Gross-Pitaevskii Theory of the Rotating Bose Gas

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Abstract

We study the Gross-Pitaevskii functional for a rotating two-dimensional Bose gas in a trap. We prove that there is a breaking of the rotational symmetry in the ground state; more precisely, for any value of the angular velocity and for large enough values of the interaction strength, the ground state of the functional is not an eigenfunction of the angular momentum. This has interesting consequences on the Bose gas with spin; in particular, the ground state energy depends non-trivially on the number of spin components, and the different components do not have the same wave function. For the special case of a harmonic trap potential, we give explicit upper and lower bounds on the critical coupling constant for symmetry breaking.

1 Introduction

We consider the Gross-Pitaevskii (GP) theory of a rotating two-dimensional Bose gas in a trap. The Bose gas is described by a single function ϕ on $\mathbb{R}^2$, the wave function of the condensate. It is confined in some trap potential

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V, and rotates around the origin at an angular velocity Ω. The strength of the interaction between the particles is measured by the positive parameter a appearing in the GP functional (1.3) below. It is related to the particle number N, the scattering length a_s of the interaction potential and the average particle density ρ via

$$a = \frac{4\pi N}{|\ln a_s^2 \rho|}. \tag{1.1}$$

Minimization of the GP functional is supposed to describe the physical properties of rotating Bose gases at very low temperatures, as considered in recent experiments. These show various interesting properties, in particular, the appearance of multiple vortices and a resulting breaking of the rotational symmetry. There have been a lot of theoretical investigations on these phenomena, based on the GP approach, either using numerical methods or various simplifying approximations, but a proof that the Gross-Pitaevskii functional indeed captures all these features is still missing.

We shall not be concerned here with the derivation of the GP functional from the basic quantum mechanical N-particle Hamiltonian. For non-rotating systems, i.e., $\Omega = 0$, this has been achieved in [1, 2]. However, the methods used there allow no simple generalization of these results to the rotating case.

We shall now describe the setting more precisely. We denote by (r, φ) radial coordinates for $\mathbf{x} = (x, y) \in \mathbb{R}^2$. In these coordinates, the angular momentum is given by $L = -i\partial/\partial\varphi$. For

$$H_0 = -\Delta - \Omega L + V(r) \tag{1.2}$$

and $\phi \in \mathcal{Q}(H_0) \cap L^4(\mathbb{R}^2, d^2\mathbf{x})$ define the *Gross-Pitaevskii energy functional* by

$$\mathcal{E}^{\rm GP}[\phi] = \langle \phi | H_0 \phi \rangle + a \int |\phi(\mathbf{x})|^4 d^2\mathbf{x}. \tag{1.3}$$

(Here $\mathcal{Q}(H_0)$ denotes the quadratic form domain of H_0, and $\langle \cdot | \cdot \rangle$ denotes the standard inner product on $L^2(\mathbb{R}^2)$). The parameter a is non-negative, and without loss of generality also $\Omega \geq 0$. We assume that $V \in L^\infty_{\rm loc}(\mathbb{R}^2)$ is a positive radial function with the property that there exists an $\Omega_c > 0$, such that

$$V(r) \geq \tilde{\Omega}^2 r^2/4 - C_{\tilde{\Omega}} \tag{1.4}$$

with $C_{\tilde{\Omega}} < \infty$ for all $0 \leq \tilde{\Omega} < \Omega_c$. We shall denote by Ω_c the largest possible value of this constant, allowing it to be infinity. Moreover, we assume that $V(r)$ is polynomially bounded at infinity, i.e, there exist constants C_1, C_2 and $2 \leq s < \infty$ such that $V(r) \leq C_1 + C_2 r^s$. For convenience, let $\inf_r V(r) = 0$.

Let $E^{\rm GP}(a,\Omega)$ be the ground state energy of $\mathcal{E}^{\rm GP}$, i.e,

$$E^{\rm GP}(a,\Omega) = \inf\left\{\mathcal{E}^{\rm GP}[\phi], \phi \in \mathcal{Q}(H_0) \cap L^4(\mathbb{R}^2), \|\phi\|_2 = 1\right\}, \tag{1.5}$$

being finite for $|\Omega| < \Omega_c$ and $a \geq 0$. Using standard methods (see e.g. [1]) one can show that there exists a minimizer $\phi^{\rm GP}$ for $\mathcal{E}^{\rm GP}$ as long as $|\Omega| < \Omega_c$, i.e., the infimum is actually a minimum. For $|\Omega| > \Omega_c$ the functional $\mathcal{E}^{\rm GP}$ is not bounded from below.

The purpose of this paper is a detailed study of the GP functional (1.3). One of our main results is that for any $\Omega > 0$ and for large enough interaction strength a, any minimizer of $\mathcal{E}^{\rm GP}$ is *not* an eigenfunction of the angular momentum L, although $\mathcal{E}^{\rm GP}$ is invariant under rotation of ϕ; i.e., the rotational symmetry is broken in the ground state. This has interesting consequences on the multi-component Bose gas (or equivalently, Bose gas with spin). In particular, we will show that the ground state energy (of the natural generalization of the GP functional to multi-component systems) depends non-trivially on the number of spin components, and the different components necessarily have a different wave function in the symmetry breaking regime.

