Semiclassical limit for multistate Klein-Gordon systems: almost invariant subspaces and scattering theory

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Abstract: Using the method of Helffer and Sjöstrand we construct almost invariant subspaces for multicomponent Klein-Gordon hamiltonians. As an application, we prove that, in the semiclassical limit and for analytic potentials, the scattering operator is block diagonal up to exponentially small errors.

1 Introduction

In this paper we consider the semiclassical limit for matricial Klein-Gordon operators; the problem we shall consider is the extension of the reduction scheme of regular perturbation theory, which in our case amounts to reduction of the spectral and scattering problems for the matricial case to the corresponding ones for "scalar" effective hamiltonians. In regular perturbation theory [Ka], the main step of the reduction scheme is the construction of perturbed invariant subspaces corresponding to the isolated parts of the spectrum of the unperturbed hamiltonian. Although the semiclassical limit can be viewed as a perturbation theory, it is a highly singular one so perturbed invariant subspaces may not exist and then one has to look for *almost* invariant subspaces as in the case of singular perturbation theory or adiabatic expansions [Ne1, Ne2]. As well known, the natural framework for the semiclassical limit is the theory of pseudodifferential operators and indeed our construction of the almost invariant subspaces rely heavily on pseudodifferential operators techniques, more exactly on the construction of Moyal projections i.e. of orthogonal projections in the algebra of formal symbols which (at the formal symbols level) commutes with the symbol of the hamiltonian. There are many methods of constructing Moyal projections [HeSj, EmWe, BrNo] and we shall follow the method of [HeSj] which is close in spirit with the Rellich-Kato perturbation theory. In the construction out of the formal Moyal projection of a bona fide orthogonal projection which almost commute with the hamiltonian we follow the procedure [Ne1, Ne2] used in singular perturbation theory.

We use the existence of almost invariant subspaces to prove the main result of the paper: under appropriate analytic and decay conditions on the potential, the matrix elements of the scattering operator have, up to exponentially small errors, a block diagonal structure. Results on almost diagonal structure of the scattering matrix at *fixed* energy were

obtained before in [Be, BeMa, MNS, Na] both for Klein-Gordon and Schrodinger cases, by the method of phase space tunnelling developed by Martinez [Ma](see also [MarNe] where almost invariant subspaces at fixed at fixed energies were

constructed in the case of the one-dimensional Schrodinger operator, for

energies above the potential barrier). Aside from more restrictive conditions on the potential the results quoted above were proved only for non-trapping energies a condition which implies the absence of resonances close to real axis and it is not valid in general at low energies (which are the most interesting from the physical point of view). Our results are valid also for the trapping energies but are weaker in the sense that involve an averaging over intervals of energies (see Section 3 for precise formulation and a heuristic discussion of its significance).

Our result on the existence of the almost invariant subspaces gives also the *existence* and the control on the (exponentially long) life time of metastable states (quasi-modes) in the semiclassical limit for trapping energies. Let us mention here that in many instances (see e.g.[St]) one can make the connection with the resonances defined as poles of the resolvent of scattering operator. In the cases when this connection can be made this implies that the imaginary part of the resonances is exponentially small.

Although, in order to keep the length of the paper at a reasonable length, we consider here only the analytic Klein-Gordon case we expect the methods developed here to be relevant to a much broader class of problems such as time dependent Born-Oppenheimer approximation [HJ1, HJ2], dynamics of crystal electrons in weak external fields [HST, Ne3], dynamics of Pauli-Fierz model [TS] etc.

The content of the paper is as follows. In Section 2 we give, at a general heuristic level, the construction of almost invariant subspaces.

Section 3 contains the technicalities in the case of analytic Klein-Gordon matricial operators as well as applications to their spectral and scattering theory. Finally, in the appendix we recall some facts from the theory of pseudodifferential operators.

2 Generalities. Moyal projections, almost invariant subspaces and block diagonalization.

We start by fixing some terminology and notations. In what follows, $x, \xi \in \mathbf{R}^n$, $n \geq 1$, $\epsilon \in [0, \epsilon_0]$, $\epsilon_0 > 0$ and $h(x, \xi; \epsilon)$ are C^∞, $m \times m$ hermitian matrix valued functions on $\mathbf{R}^{2n} \times [0, \epsilon_0]$ (actually by adding the necessary technicalities one can consider the case when $h(x, \xi; \epsilon)$ are C^∞ selfadjoint operator valued functions). If h and its derivatives satisfies suitable estimates (see Appendix), then associated to $h(x, \xi; \epsilon)$ one can consider the (essentially selfadjoint under appropriate conditions) operator H_ϵ obtained as the Weyl quantization of $h(x, \xi; \epsilon)$ (called the symbol of H_ϵ) i.e. for $\phi \in (\mathcal{S}(\mathbf{R}^n))^{\oplus m}$

$$(H_\epsilon \phi)(x) = \left(\mathrm{Op}^w_\epsilon(h)\phi\right)(x) = \frac{1}{(2\pi\epsilon)^n} \int e^{i(x-y)\xi/\epsilon} h(\frac{x+y}{2}, \xi; \epsilon)\phi(y)\, dy\, d\xi.$$

If $h(x, \xi; \epsilon)$ has an asymptotic expansion (as $\epsilon \to 0$)

$$h(x, \xi; \epsilon) \sim \sum_{j=0}^{\infty} h_j(x, \xi)\epsilon^j \tag{2.1}$$

we say that $\sum_{j=0}^{\infty} h_j(x,\xi)\epsilon^j$ is the formal symbol of H_ϵ. Given another symbol b and denoting by $B_\epsilon = \mathrm{Op}_\epsilon^w(b)$, then as well known, the operator multiplication corresponds to the Moyal product of symbols

$$H_\epsilon B_\epsilon = \mathrm{Op}_\epsilon^w(h\tilde{\#}b)$$

where $h\tilde{\#}b$ is given by the oscillatory integral

$$(h\tilde{\#}b)(x,\xi;\epsilon) = \frac{1}{(2\pi\epsilon)^{2n}}\int e^{i\varphi(x,y,v,\xi,\eta,\theta)/\epsilon} h(\frac{x+u}{2},\eta;\epsilon)b(\frac{x+v}{2},\theta;\epsilon)dudvd\eta d\theta \tag{2.2}$$

with $\varphi(x,y,v,\xi,\eta,\theta) = (\xi-\eta)(v-x)+(\xi-\theta)(x-v)$.

At the level of formal symbols by expanding the oscillatory integral (2.2), one obtain that $h\tilde{\#}b \sim \sum_{j=0}^{\infty}(h\#b)_j\epsilon^j$ where

$$(h\#b)_j(x,\xi) = \sum_{|\alpha|+|\beta|+k+l=j} \Gamma(\alpha,\beta)\partial_\xi^\alpha D_x^\beta h_k(x,\xi)\partial_\xi^\beta D_x^\alpha$$

b$_l(x,\xi)$(2.3)and $\Gamma(\alpha,\beta) := \dfrac{(-1)^{|\beta|}}{\alpha!\beta!2^{|\alpha|}2^{|\beta|}}$.

Here we are looking for almost invariant subspace for H_ϵ in the semiclassical limit, $\epsilon \to 0$. The basic idea is to construct first Moyal projections i.e. formal symbols

$$\pi(x,\xi;\epsilon) \sim \sum_{j=0}^{\infty} \pi_j(x,\xi;\epsilon)\epsilon^j \tag{2.4}$$

satisfying (at a formal series level)

$$\pi \sim \pi^* \sim \pi\#\pi \tag{2.5}$$

$$\pi\#h - h\#\pi \sim 0 \tag{2.6}$$

(here by $\pi^*(x,\xi;\epsilon)$ we mean the adjoint of the matrix $\pi(x,\xi;\epsilon)$). The proposition below says that one can always construct Moyal projections corresponding to isolated parts of the spectrum of the principal symbol $h_0(x,\xi)$ of $h(x,\xi;\epsilon)$

$$h(x,\xi;\epsilon) = h_0(x,\xi) + \mathcal{O}(\epsilon). \tag{2.7}$$

Let $\sigma(x,\xi)$ be the spectrum of $h_0(x,\xi)$ and suppose that, for some $(x_0,\xi_0)\in\mathbf{R}^{2n}$,

$$\begin{aligned}&\sigma(x_0,\xi_0)=\sigma_1(x_0,\xi_0)\cup\sigma_2(x_0,\xi_0)\\&\mathrm{dist}\,(\sigma_1(x_0,\xi_0),\sigma_2(x_0,\xi_0)):=d(x_0,\xi_0)>0.\end{aligned}\tag{2.8}$$

Then by perturbation theory there exists a neighborhood $\mathcal{U}(x_0,\xi_0)$ of (x_0,ξ_0) such that, for $(x,\xi)\in\mathcal{U}(x_0,\xi_0)$ the spectrum of $h_0(x,\xi)$ is well separated i.e. (2.8) is satisfied on $\mathcal{U}(x_0,\xi_0)$ with $d(x,\xi)\geq d(x_0,\xi_0)/2$, and there exists a contour Γ_{x_0,ξ_0} (not depending upon $(x,\xi)\in\mathcal{U}(x_0,\xi_0)$) enclosing $\sigma_1(x,\xi)$ and satisfying

$$\mathrm{dist}\,(\Gamma_{x_0,\xi_0},\sigma(x,\xi))\geq d(x_0,\xi_0)/2.\tag{2.9}$$

Proposition 2.1 *Suppose (2.8) holds true. Then, for $(x,\xi)\in\mathcal{U}(x_0,\xi_0)$, there exist unique $\pi_j(x,\xi)$, $j=0,1,..$, such that $\pi_0(x,\xi)$ is the spectral projection of $h_0(x,\xi)$ corresponding to $\sigma_1(x,\xi)$ and the formal symbol $\sum_{j=0}^{\infty}\pi_j(x,\xi)\epsilon^j$ satisfies (2.5), (2.6).*

Proposition 2.1 appeared many times in the literature (see: [HeSj, EmWe, BrNo] and [Sj2] where the earlier construction [Ne1, JP] of adiabatic projections corresponding to the case $h(x,\xi;\epsilon)=\xi+H_\epsilon(x)$ was obtained in the framework of the theory of pseudodifferential operators). There are essentially two methods of proof. The first one [Ne1, EmWe, BrNo], is a recurrent construction for π_j solving the equations coming from (2.5), (2.6). The second one [HeSj, Sj2] is close in spirit to Rellich-Kato perturbation theory [Ka]: due to (2.9), for all $z\in\Gamma_{x_0,\xi_0}$, one can construct (see e.g. [Ro]) the parametrix

$$q(x,\xi;\epsilon,z)\sim\sum_{j=0}^{\infty}q_j(x,\xi;z)\epsilon^j$$

satisfying (at a formal series level)

$$q\#(h-z)\sim(h-z)\#q\sim 1\tag{2.10}$$

and then obtain $\pi_j(x,\xi)$ from Riesz formula for spectral projection

$$\pi(x,\xi;\epsilon)=\frac{i}{2\pi}\int_{\Gamma_{x_0,\xi_0}}q(x,\xi;\epsilon,z)\,dz\tag{2.11}$$

i.e.

$$\pi_j(x,\xi) = \frac{i}{2\pi}\int_{\Gamma_{x_0,\xi_0}} q_j(x,\xi;z)\, dz. \tag{2.12}$$

In what follows, we shall use the second method since, on the one hand, it gives explicit formulae for π_j and, on the other hand, allows a much easier control on $\pi_j(x,\xi)$ and their derivatives than the recurrent construction (compare for example the estimation of [Ne1] with those of [MaNe] in the adiabatic case).

