning trees). The fact that these formulas should be obtainable by a suitable generalization of Lagrange's inversion formula was first remarked by Vittot [Vi].

When k is the field of formal Laurent series $\mathbb{C}((z))$, we consider the vector space: $\mathbb{C}^n\,((z_1,\dots,z_n))$; the non-analytic inversion problem can be applied to obtain the solution of semilinear differential or q–difference equations in an explicit (i.e. not recursive) form. Our results are formulated so as to include general first–order U–differential semilinear equations[Du] and semilinear convolution equations. In particular we will study (section 4) the Siegel center problem [He, Yo] for analytic and non–analytic germs of $(\mathbb{C}^n,0)$, $n\geq 1$, and (section 5) the Problem of linearization of analytic [Br] and non–analytic vector fields of $\mathbb{C}^n$, $n\geq 1$. The reader interested only in the Siegel center problem may find useful to assume Proposition 4.2 and to skip the reading of the whole of sections 2 and 3. The same is true for those interested in the linearization of vector fields, assuming Proposition 5.1 and reading the rest of section 5, even if they will find several useful definitions in section 4.

In [CM] authors began the study of the Siegel center problem in some ultradifferentiable algebras of $\mathbb{C}((z))$, here we generalize these results to dimension $n\geq 1$.

Consider two Classes of formal power series $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathbb{C}^n\,[[z_1,\dots,z_n]]$ closed with respect to the composition. For example the Class of germs of analytic functions of $(\mathbb{C}^n,0)$ or Gevrey–s Classes, $s>0$ (i.e. series $\mathbf{F}=\sum_{\alpha\in\mathbb{N}^n}\mathbf{f}_\alpha z^\alpha$ such that there exist $c_1,c_2>0$ such that $|\mathbf{f}_\alpha|\leq c_1 c_2^{|\alpha|}(|\alpha|!)^s$, for all $\alpha\in\mathbb{N}^n$). Let $A\in GL(n,\mathbb{C})$ and $\mathbf{F}\in\mathcal{C}_1$ such that $\mathbf{F}(\mathbf{z}) = A\mathbf{z}+\dots$, we say that $\mathbf{F}$ is *linearizable* in $\mathcal{C}_2$ if there exists $\mathbf{H}\in\mathcal{C}_2$, tangent to the identity, such that:

$$\mathbf{F}\circ\mathbf{H}(\mathbf{z}) = \mathbf{H}(A\mathbf{z})$$

When A is in the Poincaré domain, the results of Poincaré [Po] and Koenigs [Ko] assure that $\mathbf{F}$ is linearizable in $\mathcal{C}_2$. When A is in the Siegel domain, the problem is harder, the only trivial case is $\mathcal{C}_2 = \mathbb{C}^n[[z_1, \dots, z_n]]$ (*formal linearization*) for which one only needs to assume A to be *non–resonant*.

In the analytic case we recover the results of Bruno [Br] and Rüssmann [Rü], whereas in the non–analytic case new arithmetical conditions are introduced (Theorem 4.6). Consider the general case where both $\mathcal{C}_1$ and $\mathcal{C}_2$ are different from the Class of germs of analytic function of $(\mathbb{C}^n, 0)$, if one requires $\mathcal{C}_1 = \mathcal{C}_2$, once again the Bruno condition is sufficient, otherwise if $\mathcal{C}_1 \subset \mathcal{C}_2$ one finds new arithmetical conditions, weaker than the Bruno one.

In section 5 we will consider the following differential equation:

$$\dot{\mathbf{z}} = \frac{d\mathbf{z}}{dt} = \mathbf{F}(\mathbf{z}), \tag{1.3}$$

where t is the time variable and $\mathbf{F}$ is a formal power series in the $n \geq 1$ variables $z_1, \dots, z_n$, with coefficients in $\mathbb{C}^n$, without constant term: $\mathbf{F} = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \geq 1} \mathbf{F}_\alpha z^\alpha$, and we are interested in the behavior of the solutions near the singular point $\mathbf{z} = 0$.

A basic but clever idea has been introduced by Poincaré (1879), which consists in reducing the system (1.3) with an appropriate change of variables, to a simpler form: the *normal form*. In [Br] several results are presented in the analytic case (namely $\mathbf{F}$ is a convergent power series). Here we generalize such kind of results to the case of non–analytic $\mathbf{F}$, with a diagonal, non–resonant linear part. More precisely considering the same Classes of formal power series as we did for the Siegel Center Problem , we take an element $\mathbf{F} \in \mathcal{C}_1$ with a diagonal, non–resonant linear part, $A\mathbf{z}$, and we look for sufficient conditions on A to ensure the existence of a change of variables $\mathbf{H} \in \mathcal{C}_2$ (the *linearization*), such that in the new variables the vector field reduces to its linear part. We will show that the Bruno condition is sufficient to linearize in the same class of the given vector field, whereas in the general case, $\mathcal{C}_1 \subset \mathcal{C}_2$, new arithmetical conditions, weaker than the Bruno one, are introduced (Theorem 5.2). Finally in the case of analytic vector field of $\mathbb{C}^2$, the use of the continued fraction and of a best description of the accumulation of small divisors (due to the Davie counting function [Da]), allows us to improve (Theorem 5.5) the results of Theorem 5.2, giving rise to (we conjecture) an optimal estimate concerning the domain of analyticity of the linearization. This gives a quantitative estimate of some previous results of [MM] and [PMY].

In our formulation we emphasize the strong similarities existing between this problem and the Siegel Center Problem, which becomes essentially the same problem; in fact once we reduced each problem to an inversion Lagrange formula (on some appropriate setting) we get the same functional equation to solve.

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2. The Lagrange inversion formula on non–Archimedean fields

In this section we generalize the Lagrange inversion formula for analytic and non–analytic functions on complete, ultrametric fields of characteristic zero. In the first part we give for completeness some basic definitions and properties of non–Archimedean fields, referring to Appendix A and to [Se, Chr, Che] for a more detailed discussion. We end the section introducing some elementary facts concerning trees.

2.1. **Statement of the Problem.** Let $(k, |\ |)$ be a non–Archimedean field of characteristic zero, where $|\ |$ is a ultrametric absolute value : $|x+y| \leq \sup(|x|, |y|)$ for all $x, y \in k$. Moreover we assume that k is complete and the norm is *non–trivial.* Let a be a real number such that $0 < a < 1$, given any $x \in k$ we define the real number $v(x)$ by: $|x| = a^{v(x)}$, the *valuation*[2] of x.

Since k is non–Archimedean one has the following elementary but fundamental result:

Proposition 2.1. *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence with $x_n \in k$. Then $\sum x_n$ converges if and only if $x_n \to 0$.*

Let $n \in \mathbb{N}$, we introduce the n–dimensional vector space k^n and using the ultrametric absolute value, defined on k, we introduce a *norm* $||\cdot|| : k^n \to \mathbb{R}_+$

$$||x|| = \sup_{1 \leq i \leq n} |x_i| \quad x = (x_1, \dots, x_n) \in k^n\,, \tag{2.1}$$

which results an ultrametric one and verifies a Schwartz–like inequality, for $x, y \in k^n$ then: $|x \cdot y| \leq ||x||\,||y||$, where $x \cdot y = \sum_{i=1}^n x_i y_i$ is a scalar product.

This norm induces a topology, where the open balls are defined[3] by

$$B_0(x, r) = \{y \in k^n : ||x - y|| < r\} \tag{2.2}$$

for $x \in k^n$ and $r \in \mathbb{R}_+$. We will denote the closed ball with $B(x, r) = \overline{B}_0(x, r)$.

Let $r > 0$ and let us consider a function $G : B_0(0, r) \subset k^n \to k^{n \times l}$, i.e. for $x \in B_0(0, r)$ and $\forall\ 1 \leq i \leq n, 1 \leq j \leq l$:

$$G(x) = \left(G_{ij}(x)\right)_{ij}\,, \quad G_{ij}(x) \in k.$$

Given $w \in k^n$, $u \in k^l$ and G as above, we consider the following problem:

> Solve with respect to $h \in k^n$, the *multidimensional non–analytic Lagrange inversion problem*:
>
> $$h = \Lambda\left[w + G(h) \cdot u\right], \tag{2.3}$$
>
> where Λ is a k^n–additive, k'–linear, non–expanding operator (i.e. $||\Lambda w|| \leq ||w||$ for all $w \in k^n$).

We will prove the existence of a solution of (2.3) using *trees.* We will now recall some elementary facts concerning trees; we refer to [Ha] for a more complete description.

2.2. **The Tree formalism.** A tree is a connected acyclic graph, composed by *nodes*, and *lines* connecting together two or more nodes. Among trees we consider *rooted trees*, namely trees with an extra node, not included in the set of nodes of the tree, called the *earth*, and an extra line connecting the earth to the tree, the *root line.* We will call *root* the only node to which the earth is linked. The existence of

[2]From the properties of $|\ |$ it follows that the valuation satisfies, for all $x, y \in k$: $v(x) = +\infty$ if and only if $x = 0$; $v(xy) = v(x) + v(y)$; $v(1) = 0$; $v(x+y) \geq \inf(v(x), v(y))$.

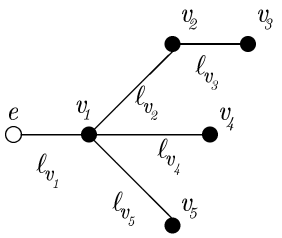
[3]One could define [Se] the open polydisks: $P_0(x, \rho) = \{y \in k^n : \forall i, 1 \leq i \leq n : |x_i - y_i| < \rho_i\}$, for some $x \in k^n$ and $\rho \in \mathbb{R}^n_+$. Clearly the induced topology is equivalent to the previously defined one.

the root introduces a *partial ordering* in the tree: given any two nodes[4] v, v', then $v \geq v'$ if the (only) path connecting the root v_1 with v', contains v. The *order* of a tree is the number of its nodes. The *forest* $\mathcal{T}_N$ is the disjoint union of all trees [5] with the same order N.

The *degree of a node*, $\deg v$, is the number of incident lines with the node. Let $m_v = \deg v - 1$, that is the number of lines entering into the node v w.r.t. the partial ordering, if $m_v = 0$ we will say that v is an *end node*; for the root v_1, because the root line doesn't belong to the lines of the tree, we define $m_{v_1} = \deg v_1$, in this way m_{v_1} also represents the number of lines entering in the root. Let ϑ be a rooted tree, for any $v \in \vartheta$ we denote by L_v the *set of lines entering into* v; if v is an end node we will set $L_v = \emptyset$.

Given a rooted tree ϑ of order N, we can view it as the union of its root and the subtrees ϑ^i obtained from ϑ by detaching the root. Let v_1 be the root of ϑ and $t = m_{v_1}$, we define the *standard decomposition* of ϑ as: $\vartheta = \{v_1\} \cup \vartheta^1 \cup \dots \cup \vartheta^t$, where $\vartheta^i \in \mathcal{T}_{N_i}$ with $N_1 + \ldots + N_t = N - 1$.

Using the definition of m_v we can associate uniquely to a rooted tree of order N a vector of $\mathbb{N}^N$, whose components are just m_v with v in the tree [Vi]. Thus $\mathcal{T}_N = \{(m_1, \dots, m_N) \in \mathbb{N}^N : \sum_{i=1}^N m_i = N - 1, \quad \sum_{i=j}^N m_i \leq N - j \quad \forall j = 1, \cdots, N\}$. We can then rewrite the standard decomposition of ϑ as: $\vartheta = \left(t, \vartheta^1, \dots, \vartheta^t\right)$ where the subtrees satisfy: $\vartheta^i \in \mathcal{T}_{N_i}$ with $N_1 + \ldots + N_t = N - 1$.



$\Leftrightarrow \quad (3, 1, 0, 0, 0)$

FIGURE 1. A rooted tree of order 5, v_i, $i = 1, \dots, 5$, are the nodes whereas ℓ_{v_i}, $i = 1, \dots, 5$, are the lines. The earth is denoted by the letter e, ℓ_{v_1} is the root line and v_3, v_4, v_5 are end nodes. On the right we show the standard decomposition of this tree.

In the following we will also use *labeled rooted trees*. A labeled rooted tree of order N is an element of $\mathcal{T}_N$ together with N labels: $\alpha_1, \dots, \alpha_N$. We can think that the label α_i is attached to the i–th node of the standard decomposition of the tree. The label is nothing else that a function from the set of nodes of a tree to some set, usually a subset of $\mathbb{Z}^m$ for some integer m. When needed we denote a labeled rooted tree of order N with the couple (ϑ, α), where $\vartheta \in \mathcal{T}_N$ and $\alpha = (\alpha_1, \dots, \alpha_N)$ is the vector label.