The paper is organized as follows: In Section 2 we study stationary points of the GP functional, in particular minimizers of $\mathcal{E}^{\rm GP}$ restricted to the subspace of eigenfunctions of the angular momentum with fixed eigenvalue n, so-called vortex states. We show that for large enough angular momentum, these vortex states can never be the absolute minimizer of the GP functional, uniformly in the coupling constant a, which will be crucial in the proof of symmetry breaking. In Section 3 we study the critical values of the angular velocity Ω for that an $n+1$-vortex becomes energetically favorable to an n-vortex. These critical velocities all tend to zero as a goes to infinity, which will allow us to conclude that all vortex states with angular momentum smaller than a certain value cannot be the actual minimizers of $\mathcal{E}^{\rm GP}$. In Section 4 we will use these results to prove symmetry breaking. Section 5 is devoted to the study of a GP density matrix functional, which will be useful in investigations on the multi-component Bose gas in Section 6. There we show that in the symmetry breaking regime, the GP energy depends non-trivially on the number of spin components. In Section 7 we finally consider the special case of an harmonic potential $V(r) = r^2$, where we derive explicit upper and lower bounds on the critical coupling constant for symmetry breaking.

2 Vortex states

Given any stationary state of $\mathcal{E}^{\rm GP}$, i.e., a function ϕ with $\|\phi\|_2 = 1$ satisfying

$$\left(H_0 + 2a|\phi|^2 - \mu\right)\phi = 0 \tag{2.1}$$

for some $\mu \in \mathbb{R}$, we define the (real) quadratic form $Q(w)$ by the perturbation

$$\mathcal{E}_\mu^{\rm GP}[\phi + \varepsilon w] - \mathcal{E}_\mu^{\rm GP}[\phi] = \varepsilon^2 Q(w) + O(\varepsilon^3) \tag{2.2}$$

as $\varepsilon \to 0$, where $\mathcal{E}_\mu^{\rm GP}[\phi] = \mathcal{E}^{\rm GP}[\phi] - \mu \int |\phi|^2$. A simple calculation shows that

$$Q(w) = \langle w | H_0 + 4a|\phi|^2 - \mu | w\rangle + 2a\,\Re \int \overline{\phi^2} w^2. \tag{2.3}$$

Multiplying (2.1) with ϕ and integrating shows that

$$\mu = \mathcal{E}^{\rm GP}[\phi] + a \int |\phi|^4. \tag{2.4}$$

DEFINITION 1 (Stability). We say that a stationary state ϕ is stable if and only if $Q(w) \geq 0$ for all $w \in \mathcal{Q}(H_0) \cap L^4(\mathbb{R}^2)$.

We have $Q(i\phi) = 0$, which corresponds to a simple phase change in (2.2). Moreover, $Q(\partial\phi/\partial\varphi) = 0$ because of rotational invariance of $\mathcal{E}^{\rm GP}$. Note that, by definition, an absolute minimizer of $\mathcal{E}^{\rm GP}$ is necessarily stable.

We now look for special solutions to (2.1) of the form

$$\phi(\mathbf{x}) = f(r)e^{in\varphi} \tag{2.5}$$

for some $n \in \mathbb{N}$, a so-called *n-vortex*. Here f is a real radial function. Since $\mathcal{E}^{\rm GP}[fe^{-in\varphi}] = \mathcal{E}^{\rm GP}[fe^{in\phi}] + 2n\Omega$ we can restrict ourselves to non-negative n without loss of generality. At least one solution of the form (2.5) for each n always exists, as one easily sees by minimizing the functional $\mathcal{E}^{\rm GP}$ in the subspace of functions with $L\phi = n\phi$. For ϕ of the form (2.5) there is the following direct sum decomposition of Q: Writing $w(\mathbf{x}) = \sum_{m\geq 0} w_m(\mathbf{x})$ with

$$w_m(\mathbf{x}) = A_m(r)e^{i(n-m)\varphi} + B_m(r)e^{i(n+m)\varphi} \tag{2.6}$$

one easily sees that $Q(w) = \sum_{m\geq 0} Q(w_m)$.

Considering the stability of vortices, we restrict ourselves to the case $\Omega < \Omega_c$, since for $\Omega > \Omega_c$ all states are certainly unstable. (The case $\Omega = \Omega_c$ depends on the particular form of the potential V.) First of all, ϕ has only a chance of being stable if f has no zeros away from $r = 0$. More precisely, the following proposition holds.