Remark 2.2

If $h_0(x,\xi)$ has a part of the spectrum which remains isolated for all $(x,\xi) \in \mathbf{R}^{2n}$, then, from Proposition 2.1, one obtain $\pi_j(x,\xi)$ defined on all $\mathbf{R}^{2n}$. Moreover $\pi_j(x,\xi)$ are C^∞ and if $h_j(x,\xi)$ are real analytic $\pi_j(x,\xi)$ are also real analytic.

∎

Remark 2.3

It is well known (see e.g.[HeSj]) that the existence of $\pi(x,\xi)$ satisfying (2.5), (2.6) is equivalent with the block diagonalization of $h(x,\xi;\epsilon)$ at a formal level (see [Ta]) i.e. with the existence of a formal symbol

$$u(x,\xi;\epsilon) \sim \sum_{j=0}^{\infty} u_j(x,\xi)\epsilon^j$$

satisfying (at the formal symbol level)

$$u\#u^* \sim u^*\#u \sim 1$$

such that, setting

$$\tilde{h} :\sim u^*\#h\#u,$$

then

$$\pi_0\#\tilde{h} - \tilde{h}\#\pi_0 \sim 0.$$

Let us remark that, unlike π_j, u_j (and then u and $\tilde{h}$) are not unique. A somewhat canonical choice is the one mimicing the reduction scheme of Rellich-Kato perturbation theory, i.e. writing $u(x,\xi;\epsilon)$ as the Nagy matrix (see (2.25)) corresponding to the pair $\pi(x,\xi\ \epsilon), \pi_0(x,\xi)$:

$$u \sim (\pi\#\pi_0 + (1-\pi)\#(1-\pi_0))\,\#\left(1+\sum_{j=1}^{\infty}\frac{(2j-1)!!}{2^j\ j!}(\pi-\pi_0)^{\#2j}\right).$$

∎

Remark 2.4
Let us stress that Proposition 2.1 gives $\pi(x, \xi; \epsilon)$ only at the formal symbol level. By resummation , there exist symbols $\hat{\pi}(x, \xi; \epsilon)$ (unique modulo $\mathcal{O}(\epsilon^\infty)$) which are asymptotically equivalent with the formal symbol $\sum_{j=0}^{\infty} \pi_j \epsilon^j$. Notice that in general $\hat{\pi}$ satisfy (2.5), (2.6) (where now # means the Weyl product in (3.6)) only up of errors smaller than any power of ϵ i.e. (2.5), (2.6) hold modulo $\mathcal{O}(\epsilon^\infty)$. ∎

We turn now to the construction, out of the formal symbol $\sum_{j=0}^{\infty} \pi_j \epsilon^j$ of almost invariant subspaces for the H_ϵ. Suppose that $\pi_j(x, \xi)$ are globally defined on $\mathbf{R}^{2n}$ and are uniformly bounded together with their derivatives. Then, by resummation, one can obtain (unique up to $\mathcal{O}(\epsilon^\infty)$) $\hat{\pi}(x, \xi; \epsilon)$ which is also bounded together with its derivatives uniformly on $\mathbf{R}^{2n} \times [0, \epsilon_0]$, for some $\epsilon_0 > 0$. Then, by Calderon-Vaillancourt theorem (see Appendix)

$$\hat{\Pi}_\epsilon = \mathrm{Op}_\epsilon^w(\hat{\pi}(x, \xi; \epsilon)) \tag{2.13}$$

is bounded and satisfy

$$\begin{aligned} \hat{\Pi}_\epsilon^2 - \hat{\Pi}_\epsilon &= \mathcal{O}(\epsilon^\infty) \\ \hat{\Pi}_\epsilon^* - \hat{\Pi}_\epsilon &= \mathcal{O}(\epsilon^\infty). \end{aligned} \tag{2.14}$$

For applications to spectral and scattering theory one would like to construct almost invariant subspaces which amounts to construct out of π_j an orthogonal projection Π_ϵ

$$\Pi_\epsilon{}^2 = \Pi_\epsilon = \Pi_\epsilon^* \tag{2.15}$$

satisfying

$$\|[\Pi_\epsilon, H_\epsilon]\| \leq s(\epsilon) \tag{2.16}$$

with $s(\epsilon) \geq 0$ as small as possible (if the series in (2.4) is non convergent one cannot have in general $s(\epsilon) = 0$ in (2.16)). Of course such a constructions is not unique and we shall follow the one of [Ne1, Ne2]. Since $\hat{\Pi}_\epsilon^2 - \hat{\Pi}_\epsilon = \mathcal{O}(\epsilon^\infty)$, *for ϵ sufficiently small the spectrum of $\hat{\Pi}_\epsilon$ is concentrate near* 0 *and* 1 *and then one can define Π_ϵ by*

$$\Pi_\epsilon := \frac{i}{2\pi} \int_{|z-1|=1/2} \left(\hat{\Pi}_\epsilon - z\right)^{-1} \, dz \tag{2.17}$$

Obviously Π_ϵ given by (2.17) satisfies (2.15). By an straightforward computation ([Ne1], [Ne2]), it is easy to see that, for ϵ sufficiently small, we have

$$\|[\Pi_\epsilon, H_\epsilon]\| \leq 2\|[\hat{\Pi}_\epsilon, H_\epsilon]\| \tag{2.18}$$

Since it turns out that (2.6) implies that

$$[\hat{\Pi}_\epsilon, H_\epsilon] = \mathcal{O}(\epsilon^\infty) \tag{2.19}$$

we obtain almost invariant subspace modulo $\mathcal{O}(\epsilon^\infty)$. It turns out that

$$\|\hat{\Pi}_\epsilon - \Pi_\epsilon\| = \mathcal{O}(\epsilon^\infty)$$

so Π_ϵ is close to a pseudodifferential operator. Actually, the results in [DiSj] implies that Π_ϵ is a pseudodifferential operator. Moreover, under appropriate analyticity properties of $h(x, \xi; \epsilon)$, one can chose a resummation of the analytic formal symbol $\sum_{j=0}^{\infty} \pi_j(x, \xi)\epsilon^j$ (see Appendix) in order to obtain an almost invariant subspace modulo $\mathcal{O}(e^{-c/\epsilon})$ for some positive constant $c > 0$ independent of ϵ i.e. we can take $s(\epsilon) = Ce^{-c/\epsilon}$ in (2.16). With respect to the decomposition

$$L^2(\mathbf{R}^n)^{\oplus m} = \Pi_\epsilon(L^2(\mathbf{R}^n)^{\oplus m}) \oplus (1 - \Pi_\epsilon)\,(L^2(\mathbf{R}^n)^{\oplus m})$$

H_ϵ takes an almost diagonal form

$$H_\epsilon = {H_\epsilon}^D + {H_\epsilon}^{OD} \tag{2.20}$$

where

$$\begin{aligned} {H_\epsilon}^D &= \Pi_\epsilon H_\epsilon \Pi_\epsilon + (1 - \Pi_\epsilon)\, H_\epsilon\, (1 - \Pi_\epsilon) && (2.21) \\ {H_\epsilon}^{OD} &= (1 - 2\Pi_\epsilon)\, [\Pi_\epsilon, H_\epsilon] && (2.22) \end{aligned}$$

with

$$\|{H_\epsilon}^{OD}\| \leq s(\epsilon) \tag{2.23}$$

If $\pi_0(x,\xi)=\pi_0(x)$ does not depend upon ξ (as is the case of the Klein-Gordon or Schrödinger operator) then $\Pi_0 := \mathrm{Op}_\epsilon^w(\pi_0)$ is just a multiplication matrix valued operator. Since

$$\lim_{\epsilon\to 0} \|\Pi_\epsilon - \Pi_0\| = 0 \tag{2.24}$$

for sufficiently small ϵ, one can write down the Nagy transformation matrix corresponding to the pair Π_ϵ, Π_0 ([Ka]):

$$U_\epsilon = \left(1-(\Pi_\epsilon-\Pi_0)^2\right)^{-1/2}(\Pi_\epsilon\Pi_0+(1-\Pi_\epsilon)(1-\Pi_0)) \tag{2.25}$$

having the properties

$$U_\epsilon{}^* U_\epsilon = U_\epsilon U_\epsilon{}^* = 1 \tag{2.26}$$

$$\Pi_\epsilon = U_\epsilon \Pi_0 U_\epsilon{}^*. \tag{2.27}$$

¿From (2.20), (2.21), (2.22), (2.23) and (2.26), (2.27) one gets that

$$\tilde{H}_\epsilon = U_\epsilon{}^* H_\epsilon U_\epsilon \tag{2.28}$$

has an almost diagonal form $\tilde{H}_\epsilon := \tilde{H}_\epsilon{}^D + \tilde{H}_\epsilon{}^{OD}$ with respect to the ϵ independent decomposition

$$L^2(\mathbf{R}^n)^{\oplus m} = \Pi_0(L^2(\mathbf{R}^n)^{\oplus m}) \oplus (1-\Pi_0)\,(L^2(\mathbf{R}^n)^{\oplus m})$$

with

$$\|\tilde{H}_\epsilon{}^{OD}\| \le s(\epsilon). \tag{2.29}$$

Further, since $\mathbf{R}^n$ is contractible, one can find $Q(x)$ (see Lemma 3.6 for an explicit construction in our case) unitary in $\mathbf{C}^n$ and satisfying

$$\Pi_0(x) = Q(x) P_0 Q^*(x)$$

with $P_0 = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & 0 \end{pmatrix}$ where $p = \dim \Pi_0(x)$ (which does not depend upon x) and then

$$H_\epsilon^{eff} := Q\tilde{H}_\epsilon Q^* \tag{2.30}$$

has a almost block diagonal forma with respect to the decomposition

$$L^2(\mathbf{R}^n)^{\oplus m} = L^2(\mathbf{R}^n)^{\oplus p} \oplus L^2(\mathbf{R}^n)^{\oplus(m-p)}.$$