[4] To denote nodes we will use letters: $u, v, w, \dots$, with possible sub-indices. Lines will be denoted by ℓ, the line exiting from the node u will be denoted by ℓ_u.

[5] Here we consider only *semitopological* trees (see [BG2]), we refer to [G1] for the definition of *topological trees*.

2.3. **The non–analytic Lagrange inversion formula.** We are now able to extend equation (1.1), the classical analytic Lagrange inversion formula, to the setting of paragraph 2.1. We refer the reader to Appendix A for a brief introduction to the theory of analytic functions on k^n (norms, Cauchy estimates, etc ...).

Let $N \in \mathbb{N}^*$, U and V be open subsets of, respectively, k^l and k^n, and $\vartheta \in \mathcal{T}_N$. We define the function $Val_\Lambda : \mathcal{T}_N \times U \times V \ni (\vartheta, u, w) \mapsto Val_\Lambda(\vartheta)(u, w) \in k^n$ as follows

(2.4)
$$Val_\Lambda(\vartheta)(u,w) = \begin{cases} \Lambda\left(G\left(\Lambda w\right)\cdot u\right) & \text{if } \vartheta \in \mathcal{T}_1 \\ \frac{1}{t!}\Lambda\left[d^t G\left(\Lambda w\right)\left(Val_\Lambda(\vartheta^1)(u,w), \dots, Val_\Lambda(\vartheta^t)(u,w)\right)\cdot u\right] & \text{otherwise} \end{cases}$$

where $\vartheta = (t, \vartheta^1, \dots, \vartheta^t)$ is the standard decomposition of the tree and $\Lambda : k^n \to k^n$.

Remark 2.2. *For $t \geq 1$ and $v^{(1)}, \dots, v^{(t)} \in k^n$ we recall that*

$$d^t G_{ij}(w)\left(v^{(1)}, \dots, v^{(t)}\right) = \sum_{l_1, \dots, l_t = 1}^{n} D_{l_1} \dots D_{l_t} G_{ij}(w)\, v^{(1)}_{l_1} \dots v^{(t)}_{l_t},$$

with $v^{(i)} = (v^{(i)}_1, \dots, v^{(i)}_n)$ and D_{l_i} are the l_i–th partial derivatives of G at w (see Appendix A).

We can then state the following existence Theorem:

Theorem 2.3 (Non–analytic case)**.** *Let n, l be positive integers. Let $u \in k^l$, $w \in k^n$ and for $1 \leq i \leq n$ let $G_i = (G_{i1}, \dots, G_{il})$, with $G_i \in k^n[[X_1, \dots, X_n]]$. Let $\Lambda : k^n \to k^n$ be a k^n–additive, k'–linear and non–expanding operator. Assume that for $(r_1, \dots, r_n) \in \mathbb{R}^n_+$ the G_i's are convergent in $B(0, r_i)$, set[6] $M_i = ||G_i||_{r_i} > 0$, $r = \min_i r_i$ and $M = \max_i M_i$. Then equation (2.3) has the unique solution*

$$H(u,w) = \Lambda w + \sum_{N \geq 1} \sum_{\vartheta \in \mathcal{T}_N} Val_\Lambda(\vartheta)(u,w), \tag{2.5}$$

where $Val_\Lambda : \mathcal{T}_N \times B_0(0, r/M) \times B_0(0, r) \to B_0(0, r)$, has been defined in (2.4). Moreover for any fixed ϑ the function Val_Λ is continuous, series (2.5) converges on $B_0(0, r/M) \times B_0(0, r)$ and the map $(u, w) \mapsto H(u, w) \in B_0(0, r)$ is continuous.

Remark 2.4. *Since Λ is not k^n–linear, Val_Λ and H cannot be analytic. However the non–expanding condition implies that Λ is Lipschitz continuous from which the regularity properties of Val_Λ and H follow.*

If Λ is k^n–linear then we have the following Corollary (particular case of the previous Theorem) with w' instead of Λw and $G' \cdot u'$ instead of $\Lambda(G \cdot u)$, which extends the analytic Lagrange inversion formula (1.1).

Corollary 2.5 (Analytic case)**.** *Let n, l be positive integers. Let $u' \in k^l$, $w' \in k^n$ and for $1 \leq i \leq n$ let $G'_i = (G'_{i1}, \dots, G'_{il})$, with $G'_i \in k^n[[X_1, \dots, X_n]]$. Assume that for $(r_1, \dots, r_n) \in \mathbb{R}^n_+$ the G'_i's are convergent in $B(0, r_i)$, let $M_i = ||G'_i||_{r_i} > 0$, $r = \min_i r_i$ and $M = \max_i M_i$. Then*

$$H(u', w') = w' + \sum_{N \geq 1} \sum_{\vartheta \in \mathcal{T}_N} Val(\vartheta)(u', w'), \tag{2.6}$$

[6] See AppendixA for the definition $|| \cdot ||_r$.

is the unique solution of (2.3) *with* $\Lambda(G \cdot u) = G' \cdot u'$ *and* $\Lambda w = w'$. *The function* $Val\,(\vartheta)\,(u', w')$ *is nothing else that the function* $Val_\Lambda(\vartheta)\,(u, w)$ *with* $\Lambda(G \cdot u) = G' \cdot u'$ *and* $\Lambda w = w'$. *Moreover* H *is analytic in* $B_0\left(0, \frac{r}{M}\right) \times B_0\,(0, r)$.

3. Proofs.

This section is devoted to the proof of Theorem 2.3 and Corollary 2.5.

Proof of Theorem 2.3. Using the fact that Λ is non–expanding the uniqueness of the solution can be proved easily. Let H_1 and H_2 be two solutions of (2.3), then

$$||H_1 - H_2|| = ||\Lambda\left[\left(G\,(H_1) - G\,(H_2)\right) \cdot u\right]|| \leq ||\left(G\,(H_1) - G\,(H_2)\right) \cdot u||,$$

but for all $i = 1, \ldots, n$:

$$||\left(G\,(H_1) - G\,(H_2)\right) \cdot u|| \leq ||G_i\,(H_1) - G_i\,(H_2)||\,||u||, \tag{3.1}$$

and by Proposition A.1 $||G_i\,(H_1) - G_i\,(H_2)|| \leq \frac{||G_i||_{r_i}}{r_i}||H_1 - H_2||$. Then setting $\mu = ||u||\frac{\max M_i}{\min r_i}$, from (3.1) we conclude that

$$||H_1 - H_2|| \leq \mu||H_1 - H_2||.$$

By hypothesis $||u|| < \frac{r}{M}$, then $\mu < 1$, from which we conclude that $H_1 = H_2$.

We now prove existence. Since G_i are convergent, $D^\alpha G_i$ also are convergent and Proposition A.2 gives the following estimate

$$||D^\alpha G_i||_{r_i} \leq \frac{M_i}{r_i^{|\alpha|}}, \quad \forall \alpha \in \mathbb{N}^n\,, \tag{3.2}$$

which together with the non–expanding property of Λ allows us to prove that for all $N \geq 1$ and all $\vartheta \in \mathcal{T}_N$:

$$\left|\left|\, Val_\Lambda(\vartheta)\,(u, w)\,\right|\right| \leq ||u||^N \frac{M^N}{r^{N-1}}\,.$$

Let us define $H^{(0)}\,(u, w) = \Lambda w$ and $H^{(j)}\,(u, w) = \Lambda w + \sum_{N=1}^{j} \sum_{\vartheta \in \mathcal{T}_N} Val_\Lambda(\vartheta)\,(u, w)$, clearly $H^{(j)} \to H$ as $j \to \infty$ and it is easy to check that:

$$\left|\left|H^{(j)} - \Lambda\left[w - G\left(H^{(j)}\right) \cdot u\right]\right|\right| \leq r\left(||u||\frac{M}{r}\right)^{j+1}$$

which tends to 0 as $j \to \infty$.

□

We give now the proof of Corollary 2.5. This one follows closely the one of Theorem 2.3 in particular the uniqueness statement, so we will outline only the main differences w.r.t to the previous proof.

Proof. The hypothesis on G'_i gives an estimate similar to (3.2), then by induction on N it is easy to prove that for all $N \geq 1$ and $\vartheta \in \mathcal{T}_N$ one has

$$\left|\left|\, Val\,(\vartheta)\,(u', w')\,\right|\right| \leq ||u'||^N \frac{M^N}{r^{N-1}}.$$

Then if $||u'|| < \frac{r}{M}$ series (2.6) converges and if $||w'|| \leq r$, $H\,(u', w') \in B_0\,(0, r)$, in fact

$$||H|| = \left|\left|w' + \sum_N \sum_{\vartheta \in \mathcal{T}_N} Val\,(\vartheta)\,(u', w')\right|\right| \leq \sup\left\{||w'||, \sup_{N, \vartheta \in \mathcal{T}_N}\left|\left|\, Val\,(\vartheta)\,(u', w')\right|\right|\right\} < r.$$

Now introducing $H^{(0)}(u', w') = w$ and $H^{(j)}(u', w') = w' + \sum_{N=1}^{j} \sum_{\vartheta \in \mathcal{T}_N} Val(\vartheta)(u', w')$, one can easily prove that $H^{(j)} \to H$ as $j \to \infty$. $\square$

Remark 3.1. *In the simplest case $n = l = 1$, namely $u, w \in k$ and $G \in k[[X]]$, the solution given by* (2.6) *coincides with the classical one of Lagrange* (1.2). *One can prove this fact either using the uniqueness of the Taylor development or by direct calculation showing that for all positive integer $N \geq 1$ we have*

$$\sum_{\vartheta \in \mathcal{T}_N} Val(\vartheta)(u, w) = \frac{u^N}{N!} \frac{d^{N-1}}{dw^{N-1}} [G(w)]^N .$$

In the other cases formula (2.6) *is the natural generalization of* (1.2).

Remark 3.2. *Series* (2.6) *is an analytic function of u, w, but it is not explicitly written as u–power series. We claim that introducing* labeled rooted trees *we can rewrite* (2.6) *explicitly as a u–power series.*

4. The Non–analytic Siegel center problem.

In this part we show that the problem of the conjugation a (formal) germ of a given function with its linear part near a fixed point (the so called *Siegel Center Problem*) can be solved applying Theorem 2.3 to the field of (formal) power series. The Siegel Center Problem is a particular case of first order semilinear q–difference equation, but our results apply to general first order semilinear q–difference and differential equations (see next section).

4.1. Notations and Statement of the Problem.

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ with $\lambda_i \neq \lambda_j$ if $i \neq j$, we will use the compact notation $\lambda^\alpha = \lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$; we will denote the diagonal $n \times n$ matrix with λ_i at the (i, i)–th place with $diag(\lambda_1, \dots, \lambda_n)$.

Let $V = \mathbb{C}^n [[z_1, \dots, z_n]]$ be the vector space of the formal power series in the n variables $z_1, \dots, z_n$ with coefficients in $\mathbb{C}^n$: $\mathbf{f} \in V$ if

$$\mathbf{f} = \sum_{\alpha \in \mathbb{N}^n} \mathbf{f}_\alpha z^\alpha, \quad \mathbf{f}_\alpha \in \mathbb{C}^n \text{ and } z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n} \ \forall\, \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n .$$

We consider V endowed with the ultrametric absolute norm induced by the $\mathbf{z}$–adic valuation ($\mathbf{z} = (z_1, \dots, z_n)$): $||\mathbf{f}|| = 2^{-v(\mathbf{f})}$, where $v(\mathbf{f}) = \inf\{|\alpha|, \alpha \in \mathbb{N}^n : \mathbf{f}_\alpha \neq 0\}$, and for any positive integer j we denote by $V_j = \{\mathbf{f} \in V : v(\mathbf{f}) > j\}$.

Let $\mathcal{C}$ be a Class (that we will define later, see paragraph 4.3) of formal power series, closed w.r.t. the (formal) derivation, the composition and where, roughly speaking, the (formal) Taylor series makes sense. One can think for example to the Class of germs of analytic diffeomorphism of $(\mathbb{C}^n, 0)$ or Gevrey–s Classes, in fact we will see that our classes will contain these special cases.

Let $A \in GL(n, \mathbb{C})$ and assume A to be diagonal [7] with all the eigenvalues distinct. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes as stated before, then the *Siegel center problem* can be formulated as follow [He, CM]:

[7] The case A non–diagonal need some special attentions, see [He] Proposition 3 page 143 and [Yo] Appendix 1.