PROPOSITION 1 (Instability for f's with zeros). *Assume that either $n = 0$ and f has some zero, or $n \geq 1$ and f has some zero away from $r = 0$. Then ϕ is unstable.*

Proof. For some real h we choose $w(\mathbf{x}) = ih(r)e^{in\varphi}$ as a trial function for Q. We get

$$Q(w) = \langle h| -\Delta + \frac{n^2}{r^2} - n\Omega + V(r) + 2af^2 - \mu |h\rangle \equiv \langle h|\widetilde{H}|h\rangle. \tag{2.7}$$

We know that $\widetilde{H}f = 0$, but because of its zeros, f cannot be the ground state of $\widetilde{H}$, so there exists a h with $\langle h|\widetilde{H}|h\rangle < 0$. □

Now let ϕ be the minimizer of $\mathcal{E}^{\rm GP}$ in the subspace with angular momentum n. Then f defined in (2.5) minimizes the energy functional

$$\mathcal{E}_n[f] = \langle f| -\Delta + \frac{n^2}{r^2} + V(r)|f\rangle + 2\pi a\int_0^\infty |f(r)|^4 r dr \tag{2.8}$$

under the condition $2\pi\int |f(r)|^2 r dr = 1$, with corresponding energy

$$E_n(a) = \mathcal{E}_n[f] = \mathcal{E}^{\rm GP}[fe^{in\varphi}] + \Omega n. \tag{2.9}$$

In the following, we will study $\mathcal{E}_n$ for all $n \geq 0$, not only for integers. Denote

$$\tilde\mu \equiv \mu + n\Omega, \tag{2.10}$$

which is independent of Ω. The minimizer f of $\mathcal{E}_n$ has the following properties.

LEMMA 1 (Properties of f). *$f(r) > 0$ for $r > 0$, $f \in C^\infty(\mathbb{R}_+)$ if $V \in C^\infty$, and $f(r) = O(r^n)$ as $r \to 0$. Moreover, $f \in L^\infty(\mathbb{R}_+)$, and*

$$\|f\|_\infty^2 \leq \frac{\tilde\mu}{2a}. \tag{2.11}$$

Proof. The regularity and strict positivity follow in a standard way from the variational equation for f. Writing $f(r) = r^n g(r)$ we see that g minimizes the functional

$$\tilde{\mathcal{E}}[g] = \int_0^\infty r^{2n+1} dr g(r) \left(-g''(r) - \frac{2n+1}{r} g'(r) + V(r)g(r) + ar^{2n} g(r)^3 \right) \tag{2.12}$$

under the condition $2\pi \int g(r)^2 r^{2n+1} dr = 1$, from which we conclude that g is a bounded, strictly positive function.

The bound (2.11) is proved analogously to Lemma 2.1 in [2]: Let $\mathcal{B} = \{\mathbf{x}, 2af(|\mathbf{x}|)^2 > \tilde{\mu}\}$. We see that $-\Delta f < 0$ on $\mathcal{B}$, i.e., f is subharmonic on $\mathcal{B}$ and therefore achieves its maximum on the boundary of $\mathcal{B}$. Hence $\mathcal{B}$ is empty. □

We remark that all the properties of f stated in Lemma 1, except for the positivity, hold for all n-vortices and not only for minimizers. Also the following lemma holds true for arbitrary vortex states.

LEMMA 2 (Properties of g). *Let $f(r) = r^n g(r)$ be a stationary point of $\mathcal{E}_n$, for $n \in \mathbb{R}_+$. Then*

$$\|g\|_\infty \le \|f\|_\infty \left(c_n^2 \tilde{\mu} \right)^{n/2}, \tag{2.13}$$

where

$$c_n = \left(\frac{2^{-n} \left(\frac{2-n}{n}\right)^{n/2} \pi \operatorname{Csc}\left(\frac{n\pi}{2}\right)}{(2-n)\Gamma(n)} \right)^{1/n} \quad \text{for } n \le 1$$

$$c_n = \frac{\sqrt{\pi}}{n} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)} \quad \text{for } n \ge 1. \tag{2.14}$$

If f is the minimizer of $\mathcal{E}_n$, and if V is monotone increasing, then g is a monotone decreasing function.

Proof. By a rearrangement argument one sees from (2.12) that the minimizer of $\tilde{\mathcal{E}}$ is monotone decreasing if V is monotone increasing. For a general n-vortex, g fulfills the equation

$$-g''(r) - \frac{2n+1}{r} g'(r) + V(r)g(r) + 2ar^{2n} g(r)^3 = \tilde{\mu} g(r). \tag{2.15}$$

Kato's inequality and the positivity of V imply that

$$-|g(r)|'' - \frac{2n+1}{r}|g(r)|' \leq \tilde{\mu}|g(r)| \tag{2.16}$$

in the sense of distributions. Now let $\chi_n(r,s)$ be the kernel of the operator

$$\left(-\frac{d^2}{dr^2} - \frac{2n+1}{r}\frac{d}{dr} + 1\right)^{-1}, \tag{2.17}$$

acting on $L^2(\mathbb{R}_+, r^{2n+1}dr)$. It is given by

$$\chi_n(r,s) = \frac{1}{(rs)^n}\begin{cases} I_n(r)K_n(s) & \text{for } r \leq s \\ K_n(r)I_n(s) & \text{for } r \geq s, \end{cases} \tag{2.18}$$

where I_n and K_n denote the usual modified Bessel functions. Note that both I_n and K_n are positive, so χ_n is positive. By scaling, the integral kernel of (2.17) with $+1$ replaced by $+t^2$ is $t^{2n}\chi_n(rt, st)$. Therefore (2.16) implies that, for $t > 0$ and $0 \leq \alpha \leq 1$,

$$\begin{aligned} |g(r)| &\leq (\tilde{\mu} + t^2)t^{2n} \int_0^\infty \chi_n(rt, st)|g(s)|s^{2n+1}ds \\ &\leq (\tilde{\mu} + t^2)t^{2n}\|g\|_\infty^{1-\alpha}\|f\|_\infty^\alpha \int_0^\infty \chi_n(rt, st)s^{2n+1-n\alpha}ds. \end{aligned} \tag{2.19}$$