In this way, up to a unitary transformation and errors of order $s(\epsilon)$ the study of H_ϵ in $L^2(\mathbf{R}^n)^{\oplus m}$ is reduced to the study of two operators in $(L^2(\mathbf{R}^n))^{\oplus p}$ and $(L^2(\mathbf{R}^n))^{\oplus(m-p)}$ respectively. If one assume that all the m eigenvalues of $h_0(x,\xi)$ remain isolated over all $\mathbf{R}^{2n}$, one can make a complete diagonalization which reduces (up to errors of order $s(\epsilon)$) the study of the m-states hamiltonian H_ϵ to the study of m scalar operators in $L^2(\mathbf{R}^n)$.

Remark 2.5

For the scattering theory the estimates (2.16) or (2.34) have to be improved at $|x| \to \infty$; for example in the Klein-Gordon case one has to replace (2.16) by

$$\|\langle x\rangle^{t\delta}[\Pi_\epsilon, H_\epsilon]\langle x\rangle^{(1-t)\delta}\| \leq s(\epsilon), \quad t \in [0,1] \tag{2.31}$$

for some $\delta > 1$. At the technical level this amounts to show that the decay of $V(x)$ (see (3.3) below) "propagates" through the construction of Π_ϵ. ∎

Remark 2.6

The above construction works nicely if $\pi_j(x,\xi)$ are bounded as it is the case for Klein-Gordon or Dirac operators. Unfortunately, this is not the case in the Schrödinger case where $\pi_j(x,\xi)$ behaves like $\langle\xi\rangle^j$ as $|\xi| \to \infty$ irrespective of how nice the potential is. As a consequence a block diagonalization valid on the whole $L^2(\mathbf{R}^n)^{\oplus m}$ seem not to exists (in concordance with the physical arguments saying that the adiabatic decoupling becomes poorer and poorer as energy increases (see also the remarks in [MarNe])). However one can still hope to have it as far as the energy (and then $|\xi|$) remains bounded. One can show (see: [So]) that this is indeed the case and the idea is to replace $\hat{\Pi}_\epsilon$ with

$$\hat{\Pi}_\epsilon^\Phi = \Phi(H_\epsilon)\hat{\Pi}_\epsilon + (1-\Phi(H_\epsilon))\hat{\Pi}_\epsilon\Phi(H_\epsilon) +$$

$(1\text{-}\Phi(H_\epsilon))\Pi_0(1-\Phi(H_\epsilon))$(2.32)*where Φ is a smooth cutoff function on $\mathbf{R}$. Then the whole procedure works and, defining*

$$\Pi_\epsilon{}^\Phi := \frac{i}{2\pi}\int_{|z-1|=1/2}\left(\hat{\Pi}_\epsilon^\Phi - z\right)^{-1}\, dz, \tag{2.33}$$

gives instead of (2.16)

$$\|\chi(H_\epsilon)[\Pi_\epsilon{}^\Phi, H_\epsilon]\| \leq s(\epsilon) \tag{2.34}$$

for any cutoff function χ such that $\chi\Phi = \chi$. ∎

3 Two component Klein-Gordon systems. Exponential estimates

In this section we prove the existence of invariant subspaces up to exponential small errors as $\epsilon \to 0$*, for two component Klein-Gordon systems, and apply this to spectral and scattering theory. Consider the two-component Klein-Gordon hamiltonian*

$$H_\epsilon = a(\epsilon D_x)\mathbf{1}_2 + V(x) \tag{3.1}$$

acting on $\mathcal{H} := L^2(\mathbf{R}^n)^{\oplus 2} := L^2(\mathbf{R}^n) \oplus L^2(\mathbf{R}^n)$.

Here $a(\xi) = (\sqrt{\xi^2+1} - 1)$, $\mathbf{1}_2$ *is the* 2×2 *identity matrix and* $V(x)$ *is a* 2×2 *hermitian matrix valued function that admits an analytic extension in some strip*

$$\Gamma_a = \{x \in \mathbf{R}^n \ ; \ |\mathrm{Im} x| < a\}, a > 0. \tag{3.2}$$

Moreover, since we have in mind application to scattering theory we assume that $V(x)$ *to be " short range" i.e. there exists a hermitian* 2×2 *matrix* V_∞ *such that*

$$\operatorname*{Sup}_{x \in \Gamma_a} \langle x \rangle^\delta |V(x) - V_\infty| < \infty \tag{3.3}$$

uniformly on Γ_a *with* $\delta > 1$*. Without loss of generality we can assume that*

$$V_\infty := \begin{pmatrix} \lambda_{1,\infty} & 0 \\ 0 & \lambda_{2,\infty} \end{pmatrix}. \tag{3.4}$$

Concerning the spectrum of $V(x)$*, we assume that the two real eigenvalues* $\lambda_i(x)$*, satisfy*

$$\operatorname*{Inf}_{x \in \mathbf{R}^n} |\lambda_1(x) - \lambda_2(x)| := 2d > 0. \tag{3.5}$$

By eventually shrinking Γ_a*, one can suppose that*

$$\operatorname*{Inf}_{x \in \Gamma_a} |\lambda_1(x) - \lambda_2(x)| \geq d. \tag{3.6}$$

It follows that on Γ_a

$$V(x) = \lambda_1(x)\pi_0(x) + \lambda_2(x)\left(1 - \pi_0(x)\right)$$

where $\pi_0(x)$ *is a bounded projection in* $\mathbf{C}^2$*, analytic in* Γ_a *and selfadjoint for* $x \in \mathbf{R}^n$*. Due to (3.4),* $\pi_0(\infty) := \lim_{|x| \to +\infty} \pi_0(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$*, and from (3.3)*

$$\operatorname*{Sup}_{x \in \Gamma_a} \langle x \rangle^\delta |\pi_0(x) - \pi_0(\infty)| < \infty \tag{3.7}$$

In what follows, "for ϵ sufficiently small" is a shorthand for "there exists $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$". A finite number of finite, strictly positive constants independent of ϵ will appear in this section; they are all denoted by c or C. For an operator A and $\eta > 0$ we shall denote

$$\| A \|_\eta = \operatorname*{Sup}_{t\in[0,1]} \| \langle x\rangle^{t\eta} A \langle x\rangle^{(1-t)\eta} \|$$

In the following we denote by Π_0 the operator in $\mathcal{H}$ acting as multiplication by the matrix $\pi_0(x)$.

Theorem 3.1 *Assume (3.2), (3.3) and (3.5) hold true. Then, for sufficiently small ϵ, there exists an orthogonal projection Π_ϵ in $\mathcal{H}$ satisfying:*

$$\|[\Pi_\epsilon, H_\epsilon]\|_\delta = \mathcal{O}(e^{-c/\epsilon}), \tag{3.8}$$

$$\epsilon^{-1}\|\Pi_\epsilon - \Pi_0\|_\delta = \mathcal{O}(1) \tag{3.9}$$

uniformly with respect to $\epsilon \in (0, \epsilon_0]$

Proof. *As already said in Section 2, the main step in the proof of Theorem 3.1 is the construction of Moyal projections for*

$$h_0(\xi, x) = a(\xi)\mathbf{1}_2 + V(x).$$

Lemma 3.2 *There exists a formal analytic symbol*

$$\pi(x,\xi;\epsilon) = \pi_0(x) + \sum_{j=1}^{\infty} \pi(x,\xi)\epsilon^j, \tag{3.10}$$

that satisfies (2.5), (2.6) and, in addition, such that $\pi(x,\xi;\epsilon) - \pi_0(x) \in S^0_A(\langle x\rangle^{-\delta})$. (For the definition of $S^0_A(1)$ and $S^0_A(\langle x\rangle^{-\delta})$ see Appendix).

Proof: *Let $x_0, \xi_0 \in \mathbf{R}^n$, $0 < \eta < d/4$ and $\Gamma_{x_0,\xi_0,d,\eta} = \{z \in \mathbf{C}\ ;\ d/2 - \eta < |z - a(\xi_0) - \lambda_1(x_0)| < d/2 + \eta\}$. Since $V(x)$ is analytic and uniformly bounded on Γ_a, $a(\xi)$ is analytic on Γ_1 and in addition $\partial_\xi^\alpha a(\xi)$, $|\alpha| = 1$ are uniformly bounded on Γ_1, there exists $b > 0$, $M > 0$ independent on $x_0, \xi_0 \in \mathbf{R}^n$ such that for all $z \in \Gamma_{x_0,\xi_0,d,\eta}$*

$$q_0(x,\xi,z) := (h_0(x,\xi) - z)^{-1}.$$

is analytic in $|x - x_0| < b,\ |\xi - \xi_0| < b$ *and in addition*

$$\operatorname*{Sup}_{|x-x_0|<b,\ |\xi-\xi_0|<b, z\in\Gamma_{x,\xi,d,\eta}} |q_0(x,\xi,z)| < M. \tag{3.11}$$

Then from Cauchy inequalities there exists $C > 0$ *such that for any* $\alpha, \beta \in \mathbf{N}^n$

$$\operatorname*{Sup}_{x,\xi\in\mathbf{R}^n, z\in\Gamma_{x,\xi,d,\eta}} |\partial_\xi^\alpha \partial_x^\beta q_0(x,\xi,z)| \leq C^{|\alpha|+|\beta|+1}\alpha!\beta! \tag{3.12}$$

and in addition (see (3.3)), if $|\beta| \neq 0$

$$\operatorname*{Sup}_{x,\xi\in\mathbf{R}^n, z\in\Gamma_{x,\xi,d,\eta}} |\langle x\rangle^\delta \partial_\xi^\alpha \partial_x^\beta q_0(x,\xi,z)| \leq C^{|\alpha|+|\beta|+1}\alpha!\beta! \tag{3.13}$$