Let $\mathbf{F}(\mathbf{z}) = A\mathbf{z}+\mathbf{f}(\mathbf{z}) \in \mathcal{C}_1$, $\mathbf{f} \in \mathcal{C}_1 \cap V_1$, find necessary and sufficient conditions on A to *linearize* in $\mathcal{C}_2$ $\mathbf{F}$, namely find $\mathbf{H} \in \mathcal{C}_2 \cap V_0$ (the *linearization*) solving:

$$\mathbf{F} \circ \mathbf{H}(\mathbf{z}) = \mathbf{H}(A\mathbf{z})\,. \tag{4.1}$$

Let $A = diag(\lambda_1, \dots, \lambda_n)$ we introduce the operator $D_\lambda : V \to V$:

$$D_\lambda \mathbf{g}(\mathbf{z}) = \mathbf{g}(A\mathbf{z}) - A\mathbf{g}(\mathbf{z})\,, \tag{4.2}$$

for any $\mathbf{g}(\mathbf{z}) \in V$. We remark that the action of D_λ on the monomial $\mathbf{v}z^\alpha$, for any $\mathbf{v} \in \mathbb{C}^n$ and any $\alpha \in \mathbb{N}^n$, is given by:

$$D_\lambda \mathbf{v}(z^\alpha) = (\Omega_\alpha \mathbf{v})z^\alpha\,, \tag{4.3}$$

where the matrix Ω_α is defined by $\Omega_\alpha = diag(\lambda^\alpha - \lambda_1, \dots, \lambda^\alpha - \lambda_n)$ and $(\Omega_\alpha \mathbf{v}) = \sum_{i=1}^n (\lambda^\alpha - \lambda_i)v_i$ is the matrix–vector product.

Let $A = diag(\lambda_1, \dots, \lambda_n) \in GL(n, \mathbb{C})$, we say that A is *resonant* if there exist $\alpha \in \mathbb{N}^n$, $|\alpha| \geq 2$ and $j \in \{1, \dots, n\}$ such that:

$$\lambda^\alpha - \lambda_j = 0\,. \tag{4.4}$$

If (4.4) doesn't hold, we say that A is *non–resonant*.

If A is non–resonant then D_λ is invertible on V_1 (namely $|\alpha| \geq 2$), it is non–expanding ($||D_\lambda \mathbf{g}|| \leq ||\mathbf{g}||$), and clearly V–additive ($D_\lambda(\mathbf{f}+\mathbf{g}) = D_\lambda \mathbf{f} + D_\lambda \mathbf{g}$) and $\mathbb{C}$–linear. Then we claim that the Siegel center problem (4.1) is equivalent to solve the functional equation:

$$D_\lambda \mathbf{h} = \mathbf{f} \circ (\mathbf{z} + \mathbf{h}) \tag{4.5}$$

where $\mathbf{z}$ is the identity formal power series, $\mathbf{f} \in \mathcal{C}_1 \cap V_1$ and $\mathbf{h} \in \mathcal{C}_2 \cap V_1$. In fact from (4.1) we see that the linear part of $\mathbf{H}$ doesn't play any role, so we can choose $\mathbf{H}$ tangent to the identity (this normalization assures the uniqueness of the linearization): $\mathbf{H}(\mathbf{z}) = \mathbf{z} + \mathbf{h}(\mathbf{z})$, with $\mathbf{h} \in \mathcal{C}_2 \cap V_1$. But then (4.1) can be rewritten as:

$$\mathbf{h}(A\mathbf{z}) - A\mathbf{h}(\mathbf{z}) = \mathbf{f} \circ (\mathbf{z} + \mathbf{h}(\mathbf{z}))\,, \tag{4.6}$$

and replacing the left hand side with the operator $D_\lambda \mathbf{h}$ we obtain (4.5).

Given $\mathbf{f} \in V_1$ we consider the function $G_\mathbf{f}(\mathbf{g}) = \mathbf{f} \circ (\mathbf{z}+\mathbf{g})$, for all $\mathbf{g} \in V_1$, assume that we can invert the operator D_λ (because $v(\mathbf{f}(\mathbf{z}+\mathbf{g})) \geq v(\mathbf{f})$ we can invert D_λ whenever $\mathbf{f} \in V_1$), we can rewrite (4.5) as:

$$\mathbf{h} = D_\lambda^{-1}\left(G_\mathbf{f}(\mathbf{h})\right)\,, \tag{4.7}$$

which is a particular case of the non–analytic multidimensional Lagrange inversion formula (2.3) with $u = 1$, $w = 0$ and $\Lambda = D_\lambda^{-1}$. The following Lemma assures that $G_\mathbf{f}$ verifies the hypotheses of Theorem 2.3.

Lemma 4.1. *Given* $\mathbf{f} \in V_1$ *the composition* $\mathbf{f} \circ (\mathbf{z} + \mathbf{v})$ *defines a power series* $G_\mathbf{f} \in V_1[[v_1, \dots, v_n]]$ *convergent on* $B(0, 1/2) = V_0$ *and*

$$G_\mathbf{f}(\mathbf{v}) = \sum_{\beta \in \mathbb{N}^n} \mathbf{g}_\beta(\mathbf{f})v^\beta \tag{4.8}$$

where (if we define $\mathbf{f}_\beta = 0$ *for* $|\beta| = 0$ *and* $|\beta| = 1$*) series* $\mathbf{g}_\beta(\mathbf{f}) \in V$ *are given by*

$$\mathbf{g}_\beta(\mathbf{f})(\mathbf{z}) = \sum_{\alpha \in \mathbb{N}^n} \mathbf{f}_{\alpha+\beta}\binom{\alpha+\beta}{\beta}z^\alpha \tag{4.9}$$

Here we used the compact notations $\alpha! = \alpha_1! \dots \alpha_n!$ and $\binom{\alpha+\beta}{\beta} = \frac{(\alpha+\beta)!}{\alpha!\beta!}$, for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^n$. Moreover one has

$$||\mathbf{g}_0|| = ||\mathbf{f}||, \; ||\mathbf{g}_\beta|| = 2||\mathbf{f}|| \text{ for any } |\beta| = 1 \text{ and } ||\mathbf{g}_\beta|| \leq 4||\mathbf{f}|| \text{ for any } |\beta| \geq 2; \tag{4.10}$$

thus

$$||G_{\mathbf{f}}||_{1/2} \leq ||\mathbf{f}||. \tag{4.11}$$

The proof is straightforward and we omit it.

We can thus apply Theorem 2.3 to solve (4.7) with $u = 1$, $w = 0$, $G = G_{\mathbf{f}}$, $\Lambda = D_\lambda^{-1}$, $r = 1/2$ and $M = 1/4$, and the unique solution of (4.7) is given by

$$\mathbf{h} = \sum_{N \geq 1} \sum_{\vartheta \in \mathcal{T}_N} Val_{D_\lambda^{-1}}(\vartheta)\,(1,0)\,. \tag{4.12}$$

An explicit expression for the power series coefficients of $\mathbf{h}$ can be obtained introducing *labeled rooted trees* (see Remark 3.2). Let us now explain how to do this. Let $N \geq 1$ and let ϑ be a rooted tree of order N; to any $v \in \vartheta$ we associate a *(node)–label* $\alpha_v \in \mathbb{N}^n$ s.t. $|\alpha_v| \geq 2$ and to the line ℓ_v (exiting w.r.t. the partial order from v) we associate a *(line)–label* $\beta^{\ell_v} \in \mathbb{N}^n$ s.t. $|\beta^{\ell_v}| = 1$. We define $\beta_v = \sum_{\ell \in L_v} \beta^\ell$ (so $\beta_v \in \mathbb{N}^n$ and $|\beta_v| = m_v$) and the *momentum* flowing through the line ℓ_v: $\nu_{\ell_v} = \sum_{w \in \vartheta : w \leq v}(\alpha_w - \beta_w)$. When v will be root of the tree, we will also use the symbol ν_ϑ (the *total momentum* of the tree) instead of ν_{ℓ_v}. It is trivial to show that the momentum function is increasing (w.r.t. the partial order of the tree), namely if v is the root of a rooted labeled tree and if v_i is any of the immediate predecessor of v, $v > v_i$, we have: $|\nu_v| > |\nu_{v_i}|$). Recalling that the order of ϑ is N we have: $|\nu_\vartheta| \geq N + 1$.

Let $N \geq 1$, $j \in \{1, \dots, n\}$ and $\alpha \in \mathbb{N}^n$, we finally define $\mathcal{T}_{N,\alpha,j}$ the *forest of rooted labeled trees* of order N with total momentum $\nu_\vartheta = \alpha$ and $\beta^{\ell_{v_1}} = e_j$ (being e_j the vector with all zero entries but the j–th which is set equal to 1) for the root line ℓ_{v_1}.

We are now able to prove the following

Proposition 4.2. *Equation* (4.7) *admits a unique solution* $\mathbf{h} \in V_1$, $\mathbf{h} = \sum_{\alpha \in \mathbb{N}^n} \mathbf{h}_\alpha z^\alpha$. *For $|\alpha| \geq 2$ the j–th component of the coefficient $\mathbf{h}_\alpha$ is given by* [8]*:*

$$h_{\alpha,j} = \sum_{N=1}^{|\alpha|-1} \sum_{\vartheta \in \mathcal{T}_{N,\alpha,j}} ((\Omega^{-1}_{\nu_{\ell_{v_1}}} \mathbf{f}_{\alpha_{v_1}}) \cdot \beta^{\ell_{v_1}}) \prod_{v \in \vartheta} \binom{\alpha_v}{\beta_v} \prod_{\ell_w \in L_v} ((\Omega^{-1}_{\nu_{\ell_w}} \mathbf{f}_{\alpha_w}) \cdot \beta^{\ell_w})\,, \tag{4.13}$$

where the last product has to be set equal to 1 *whenever v is an end node ($L_v = \emptyset$).*

Remark 4.3. *By definition β^{ℓ_w}, for any $w \in \vartheta$, has length* 1*, so it coincides with an element of the canonical base. Then for $w \in \vartheta$ and any choice of the labels, such that $\beta^{\ell_w} = e_i$, the term $((\Omega^{-1}_{\nu_{\ell_w}} \mathbf{f}_{\alpha_w}) \cdot \beta^{\ell_w})$ is nothing else that*

$$((\Omega^{-1}_{\nu_{\ell_w}} \mathbf{f}_{\alpha_w}) \cdot \beta^{\ell_w}) = \frac{1}{\lambda^{\nu_{\ell_w}} - \lambda_i} f_{\alpha_w, i}\,.$$

Remark 4.4. *Even if all the nodes labels α_v have non–negative components, the momenta can have (several) negative components. More precisely, let $\vartheta \in \mathcal{T}_N$, if the order of the tree if big enough w.r.t. the dimension n ($N \geq n$) then ν_ϑ can have*

[8] Compare this expression with equation (3.7) of Proposition 3.1 in [CF].

$n-1$ negative components and their sum can be equal to $1-N$, but we always have $|\nu_\vartheta| \geq N+1$. This fact reminds the definitions of the sets $\mathbf{N}_i$, $\mathbf{N}^{(m)}$ and $\mathbf{N}_+^{(m)}$ of [Br].

Proof. Let $\vartheta \in \mathcal{T}_N$, for $N \geq 1$ and $\alpha \in \mathbb{N}^n$ such that $|\alpha| \geq N+1$. Let us define $Val(\vartheta) = ((\Omega^{-1}_{\nu_{\ell_{v_1}}} \mathbf{f}_{\alpha_{v_1}}) \cdot \beta^{\ell_{v_1}}) \prod_{v\in\vartheta} \binom{\alpha_v}{\beta_v} \prod_{\ell_w \in L_v} ((\Omega^{-1}_{\nu_{\ell_w}} \mathbf{f}_{\alpha_w}) \cdot \beta^{\ell_w})$ and

$$h^\vartheta_{|\alpha|,N,j} = \sum_{|\nu_\vartheta|=|\alpha|} Val(\vartheta)\,, \tag{4.14}$$

namely for a fixed tree, sum over all possible labels α_{v_i} and $\beta^{\ell_{v_i}}$, with v_i in the tree, in such a way that the total momentum is fixed to α and the root line has label $\beta^{\ell_{v_1}} = e_j$. It is clear that

$$\mathbf{h} = \sum_{|\alpha|\geq 2} z^\alpha \sum_{N=1}^{|\alpha|-1} \sum_{\vartheta\in\mathcal{T}_N} \mathbf{h}^\vartheta_{|\alpha|,N}\,, \tag{4.15}$$

being $\mathbf{h}^\vartheta_{|\alpha|,N}$ the vector whose j–th component is $h^\vartheta_{|\alpha|,N,j}$. Let $\mathbf{h}^\vartheta_{N+1} = \sum_{|\alpha|\geq N+1} z^\alpha \mathbf{h}^\vartheta_{|\alpha|,N}$, clearly $||\mathbf{h}^\vartheta_{N+1}|| \leq 2^{-(N+1)}$ and

$$\mathbf{h} = \sum_{N\geq 1} \sum_{\vartheta\in\mathcal{T}_N} \mathbf{h}^\vartheta_{N+1}\,. \tag{4.16}$$

Actually in (4.15) $\mathbf{h}$ is ordered with increasing powers of $\mathbf{z}$ whereas in (4.16) with increasing order of trees. Convergence in V_1 for (4.16) is assured from the estimate $||\mathbf{h}^\vartheta_{N+1}|| \leq 2^{-(N+1)}$ and uniform convergence assures that (4.15) and (4.16) coincide.