We now claim that $\int \chi_n(r,s)h(s)s^{2n+1}ds$ is monotone decreasing in r if h is a positive, monotone decreasing function. To prove this, it suffices to consider a step function $h(s) = \Theta(R-s)$, $R > 0$. A simple calculation yields

$$\int_0^R \chi_n(r,s)s^{2n+1}ds = \begin{cases} 1 - R^{n+1}K_{n+1}(R)\frac{I_n(r)}{r^n} & \text{for } r \leq R \\ R^{n+1}I_{n+1}(R)\frac{K_n(r)}{r^n} & \text{for } r \geq R, \end{cases} \tag{2.20}$$

which proves the claim, since $I_n(r)/r^n$ and $K_n(r)/r^n$ are monotone increasing and decreasing, respectively. Therefore the maximum on the right hand side of (2.19) is achieved for $r = 0$, and we get

$$\|g\|_\infty \leq \frac{\tilde{\mu} + t^2}{t^{2-n\alpha}}\|g\|_\infty^{1-\alpha}\|f\|_\infty^\alpha \frac{2^{-n}}{\Gamma(n+1)}\int_0^\infty K_n(s)s^{n+1-n\alpha}ds. \tag{2.21}$$

The last integral can be evaluated explicitly, if $n\alpha < 2$. Choosing $\alpha = 1$ for $n \leq 1$ and $\alpha = 1/n$ for $n \geq 1$ and optimizing over t yields the desired result. □

Equation (2.13) effectively gives a lower bound on s, the size of the vortex core, defined by $|f(r)| \sim \|f\|_\infty (r/s)^n$ as $r \to 0$, i.e.,

$$s = \left(\lim_{r\to 0} \frac{|f(r)|}{r^n \|f\|_\infty} \right)^{-1/n} \geq \left(\frac{\|f\|_\infty}{\|g\|_\infty} \right)^{1/n} \geq \frac{1}{\tilde\mu^{1/2} c_n}. \tag{2.22}$$

Note that $c_n^n \to 1$ as $n \to 0$, and $c_n = O(n^{-1/2})$ for large n. The latter fact will be important in the proof of the following theorem.

THEOREM 1 (Instability for large n). *For all $0 \leq \Omega < \Omega_c$ there exists an $N_\Omega < \infty$ (independent of a!) such that all vortices with $n \geq N_\Omega$ are unstable.*

Proof. Let $n \geq 1$, and let $w_1 \in H^1(\mathbb{R}^2)$ be radial and normalized, with support in the ball of radius 1. Let $X = \int |w_1(r)|^2 V(r/c_n\tilde\mu^{1/2}) d^2\mathbf{x}$, $T = \langle w_1 | -\Delta w_1 \rangle$, and define w by

$$w(\mathbf{x}) = c_n \tilde\mu^{1/2} w_1(c_n \tilde\mu^{1/2} r), \tag{2.23}$$

with c_n given in (2.14). We have, using (2.11) and (2.13),

$$Q(w) \leq n\Omega + \tilde\mu \left(c_n^2 T + \frac{1}{\tilde\mu} X - 1 + 2\int |w(\mathbf{x})|^2 \min\{1, (r^2 \tilde\mu c_n^2)^n\} d^2\mathbf{x} \right). \tag{2.24}$$

With $M = \int |w_1|^2 r^{2n}$ this gives

$$Q(w) \leq n\Omega + \tilde\mu \left(c_n^2 T - 1 + 2M \right) + X. \tag{2.25}$$

Now $\tilde\mu$ is larger than $e_n^{(0)} \equiv \inf \text{spec } (-\Delta + V) \restriction_{L=n}$, which can be bounded below as follows. By assumption (1.4), $V(r) \geq -C_{\tilde\Omega} + \tilde\Omega^2 r^2/4$ for some constant $C_{\tilde\Omega}$ and $\Omega < \tilde\Omega < \Omega_c$. Denoting by $\psi_n^{(0)}$ the eigenfunction corresponding to $e_n^{(0)}$, we have

$$e_n^{(0)} \geq \langle \psi_n^{(0)} | \frac{n^2}{r^2} + V(r) | \psi_n^{(0)} \rangle \geq \langle \psi_n^{(0)} | -C_{\tilde\Omega} + \frac{n^2}{r^2} + \frac{\tilde\Omega^2}{4} r^2 | \psi_n^{(0)} \rangle \geq -C_{\tilde\Omega} + n\tilde\Omega. \tag{2.26}$$

Now $c_n^2 = O(n^{-1})$ as $n \to \infty$, and the same holds for M. X can be bounded by $\sup_{r \leq (c_n^2\tilde\mu)^{-1/2}} V(r)$, which is, for fixed Ω, bounded independent of n and a by the considerations above. Therefore $Q(w) < 0$ for n large enough. □

For a special class of potentials, we can extend the previous result in the following way:

THEOREM 2 (Instability for special V's). *Assume that for some $d \in \mathbb{N}$, $d \geq 2$,*

$$\left(r\left(\frac{V(r)}{r^{2(d-1)}}\right)'\right)' \leq 0 \tag{2.27}$$

for all r. Assume also that $n \geq d$ and

$$\tilde{\mu} > n\Omega\left(1 + \frac{2}{d-1}\right). \tag{2.28}$$

Then ϕ is unstable.