Define now for $x, \xi \in \mathbf{R}^n, z \in \Gamma_{x,\xi,d,\eta}$ *the formal analytic symbol*

$$r = \sum_{j=1}^{\infty} r_j(x,\xi;z)\epsilon^j$$

by

$$(h_0(x,\xi) - z)\,\#q_0(x,\xi;z) = 1 - r(x,\xi;\epsilon,z).$$

Notice that $r_0(x,\xi) = 0$. *Taking into account (3.12)-(3.13) and that, for all* $\alpha, \beta \in \mathbf{N}^n$, $|\beta| \neq 0$

$$\operatorname*{Sup}_{(x,\xi)\in\mathbf{R}^n} |\langle x\rangle^\delta \partial_\xi^\alpha \partial_x^\beta h_0(x,\xi)| \leq$$

$C^{|\alpha|+|\beta|+1}\alpha!\beta!, (3.14)$*from the composition rule and Leibnitz formula (see also Lemma 3.4) one obtains that,*

for all $j = 0, 1, ..,$ *and* $\alpha, \beta \in \mathbf{N}^n$,

$$\operatorname*{Sup}_{x,\xi\in\mathbf{R}^n, z\in\Gamma_{x,\xi,d,\eta}} |\langle x\rangle^\delta \partial_\xi^\alpha \partial_x^\beta r_j(x,\xi,z)| \leq C^{j+|\alpha|+|\beta|+1}\alpha!\beta!j! \tag{3.15}$$

If we define the formal symbol $q(x,\xi;\epsilon,z)$ *by*

$$q = q_0 + q_0\# \sum_{j=1}^{\infty} r^{\#j} \tag{3.16}$$

then q *satisfy an estimate like (3.15). More exactly:*

Lemma 3.3 *There exists* $K > 0$ *such that for all* $j = 1, 2, ...,$ *and* $\alpha, \beta \in \mathbf{N}^n$

$$\operatorname*{Sup}_{x,\xi\in\mathbf{R}^n, z\in\Gamma_{x,\xi,d,\eta}} |\langle x\rangle^\delta \partial_\xi^\alpha \partial_x^\beta q_j(x,\xi,z)| \leq K^{j+|\alpha|+|\beta|+1}\alpha!\beta!j! \tag{3.17}$$

Proof. *Replace the supremum over* $(x,\xi) \in \Gamma_{a'} \times \Gamma_{b'}$ *in (3.5) with the supremum over* $x, \xi \in \mathbf{R}^n, z \in \Gamma_{x,\xi,d,\eta}$ *and mimic the proof of Lemmae 3.4 and 3.5 in Appendix .* ∎

¿From (2.10) it follows that, for $x, \xi \in \mathbf{R}^n, z \in \Gamma_{x,\xi,d,\eta}$, $q(x,\xi;\epsilon,z)$ *satisfies the "resolvent equation"*

$$q(x,\xi;\epsilon,z) - q(x,\xi;\epsilon,z') = (z-z')q(x,\xi;\epsilon,z)\#q(x,\xi;\epsilon,z'). \tag{3.18}$$

Mimicing the proof of the Riesz formula for the spectral projection [Ka] one obtain that the formal symbol displaystyle $\sum_{j=0}^{\infty} \pi_j(x,\xi)\epsilon^j$ *defined by*

$$\pi_j(x,\xi) = \frac{i}{2i\pi} \int_{\Gamma_{x,\xi}} q_j(x,\xi,z)\ dz \tag{3.19}$$

where $\Gamma_{x,\xi} = \{z \in \mathbf{C}\ ;\ |z - a(\xi) - \lambda_1(x)| = d/2\}$ *satisfies (2.5) and (2.6). Plugging the estimation (3.17) into (3.19) finishes the proof of Lemma 3.2.* ∎

End of the proof of Theorem 3.1. *Let* $\hat{\pi}_N$ *be a resummation of the formal analytic symbol as given in the Appendix i.e.* $\hat{\pi}_N(x,\xi;\epsilon) = \displaystyle\sum_{j=0}^{N} \pi_j(x,\xi)\epsilon^j$; *with* $N = [1/Bh]$, $B > 0$. *¿From Lemmae 3.6 and 3.2, it follows that*

$$\hat{\pi}_N \tilde{\#} \hat{\pi}_N(x,\xi;\epsilon) = \hat{\pi}_N(x,\xi;\epsilon) +$$

$r_N(x,\xi;\epsilon)(3.20)$*with*

$$\|r_N\|_{p,\delta} \leq (C\epsilon)^{N+1}(N+1)!.$$

for a new constant $C > 0$ *(see (A.1) for the definition of* $\|\cdot\|_{p,0\delta}$*). By a similar argument:*

$$\|h\tilde{\#}\hat{\pi}_N - \hat{\pi}_N\tilde{\#}h\|_{p,\delta} \leq (C\epsilon)^{N+1}(N+1)!.$$

Then, taking $N = [1/B\epsilon]$ *with* B *sufficiently large and denoting*

$$\hat{\pi}^B := \sum_{j=0}^{[1/B\epsilon]} \pi_j \epsilon_j,$$

by Stirling formula, one has

$$\|\hat{\pi}^B \tilde{\#} \hat{\pi}^B - \hat{\pi}^B\|_{p,\delta} + \|h\tilde{\#}\hat{\pi}^B - \hat{\pi}^B\#h\|_{p,\delta} \leq Ce^{c/h} \tag{3.21}$$

for some constant $c, C > 0$ independent of ϵ. Let now

$$\hat{\Pi}_\epsilon = Op_\epsilon^w(\hat{\pi}^B). \tag{3.22}$$

Then from (3.21), Lemma 3.6 and Calderon-Vaillancourt theorem (see Theorem 3.7) $\hat{\Pi}_\epsilon$ is bounded uniformly with respect to ϵ, and in addition:

$$\|\hat{\Pi}_\epsilon^2 - \hat{\Pi}_\epsilon\|_\delta = \mathcal{O}(e^{-c/\epsilon}) \tag{3.23}$$

$$\|\left[\hat{\Pi}_\epsilon, H_\epsilon\right]\|_\delta = \mathcal{O}(e^{-c/\epsilon}) \tag{3.24}$$

for some constant $c > 0$ independent of $\epsilon \in (0, \epsilon_0]$. Then (see Section 2 and [Ne1, Ne2] for details), if Π_ϵ is given by

$$\Pi_\epsilon = \frac{i}{2\pi} \int_{|z-1|=1/2} \left(\hat{\Pi}_\epsilon - z\right)^{-1} \, dz$$

the only thing to be verified is the $\langle x \rangle^{-\delta}$ decay as $|x| \to \infty$. By a simple computation (see Appendix in [Ne2])

$$(\hat{\Pi}_\epsilon - z)^{-1} = \frac{\hat{\Pi}_\epsilon + z - 1}{z(z-1)} \left[1 + \frac{\Delta_\epsilon}{z(1-z)}\right]^{-1} \tag{3.25}$$

where

$$\Delta_\epsilon = \hat{\Pi}_\epsilon^2 - \hat{\Pi}_\epsilon. \tag{3.26}$$

Expanding in (3.25) and using (3.23), one has uniformly for sufficiently small ϵ:

$$\|(\hat{\Pi}_\epsilon - z)^{-1}\|_\delta = \mathcal{O}(1). \tag{3.27}$$

Again by a simple computation

$$\Pi_\epsilon - \hat{\Pi}_\epsilon = -\Delta_\epsilon \frac{i}{2\pi} \oint_{|z-1|=1/2} \frac{1}{z(1-z)} \left(\frac{\hat{\Pi}_\epsilon}{z(1-z)} + 1\right) \left(1 + \frac{\Delta_\epsilon}{z(1-z)}\right)^{-1} dz$$

which together with (3.23) and (3.27) gives

$$\|\Pi_\epsilon - \hat{\Pi}_\epsilon\|_\delta \le C e^{-c/\epsilon} \tag{3.28}$$

which implies (3.9). For (3.8) use

$$[H_\epsilon, \Pi_\epsilon] = -\frac{i}{2\pi} \oint_{|z-1|=1/2} (\hat{\Pi}_\epsilon - z)^{-1} [H_\epsilon, \hat{\Pi}_\epsilon] (\hat{\Pi}_\epsilon - z)^{-1} dz$$

(3.27) and (3.24). ∎

Remark 3.4

¿From (3.28), Π_ϵ *is exponentially close to* $\hat{\Pi}_\epsilon$ *which is a pseudodifferential operator with an analytic symbol. Actually, by using the results in [DiSj] on functional calculus for pseudodifferential operators, one can show that* Π_ϵ *is a pseudodifferential operator with symbol* $\hat{\pi} \in S^0_A(1)$ *given by (3.10) and such that* $\hat{\pi} - \pi_0 \in S^0_A(\langle x\rangle^{-\delta})$ ∎

We turn now to applications to spectral and scattering theory for H_ϵ. *Consider the scattering problem for the pair* H_ϵ *and*

$$H_{\epsilon,0} := a(\epsilon D_x)\mathbf{1}_2 + \begin{pmatrix} \lambda_{1,\infty} & 0 \\ 0 & \lambda_{2,\infty} \end{pmatrix}. \tag{3.29}$$

We need the following elementary propagation estimate:

Lemma 3.5 *Let* $\phi \in \mathcal{H}$, $\hat{\phi}$*(:= the Fourier transform of* ϕ*)* $\in \mathcal{C}_0^\infty(\mathbf{R}^n \backslash \{0\})^{\oplus 2}$ *Then there exists* $K(\phi) > 0$ *such that, uniformly for sufficiently small* ϵ,

$$\|\langle x\rangle^{-\delta} e^{-itH_{\epsilon,0}}\phi\| \leq K(\phi)\epsilon^{-2\delta}\langle t\rangle^{-\delta}. \tag{3.30}$$