We claim that by induction on the order of the tree, we can prove that for all $N \geq 1$ and all $\vartheta \in \mathcal{T}_N$:

$$\mathbf{h}^\vartheta_{N+1} = Val_{D_\lambda^{-1}}(\vartheta)\,(1,0) \tag{4.17}$$

where $Val_{D_\lambda^{-1}}(\vartheta)\,(1,0)$ has been defined in (2.4), taking $\Lambda = D_\lambda^{-1}$ and $G = G_{\mathbf{f}}$, thus establishing the equivalence of (4.13) and (4.12). ☐

4.2. **Some known results.** If both Classes $\mathcal{C}_1$ and $\mathcal{C}_2$ are V_1 (which verify the hypotheses of stability w.r.t. the derivation, closeness w.r.t. the composition, and the formal Taylor series makes sense), then the *Formal* Siegel Center Problem has a solution if the linear part of $\mathbf{F}$ is non–resonant.

A matrix $A = diag(\lambda_1, \dots, \lambda_n) \in GL(n, \mathbb{C})$, is in the *Poincaré domain* if

$$\sup_{1\leq j\leq n} |\lambda_j| < 1 \quad \text{or} \quad \sup_{1\leq j\leq n} |\lambda_j^{-1}| > 1\,, \tag{4.18}$$

if A doesn't belong to the Poincaré domain it will be in the *Siegel domain*.

In the *Analytic* case (both $\mathcal{C}_1$ and $\mathcal{C}_2$ are the ring of the germs of analytic diffeomorphism of $(\mathbb{C}^n, 0)$) , let A be the derivative of $\mathbf{F}$ at the origin, then if A is non–resonant and it is in the *Poincaré domain*, the Analytic Siegel Center Problem has a solution [Po, Ko] (see also [He] and references therein). Moreover if A is resonant and in the Poincaré domain, but $\mathbf{F}$ is formally linearizable, then $\mathbf{F}$ is analytically linearizable.

If A is in the Siegel domain the problem is harder, but we can nevertheless have a solution of the Analytic Siegel Center Problem, introducing some new condition on A. Let $p \in \mathbb{N}$, $p \geq 2$, and let us define

$$\tilde{\Omega}(p) = \min_{1 \leq j \leq n} \inf_{\alpha \in \mathbb{Z}^n : |\alpha| < p} |\lambda^\alpha - \lambda_j| \,, \tag{4.19}$$

we remark that even if A is non–resonant, but in the Siegel domain, one has $\lim_{p \to \infty} \tilde{\Omega}(p) = 0$; this is the so called *small divisors problem*, the main obstruction to the solution of equation (4.1). A non–resonant matrix A verifies a *Bruno condition* [9] if there exists an increasing sequence of natural numbers $(p_k)_k$ such that

$$\sum_{k=0}^{+\infty} \frac{\log \tilde{\Omega}^{-1}(p_{k+1})}{p_k} < +\infty \,. \tag{4.20}$$

Then if A satisfies a Bruno Condition, the germ is analytically linearizable [Br, Rü].

For the 1–dimensional Analytic Siegel Center Problem, Yoccoz [Yo] proved that the Bruno condition is necessary and sufficient to linearize analytically, in this case the Bruno condition reduces to the convergence of the series

$$\sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k} < +\infty \,, \tag{4.21}$$

where $(q_k)_k$ is the sequence of the convergent to $\omega \in \mathbb{R} \setminus \mathbb{Q}$ such that $\lambda = e^{2\pi i \omega}$.

4.3. **A new result: ultradifferentiable Classes.** Let $(M_k)_{k \geq 1}$ be a sequence of positive real numbers such that:

0) $\inf_{k \geq 1} M_k^{1/k} > 0$;

1) There exists $C_1 > 0$ such that $M_{k+1} \leq C_1^{k+1} M_k$ for all $k \geq 1$;

2) The sequence $(M_k)_{k \geq 1}$ is logarithmically convex;

3) $M_k M_l \leq M_{k+l-1}$ for all $k, l \geq 1$.

We define the class $\mathcal{C}_{(M_k)} \subset \mathbb{C}^n [[z_1, \dots, z_n]]$ as the set of formal power series $\mathbf{f} = \sum \mathbf{f}_\alpha z^\alpha$ such that there exist A, B positive constant, such that:

$$|\mathbf{f}_\alpha| \leq A B^{|\alpha|} M_{|\alpha|} \quad \forall \alpha \in \mathbb{N}^n \,. \tag{4.22}$$

The hypotheses on the sequence $(M_k)_k$ assure that $\mathcal{C}_{(M_k)}$ is stable w.r.t. the (formal) derivation, w.r.t. the composition of formal power series and for every tensor built with element of the class, its contraction[10] gives again an element of the class. For example if $\mathbf{f}, \mathbf{g} \in \mathcal{C}_{(M_k)}$ then also $d\mathbf{f}(\mathbf{z})(\mathbf{g}(\mathbf{z}))$ belongs to the same class.

Remark 4.5. *Our classes include the Class of Gevrey–s power series as a special case: $M_k = (k!)^s$. Also the ring of convergent (analytic) power series are trivially included.*

In [CM] a similar problem was studied in the 1–dimensional case. Here we will extend the results contained there to the case of dimension $n \geq 1$. The main result will be the following Theorem

Theorem 4.6. *Let $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and let $A = diag(\lambda_1, \dots, \lambda_n)$ be non–resonant, $(M_k)_k$ and $(N_k)_k$ be sequences verifying hypotheses 0)–3). Let $\mathbf{F} \in V_0$, s.t. $\mathbf{F}(\mathbf{z}) = A\mathbf{z} + \mathbf{f}(\mathbf{z})$ where $\mathbf{f} \in V_1$. Then*

1. *If moreover $\mathbf{F} \in \mathcal{C}_{(M_k)} \cap V_0$ and A verifies a Bruno condition* (4.20)*, then also the linearization $\mathbf{H}$ belongs to $\mathcal{C}_{(M_k)} \cap V_0$.*

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[9] See [Br, Rü] for various equivalent formulations of this condition.

[10] This assures that any term of the Taylor series is well defined.

2. *If* $\mathbf{F}$ *is a germ of analytic diffeomorphism of* $(\mathbb{C}^n, 0)$ *and there exists an increasing sequence of integer numbers* $(p_k)_k$ *such that* A *verifies:*

$$\limsup_{|\alpha|\to+\infty}\left(2\sum_{m=0}^{\kappa(\alpha)}\frac{\log\Omega^{-1}(p_{m+1})}{p_m}-\frac{1}{|\alpha|}\log N_{|\alpha|}\right)\,, \tag{4.23}$$

where $\kappa(\alpha)$ *is the integer defined by:* $p_{\kappa(\alpha)}\leq|\alpha|<p_{\kappa(\alpha)+1}$*, then the linearization* $\mathbf{H}$ *belongs to* $\mathcal{C}_{(N_k)}\cap V_0$*.*

3. *If* $\mathbf{F}\in\mathcal{C}_{(M_k)}\cap V_0$*, the sequence* $(M_k)_k$ *is asymptotically bounded by the sequence* $(N_k)_k$ *(namely* $N_k\geq M_k$ *for all sufficiently large* k*) and there exists an increasing sequence of integer numbers* $(p_k)_k$ *such that* A *verifies:*

$$\limsup_{|\alpha|\to+\infty}\left(2\sum_{m=0}^{\kappa(\alpha)}\frac{\log\Omega^{-1}(p_{m+1})}{p_m}-\frac{1}{|\alpha|}\log\frac{N_{|\alpha|}}{M_{|\alpha|}}\right)\,, \tag{4.24}$$

where $\kappa(\alpha)$ *is the integer defined by:* $p_{\kappa(\alpha)}\leq|\alpha|<p_{\kappa(\alpha)+1}$*, then the linearization* $\mathbf{H}$ *belongs to* $\mathcal{C}_{(N_k)}\cap V_0$*.*

The proof of Theorem 4.6 will be done in section 4.3.2, before we make some remarks and we prove some preliminaries lemmata.

Remark 4.7. *The new arithmetical conditions* (4.23) *and* (4.24) *are generally weaker than the Bruno condition* (4.20)*. Theorem 4.6 is the natural generalization of results of* [CM] *(compare it with Theorem 2.3 of* [CM]*): condition* (4.24) *(respectively* (4.23)*) reduces to condition (2.10) (respectively condition (2.9)) of* [CM] *except for the factor in front of the sum: here we have* 2 *instead of* 1 *in* [CM]*. This is due to the better control of small denominators one can achieve using continued fractions and Davie's counting lemma* [Da] *as explained in* [CM].

4.3.1. *Some preliminaries.* Let $\omega_j\in(-1/2,1/2)\setminus\mathbb{Q}$ for $j=1,\dots,n$ and assume $\lambda=(\lambda_1,\dots,\lambda_n)$ with $\lambda_j=e^{2\pi i\omega_j}$. Let ν_ℓ, be the momentum of a line of a rooted labeled tree which contributes with a small denominator of the form: $|\lambda^{\nu_\ell}-\lambda_j|=2|\sin\pi(\nu_\ell\cdot\omega-\omega_j)|$, where $\nu_\ell\cdot\omega=\sum_{j=1}^n\nu_{\ell j}\omega_j$. Then using

$$2|x|\leq|\sin\pi x|\leq\pi|x|\quad\forall|x|\leq\frac{1}{2}\,,$$

we claim that the contribution of the small denominator is equivalent to $\{\nu_\ell\cdot\omega-\omega_j\}$, where $\{x\}$ denotes the distance of x from its nearest integer.

Let $p\in\mathbb{N}$, $p\geq2$, and $\Omega(p)=\min\{\{\nu\cdot\omega\},\nu\in\mathbb{Z}^n:0\leq|\nu|\leq p\}$. Let $(p_k)_k$ be an increasing sequence of positive integer and define (see [Br])

$$\Phi^{(k)}(\nu)=\begin{cases}1 & \text{if } \{\nu\cdot\omega\}<\frac{1}{2}\Omega(p_k)\\ 0 & \text{if } \{\nu\cdot\omega\}\geq\frac{1}{2}\Omega(p_k)\,,\end{cases} \tag{4.25}$$

for any $\nu\in\mathbb{Z}^n\setminus 0$. By definition we trivially have $\Phi^{(k)}(\nu)=0$ for all $0\leq|\nu|\leq p_k$.

We define the following Bruno condition:

$$\sum_{k=0}^{+\infty}\frac{\log\Omega^{-1}(p_{k+1})}{p_k}<+\infty\,. \tag{4.26}$$

and we claim that it is equivalent to (4.20). We can prove the following

Lemma 4.8. *Let $\nu_1 \in \mathbb{Z}^n$ such that $\Phi^{(k)}(\nu_1) = 1$, for some k. Then for all $\nu_2 \in \mathbb{Z}^n$, such that $0 < |\nu_2| \leq p_k$, we have $\Phi^{(k)}(\nu_1 - \nu_2) = 0$.*

The proof follows closely the one of Lemma 10 p.218 of [Br] and we don't prove it.

Let ℓ be a line of a rooted labeled tree, ϑ, let us introduce the notion of *scale* of the line ℓ. Let ν_ℓ be the momentum of the line and let us define $\bar{\nu}_\ell = \nu_\ell - \beta^\ell$. Let $(p_k)_k$ be an increasing sequence of positive integer, then for any $k \geq 0$ we define:

$$s_\ell(k) = \begin{cases} 1 & \text{if } \frac{1}{2}\Omega(p_{k+1}) \leq \{\bar{\nu}_\ell \cdot \omega\} < \frac{1}{2}\Omega(p_k) \\ 0 & \text{otherwise .} \end{cases} \tag{4.27}$$

For short we will say that a line ℓ is on scale k if $s_\ell(k) = 1$. Let us define $N_k(\vartheta)$ be the number of line on scale 1 in the rooted labeled tree ϑ. We can now prove the following Lemma, which roughly speaking says that the number of "bad" (too small) denominators is not too big, whereas Lemma 4.8 says that they do not occur so often.