Proof. For $1 \leq d \leq n$ we choose as a trial function

$$w(\mathbf{x}) = (A(r) + B(r))\, e^{i(n-d)\varphi} + (A(r) - B(r))\, e^{i(n+d)\varphi}. \tag{2.29}$$

Then $Q(w)$ can be written as

$$Q(w) = 2\left\langle \begin{array}{c} A \\ B \end{array} \middle| \mathbf{H} \middle| \begin{array}{c} A \\ B \end{array} \right\rangle, \tag{2.30}$$

where

$$\mathbf{H} = \begin{pmatrix} H_0 + \frac{n^2+d^2}{r^2} - n\Omega - \mu + 6af^2 & d\Omega - \frac{2nd}{r^2} \\ d\Omega - \frac{2nd}{r^2} & H_0 + \frac{n^2+d^2}{r^2} - n\Omega - \mu + 2af^2 \end{pmatrix}. \tag{2.31}$$

We now choose $A(r) = f'(r)/r^{d-1}$ and $B(r) = nf(r)/r^d$. Note that $A - B = r^{n-d+1}(f/r^n)' \leq O(r)$ as $r \to 0$, so $w \in H^1(\mathbb{R}^2)$. A straightforward calculation using Equation (2.1) yields

$$\begin{aligned} Q(w) &= 8\pi \int_0^\infty dr \frac{f'(r)f(r)}{r^{2(d-1)}} \Big(-\mu(d-1) + 2a(d-1)f(r)^2 + n\Omega \\ &\qquad + (d-1)V(r) - \frac{r}{2}V'(r)\Big) \\ &= 8\pi \int_0^\infty dr \frac{f(r)^2}{r^{2(d-1)+1}} \Big(-\mu(d-1)^2 + a(d-1)^2 f(r)^2 + (d-1)n\Omega \\ &\qquad + \frac{1}{4} r^{2(d-1)+1}\left(r\left(\frac{V(r)}{r^{2(d-1)}}\right)'\right)'\Big), \end{aligned} \tag{2.32}$$

where we used partial integration in the last step. Estimating $af(r)^2$ by (2.11) this shows the negativity of $Q(w)$ as long as (2.27) and (2.28) are satisfied. $\square$

In the case of a homogeneous potential $V(r) = r^\nu$, $2 \leq \nu < \infty$, the condition (2.27) is fulfilled for $\nu = 2(d-1)$, $d \in \mathbb{N}$, showing that in this case every vortex with $n \geq d = \frac{1}{2}\nu + 1$ is unstable, if (2.28) is fulfilled, i.e., if $\tilde{\mu}$ is large enough.

REMARK 1 (Translational stability). The calculation in the proof of Theorem 2 shows that any vortex with $n \geq 1$ is stable against translations, if the opposite of the assumption on V is true for $d = 1$. More precisely, the function $w(\mathbf{x})$ defined above is, for $d = 1$, equal to $\partial\phi/\partial x$. Looking at (2.32) we see that this expression is always positive for $d = 1$, if $(rV'(r))' \geq 0$. This implies that $Q(\partial\phi/\partial x) \geq 0$, and the same conclusion holds for $\partial\phi/\partial y$. Note that the condition on V is in particular fulfilled for any homogeneous potential $V(r) = r^\nu$.

The choice of the test function in the proof of Thm. 2 is motivated by analogous considerations in [3] (see also [4] for a treatment of the Ginzburg-Landau model).

3 The critical frequencies

From (2.9) one sees that an $n+1$-vortex becomes energetically favorable to an n-vortex if $\Omega > \Omega_n$, where the critical frequency is given by

$$\Omega_n(a) = E_{n+1}(a) - E_n(a) > 0, \tag{3.1}$$

with $E_n(a)$ defined in (2.9). In the following we will study the properties of the Ω_n's, in particular their behavior for large a. This will be important in the proof of symmetry breaking in the ground state of the GP functional.