P*roof. The proof is standard stationary phase estimates (see e.g. Appendix to XI.3 in [ReSi1]) and the only thing to do is to control their* ϵ *dependence. For it, write*

$$\sqrt{\epsilon^2 k^2 + 1} - 1 = \epsilon^2 K_\epsilon(k)$$

and observe that

$$\mathrm{Inf}_{\epsilon \in (0,\epsilon_0]} |\mathbf{grad} K_\epsilon(k)| = \frac{|k|}{\sqrt{\epsilon_0^2 k^2 + 1}}$$

and

$$K_\epsilon(k) = \frac{k^2}{2} + \mathcal{O}(\epsilon^2 k^4).$$

∎

¿From Lemma 3.5, by the standard Cook argument, the wave operators

$$W_\pm(H_\epsilon, H_{\epsilon,0}) = s - \lim_{t \to \pm\infty} e^{itH_\epsilon} e^{-itH_{\epsilon,0}} \tag{3.31}$$

exist for $\delta > 1$. *As for completeness, if* $\delta > n$, *it follows by a direct application of Birman's theorem [ReSi1], Thm. XI.10 (mimic the proof of Thm. XI.30* *in [ReSi1]). Actually the completeness holds true for $\delta > 1$; one can prove it [Si] using Enss method. The same argument applies to all the wave operators appearing in this section so in what follows we shall assume that all of them exist and are complete. Let S be the scattering operator corresponding to the pair $H_\epsilon, H_{\epsilon,0}$:*

$$S = W_+(H_\epsilon, H_{\epsilon,0})^* W_-(H_\epsilon, H_{\epsilon,0}). \tag{3.32}$$

One expects that (3.8) implies that S has an almost block diagonal structure with respect to the decomposition given by Π_ϵ, and indeed one can prove (see the proof of Theorem 3.10 below) that if $\phi_1 \in \mathcal{H}$, $\phi_2 \in (1-\Pi_\epsilon)(\mathcal{H})$ and $\hat{\phi}_j \in \mathcal{C}_0^\infty(\mathbf{R}^n \backslash \{0\})^{\oplus 2}$ then

$$|(\phi_2, S\phi_1)| \le K(\phi_1, \phi_2) e^{-c/\epsilon}.$$

What we shall prove below (under an additional condition on $V(x)$, see Lemma 3.6 below) is that S has an almost diagonal form with respect to the canonical decomposition of $\mathcal{H}$ i.e. the decomposition given by

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We start with the following elementary lemma

Lemma 3.6 *Suppose that either*

$$\operatorname*{Sup}_{x\in\Gamma_a} \|\pi_0(x) - P_0\| < 1 \tag{3.33}$$

or, for $x \in \mathbf{R}^n$, that

$$V_{ij}(x) = \overline{V_{ij}(x)} \tag{3.34}$$

i.e. $V(x)$ has real entries for $x \in \mathbf{R}^n$. Then there exists a matrix $Q(x)$ bounded with bounded inverse uniformly in Γ_a, unitary for $x \in \mathbf{R}^n$ and satisfying:

$$\pi_0(x) = Q(x) P_0 Q(x)^{-1} \tag{3.35}$$

and

$$\operatorname*{Sup}_{x\in\mathbf{R}^n} |(Q(x) - \mathbf{1}_2)\langle x\rangle^\delta| < \infty. \tag{3.36}$$

Proof: If (3.33) holds true, one can take as $Q(x)$ the Nagy transformation matrix corresponding to the pair $\pi_0(x), \pi_0(\infty) = P_0$.

Suppose now (3.34) holds true. For $x = (x', x_n) \in \Gamma_a$, let us set $x_n = y_n + i\eta$, $y_n \in \mathbf{R}$ and define $Q(x)$ by

$$Q(x) = 1 + \int_{-\infty}^{y_n} (1 - 2\pi_0(x', s + i\eta))\,(\frac{d}{ds}\pi_0(x', s+i\eta))Q(x', s+i\eta)ds. \quad (3.37)$$

Notice that the integral in (3.37) is convergent since $\frac{d}{ds}\pi_0(x', s + i\eta) = \mathcal{O}(\langle s\rangle^{-\delta})$. Now (see e.g. [Ka]) $Q(x)$ is analytic and uniformly bounded in Γ_a and satisfies (3.35) since

$$\lim_{y_n \to +\infty} \pi_0(x', y_n + i\eta) = \pi(\infty) = P_0 \quad (3.38)$$

Since (3.37) implies that for $x \in \mathbf{R}^n$, $Q(x) = \overline{Q(x)}$, from (3.35) and (3.7) one obtains

$$Q^2_{1,1}(x) - 1 = \mathcal{O}(\langle x\rangle^{-\delta}), \quad Q_{1,1}(x)Q_{1,2}(x) = \mathcal{O}(\langle x\rangle^{-\delta}) \quad (3.39)$$

This implies that as $|x| \to +\infty$, $Q_{1,1}(x) \to \pm 1$. By continuity the choice is the same for all directions. Since, for $x_n \to -\infty$, $Q(x) \to \mathbf{1}_2$, it follows that $\lim_{|x|\to+\infty} Q_{1,1}(x) = 1$ which together with (3.39) finishes the proof. ∎

Let now U_ϵ be the Nagy transformation matrix corresponding to the pair Π_ϵ, Π_0 (see (2.25)). The results of [DiSj] implies that U_ϵ is also a pseudodifferential operator. Let us consider the rotated hamiltonian (see (2.28), (2.30)):

$$H_\epsilon^{eff} := Q^* {U_\epsilon}^* H U_\epsilon Q \quad (3.40)$$

where Q is the operator of multiplication with $Q(x)$.

Corollary 3.7 *Under assumptions (3.2),(3.3),(3.5), (3.33) or (3.34) we have*

$$H_\epsilon^{eff} = \begin{pmatrix} a(\epsilon D_x) + \lambda_1(x) & 0 \\ 0 & a(\epsilon D_x) + \lambda_2(x) \end{pmatrix} + \epsilon B_\epsilon \quad (3.41)$$

where $B_{\epsilon,ii}$; $i,j = 1,2$ are bounded operators satisfying

$$\|B_{\epsilon,ii}\|_\delta = \mathcal{O}(1) \quad (3.42)$$

$$\|B_{\epsilon,12}\|_\delta = \|B_{\epsilon,21}\|_\delta = \mathcal{O}(e^{-c/\epsilon}) \quad (3.43)$$

uniformly with respect to sufficiently small ϵ.

Proof. *Write*

$$H_\epsilon^{eff} = Q^*(H_\epsilon + U_\epsilon{}^*[H_\epsilon, U_\epsilon]Q) =$$

$$\begin{pmatrix} a(\epsilon D_x) + \lambda_1(x) & 0 \\ 0 & a(\epsilon D_x) + \lambda_2(x) \end{pmatrix} + Q^*[a(\epsilon D_x)\mathbf{1}_2, Q] + Q^*U_\epsilon{}^*[H_\epsilon, U_\epsilon]Q \tag{3.44}$$

The fact that, uniformly for ϵ sufficiently small,

$$\epsilon^{-1}\|O^*[a(\epsilon D_x)\mathbf{1}_2, Q]\|_\delta = \mathcal{O}(1) \tag{3.45}$$

follows from the fact that $\langle x\rangle^\delta$ commutes with Q, (3.36) and Lemma 3.6. Analogously

$$\epsilon^{-1}\|[H_\epsilon, \Pi_0]\|_\delta = \mathcal{O}(1) \tag{3.46}$$

Since $\langle x\rangle^\delta$ commutes with Π_0 from (3.9) one has

$$\|\langle x\rangle^\delta \Pi_\epsilon \langle x\rangle^{-\delta}\| = \mathcal{O}(1). \tag{3.47}$$

Hence, from the Nagy formula for U_ϵ, (3.8), (3.46) and (3.47) one has:

$$\epsilon^{-1}\|[H_\epsilon, U_\epsilon]\|_\delta = \mathcal{O}(1) \tag{3.48}$$

$$\epsilon^{-1}\|U_\epsilon - 1\|_\delta = \mathcal{O}(1) \tag{3.49}$$

uniformly for small ϵ. ¿From (3.48) and (3.49)

$$\epsilon^{-1}\|Q^*U_\epsilon{}^*[H_\epsilon, U_\epsilon]Q\|_\delta \tag{3.50}$$

which together with (3.45) finishes the proof of (3.42). Now from (3.35) and $\Pi_\epsilon = U_\epsilon \Pi_0 U_\epsilon{}^*$ it follows that

$$\Pi_\epsilon = U_\epsilon Q P_0 Q^* U_\epsilon{}^*$$

and then

$$[H_\epsilon^{eff}, P_0] = Q^*U_\epsilon{}^*[H_\epsilon, \Pi_\epsilon]U_\epsilon Q \tag{3.51}$$

which together with (3.8) and (3.49) gives

$$\|[H_\epsilon^{eff}, P_0]\|_\delta = \mathcal{O}(e^{-c/\epsilon}). \tag{3.52}$$

Taking into account that

$$\epsilon B_{\epsilon,12} + \epsilon B_{\epsilon,21} = (1 - 2P_0)[H_\epsilon^{eff}, P_0],$$

(3.52) implies (3.43). ∎

Remark 3.8

As in the case of Π_ϵ*, from (3.44) one sees that* H_ϵ^{eff} *is exponentially close to a pseudodifferential operator and again the results in [DiSj] enables to show that in fact* H_ϵ^{eff} *is a pseudodifferential operator. Let us stress here that the constructive proof above provides the means to compute effectively the formal symbol of* H_ϵ^{eff}*.* ∎

The following lemma allows to replace H_ϵ *by* H_ϵ^{eff} *when computing the scattering operator.*

Lemma 3.9 *Let* $\tilde{S}$ *be the scattering operator corresponding to the pair* $H_\epsilon^{eff}, H_{\epsilon,0}$*. Then for sufficiently small* ϵ*:*

$$S = \tilde{S} \tag{3.53}$$

Proof. Let $\Omega_\epsilon := QU_\epsilon$*. ¿From (3.36) and (3.49)*

$$\|(\Omega_\epsilon - 1)\langle x\rangle^\delta\| = \mathcal{O}(1) \tag{3.54}$$

Now (see (3.40))

$$\begin{aligned} W_\pm(H_\epsilon^{eff}, H_{\epsilon,0}) &= {\Omega_\epsilon}^*(s - \lim_{t\to\pm\infty} \Omega_\epsilon e^{itH_\epsilon^{eff}} e^{-itH_{\epsilon,0}}) \\ &= {\Omega_\epsilon}^* W_\pm(H_\epsilon, H_{\epsilon,0}) + (s - \lim_{t\to\pm\infty} {\Omega_\epsilon}^* e^{itH_\epsilon}(\Omega_\epsilon - 1)e^{-itH_{\epsilon,0}}). \end{aligned}$$