Lemma 4.9 (Bruno's Counting lemma)**.** *The number of lines on scale k in a rooted labeled tree verifies the following bound:*

$$N_k(\vartheta) \leq \begin{cases} 0 & \text{if } |\bar{\nu}_\vartheta| < p_k \\ 2\left\lfloor \frac{|\bar{\nu}_\vartheta|}{p_k} \right\rfloor - 1 & \text{if } |\bar{\nu}_\vartheta| \geq p_k \, . \end{cases} \tag{4.28}$$

where $\lfloor x \rfloor$ denotes the integer part of the real number x. We recall that ν_ϑ is the total momentum of the tree and $\bar{\nu}_\vartheta = \nu_\vartheta - \beta^{\ell_{v_1}}$, being ℓ_{v_1} the root line of ϑ.

Proof. Consider firstly the case $|\bar{\nu}_\vartheta| < p_k$. Because for any $\ell_w \in \vartheta$ we have: $|\nu_\vartheta| \geq |\nu_{\ell_w}|$, we conclude that no one line is on scale k: $N_k(\vartheta) = 0$.

Consider now the case $|\bar{\nu}_\vartheta| \geq p_k$. If $\vartheta \in \mathcal{T}_N$ for $N \geq 1$ then using the standard decomposition of the tree $\vartheta = (t, \vartheta^1, \dots, \vartheta^t)$, being t the degree of the root v_1, $\vartheta^i \in \mathcal{T}_{N_i}$ and $N_1 + \dots + N_t = N - 1$, we have

$$N_k(\vartheta) = s_{\ell_{v_1}}(k) + N_k(\vartheta^1) + \dots + N_k(\vartheta^t) \tag{4.29}$$

$$\nu_\vartheta = \alpha_{v_1} - \beta_{v_1} + \nu_{\vartheta^1} + \dots + \nu_{\vartheta^t} \, . \tag{4.30}$$

We will prove (4.28) by induction of the total momentum of the tree. We will distinguish several cases.

A: $\mathbf{s}_{\ell_{v_1}}(k) = 0$ Because $|\nu_{\vartheta_i}| < |\nu_\vartheta|$ for all $i = 1, \dots, t$ we can use the induction hypothesis and from (4.29) we get

$$\begin{aligned} N_k(\vartheta) &= N_k(\vartheta^1) + \dots + N_k(\vartheta^t) \\ &\leq 2\left\lfloor \frac{|\bar{\nu}_{\vartheta^1}|}{p_k} \right\rfloor - 1 + \dots + 2\left\lfloor \frac{|\bar{\nu}_{\vartheta^t}|}{p_k} \right\rfloor - 1 \, , \end{aligned}$$

from which the thesis follows using (4.30) and the hypothesis $|\bar{\nu}_\vartheta| \geq p_k$.

B: $\mathbf{s}_{\ell_{v_1}}(k) = 1$ We now consider 3 subcases.

B.1: $t = 0$ Then (4.29) gives $N_k(\vartheta) = 1$ and the thesis follows recalling that $|\bar{\nu}_\vartheta| \geq p_k$.

B.2: $t \geq 2$ Then (4.29) gives

$$N_k(\vartheta) \leq 1 + 2\left\lfloor \frac{|\bar{\nu}_{\vartheta^{i_1}}|}{p_k} \right\rfloor - 1 + 2\left\lfloor \frac{|\bar{\nu}_{\vartheta^{i_2}}|}{p_k} \right\rfloor - 1 \, ,$$

and again the thesis follows using $|\bar{\nu}_\vartheta| \geq p_k$.

B.3: $t = 1$ Equation (4.29) gives $N_k(\vartheta) \leq 2\left\lfloor \frac{|\bar{\nu}_{\vartheta^1}|}{p_k} \right\rfloor$, so if $\left\lfloor \frac{|\bar{\nu}_{\vartheta^1}|}{p_k} \right\rfloor < \left\lfloor \frac{|\bar{\nu}_{\vartheta}|}{p_k} \right\rfloor$ the thesis follows again. It remains the case $\left\lfloor \frac{|\bar{\nu}_{\vartheta^1}|}{p_k} \right\rfloor = \left\lfloor \frac{|\bar{\nu}_{\vartheta}|}{p_k} \right\rfloor$, namely $|\bar{\nu}_{\vartheta}| - |\bar{\nu}_{\vartheta^1}| < p_k$. Let v_1' be the root of ϑ^1, let t' be its degree and consider the standard decomposition of the subtree $\vartheta^1 = (t', \vartheta_1^1, \dots, \vartheta_1^{t'})$. Lemma 4.8 assures that $s_{\ell_{v_1'}}(k) = 0$, so (4.29) (written for the standard decomposition of ϑ^1) reduces to:

$$N_k(\vartheta) = 1 + N_k(\vartheta_1^1) + \cdots + N_k(\vartheta_1^{t'}) \,.$$

We now consider the cases B.1, B.2 and B.3 for the subtrees ϑ_i^1. We claim that if case B.1 or B.2 holds the proof is done, whereas in case B.3 the proof is achieved only if the first subcase holds, but the remaining case can happen only a finite number of times, and so in this case too, the proof is done.

□

4.3.2. *Proof of Theorem 4.6.* We are now able to prove the main Theorem, we will prove only point 3 which clearly implies point 1 (choosing $M_k = N_k$ for all k) and point 2 (choosing $M_k = C^k$ for all k and some constant $C > 0$).

Let us then assume that $\mathbf{F} \in V_0$, is of the form $\mathbf{F}(\mathbf{z}) = A\mathbf{z} + \mathbf{f}(\mathbf{z})$ where $\lambda \in \mathbb{C}^n$, $A = diag(\lambda_1, \dots, \lambda_n)$ and $\mathbf{f} \in \mathcal{C}_{(M_k)} \cap V_1$.

For a fixed rooted labeled tree of order $N \geq 1$ with total momentum equals to $\alpha \in \mathbb{N}^n$, $|\alpha| \geq 2$, we consider the following term of equation (4.13):

$$((\Omega^{-1}_{\nu_{\ell_{v_1}}} \mathbf{f}_{\alpha_{v_1}}) \cdot \beta^{\ell_{v_1}}) \prod_{v \in \vartheta} \prod_{\ell_w \in L_v} ((\Omega^{-1}_{\nu_{\ell_w}} \mathbf{f}_{\alpha_w}) \cdot \beta^{\ell_w}) \tag{4.31}$$

Recalling the definition of scale and the definition on number of lines on scale k we can bound (4.31) with

$$\left| ((\Omega^{-1}_{\nu_{\ell_{v_1}}} \mathbf{f}_{\alpha_{v_1}}) \cdot \beta^{\ell_{v_1}}) \prod_{v \in \vartheta} \prod_{\ell_w \in L_v} ((\Omega^{-1}_{\nu_{\ell_w}} \mathbf{f}_{\alpha_w}) \cdot \beta^{\ell_w}) \right| \leq \prod_{m=0}^{\kappa(\alpha)} \left[2\Omega^{-1}(p_{m+1}) \right]^{N_m(\vartheta)} \prod_{v \in \vartheta} |\mathbf{f}_{\alpha_v}|$$

where $\kappa(\alpha)$ is the integer defined by: $p_{\kappa(\alpha)} \leq |\alpha| < p_{\kappa(\alpha)+1}$.

Using hypothesis 3. of paragraph 4.3 on the sequence $(M_k)_k$ and the hypothesis $\mathbf{f} \in \mathcal{C}_{(M_k)} \cap V_1$, we will obtain for some positive constant A, B the bound:

$$\prod_{v \in \vartheta} |\mathbf{f}_{\alpha_v}| \leq \prod_{v \in \vartheta} AB^{|\alpha_v|} M_{|\alpha_v|} \leq A^N B^{\sum_{v \in \vartheta} |\alpha_v|} M_{\sum_{v \in \vartheta} |\alpha_v| - (N-1)} \,,$$

by definition of total momentum: $\sum_{v \in \vartheta} |\alpha_v| - (N-1) = |\nu_\vartheta|$, which has been fixed to $|\alpha|$, then

$$\prod_{v \in \vartheta} |\mathbf{f}_{\alpha_v}| \leq B^{-1} (AB)^N B^{|\alpha|} M_{|\alpha|} \,.$$

Using finally the bound of the Counting Lemma 4.9 we get:

$$\log \frac{|\mathbf{h}_\alpha|}{N_{|\alpha|}} \leq |\alpha| \log C + \log \frac{M_{|\alpha|}}{N_{|\alpha|}} + 2|\alpha| \sum_{m=0}^{\kappa(\alpha)} \frac{\log \Omega^{-1}(p_{m+1})}{p_m} \,, \tag{4.32}$$

for some positive constant C. Dividing (4.32) by $|\alpha|$ and passing to the limit superior we get the thesis.

4.4. **A 1–dimensional particular case.** In this part we give a partial analytical answer of some numerical results showed in Figures 5 and 7 of [Ma], more precisely we prove that the 1–dimensional Siegel problem with the given function F of the form: $F_{(k)}(z)=\lambda z(1-\frac{z^k}{k})$, for some integer k, $\lambda=e^{2\pi i\omega}$, $\omega\in\mathbb{R}\setminus\mathbb{Q}$, the radius of convergence, $R_{(k)}(\omega)$, of the unique linearization associate to $F_{(k)}$, $H_{(k)}$, is periodic of period $1/k$ as a function of ω.

The first step is always formal: an analogue of Proposition 4.2 holds in the case $n=1$:

Proposition 4.10. *Let $\lambda\in\mathbb{C}^*$ non–resonant and $F=\lambda z+f$, with $f=\sum_{m\geq 2}f_m z^m\in V_1$. The coefficients of the solution $h=\sum_{m\geq 2}h_m z^m$ of the functional equation*

$$h=D_\lambda^{-1}f(id+h)\,,$$

are given by

$$h_m=\sum_{n=1}^{m-1}\sum_{\vartheta\in\mathcal{T}_{n,m}}\prod_{v\in\vartheta}\binom{\alpha_v}{m_v}(\lambda^{\nu_{\ell_v}}-\lambda)^{-1}f_{\alpha_v}\,, \tag{4.33}$$

where to any node we associate a label $\alpha_v\in\mathbb{N}$, $\alpha_v\geq 2$, and to the line ℓ_v the momentum $\nu_{\ell_v}=\sum_{w\leq v}(\alpha_w-m_w)$. Being $\mathcal{T}_{n,m}$ the forest of rooted labeled trees of order n and total momentum m.

Then we can prove:

Theorem 4.11. *There exists an universal positive constant C_k, depending on k but independent of ω, such that*

$$\log R_{(k)}(\omega)\geq -\frac{1}{k}B(k\omega)+\frac{\log k-C_k}{k} \tag{4.34}$$

The periodicity $1/k$ follows immediately from the properties of the Bruno function [MMY].

Proof. The m–th Taylor coefficient of the linearization $H_{(k)}$ is given by (4.33). Because $F_{(k)}$ has only one non–vanishing Taylor coefficient ($f_{k+1}=-\frac{\lambda}{k}$) we will have contributions only from trees whose labels verify $\alpha_v=k+1$ for all $v\in\vartheta$. Moreover the binomial factor implies that only trees with $m_v\leq\alpha_v=k+1$ will contribute.

Let ϑ be a rooted labeled tree, for any node $v\in\vartheta$ we denote by $\hat{v}$ the number of its predecessors w.r.t the partial order, namely $\hat{v}=\sum_{w\leq v}1$, and by definition of tree $\sum_{w\leq v}m_w=\hat{v}-1$. Then for every line ℓ_v we get $\ell_v=\sum_{w\leq v}(\alpha_w-m_w)=(k+1)\hat{v}-\hat{v}+1=1+k\hat{v}$, so the small divisors are all of the form: $\frac{1}{\lambda(\lambda^{kj}-1)}$ for some integer j.

The total momentum associate to a tree of order n, is $\nu_\vartheta=nk+1$, and by hypothesis it is fixed to m, so $m-1$ should be a multiple of k, and more important for a fixed $m-1\in k\mathbb{N}^*$, only trees with order $n=\frac{m-1}{k}$ will contribute.

Let l be an integer bigger than 1, fix $m-1=kl$ and $n=l$, then equation (4.33) can be rewritten as follows

$$h_{lk+1}=\sum_{\vartheta\in\mathcal{T}_l}{}'\left(\frac{-1}{k}\right)^l\prod_{v\in\vartheta}\binom{k+1}{m_v}(\tilde{\lambda}^{\hat{v}}-1)^{-1} \tag{4.35}$$

where $\tilde{\lambda} = e^{2\pi i k\omega}$ and the "prime" on the sum recall that only trees with $m_v \leq k+1$ must be considered.