LEMMA 3 (Relation between Ω_n's). *For $n \geq 0$*

$$\Omega_{n+1} \leq \frac{2n+3}{2n+1}\Omega_n. \tag{3.2}$$

Proof. Using f_{n+1}, the minimizer for $\mathcal{E}_{n+1}$, as a trial function for $\mathcal{E}_{n+2}$ and $\mathcal{E}_n$, respectively, we get

$$\Omega_{n+1} \leq (2n+3) \int \frac{f_{n+1}(r)^2}{r^2} d^2\mathbf{x} \tag{3.3}$$

and

$$\Omega_n \geq (2n+1) \int \frac{f_{n+1}(r)^2}{r^2} d^2\mathbf{x}. \tag{3.4}$$

□

THEOREM 3 (Decrease of Ω_n with a). *For all $n \in \mathbb{N}_0$*

$$\Omega_n(a) \leq (2n+1)\frac{2\pi e}{a} E_1(a) \left(3 + \left[\ln\left(\frac{a}{2\pi e^2}\right)\right]_+\right), \tag{3.5}$$

$$\Omega_n(a) \geq (2n+1)\frac{1}{4}\frac{\tilde{\Omega}^2}{C_{\tilde{\Omega}} + E_{n+1}(a)} \quad \text{for all } 0 \leq \tilde{\Omega} < \Omega_c. \tag{3.6}$$

Proof. The concavity of $E_n(a)$ in n^2 implies that the right and left derivatives of E_n with respect to n exist, and from the existence of a unique minimizer for $\mathcal{E}_n$ for all n we conclude that E_n is in fact differentiable in n. Therefore we have

$$\Omega_n = \left.\frac{\partial E_n}{\partial n}\right|_{n=n_0} \tag{3.7}$$

for some $n_0 \in (n, n+1)$. Now let $f(r) = r^n g(r)$ be the minimizer of $\mathcal{E}_n$. To obtain the upper bound, we estimate, for $0 < \alpha \leq 1$,

$$\begin{aligned}
\frac{\partial E_n}{\partial n} &= 2n \int \frac{f(r)^2}{r^2} d^2\mathbf{x} \\
&\leq 2n \left(\|f\|_4^2 \left(\int_{|\mathbf{x}|\geq R} r^{-4} d^2\mathbf{x} \right)^{1/2} + \|g\|_\infty^{2\alpha} \|f\|_\infty^{2(1-\alpha)} \int_{|\mathbf{x}|\leq R} r^{2n\alpha-2} d^2\mathbf{x} \right) \\
&\leq 2n \left(\|f\|_\infty \frac{\sqrt{\pi}}{R} + \|g\|_\infty^{2\alpha} \|f\|_\infty^{2(1-\alpha)} \frac{\pi}{n\alpha} R^{2n\alpha} \right)
\end{aligned} \tag{3.8}$$

for all $R > 0$. Optimizing over R yields

$$\frac{\partial E_n}{\partial n} \leq \left(2n + \frac{1}{\alpha}\right) \left(2\pi^{1+n\alpha} \|f\|_\infty^{2n\alpha+2(1-\alpha)} \|g\|_\infty^{2\alpha}\right)^{1/(2n\alpha+1)}, \tag{3.9}$$

and by (2.13),

$$\frac{\partial E_n}{\partial n} \leq \left(2n + \frac{1}{\alpha}\right) \left(2\pi^{1+n\alpha} c_n^{2n\alpha} \|f\|_\infty^{2n\alpha+2} \tilde{\mu}^{\alpha n}\right)^{1/(2n\alpha+1)} . \tag{3.10}$$

Next we choose

$$\alpha = \min \left\{ 1, \frac{1}{n} \left[\ln \left(\frac{c_n^2 \tilde{\mu}}{4\pi e^2 \|f\|_\infty^2} \right) \right]_+^{-1} \right\} \tag{3.11}$$

and use (2.11), which yields

$$\frac{\partial E_n}{\partial n} \leq \frac{\pi e}{a} \tilde{\mu} \max \left\{ 2n+1, n \ln \left(\frac{c_n^2 a}{2\pi} \right) \right\} . \tag{3.12}$$

Using $\tilde{\mu} \leq 2E_n(a)$ and $c_n^{2n} \leq e$ this gives, together with (3.7), for Ω_0

$$\Omega_0(a) \leq \frac{2\pi e}{a} E_1(a) \max \left\{ 3, 1 + \ln \left(\frac{a}{2\pi} \right) \right\} . \tag{3.13}$$

Now $\Omega_n \leq (2n+1)\Omega_0$ by Lemma 3, which finishes the proof of the upper bound.

To obtain the lower bound, we use (3.4) and

$$\int \frac{f_{n+1}^2}{r^2} d^2\mathbf{x} \geq \frac{1}{\int f_{n+1}^2 r^2 d^2\mathbf{x}} \geq \frac{\tilde{\Omega}^2}{4} \frac{1}{C_{\tilde{\Omega}} + E_{n+1}(a)} \tag{3.14}$$

because of (1.4), for all $0 \leq \tilde{\Omega} < \Omega_c$. □

Note that since $V(r) \leq C_1 + C_2 r^s$ for some $2 \leq s < \infty$ by assumption, a simple trial wave function shows that $E_n(a) \leq O(a^{s/(s+2)})$ as $a \to \infty$, implying that Ω_n behaves at most as $a^{-2/(s+2)} \ln a$ for large a. In particular, $\lim_{a\to\infty} \Omega_n(a) = 0$ for all n.

4 Symmetry breaking in the ground state

We now have the necessary tools to prove symmetry breaking. With the results of Theorems 1 and 3 the following is easily shown.