Let now $\phi \in \mathcal{H}$*,* $\hat{\phi} \in \mathcal{C}_0^\infty(\mathbf{R}^n\backslash\{0\})^{\oplus 2}$*. then from Lemma 3.5 and (3.54)*

$$\|{\Omega_\epsilon}^* e^{itH_\epsilon}(\Omega_\epsilon - 1)e^{-itH_{\epsilon,0}}\phi\| \leq C(\phi)\epsilon^{-2\delta}\langle t\rangle^{-\delta}.$$

Sending $|t| \to \infty$ *(at fixed* ϵ*!), by density we get*

$$s - \lim_{t\to\pm\infty} {\Omega_\epsilon}^* e^{itH_\epsilon}(\Omega_\epsilon - 1)e^{-itH_{\epsilon,0}} = 0$$

which gives

$$W_\pm(H_\epsilon^{eff}, H_{\epsilon,0}) = {\Omega_\epsilon}^* W_\pm(H_\epsilon, H_{\epsilon,0})$$

and the proof of the lemma is finished. ∎

We are now in position to state the main result of the paper

Theorem 3.10 *Let, for $j = 1, 2$:*

$$H_{\epsilon,j}^{eff} := a(\epsilon D_x) + \lambda_j(x) + \epsilon B_{\epsilon,ii}; \quad H_{\epsilon,j,0} := a(\epsilon D_x) + \lambda_{j,\infty} \tag{3.55}$$

acting in $L^2(\mathbf{R}^n)$ and S_j the scattering operator corresponding to the pair $H_{\epsilon,j}^{eff}, H_{\epsilon,j,0}$. Then, there exists $c > 0$ independent of ϵ, such that, for any $\phi_k \in \mathcal{H}$, $\hat{\phi}_k \in \mathcal{C}_0^\infty(\mathbf{R}^n \backslash \{0\})^{\oplus 2}, k = 1, 2$ and $\epsilon \in (0, \epsilon_0]$ sufficiently small, we have

$$|(\phi_2, \left(S - \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}\right) \phi_1)| \leq C(\phi_1, \phi_2) e^{-c/\epsilon}. \tag{3.56}$$

for some constant $C(\phi_1, \Phi_2) > 0$ depending on ϕ_1, Φ_2 but not on ϵ.

Proof. Write

$$D_\epsilon := \begin{pmatrix} H_{\epsilon,1}^{eff} & 0 \\ 0 & H_{\epsilon,2}^{eff} \end{pmatrix}, \quad R_\epsilon := H_\epsilon^{eff} - D_\epsilon.$$

and let $\phi \in \mathcal{H}$, $\hat{\phi} \in \mathcal{C}_0^\infty(\mathbf{R}^n \backslash \{0\})^{\oplus 2}$. By the usual Cook argument, for $T > 0$, we have

$$\begin{aligned} &\|(W_+(H_\epsilon^{eff}, H_{\epsilon,0}) - e^{iTH_\epsilon^{eff}} e^{-iTH_{\epsilon,0}}\phi\| \\ &= \| \int_T^\infty e^{itH_\epsilon^{eff}} (H_\epsilon^{eff} - H_{\epsilon,0}) e^{-itH_{\epsilon,0}}\phi\| \leq C(\phi)\epsilon^{-2\delta+1} T^{1-\delta}. \end{aligned} \tag{3.57}$$

and

$$\begin{aligned} &\|(W_+(D_\epsilon, H_{\epsilon,0}) - e^{iTD_\epsilon} e^{-iTH_{\epsilon,0}}\phi\| \\ &= \| \int_T^\infty e^{itD_\epsilon} (D_\epsilon - H_{\epsilon,0}) e^{-itH_{\epsilon,0}}\phi\| \leq C(\phi)\epsilon^{-2\delta+1} T^{1-\delta}. \end{aligned} \tag{3.58}$$

For the last inequality we used Lemma 3.5 and the fact that (see Corollary 3.7) $\|H_\epsilon^{eff} - H_{\epsilon,0}\|_\delta + \|D_\epsilon - H_{\epsilon,0}\|_\delta = \mathcal{O}(\epsilon)$. On the other hand from Duhamel's formula:

$$e^{iTH_\epsilon^{eff}} = e^{iTD_\epsilon} + i\int_0^T e^{i(T-s)D} R e^{isH_\epsilon^{eff}} ds \tag{3.59}$$

which together with (3.43) gives

$$\|e^{iTH_\epsilon^{eff}} e^{-iTD_\epsilon} - \mathbf{1}_2\| \leq C'\epsilon T e^{-c/\epsilon}. \tag{3.60}$$

¿From (3.57), (3.58) and (3.60) one gets:

$$\| \left(W_+(H_\epsilon^{eff}, H_{\epsilon,0}) - W_+(D_\epsilon, H_{\epsilon,0}) \right) \phi \| \leq 2C(\phi)\epsilon^{-2\delta+1} T^{1-\delta} + C'\epsilon T e^{-c/\epsilon}. \tag{3.61}$$

By choosing $T = \epsilon^{-2-1/\delta} e^{c/\delta\epsilon}$ *in (3.61) one obtains*

$$\| \left(W_+(H_\epsilon^{eff}, H_{\epsilon,0}) - W_+(D_\epsilon, H_{\epsilon,0}) \right) \phi \| \leq C_1(\phi) e^{-c/\epsilon}. \tag{3.62}$$

for some c', $0 < c' \leq c$, *and* $C = C(\phi) > 0$ *independent of* ϵ. *In the same manner*

$$\| \left(W_-(H_\epsilon^{eff}, H_{\epsilon,0}) - W_-(D_\epsilon, H_{\epsilon,0}) \right) \phi \| \leq C_2(\phi) e^{-c/\epsilon}. \tag{3.63}$$

Writing

$$\begin{aligned} &S(H_\epsilon^{eff}, H_{\epsilon,0}) - S(D_\epsilon, H_{\epsilon,0}) \\ &\quad = \left(W_+(H_\epsilon^{eff}, H_{\epsilon,0})^* - W_+(D_\epsilon, H_{\epsilon,0})^* \right) W_+(H_\epsilon^{eff}, H_{\epsilon,0}) \\ &\quad + W_+(D_\epsilon, H_{\epsilon,0})^* \left(W_+(H_\epsilon^{eff}, H_{\epsilon,0}) - W_+(D_\epsilon, H_{\epsilon,0}) \right) \end{aligned}$$

and using (3.62) and (3.63) and Lemma 3.9, (3.56) follows. ∎

Since $\tilde{S}$ *commutes with* $H_{\epsilon,0}$, *one can define the scattering matrix at fixed energy* $\tilde{S}(\lambda)$. *Let us stress that Theorem 3.10 does not imply the corresponding result for* $\tilde{S}(\lambda)$; *when shrinking the supports of* ϕ_1 *and* ϕ_2 *the constant* $C(\phi_1, \phi_2)$ *may blow up. At the heuristic level the reason for this blowing up is clear: Corollary 3.7 (via the Duhamel's formula) implies that the transitions* per unit time *between the two levels is exponentially small, but if, at the given energy, there is a resonance very (exponentially) close to the real axis, it has an exponentially long life time and during this time considerable transitions can take place. Accordingly, one expects exponentially block diagonalization of* $\tilde{S}(\lambda_0)$ *only if there are no resonances around* λ_0 *or in other words if* $\tilde{S}(\lambda)$ *is smooth in a neighborhood of* λ_0. *Under some more restrictive conditions on the potential, results of this type were obtained in [Be, BeMa, MNS, Na] both for Klein-Gordon and Schrodinger cases, by the method of phase space tunnelling [Ma]. See also [MarNe] where almost invariant subspaces at fixed* λ_0 *were constructed in the case of the one-dimensional Schrodinger operator, for* *energies above the potential barrier. One can obtain exponential estimates on* $\tilde{S}(\lambda_0)$ *(see: [So]), by using Corollary 3.7 and resolvent estimates [GeMa, JeNa, Is, Wa].*

We turn now to some direct applications of Corollary 3.7 to spectral theory. We shall discuss H_ϵ^{eff} *but all the results apply to* H_ϵ *also, since they are unitarily equivalent. If*

$$\operatorname*{Inf}_{x\in\mathbf{R}^n} \lambda_2(x) := m_2 < \lambda_{2,\infty}.$$

then H_ϵ^{eff} *has discrete spectrum in the interval* $(m_2, \lambda_{2,\infty})$ *which, up to exponentially small errors, equals the union of the (discrete) spectra of* $H_{\epsilon,j}^{eff}$ *in this interval (see [Ne2] for details). Suppose now*

$$\operatorname*{Inf}_{x\in\mathbf{R}^n} \lambda_1(x) := m_1 < \lambda_{1,\infty}.$$

Then $H_{\epsilon,1}^{eff}$ *has bound states in the interval* $(m_1, \lambda_{1,\infty})$ *and as a consequence,* D_ϵ *has bounded states embedded in the continuum spectrum in the interval* $(\mathrm{Max}\{m_1, \lambda_{2,\infty}\}, \lambda_{1,\infty})$*. The small off-diagonal term ,*R_ϵ*, turns (generically) these bound (stable) states into metastable states. Let* Ψ *be an eigenfunction of such a bound state i.e.*

$$H_{\epsilon,1}^{eff}\Psi = E\Psi, \quad E \in (\mathrm{Max}\{m_1, \lambda_{2,\infty}\}, \lambda_{1,\infty}).$$

Then from Corollary 3.7

$$\|(H_\epsilon^{eff} - E)\Psi\| \le Ce^{-c/\epsilon},$$

i.e. Ψ, E *are pseudo-eigenvectors and pseudo-eigenvalues (see [Ka]) of exponential order for* H_ϵ^{eff} *(the name quasi-modes is also used). A more physical picture is given by Duhamel formula (3.59) which together with Corollary 3.7 gives*