Following closely the proof of Theorem 4.6 (in the analytic case), we conclude that there exists a positive constant independent of ω, depending on k such that

$$\log |h_{lk+1}| \leq l \log \frac{1}{k} + l \log C_k + l \sum_{j=0}^{\kappa(lk+1)} \frac{\log q_{j+1}}{q_j} \, ,$$

where $(q_j)_j$ are the convergent to $k\omega$. Then dividing by $lk+1$ and passing to the limit we get the thesis. □

5. Linearization of non–analytic vector fields

The aim of this section is to extend the analytic results of Bruno about the linearization of an analytic vector field near a fixed point, to the case of ultradifferentiable vector fields. We will show that this problem can be put in the framework of Theorem 2.3 and then obtaining an explicit (i.e. non–recursive) expression for the change of variables (the *linearization*) in which the vector field has a simpler form.

Our aim is also to point out the strong similarities of this problem with the Siegel Center Problem, previously studied. In particular when both problems are put in the framework of the multidimensional non–analytic Lagrange inversion formula on the field of formal Laurent series, they give rise to (essentially) the same problem. For this reason most results will only be stated without proofs, these being very close to the proofs of the previous section.

5.1. **Notation and Statement of the Problem.** In this section we will use the same notations given at the beginning of section 4.3. Let $A \in GL(n, \mathbb{C})$ and assume A to be diagonal. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes of formal power series as defined before, then the *problem of the linearization of vector fields* can be formulated as follows:

> Let $\mathbf{F}(\mathbf{z}) = A\mathbf{z} + \mathbf{f}(\mathbf{z}) \in \mathcal{C}_1$, $\mathbf{f} \in \mathcal{C}_1 \cap V_1$, and consider the following differential equation:
>
> $$\dot{\mathbf{z}} = \frac{d\mathbf{z}}{dt} = A\mathbf{z} + \mathbf{f}(\mathbf{z}) \, , \tag{5.1}$$
>
> where t denotes the time variable. Determine necessary and sufficient conditions on A to find a change of variables in $\mathcal{C}_2 \cap V_1$ (called the *linearization*) which leaves the singularity ($\mathbf{z} = 0$) fixed, doesn't change the linear part of $\mathbf{F}$ and allows to rewrite (5.1) in a simpler form[11]. Namely find $\mathbf{h} \in \mathcal{C}_2 \cap V_1$, such that $\mathbf{z} = \mathbf{w} + \mathbf{h}(\mathbf{w})$ and in the new variables $\mathbf{w}$, equation (5.1) rewrites:
>
> $$\dot{\mathbf{w}} = A\mathbf{w} \, . \tag{5.2}$$

[11]Here we don't consider the most general case of looking for a change of coordinates which put (5.1) in *normal form*, namely containing only resonant terms:

$$\dot{w}_i = w_i \sum_{\alpha : \alpha \cdot \omega = 0} g_{\alpha, i} w^{\alpha} \, .$$

Our results will concern vector fields with non–resonant linear parts, so (5.2) will be the normal form.

Let $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{C}^n$ and $A = diag(\omega_1, \dots, \omega_n)$, we will say that A is *resonant* if there exist $\alpha \in \mathbb{Z}^n$, with all positive component except at most one which can assume the value -1, $|\alpha| \geq 2$, and $j \in \{1, \dots, n\}$ such that: $\omega \cdot \alpha - \omega_j = 0$, where $\omega \cdot \alpha = \sum_{i=1}^n \omega_i \alpha_i$ is the scalar product. Let $\alpha \in \mathbb{N}^n$ we introduce the diagonal matrix $\Omega'_\alpha = diag(\omega \cdot \alpha - \omega_1, \dots, \omega \cdot \alpha - \omega_n)$.

Let us introduce the operator[12] $D'_\omega : V \to V$ as follows

$$D'_\omega \mathbf{g}(\mathbf{z}) = \sum_{\alpha \in \mathbb{N}^n} (\Omega'_\alpha \mathbf{g}_\alpha) z^\alpha \, , \tag{5.3}$$

for any $\mathbf{g}(\mathbf{z}) \in V$, where $\Omega'_\alpha \mathbf{g}_\alpha = \sum_{i=1}^n (\omega \cdot \alpha - \omega_i) g_{\alpha,i}$ is the matrix–vector product.. If A is non–resonant then D'_ω is invertible on V_1 (namely $|\alpha| \geq 2$), it is non–expanding and clearly V–additive and $\mathbb{C}$–linear. Then we claim that the linearization, $\mathbf{h}$, is solution of the functional equation:

$$D'_\omega \mathbf{h} = \mathbf{f} \circ (z + \mathbf{h}) \, , \tag{5.4}$$

where $\mathbf{f} \in \mathcal{C}_1 \cap V_1$, $\mathbf{h} \in \mathcal{C}_2 \cap V_1$ and $\mathbf{z}$ denotes the identity formal power series.

Given $\mathbf{f} \in V_1$ we consider the function $G_{\mathbf{f}}(\mathbf{h}) = \mathbf{f} \circ (z + \mathbf{h})$, assume A to be non–resonant to invert the operator D_ω, then we rewrite (5.4) as:

$$\mathbf{h} = {D'}_\omega^{-1} \left(G_{\mathbf{f}}(\mathbf{h}) \right) \, , \tag{5.5}$$

which is a particular case of the non–analytic multidimensional Lagrange inversion formula (2.3). Apart from the different operator, this equation is the same of the Siegel Center Problem (4.5). Lemma 4.1 assures that $G_{\mathbf{f}}$ verifies the hypotheses of Theorem 2.3 and thus we can apply it with: $u = 1$, $w = 0$, $G = G_{\mathbf{f}}$, $\Lambda = {D'}_\omega^{-1}$, $r = 1/2$ and $M = 1/4$. The unique solution of (5.5) is then given by:

$$\mathbf{h} = \sum_{N \geq 1} \sum_{\vartheta \in \mathcal{T}_N} Val_{{D'}_\omega^{-1}}(\vartheta) \, (1, 0) \, . \tag{5.6}$$

Once again we can give an explicit expression for the linearization $\mathbf{h}$ using rooted labeled trees. Introducing the same labels as for the Siegel center problem (see page 10) we can prove the following

Proposition 5.1. *Equation* (5.5) *admits a unique solution* $\mathbf{h} \in V_1$, $\mathbf{h} = \sum_{\alpha \in \mathbb{N}^n} \mathbf{h}_\alpha z^\alpha$. *For* $|\alpha| \geq 2$ *the* j*–th component of the coefficient* $\mathbf{h}_\alpha$ *is given by:*

$$h_{\alpha,j} = \sum_{N=1}^{|\alpha|-1} \sum_{\vartheta \in \mathcal{T}_{N,\alpha,j}} \left(({\Omega'}_{\nu_{\ell_{v_1}}}^{-1} \mathbf{f}_{\alpha_{v_1}}) \cdot \beta^{\ell_{v_1}} \right) \prod_{v \in \vartheta} \binom{\alpha_v}{\beta_v} \prod_{\ell_w \in L_v} \left(({\Omega'}_{\nu_{\ell_w}}^{-1} \mathbf{f}_{\alpha_w}) \cdot \beta^{\ell_w} \right) , \tag{5.7}$$

where the last product has to be set equal to 1 *whenever* v *is an end node* ($L_v = \emptyset$).

We don't prove this Proposition (whose proof is the same as the one of Proposition 4.2); moreover we point out that because both problems give rise to (essentially) the same multidimensional non–analytic Lagrange inversion formula, we can pass from one solution to the other with very small changes: ${\Omega'}_\alpha^{-1}$ instead of Ω_α^{-1}.

If both classes $\mathcal{C}_1$ and $\mathcal{C}_2$ are V_1 (formal case) then the linearization problem has solution if A is non–resonant. In the analytic case we distinguish again the *Poincaré domain* (the eigenvalues are non–resonant and the convex hull of the n complex points $\omega_1, \dots, \omega_n$ doesn't contain the origin) and the *Siegel domain* (if they are not in the Poincaré domain). In the first case Poincaré proved that the vector

[12] An equivalent definition will be: $D'_\omega \mathbf{g}(\mathbf{w}) = \sum_{i=1}^n (A\mathbf{w})_i \partial_{w_i} \mathbf{g}(\mathbf{w}) - A\mathbf{g}(\mathbf{w})$.

field is analytically linearizable, whereas in the second case Bruno proved analytic linearizability [Br], under the Bruno condition which reads:

$$\sum_{k\geq 0} \frac{\log \hat{\Omega}^{-1}(p_{k+1})}{p_k} < +\infty \,, \tag{5.8}$$

for some increasing sequence of integer number $(p_k)_k$, where $\hat{\Omega}(p) = \min\{|\alpha\cdot\omega| : \alpha\in\mathbb{N}^n, \alpha\cdot\omega\neq 0, 0<|\alpha|<p, \text{at most one component } \alpha_i=-1.\}$.

We now extend this kind of results to the case of ultradifferentiable vector fields. Namely we consider two classes $\mathcal{C}_{(M_k)}$ and $\mathcal{C}_{(N_k)}$, defined as in section 4.3 and we prove the following

Theorem 5.2. *Let $(\omega_1,\dots,\omega_n)\in\mathbb{C}^n$, $A=diag(\omega_1,\dots,\omega_n)$ non–resonant, $(M_k)_k$ and $(N_k)_k$ be sequences verifying hypotheses 0)–3) of section 4.3. Let $\mathbf{F}\in V_0$, s.t. $\mathbf{F}(\mathbf{z}) = A\mathbf{z}+\mathbf{f}(\mathbf{z})$ where $\mathbf{f}\in V_1$. Then*

1. *If moreover $\mathbf{F}\in\mathcal{C}_{(M_k)}\cap V_0$ and A verifies a Bruno condition* (5.8)*, then also the linearization $\mathbf{h}$ belongs to $\mathcal{C}_{(M_k)}\cap V_1$.*
2. *If $\mathbf{F}$ is a germ of analytic diffeomorphism of $(\mathbb{C}^n,0)$ and there exists an increasing sequence of integer numbers $(p_k)_k$ such that A verifies:*

$$\limsup_{|\alpha|\to+\infty}\left(2\sum_{m=0}^{\kappa(\alpha)}\frac{\log\hat{\Omega}^{-1}(p_{m+1})}{p_m}-\frac{1}{|\alpha|}\log N_{|\alpha|}\right)\,, \tag{5.9}$$

where $\kappa(\alpha)$ is the integer defined by: $p_{\kappa(\alpha)}\leq|\alpha|<p_{\kappa(\alpha)+1}$, then the linearization $\mathbf{h}$ belongs to $\mathcal{C}_{(N_k)}\cap V_1$.

3. *If $\mathbf{F}\in\mathcal{C}_{(M_k)}\cap V_0$, the sequence $(M_k)_k$ is asymptotically bounded by the sequence $(N_k)_k$ (namely $N_k\geq M_k$ for all sufficiently large k) and there exists an increasing sequence of integer numbers $(p_k)_k$ such that A verifies:*

$$\limsup_{|\alpha|\to+\infty}\left(2\sum_{m=0}^{\kappa(\alpha)}\frac{\log\hat{\Omega}^{-1}(p_{m+1})}{p_m}-\frac{1}{|\alpha|}\log \frac{N_{|\alpha|}}{M_{|\alpha|}}\right)\,, \tag{5.10}$$

where $\kappa(\alpha)$ is the integer defined by: $p_{\kappa(\alpha)}\leq|\alpha|<p_{\kappa(\alpha)+1}$, then the linearization $\mathbf{h}$ belongs to $\mathcal{C}_{(N_k)}\cap V_1$.