THEOREM 4 (Symmetry breaking). *For all $0 < \Omega < \Omega_c$ there is an a_Ω such that $a \geq a_\Omega$ implies that any ground state of the functional (1.3) is* not *an eigenfunction of the angular momentum.*

Proof. Fix $0 < \Omega < \Omega_c$. From Thm. 1 we see that there exists an N_Ω independent of a such that all vortex states with $n \geq N_\Omega$ are unstable, and therefore cannot be minimizers of $\mathcal{E}^{\rm GP}$. By the definition of the critical frequencies (3.1),

$$\min_{0\leq n<N_\Omega}\{E_n(a)-n\Omega\} > \min_{n\geq N_\Omega}\{E_n(a)-n\Omega\} \tag{4.1}$$

if

$$\Omega > \max_{0\leq j<N_\Omega}\frac{1}{N_\Omega-j}\sum_{i=j}^{N_\Omega-1}\Omega_i(a). \tag{4.2}$$

By Thm. 3 (and the remark after the proof) this can always be fulfilled for a large enough, so the ground state of $\mathcal{E}^{\rm GP}$ cannot be a vortex state. □

This shows that any minimizer of the GP functional is *not* an eigenfunction of the angular momentum. We can even show more, namely that the absolute value of a minimizer is *not* a radial function. To prove this, we need the following general lemma.

LEMMA 4 (Fourier series of $e^{ih(\varphi)}$). *Let $h:[0,2\pi]\to\mathbb{R}$ be a measurable function. Then the set of Fourier coefficients of e^{ih} contains either only one or infinitely many non-zero elements.*

Proof. Let $e^{ih(\varphi)}=\sum_n h_n e^{in\varphi}$, where the h_n's are the Fourier coefficients of e^{ih}. Let $n_0=\max\{n, h_n\neq 0\}$ and $n_1=\min\{n,h_n\neq 0\}$, assuming that both are finite. Since

$$1=\left|e^{ih(\varphi)}\right|^2=\sum_n k_n e^{in\varphi} \quad \text{with } k_n=\sum_m h_{n+m}\overline{h_m}, \tag{4.3}$$

we know that $k_n=\delta_{n0}$. But $k_{n_0-n_1}=h_{n_0}\overline{h_{n_1}}\neq 0$, so $n_0=n_1$. □

COROLLARY 1 (Symmetry breaking, part 2). *Let $0<\Omega<\Omega_c$ and $a\geq a_\Omega$, and let $\phi^{\rm GP}$ be a minimizer of $\mathcal{E}^{\rm GP}$. Then $|\phi^{\rm GP}|$ is* not *a radial function.*

Proof. Assume $|\phi^{\rm GP}|$ is radial. With $\widetilde{H}=H_0+2a|\phi^{\rm GP}|^2-\mu$ we have $\widetilde{H}\phi^{\rm GP}=0$. Because $\widetilde{H}$ commutes with L, it has eigenfunctions $h_n(r)e^{in\varphi}$ with corresponding eigenvalue 0. Therefore

$$\phi^{\rm GP}(r,\varphi)=\sum_n \lambda_n h_n(r)e^{in\varphi} \tag{4.4}$$

for some $0 \neq \lambda_n \in \mathbb{C}$, and the sum is finite, since the ground state energies of $\widetilde{H}$ restricted to subspaces of $L = n$ go to infinity as $n \to \infty$, and there can only be one eigenfunction for each n. Choosing some interval $I \in \mathbb{R}_+$ where $|\phi^{\mathrm{GP}}|$ does not vanish, we can conclude with Lemma 4 that only one h_n is unequal to zero in I. But since the h_n's do not vanish on some open set, this is true on all of $\mathbb{R}_+$. Therefore ϕ^{GP} has to be an eigenfunction of L, contradicting Thm. 4. □

Note that Thm. 4 implies in particular that the minimizer of $\mathcal{E}^{\mathrm{GP}}$ is *not* unique (up to a constant phase), and Corollary 1 shows that even the absolute value is not unique. By rotating a minimizer ϕ^{GP} one obtains again a minimizer, which is, at least except for exceptional values of the rotation angle, different from the original one.

Numerical investigations [5, 6] indicate that the symmetry breaking results from a splitting of an n-vortex into several vortices with winding number 1. I.e., one expects that ϕ^{GP} has d distinct zeros of degree 1, where $d = \deg\{\phi^{\mathrm{GP}}/|\phi^{\mathrm{GP}}|\}$ for large enough r. This property was proved for large a for the minimizer of models similar to the GP functional [7, 8, 9].

5 A density matrix functional

We now introduce a new functional, which will be convenient in the following. Firstly, to obtain a lower bound on the critical parameter a_Ω, and secondly, for studying a generalization of the GP functional to a Bose gas with several components, which will be done in the next section.

Analogously to the Gross-Pitaevskii functional we define the GP *density matrix* (DM) functional as

$$\mathcal{E}^{\mathrm{DM}}[\gamma] = \mathrm{Tr}[H_0\gamma] + a\int \rho_\gamma(\mathbf{x})^2 d^2\mathbf{x}. \tag{5.1}$$

Here γ is a one-particle density matrix, a positive trace-class operator on $L^2(\mathbb{R}^2)$, and ρ_γ denotes its density. The ground state energy, the infimum of (5.1) under the condition $\mathrm{Tr}[\gamma] = 1$, will be denoted by $E^{\mathrm{DM}}(a, \Omega)$. It is clear that $E^{\mathrm{DM}} \leq E^{\mathrm{GP}}$. Using the methods of [1] and [10] one can prove the following theorem.