Corollary 3.11

$$|(\Psi, e^{-itH_\epsilon^{eff}}\Psi)| \ge 1 - C|t|e^{-c/\epsilon}. \tag{3.64}$$

As it stands Corollary 3.11 gives the existence and the control on the (exponentially long) life time of metastable states (quasi-modes) in the semiclassical limit. However in many instances (see

e.g.[St]) one can make the connection with the resonances (defined as poles of the resolvent of scattering operator). In the cases when this connection can be made Corollary 3.11 says that the imaginary part of the resonances is exponentially small. The same argument applies to other bound states and/or resonances of D in the interval $(\lambda_{1,\infty}, \infty)$ *(as far as they exist).*

Appendix : Analytic Symbols

We recall here, in an appropriate form, some standard facts about pseudodifferential operators. For details see e.g. [Ro, DiSj, Ma, Ta, BoKr]. We consider matrix valued symbols so the product and the modulus are the standard product and matrix norm respectively. We employ the standard notations concerning the derivatives, factorial etc. Also for $y \in \mathbf{C}^n$, $\langle y\rangle = (1+|y|^2)^{1/2}$ *and for* $a, b > 0$

$$\Gamma_a \times \Gamma_b = \{(x,\xi) \in \mathbf{C}^{2n} \ ; \ |\mathrm{Im} x| < a \ |\mathrm{Im}\xi| < b\}$$

Symbols $a(x,\xi;\epsilon)$ *are smooth matrix valued functions of* $(x,\xi) \in \mathbf{R}^{2n}$ *or* Γ_a*, depending on* $\epsilon \in (0,\epsilon_0]$*, as a parameter.*

Definition A.1 *A function* $a(x,\xi;\epsilon)$ *defined on* $\mathbf{R}^{2n}\times(0,\epsilon_0]$*, is said to be a symbol in* $S^m(\langle x\rangle^{-\delta})$*, for some* $m,\delta \in \mathbf{R}$ *if and only if* a *is a smooth function of* (x,ξ) *and, for all* $\alpha,\beta \in \mathbf{N}^n$*, there exist* $C_{\alpha,\beta} > 0$ *such that*

$$\operatorname*{Sup}_{(x,\xi)\in\mathbf{R}^{2n},\epsilon\in(0,\epsilon_0]} |\langle\xi\rangle^{-m}\langle x\rangle^{\delta}\partial_x^{\alpha}\partial_{\xi}^{\beta}a(x,\xi,\epsilon)| \le C_{\alpha,\beta}.$$

In particular $\mathcal{S}^0(1)$ *is the set of* $a(x,\xi,\epsilon)$ *which are uniformly bounded together with all their derivatives. For* $a \in \mathcal{S}^0(\langle x\rangle^{-\delta})$ *and* $p \in \mathbf{N}$ *we set*

$$|\!|\!| a |\!|\!|_{p,\delta} = \operatorname*{Sup}_{(x,\xi)\in\mathbf{R}^{2n},\epsilon\in(0,\epsilon_0]} \sum_{|\alpha|+|\beta|\le p} |\langle x\rangle^{\delta}\partial_x^{\alpha}\partial_{\xi}^{\beta}a(x,\xi;\epsilon)| \tag{A.1}$$

Definition A.2 *A function* $a(x,\xi,\epsilon)$, $(x,\xi) \in \Gamma_a\times\Gamma_b$, $\epsilon \in (0,\epsilon_0]$ *is called an analytic symbol of class* $\mathcal{S}_A^m(\langle x\rangle^{-\delta})$ *for some* $m,\delta\in\mathbf{R}$ *iff it is analytic in* $\Gamma_a\times\Gamma_b$ *and, for any* $a' < a, b' < b$*, there exists* $K > 0$ *such that*

$$\operatorname*{Sup}_{(x,\xi)\in\Gamma_{a'}\times\Gamma_{b'},\epsilon\in(0,\epsilon_0]} |\langle\xi\rangle^{-m}\langle x\rangle^{\delta}a(x,\xi,\epsilon)| \le K \tag{A.2}$$

Definition A.3 *Given a family* $a_j(x,\xi) \in \mathcal{S}_A^m(\langle x\rangle^{-\delta})$, $j \in \mathbf{N}$*, analytic on* $\Gamma_a\times\Gamma_b$ *such that, for any* $a' < a, b' < b$*, there exists a constant* $C > 0$ *independent of* j *such that*

$$\operatorname*{Sup}_{(x,\xi)\in\Gamma_{a'}\times\Gamma_{b'}} |\langle\xi\rangle^{m}\langle x\rangle^{\delta}a_j(x,\xi)| \le C^{j+1}j! \tag{A.3}$$

we call $a_f = \sum_{j=0}^{\infty} \epsilon^j a_j(x,\xi)$ *a formal analytic symbol of class* $\mathcal{S}_A^0(\langle x\rangle^{-\delta})$ *on* $\Gamma_a \times \Gamma_b$.

We say that a symbol $a(x,\xi;\epsilon) \in \mathcal{S}_A^m(\langle x\rangle^{-\delta})$ *is asymptotic equivalent to the formal analytic symbol* $\sum_{j=0}^{\infty} \epsilon^j a_j(x,\xi)$ *and we write*

$$a \sim \sum_{j=0}^{\infty} \epsilon^j a_j(x,\xi)$$

iff there exists a constant $C > 0$ *such that for any* $N \in \mathbf{N}$ *and any* $a' < a, b' < b$

$$\mathrm{Sup}_{(x,\xi)\in\Gamma_{a'}\times\Gamma_{b'},\epsilon\in(0,\epsilon_0]}$$

$|\langle\xi\rangle^{-m}\langle x\rangle^{\delta}\left(a(x,\xi;\epsilon) - \sum_{j=0}^{N} \epsilon^j a_j(x,\xi)\right)| \leq$
$C^{N+1}\epsilon^{N+1}(N+1)!$(A.4)*If* $\sum_{j=0}^{\infty} \epsilon^j a_j(x,\xi)$ *is a formal analytic symbol, and* $T > 0$ *is sufficiently large then one can define* $a^T(x,\xi;\epsilon) := \sum_{j=0}^{[1/T\epsilon]} \epsilon^j a_j(x,\xi)$ *(unique modulo* $\mathcal{O}(e^{-c/\epsilon})$*) satisfying (A.4). The symbol* $a^T(x,\xi;\epsilon)$ *is called a resummation of the formal analytic symbol* $\sum_{j=0}^{\infty} \epsilon^j a_j(x,\xi)$ *and any symbol* a *that can be written as such a resummation up to error* $\mathcal{O}(e^{-c/\epsilon})$ *is an analytic representation of the formal analytic symbol* $\sum_{j=0}^{\infty} \epsilon^j a_j(x,\xi)$

Let us notice (see [BoKr]) that a formal symbol $a = \sum_{j=0}^{\infty} \epsilon^j a_j(x,\xi)$ *belongs to* $S_A^{m_1}(\langle x\rangle^{-\delta_1})$ *on* $\Gamma_a \times \Gamma_b$ *if and only if, for any* $a', b' > 0, a' < a, b' < b$, *the series*

$$N_{\delta_1}^{m_1}(a,T) = \sum_{j=0}^{\infty} T^j \left(\sum_{|\alpha|+}\right.$$

$|\beta| + 2k = j c_{k,\alpha}^{\beta} A_{k,\alpha,\delta_1}^{\beta,m_1} where A_{k,\alpha,\delta_1}^{\beta,m_1} := \mathrm{Sup}_{(x,\xi)\in\Gamma_{a'}\times\Gamma_{b'}} |\langle x\rangle^{\delta_1}\langle\xi\rangle^{-m_1}\partial_x^{\alpha}\partial_{\xi}^{\beta} a_k| \leq C^{k+|\alpha|+|\beta|+1} k!\alpha!\beta!(3.5) and$

$$c_{k,\alpha}^{\beta} = \frac{2(2n)^{-k}k!}{(k+|\alpha|)!(k+|\beta|)!}$$

has a nonzero radius of convergence as a series in T.
If $a_f = \sum_{j=0}^{\infty} \epsilon^j a_j(x,\xi)$ and $b_f = \sum_{j=0}^{\infty} \epsilon^j b_j(x,\xi)$ are two formal analytic symbols on $\Gamma_a \times \Gamma_b$, we define, as usual, their the Weyl product

$$a_f \# b_f := r_f = \sum_{m=0}^{\infty} \epsilon^m r_m \tag{3.6}$$

where

$$r_m = \sum_{|\gamma|+|\theta|+l+s=m} \Gamma(\gamma,\theta) D_x^{\gamma} \partial_{\xi}^{\theta} a_l D_x^{\theta} \partial_{\xi}^{\gamma} b_l. \tag{3.7}$$

and $\Gamma(\alpha,\beta) = \dfrac{(1)^{\beta}}{\alpha!\beta!2^{|\alpha|}2^{|\beta|}}$. Using Lemma 1.2 of [BoKr], one can prove that

$$N_{\delta_1+\delta_2}^{m_1+m_2}(r,T) << N_{\delta_1}^{m_1}(p,T) N_{\delta_2}^{m_2}(q,T)$$

where $<<$ means that the coefficient of T^j in the first member is less than the same in the second member. As a consequence, we have

Lemma 3.4 *Let $a_f = \sum_{j=0}^{\infty} \epsilon^j a_j(x,\xi)$ and $b_f = \sum_{j=0}^{\infty} \epsilon^j b_j(x,\xi)$ two formal analytic symbol on $\Gamma_a \times \Gamma_b$, Then $a_f \# b_f$ is a formal analytic symbol on $\Gamma_a \times \Gamma_b$ in the class $S_A^{m_1+m_2}(\langle x\rangle^{-\delta_1-\delta_2})$*

Moreover, we have

Lemma 3.5 *Let $r_f = \sum_{j=1}^{\infty} \epsilon^j r_j$ is a formal analytic symbol in the class $S_A^0(\langle x\rangle^{-\delta_1})$ on $\Gamma_a \times \Gamma_b$. Then $\sum_{j=1}^{\infty} r_f^{\# j}$ is a formal analytic symbol in the class $S_A^0(\langle x\rangle^{-\delta_1})$ on $\Gamma_a \times \Gamma_b$.*