To prove this Theorem we will use again the majorant series method, the main step is to control the small denominators contributions. To do this, given an increasing sequence of positive integer $(p_k)_k$ we define a new *counting function*:

$$\hat{\Phi}^{(k)}(\nu) = \begin{cases} 1 & \text{if } |\nu\cdot\omega| < \frac{1}{2}\hat{\Omega}(p_k) \\ 0 & \text{if } |\nu\cdot\omega| \geq \frac{1}{2}\hat{\Omega}(p_k)\,. \end{cases} \tag{5.11}$$

for any $\nu\in\mathbb{Z}^n\setminus 0$. By definition we trivially have $\hat{\Phi}^{(k)}(\nu)=0$ for all $0<|\nu|\leq p_k$. Then we can prove the following Lemma (which will play the role of Lemma 4.8):

Lemma 5.3. *Let $\nu_1\in\mathbb{Z}^n\setminus 0$ such that $\hat{\Phi}^{(k)}(\nu_1)=1$, for some k. Then for all $\nu_2\in\mathbb{Z}^n$, such that $0<|\nu_2|\leq p_k$, we have $\hat{\Phi}^{(k)}(\nu_1-\nu_2)=0$.*

We finally define a new notion of *scale*; let ℓ be a line of a rooted labeled tree, let ν_ℓ be its momentum and let us recall that $\bar{\nu}_\ell=\nu_\ell-\beta^\ell$, then:

$$\hat{s}_\ell(k) = \begin{cases} 1 & \text{if } \frac{1}{2}\hat{\Omega}(p_{k+1}) \leq |\bar{\nu}_\ell\cdot\omega| < \frac{1}{2}\hat{\Omega}(p_k) \\ 0 & \text{otherwise}\,, \end{cases} \tag{5.12}$$

we will say that a line ℓ is on scale k if $\hat{s}_\ell(k) = 1$. We can now prove the following Counting Lemma

Lemma 5.4 (Bruno's Counting lemma 2^{nd} version). *Let ϑ be a rooted labeled tree of order $N \geq 1$, let $k \geq 1$ be an integer and let $\hat{N}_k(\vartheta)$ be the number of line on scale k in the tree. Then the following bound holds:*

$$\hat{N}_k(\vartheta) \leq \begin{cases} 0 & \text{if } |\bar{\nu}_\vartheta| < p_k \\ 2\left\lfloor \frac{|\bar{\nu}_\vartheta|}{p_k} \right\rfloor - 1 & \text{if } |\bar{\nu}_\vartheta| \geq p_k \,. \end{cases} \tag{5.13}$$

where $\lfloor x \rfloor$ denotes the integer part of the real number x. We recall that ν_ϑ is the momentum of the root line and $\bar{\nu}_\vartheta = \nu_\vartheta - \beta^{\ell_{v_1}}$, being ℓ_{v_1} the root line.

We don't prove it because its proof is the same as the one of Lemma 4.9 except for the use of Lemma 5.3 instead of Lemma 4.8.

5.2. **Proof of Theorem 5.2.** Once again we will prove only point 3 which clearly contains points 1 and 2 as special cases.

Assume $\mathbf{F} \in V_0$ of the form $\mathbf{F}(\mathbf{z}) = A\mathbf{z} + \mathbf{f}(\mathbf{z})$ where $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{C}^n$, $A = diag(\omega_1, \dots, \omega_n)$ and $\mathbf{f} \in \mathcal{C}_{(M_k)} \cap V_1$. Consider the contribution of a rooted labeled tree of order $N \geq 1$ with total momentum $\alpha \in \mathbb{N}^n$, $|\alpha| \geq 2$, given by (5.7):

$$\left((\Omega'^{-1}_{\nu_{\ell_{v_1}}} \mathbf{f}_{\alpha_{v_1}}) \cdot \beta^{\ell_{v_1}} \right) \prod_{v \in \vartheta} \prod_{\ell_w \in L_v} \left((\Omega'^{-1}_{\nu_{\ell_w}} \mathbf{f}_{\alpha_w}) \cdot \beta^{\ell_w} \right) . \tag{5.14}$$

Follow closely the proof of Theorem 4.6 we can bound (use the definitions of $\hat{N}_k(\vartheta)$ and of $\mathcal{C}_{(M_k)}$) this contribution with:

(5.15)

$$\left| \left((\Omega'^{-1}_{\nu_{\ell_{v_1}}} \mathbf{f}_{\alpha_{v_1}}) \cdot \beta^{\ell_{v_1}} \right) \prod_{v \in \vartheta} \prod_{\ell_w \in L_v} \left((\Omega'^{-1}_{\nu_{\ell_w}} \mathbf{f}_{\alpha_w}) \cdot \beta^{\ell_w} \right) \right| \leq \prod_{m=0}^{\kappa(\alpha)} \left[2\hat{\Omega}^{-1}(p_{m+1}) \right]^{\hat{N}_k(\vartheta)} A^N B^{|\alpha|} M_{|\alpha|} \,,$$

for some positive constants A, B, and $p_{\kappa(\alpha)} \leq |\alpha| < p_{\kappa(\alpha)+1}$. Finally Lemma 5.4 gives:

$$\log \frac{|\mathbf{h}_\alpha|}{N_{|\alpha|}} \leq |\alpha| \log C - \log \frac{N_{|\alpha|}}{M_{|\alpha|}} + 2|\alpha| \sum_{m=0}^{\kappa(\alpha)} \frac{\log \Omega^{-1}(p_{m+1})}{p_m} \,, \tag{5.16}$$

for some $C > 0$ and the thesis follows dividing by $|\alpha|$ and passing to the limit superior.

5.3. **A result for some analytic vector fields of $\mathbb{C}^2$.** For 2–dimensional analytic vector field the existence of the continued fraction and the convergents allows us to improve the previous Theorem, giving an optimal (we conjecture) estimate on the "size" of the analyticity domain of the linearization.

In [PMY] authors showed, using a construction of [MM], that the problem of the linearization of analytic vector field of $\mathbb{C}^2$ is completely equivalent to the problem of the linearization of analytic germs of $(\mathbb{C}, 0)$. More precisely they proved that the analytic vector field

$$\begin{cases} \dot{z}_1 & = -z_1(1 + \dots) \\ \dot{z}_2 & = \omega z_2(1 + \dots) \,, \end{cases} \tag{5.17}$$

where $\omega > 0$ and the suspension points mean terms of order bigger than 1, has the same analytical classification that the germs of $(\mathbb{C},0)$: $f(z) = e^{2\pi i\omega}z + \mathcal{O}(z^2)$. Using the results of [Yo] they obtain as corollaries that: if ω is a Bruno number then the vector field (5.17) is analytically linearizable, whereas if ω is not a Bruno number then there exist analytic vector fields of the form (5.17) which are not analytically linearizable.

Here we push up this analogy between vector fields and germs, by proving that the linearizing function of the vector field is analytic in domain containing a ball of radius ρ which satisfies the same lower bound (in term of the Bruno function) as the radius of convergence of the linearizing function of the germ [CM, Yo] does.

To do this we must introduce some *normalization condition* for the vector field; let $\omega > 0$, we consider the family $\mathcal{F}_\omega$ of analytic vector fields $\mathbf{F} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}^2$ of the form

$$\begin{cases} F_1(z_1,z_2) &= -z_1 + \sum_{|\alpha|\geq 2} f_{\alpha,1} z^\alpha \\ F_1(z_1,z_2) &= \omega z_2 + \sum_{|\alpha|\geq 2} f_{\alpha,2} z^\alpha \,, \end{cases} \tag{5.18}$$

with $|f_{\alpha,j}| \leq 1$ for all $|\alpha| \geq 2$ and $j = 1,2$.

For power series in several complex variables the analogue of the disk of convergence is the *complete Reinhardt domain of center* 0, $\mathcal{R}_0$, by studying the distance of the origin to the boundary of this domain we can obtain informations about its "size". Fixing the non linear part of the vector field: $\mathbf{f} = \sum_{|\alpha|\geq 2} \mathbf{f}_\alpha z^\alpha$, this distance is given by $d_{\mathbf{F}} = \inf_{(z_1,z_2)\in\mathcal{R}_0} (|z_1|^2 + |z_2|^2)^{1/2}$. The family $\mathcal{F}_\omega$ is compact w.r.t. the uniform convergence on compact subsets of $\mathbb{D} \times \mathbb{D}$ (use Weierstrass Theorem and Cauchy's estimates in $\mathbb{C}^2$, see for example [Sh]) so we can define $d_\omega = \inf_{\mathbf{F}\in\mathcal{F}_\omega} d_{\mathbf{F}}$.

Let $\rho_{\mathbf{F}} > 0$ and let us introduce $P(0,\rho_{\mathbf{F}}) = \{(z_1,z_2) \in \mathbb{C}^2 : |z_i| < \rho_{\mathbf{F}}, i = 1,2\}$, the biggest polydisk of center 0 contained in $\mathcal{R}_0$, whose radius depends on the vector field $\mathbf{F}$. Trivially $\rho_{\mathbf{F}}$ and $d_{\mathbf{F}}$ are related by a coefficient depending only on the dimension: $\sqrt{2}\rho_{\mathbf{F}} = d_{\mathbf{F}}$. We can then prove

Theorem 5.5 (Lower bound on d_ω)**.** *Let $\omega > 0$ be a Bruno number, then there exists an universal constant C such that:*

$$\log d_\omega \geq -B(\omega) - C \,, \tag{5.19}$$

where $B(\omega)$ is the value of the Bruno function [MMY] *on ω.*

We don't prove this Theorem being its proof very close to the one of Theorem 5.2 case 1), we only stress that the use of the continued fraction allows us to give an "optimal" counting Lemma as done in [CM, Da], which essentially bounds the number of lines on scale k in a rooted labeled tree of order N and total momentum ν_ϑ, with $\left\lfloor \frac{\bar{\nu}_\vartheta}{q_k} \right\rfloor$, being $(q_k)_k$ the denominators of the convergent to ω.

In the case of analytic germs of $(\mathbb{C},0)$ Yoccoz [Yo] proved that the same bound holds from above for the radius of convergence of the linearization, then using the results of [PMY] we conjecture that the following bound holds

$$\log d_\omega \leq -B(\omega) + C' \,,$$

for some universal constant C'. We are not able to prove this fact but can prove that the power series obtained replacing the coefficients of the linearization with their absolute values is divergent whenever ω is not a Bruno number (a similar result has been proved in [Yo] Appendix 2 and in [CM] paragraph 2.4 for germs).

Remark 5.6 (Ultradifferentiable vector fields of $\mathbb{C}^2$). *In the more general case of ultradifferentiable vector fields of $\mathbb{C}^2$ we can improve Theorem 5.2 showing that we can linearize the vector field under weaker conditions.*

Theorem 5.7. *Let $\omega > 0$ and let $(p_k/q_k)_k$ be its convergents. Let $\mathbf{F}$ be a vector field of the form* (5.18) *(without additional hypotheses on the coefficients $\mathbf{f}_\alpha$), let $(M_n)_n$ and $(N_n)_n$ be two sequences verifying conditions 0)–3) of section 4.3. Then*

1. *If moreover $\mathbf{F}$ belongs to $\mathcal{C}_{(M_n)}$ and ω is a Bruno number then also the linearization $\mathbf{h}$ belongs to $\mathcal{C}_{(M_k)} \cap V_1$.*
2. *If $\mathbf{F}$ is a germ of analytic diffeomorphism of $(\mathbb{C}^2, 0)$ and ω verifies:*

$$\limsup_{|\alpha| \to +\infty} \left(\sum_{m=0}^{\kappa(\alpha)} \frac{\log q_{m+1}}{q_m} - \frac{1}{|\alpha|} \log N_{|\alpha|} \right) , \tag{5.20}$$

   where $\kappa(\alpha)$ is the integer defined by: $q_{\kappa(\alpha)} \leq |\alpha| < q_{\kappa(\alpha)+1}$, then the linearization $\mathbf{h}$ belongs to $\mathcal{C}_{(N_k)} \cap V_1$.
3. *If $\mathbf{F} \in \mathcal{C}_{(M_k)} \cap V_0$, the sequence $(M_k)_k$ is asymptotically bounded by the sequence $(N_k)_k$ (namely $N_k \geq M_k$ for all sufficiently large k) and ω verifies:*

$$\limsup_{|\alpha| \to +\infty} \left(\sum_{m=0}^{\kappa(\alpha)} \frac{\log q_{m+1}}{q_m} - \frac{1}{|\alpha|} \log \frac{N_{|\alpha|}}{M_{|\alpha|}} \right) , \tag{5.21}$$

   where $\kappa(\alpha)$ is the integer defined by: $q_{\kappa(\alpha)} \leq |\alpha| < q_{\kappa(\alpha)+1}$, then the linearization $\mathbf{h}$ belongs to $\mathcal{C}_{(N_k)} \cap V_1$.

The proof follows closely the one of Theorem 5.2 and the weaker arithmetical condition are obtained using the "optimal" counting function as done in the proof of Theorem 5.5 and in [CM, Da].

Appendix A. Ultrametric structures and analyticity.