Proof*: It is sufficient to observe that*

$$N_{\delta}^0(\sum_{j=1}^{\infty} r^{\# j}, T) << N_{\delta}^0(r,T) \sum_{j=0}^{\infty} N_0^0(r,T)^j = N_{\delta}^0(r,T)(1 - N_0^0(r,T))^{-1}$$

(The last series is a convergent for sufficiently small $|T|$ since $N_0^0(r,T)$ is, for sufficiently small T, a convergent series and has no constant term). ∎

We state now a well known result on composition of pseudodifferential operators. Given a symbol $a \in S_A^m(\langle x\rangle^{-\delta})$ we denote by $\mathrm{Op}_\epsilon^w(q(x,\xi;\epsilon))$ we denote the ϵ-pseudodifferential operator which, for $\phi \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^{\oplus k})$ is defined by the oscillatory integral

$$\mathrm{Op}_\epsilon^w(a(x,\xi;\epsilon))\phi = (2\pi\epsilon)^{-n}\int e^{i(x-y)\xi/\epsilon} a(\frac{x+y}{2},\xi;\epsilon)\phi(y)\,dy\,d\xi. \tag{3.8}$$

We have the following

Lemma 3.6 *Let $a(x,\xi,\epsilon) \in S^{m_1}(\langle x\rangle^{-\delta_1})$, $b(x,\xi,\epsilon) \in S^{m_2}(\langle x\rangle^{-\delta_2})$,$m_i,\delta_i$, $i = 1,2$. Then $a\tilde{\#}b$ given by the oscillatory integral (2.2) belong to $S^{m_1+m_2}(\langle x\rangle^{-(\delta_1+\delta_2)})$.*

Moreover, if $a(x,\xi,\epsilon) \sim \sum_{j=0}^\infty a_j(x,\xi)\epsilon^j \in S_A^{m_1}(\langle x\rangle^{-\delta_1})$, $b(x,\xi,\epsilon) \sim \sum_{j=0}^\infty b_j(x,\xi)\epsilon^j \in S_A^{m_2}(\langle x\rangle^{-\delta_2})$ then $a\tilde{\#}b \in S_A^{m_1+m_2}(\langle x\rangle^{-(\delta_1+\delta_2)})$ and

$$a\tilde{\#}b \sim \left(\sum_{j=0}^\infty a_j(x,\xi)\epsilon^j\right) \# \left(\sum_{j=0}^\infty b_j(x,\xi)\epsilon^j\right) \tag{3.9}$$

where # is the Weyl composition of formal symbol defined in (3.6) and (3.7).

Outline of proof: *Using the operators*

$$L_{v,\eta} \;=\; (1+|\xi-\eta|^2+|v-x|^2)^{-1}\left(1+\epsilon(\xi-\eta)D_v-\epsilon(v-x)D_\eta\right)$$

$$L_{u,\theta} = (1+|\xi-\theta|^2+|u-x|^2)^{-1}\left(1+\epsilon(\xi-\theta)D_u-\epsilon(u-x)D_\theta\right)$$

which satisfy:

$$L_{v,\eta}(e^{i(\xi-\eta)(v-x)/\epsilon}) \;=\; e^{i(\xi-\eta)(v-x)/\epsilon}$$

$$L_{u,\theta}(e^{i(\xi-\theta)(u-x)/\epsilon}) = e^{i(\xi-\theta)(u-x)/\epsilon}$$

one can rewrite the oscillatory integral in (2.2) as a convergent integral i.e. one can rewrite the r.h.s. of (2.2) as

$$\frac{1}{(2\pi\epsilon)^{2n}}\int e^{i\varphi(x,y,v,\xi,\eta,\theta)/\epsilon}({}^tL_{v,\eta})^{q_1}({}^tL_{u,\theta})^{q_2}a_1(\frac{x+u}{2},\eta;\epsilon)a_2(\frac{x+v}{2},\theta;\epsilon)du\,dv\,d\eta\,d\theta \tag{3.10}$$

Now the results follow from standard application of the stationary phase theorem. For the proof in the analytic case see [Sj1] or [Ma] ∎

The last result we collect is the Calderon-Vaillancourt theorem:

Theorem 3.7 *Let $a(x,\xi;\epsilon)\in S^0(\langle x\rangle^{-\delta})$. Then, there exists $M>0$, $p\in\mathbf{N}$ depending only on n,m,δ such that*

$$\|\mathrm{Op}_\epsilon^w(a)\|_\delta \leq M\|a\|_{p,\delta} \tag{3.11}$$

For the proof of the theorem see for example [Ro].

References

[Be] M. Benchaou, *Estimations de diffusion pour un opérateur de Klein-Gordon dépendant du temps, Bull. Soc. Math. Fr. 126, No.2, p.273-294 (1998).*

[BeMa] M. Benchaou, A.Martinez, *Estimations exponentielles en théorie de la diffusion pour des opérateurs de Schrödinger matriciels, Ann. Inst. H. Poincar'71 (6), p.561-594 (1999).*

[BoKr] L. Boutet de Monvel, P. Krée,*Pseudo-differential operators and Gevrey classes, Ann. Inst. Fourier,17 (1) p.295-323 (1967).*

[BrNo] R. Brummelhuis,J. Nourrigat, *Scattering amplitude for Dirac operators, Commun. Partial Differ. Equations 24, No.1-2, p.377-394 (1999)*

[DiSj] M.Dimassi, J.Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit, London Math. Soc. Lecture Notes Series 268 (1999).*

[EmWe] C. EMMERICH, A. WEINSTEIN, *Geometry of transport equation in multicomponent WKB approximations, Commun. Math. Phys. 176, p.701-711 (1996).*

[HJ1] G. HAGEDORN, A. JOYE, *Semiclassical dynamics with exponentially small error estimates, Commun. Math. Phys. 207, p.439-465 (2000).*

[HJ2] G. HAGEDORN, A. JOYE, *A time-dependent Born-Oppenheimer approximation with exponentially small error estimates, Prépublications de lÍnstitut Fourier 507 (2000).*

[GeMa] C. GÉRARD, A. MARTINEZ, *Principe d'absorption limite pour des opérateurs de Schrd̈inger à longue portée , C.R.Acad.Sci.Paris, t.906, Série I, p.121-123 (1988).*

[HeSj] B.HELFFER,J.SJÖSTRAND , *Analyse semiclassique pour l'equation de Harper II, Mem. Soc. Math. Fr., Nouv. Ser. 40, 139 p. (1990).*

[HST] F. HOVERMANN, H. SPOHN, S. TEUFEL, *Semiclassical limit for the Schrodinger equation with a short scale periodic potential arXiv:math-math.ph/0001042.*

[JeNa] A. JENSEN, S. NAKAMURA, *Mapping Properties for Wave and Scattering Operators for Two-Body Schrödinger Operators, Lett. Math. Phys. 24, 295-305 (1992)*

[JP] A. JOYE, CH-E. PFISTER, *Superadiabatic evolution and adiabatic transition probability between two non-degenerate levels isolated in the spectrum, J. Math. Phys. 34, p.454-479 (1993).*

[Ka] T.KATO, *Perturbation theory for linear operators, 2nd ed. Classics in Mathematics. Berlin: Springer-Verlag (1980)*

[Is] H. ISOZAKI, *Decay Rates of Scattering States for Schrödinger Operators, J. Math. Kyoto Univ. 26, p. 595-603 (1986)*

[Ma] A. MARTINEZ, *An Introduction to Semiclassical and Microlocal Analysis, To be published,,*

[MaNe] A. MARTINEZ,G.NENCIU, *On adiabatic reduction theory, Oper. Theory, Adv. Appl. 78, p.243-252 (1995).*

[MarNe] PH-A. MARTIN, G. NENCIU, *Semi-classical Inelastic S-Matrix for One-dimensional N-States Systems , Rev. Math. Phys. 7, p 193-242 (1995)*

[MNS] A. MARTINEZ, S. NAKAMURA, V.SORDONI, *Phase Space Tunneling and Multistate Scattering, in preparation*

[Na] S. NAKAMURA, *Tunneling effects in momentum space and scattering. M. Ikawa (ed.), Spectral and scattering theory. Proceedings of the Taniguchi international workshop, Lect. Notes Pure Appl. Math. M. Dekker, 161, 131-151 (1994)*

[Ne1] G.NENCIU, *Linear Adiabatic Theory. Exponential Estimates, Commun. Math. Phys. 152, No.3, p.479-496 (1993).*

[Ne2] G. NENCIU, *On asymptotic perturbation theory for quantum mechanics: almost invariant subspaces and gauge invariant magnetic perturbation theory, Preprint ESI-848-2000; submitted to Rev. Math. Phys.*

[Ne3] G. NENCIU, *Dynamics of band electrons in electric and magnetic fields: rigorous justification of the efective hamiltonians, Rev. Mod. Phys. 63, p.91-128 (1991)*

[ReSi1] M. REED, B. SIMON, *Methods of Modern Mathematical Physics III. Scattering Theory, Academic Press (1979)*

[ReSi2] M. REED, B. SIMON, *Methods of Modern Mathematical Physics IV. Analysis of Operators,*

Academic Press (1978)

[Ro] D. ROBERT, *Autour de l'approximation semi-classique, Birkhäuser (1987)*

[Si] B. SIMON, *Phase space analysis of simple scattering systems: extensions of some work by Enss, Duke Math. J. 46, p.119-168 (1979).*

[Sj1] J. SJÖSTRAND, *Singularités analytiques microlocales, Astérisque 95 (1982)*

[Sj2] J. SJÖSTRAND, *Projecteurs adiabatique du point de vue pseudodifferéntiel, C.R.Acad.Sci. Paris, t. 317, Série I, p. 217-220 (1993).*

[So] V. SORDONI, *In preparation.*

[St] P. STEFANOV , *Quasimodes and resonances: sharp lower bounds, Duke Math. J. 99, p.75-92 (1999)*

[Ta] M.E.TAYLOR, *Pseudodifferential operators Princeton Mathematical Series, 34. Princeton, New Jersey: Princeton University Press. XI (1981).*

[TS] S. TEUFEL, H. SPOHN, *Semi-classical motion of dressed elelctrons, arXiv:math-ph/0010009*

[Wa] X.P. WANG, *Time-Decay of Scattering Solutions and Resolvent Estimates for Semiclassical Schrödinger Operators, J. Diff. Eq. 71, p. 348-395 (1988).*