Let k be an ultrametric field and let v be a valuation. The ring $A_v = \{x \in k \mid v(x) \geq 0\}$ is called the *ring of the valuation* v and the sets $I'_\alpha = \{x \in A_v \mid v(x) > \alpha\}$, for $\alpha \geq 0$, are ideals of A_v. I'_0 is the maximal ideal of A_v and it is an open set in the topology induced by the ultrametric absolute value defined on k:

$$I'_0 = \{x \in A_v \mid v(x) > 0\} = \{x \in A_v \mid |x| < 1\} = B_0(0,1) ,$$

where $B_0(x,r) = \{y \in k \, , |x-y| < r\}$. The field $k' = A_v/I'_0$ is called the *residue field.*

Let n and m be positive integers, we consider the set of the formal power series with coefficients in k^n in the m variables $X_1, \dots, X_m$, $S_{n,m} = k^n[[X_1, \dots, X_m]]$, $F \in S_{n,m}$:

$$F = \sum_{\alpha \in \mathbb{N}^m} F_\alpha X^\alpha = \sum_{\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m} F_\alpha X_1^{\alpha_1} \dots X_m^{\alpha_m} ,$$

with $F_\alpha = (F_{\alpha,1}, \dots, F_{\alpha,n}) \in k^n$ for all $\alpha \in \mathbb{N}^m$. We will be interested in composition problems, so it is natural to set $m = n$ and to define the composition of two elements F, G, with $v(G) \geq 1$, as

$$F \circ G = \sum_{\alpha \in \mathbb{N}^n} F_\alpha G^\alpha = \sum_{\alpha_1, \dots, \alpha_n \in \mathbb{N}} F_{\alpha_1, \dots, \alpha_n} (G_1)^{\alpha_1} \dots (G_n)^{\alpha_n} . \tag{A.1}$$

We will set $S_{n,n} = S_n$ and we will introduce some definitions and properties of S_n, but it's clear that they also hold on $S_{n,m}$ with some small changes.

Let $F = \sum F_\alpha X^\alpha \in S_n$, then we say that F *converges* in $B(0,r)$, for some $r > 0$, if: $\sum_{\alpha\in\mathbb{N}^n} ||F_\alpha|| r^{|\alpha|} < +\infty$, where $|\alpha| = \sum_{i=1}^n \alpha_i$. F will be said convergent in $B_0(0,r)$ if it is convergent in $B(0,r')$ for all $0 < r' < r$.

One has the following result ([Se], pp. 67–68):

Proposition A.1.
1. If F is convergent in $B(0,r)$ then there exists $M > 0$ such that

$$||F_\alpha|| \leq M r^{-|\alpha|} \quad \forall \alpha \in \mathbb{N}^n \,. \tag{A.2}$$

2. If there exists $M > 0$ such that (A.2) holds for all $\alpha \in \mathbb{N}^n$, F converges in $B_0(0,r)$ and uniformly in $B(0,r')$ for all $0 < r' < r$.
3. Let $\tilde{F} : B_0(0,r) \to k^n$ denote the continuous function defined as the sum of the series $F \in S_n$ convergent in $B_0(0,r)$. Then $\tilde{F} \equiv 0 \iff F = 0$.

We can therefore identify a convergent power series F with its associated function $\tilde{F}$ and vice versa. Let U be an open set of k^n, $\tilde{G} : U \to k^n$ is said to be *analytic* in U if for all $x \in U$ there is a formal power series $G \in S_n$ and a radius $r > 0$ such that:

1. $B_0(x,r) \subset U$,
2. G converges in $B_0(0,r)$, and for all $y \in B_0(0,r)$, $\tilde{G}(x+y) = G(y)$.

With a slight abuse of notation we will omit the superscript $\tilde{}$ to distinguish analytic functions from convergent power series. If F is a convergent power series on $B(0,r)$ we denote

$$||F||_r = \sup_{\alpha\in\mathbb{N}^n} ||F_\alpha|| r^{|\alpha|} \,, \tag{A.3}$$

and we define $\mathcal{A}_r(k^n) = \{F \in S_n : ||F||_r < +\infty\}$.

Let U be an open set of k^n, x a point of U, let us consider $G : U \to V \subset k^m$. A linear function $L : k^n \to k^m$ is called a *derivative* of G at x if:

$$\lim_{\substack{||y||\to 0\\ y\neq 0}} \frac{||G(x+y) - G(x) - Ly||}{||y||} = 0.$$

Clearly if the limit exists then the derivative is unique and it will be denoted by $dG(x)$. Let $\delta_i = (0,\dots,1,\dots,0) \in k^n$ the vector with 1 at the i–th place, we call

$$D_i G(x) = dG(x)(\delta_i) \in k^m \tag{A.4}$$

the i-th partial derivative of G at x. Higher order derivatives are defined analogously.

Let $G = \sum G_\alpha X^\alpha$ be an element of S_n, then $G_\alpha = \frac{D^\alpha G(x)}{\alpha!}$, where for $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, $D^\alpha = D_1^{\alpha_1}\dots D_n^{\alpha_n}$ and $\alpha! = \alpha_1!\dots\alpha_n!$. Thus G is just the Taylor series of G at the point x.

It is not difficult to prove that any power series $G \in S_n$ convergent in $B_0(0,r)$ defines an analytic function G in $B_0(0,r)$. However one should be aware of the fact that in general the local expansion of a function G analytic on U at a point $x \in U$ such that $B_0(x,r) \subset U$ does not necessarily converge on all of $B_0(0,r)$. This is true if one assumes k to be algebraically closed.

Let $F \in S_n$, $F = \sum_{\alpha \in \mathbb{N}^n} F_\alpha X^\alpha$, let $\beta \in \mathbb{N}^n$ we define the formal derivative of F by

$$\Delta^\beta F = \sum_{\alpha \in \mathbb{N}^n, \alpha \geq \beta} F_\alpha \binom{\alpha}{\beta} X^{\alpha - \beta} \,, \tag{A.5}$$

where we used the notations: $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for $1 \leq i \leq n$, $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$, for $\alpha, \beta \in \mathbb{N}^n$ and one can then prove [Se]: $\alpha! \Delta^\alpha = D^\alpha$, $\binom{\alpha+\beta}{\alpha} \Delta^{\alpha+\beta} = \Delta^\alpha \Delta^\beta$.

Finally we note that the composition of two analytic functions is analytic ([Se], p. 70) and that the following Cauchy estimates and Taylor formula hold ([HY], pp. 421-422):

Proposition A.2. *Let $r > 0$, $s > 0$, let $F \in \mathcal{A}_r(k^n)$, and let G, H be two elements of $\mathcal{A}_s(k^n)$, with $||G||_s \leq r$ and $||H||_s \leq r$. Then the following estimates hold:*

- *(Cauchy's estimates)* $||\Delta^\alpha F||_r \leq \frac{||F||_r}{r^{|\alpha|}}$ *for all* $\alpha \in \mathbb{N}^n$*;*
- $F \circ G \in \mathcal{A}_s(k^n)$ *and* $||F \circ G||_s \leq ||F||_r$*;*
- *(Taylor's formula)*

$$||F \circ (G+H) - F \circ G||_s \leq \frac{||F||_r}{r} ||H||_s$$
$$||F \circ (G+H) - F \circ G - DF \circ G \cdot H||_s \leq \frac{||F||_r}{r^2} ||H||_s^2$$

References

[BG1] A.Berretti and G.Gentile: *Scaling properties for the radius of convergence of a Lindstedt series: the Standard Map*, J.Math.Pures.Appl. (9), **78**, no.2, (1999) pp. 159–176.

[BG2] A.Berretti and G.Gentile: *Bruno function and the Standard Map* preprint, (1998) .

[Br] A.D.Bruno: *analytical form of differential equations*, Trans. Moscow Math. Soc. **25**, (1971), pp. 131–288.

[Caj] F. Cajori: *A History of Mathematics*, Chelsea Publishing Company, New York, N.Y. (1991).

[CM] T.Carletti and S.Marmi:*Linearization of analytic and non–analytic germs of diffeomorphisms of* $(\mathbb{C}, 0)$, Bull. Soc. Math. Française, **128**, (2000), pp. 69–85.

[Che] A.Chenciner:*Courbes Algébriques Planes*, Publ.Math. de l'Univ. Paris VII **4**, Ch. VI et VII (1979).

[Chr] G.Christol: *p-adic Numbers and Ultrametricity*, in "From Number Theory to Physics", M.Waldsehmidt, P.Moussa, J.-M.Luck, C.Itzykson editors, Springer-Verlag (1992) pp. 440–475.

[CF] L. Chierchia and C. Falcolini:*A Direct Proof of a Theorem by Kolmogorov in Hamiltonian Systems*, Ann. Scuola Norm. Super. Pisa Cl. Sci. Ser. IV, **16**, (1994), pp. 541–593.

[Da] A.M.Davie: *The critical function for the semistandard map*, Nonlinearity **7**, (1994), pp. 219–229.

[Di] J.Dieudonné: *Calcul infinitésimal*, Hermann Paris Collection Méthodes.

[Du] D.Duverney:*U-Dérivation*, Annales de la Faculté des Sciences de Toulouse, vol II, **3** (1993).

[E] H. Eliasson:*Absolutely convergent series expansions for quasi–periodic motions*, Report 2 – 88, Dept. of Math., Univ. of Stockholm (1988) and Math. Phys. Electronic Journal (1996)

[G1] G.Gallavotti: *Twistless KAM tori, quasiflat homoclinic intersection and other cancellations in the perturbation series of certain completely integrable Hamiltonian systems. A review.*, Rev. Math. Phys. **6**, no. 3, (1994), pp. 343–411.

[G2] G.Gallavotti: *Twistless KAM tori*, Comm. Math. Phys. **164**, (1994), pp. 145–156.

[Ge] I.M. Gessel: *A combinatorial proof of the multivariable Lagrange inversion formula*, J. of Comb. Theory, series A, **45**, (1987), pp. 178–195.

[Go] I.J.Good: *Generalizations to several variables of Lagrange's expansion, with application to stochastic processes*, Proc. Camb. Phil. Soc. **56** (1960), pp.367–380.

[Ha] Harary:*Graph theory*, Addison-Wesley (1969).

[He] M.R.Herman:*Recent Results and Some Open Questions on Siegel's Linearization Theorem of Germs of Complex Analytic Diffeomorphisms of* $\mathbb{C}^n$ *near a Fixed Point*, Proc. VIII Int. Conf. Math. Phys. Mebkhout Seneor Eds. World Scientific, (1986), pp. 138–184.

[HY] M.R.Herman and J.-C.Yoccoz:*Generalization of some theorems of small divisors to non-archimedean fields*, in Geometric Dynamics, Proceedings of the Symposium in Rio de Janeiro 1981, Springer Lecture Notes in Mathematics **1007**, pp. 408–447.

[Ko] G. Koenigs:*Recherches sur les équations fonctionelles*, Ann. Sc. E.N.S. **1**, (1884), supplément, pp. 3–41.

[La] J.L.Lagrange:*Nouvelle méthode pour résoudre les équations littérales par le moyen des séries*, Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin, **25** (1769), published 1771, Œuvres, vol. 3, Paris, 1869 pp. 113-138.

[Ma] S. Marmi: *Critical functions for complex analytic maps.*, J. Phys. A: Math. Gen. **23**, (1990), pp. 3447 – 3474.

[MMY] S.Marmi, P.Moussa and J.-C. Yoccoz:*The Bruno functions and their regularity properties*, Communications in Mathematical Physics **186**, (1997), pp. 265–293.

[MM] J.-F. Mattei and R. Moussu: *Holonomie et intégrales preimières*, Ann. Sc. E.N.S. 4^{eme} série, t. **13**, (1980), pp. 469–523.

[PMY] R.Pérez–Marco and J.-C.Yoccoz:*Germes de feuilletages holomorphes à holonomie prescrite*, Astérisque **222**, (1994), pp. 345– 371.

[Po] H.Poincaré:*Œuvres, tome I*, Gauthier–Villars, Paris, (1917).

[Rü] H. Rüssmann:*On the one dimensional Schrödinger equation with a quasi periodic potential*, Ann. New York Acad. Sci., **357**, (1980), pp. 90–107.

[San] G.Sansone:*Lezioni sulla teoria delle funzioni di una variabile complessa*, Volume Primo, Cedam Padova (1972).

[Se] J.P.Serre:*Lie Algebras and Lie Groups*, LNM 1500 (1995) Part II, chapters 1^{st} and 2^{nd}.

[Sh] B.V. Shabat:*Introduction to complex analysis, part II: functions of several variables*, Translations of Mathematical Monographs, Vol. 110, AMS (1992).

[Si] C.L.Siegel:*Iteration of analytic functions*, Annals of Mathematics **43** (1942), pp. 807–812.

[Vi] M.Vittot:*Lindstedt perturbation series in Hamiltonian mechanics: explicit formulation via a multidimensional Burmann-Lagrange formula*, preprint CNRS-Luminy CPT–91/P.2603, Marseille (1991).

[Yo] J.-C.Yoccoz:*Théorème de Siegel, polynômes quadratiques et nombres de Bruno*, Astérisque **231**, (1995), pp. 3– 88.

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