THEOREM 5 (Minimizer of $\mathcal{E}^{\mathrm{DM}}$). *For each $0 \leq \Omega < \Omega_c$ and $a \geq 0$ there exists a minimizing density matrix for (5.1) under the condition* $\mathrm{Tr}[\gamma] =$

1. *The density corresponding to the minimizer, denoted by ρ^{DM}, is unique (and therefore a radial function), and each minimizer also minimizes the linearized functional*

$$\mathcal{E}^{\mathrm{DM}}_{\mathrm{lin}}[\gamma] = \mathrm{Tr}[(H_0 + 2a\rho^{\mathrm{DM}})\gamma]. \tag{5.2}$$

The uniqueness of the density results from the strict convexity of $\mathcal{E}^{\mathrm{DM}}$ in ρ_γ. In general, the minimizing density matrix need not be unique. However, for the functional (5.1) we can show that this is indeed the case.

THEOREM 6 (Uniqueness of γ^{DM}). *The minimizer of $\mathcal{E}^{\mathrm{DM}}$, denoted by γ^{DM}, is unique. Moreover, it has finite rank.*

Proof. Being a minimizer of (5.2), γ^{DM} can be decomposed as

$$\gamma^{\mathrm{DM}}(\mathbf{x},\mathbf{x}') = \sum_{j,k\geq 0} \lambda_{j,k} f_j(r) f_k(r') e^{i(j\varphi - k\varphi')}, \tag{5.3}$$

where the $f_k(r)e^{ik\varphi}$ are the ground states of $H_0 + 2a\rho^{\mathrm{DM}}$, and the sum is finite because of the discreteness of the spectrum of this operator, implying finite rank of γ^{DM}. Moreover, there can be only one ground state for each angular momentum, and $f_j(r) = r^j g_j(r)$ with $g_j(r)$ bounded and strictly positive. Therefore

$$\rho^{\mathrm{DM}}(r) = \sum_{j\geq 0} r^j \chi_j(r,\varphi), \tag{5.4}$$

with

$$\chi_j(r,\varphi) = \sum_k \lambda_{j-k,k} g_{j-k}(r) g_k(r) e^{i(j-2k)\varphi}. \tag{5.5}$$

Hence each χ_j has to be independent of φ as $r \to 0$, which implies, together with Lemma 4, that $\lambda_{j,k} = 0$ for $j \neq k$. Moreover, $\lambda_{j,j}$ is determined by the unique density $\rho^{\mathrm{DM}} = \sum_{j\geq 0} r^{2j} \lambda_{j,j} g_j^2$, so γ^{DM} is unique. □

Analogously to minimizers of the GP functional, the DM density has the following properties.

LEMMA 5 (Properties of ρ^{DM}). *For $r > 0$ we have $\rho^{\mathrm{DM}} > 0$ and $\rho^{\mathrm{DM}} \in C^\infty$ if $V \in C^\infty$. Moreover, $\|\rho^{\mathrm{DM}}\|_\infty \leq \mu^{\mathrm{DM}}/(2a)$, where μ^{DM} is the chemical potential of the DM theory, which is the ground state energy of $\mathcal{E}^{\mathrm{DM}}_{\mathrm{lin}}$.*

Proof. Note that $\rho^{\rm DM} = \sum_j \lambda_{j,j} f_j(r)^2$ with the notation of the proof of Theorem 6, where $\lambda_{j,j} \geq 0$. The first two properties follow from this decomposition and a bootstrap argument. Moreover, a direct computation gives

$$-\Delta\rho^{\rm DM} \leq 2\rho^{\rm DM}(\mu^{\rm DM} - 2a\rho^{\rm DM} - V(r)). \tag{5.6}$$

Since $V(r) \geq 0$ this implies that $2a\rho^{\rm DM} \leq \mu^{\rm DM}$ by a subharmonicity argument as in Lemma 1. □

An important consequence of the uniqueness of the minimizer of $\mathcal{E}^{\rm DM}$ is the following corollary.

COROLLARY 2 (Non-equivalence of $\mathcal{E}^{\rm GP}$ and $\mathcal{E}^{\rm DM}$). *Assume that the minimizer of $\mathcal{E}^{\rm GP}$ is not unique, which is in particular the case for $a \geq a_\Omega$. Then $E^{\rm GP}(a,\Omega) > E^{\rm DM}(a,\Omega)$.*

Proof. This follows immediately from the uniqueness of $\gamma^{\rm DM}$. □

Note that in the case of non-uniqueness of $\phi^{\rm GP}$, the rank of $\gamma^{\rm DM}$ is always greater or equal to two, and therefore the ground state of $H_0 + 2a\rho^{\rm DM}$ is degenerate. This holds in particular in the whole region $a \geq a_\Omega$, not only for isolated points or lines in the (a,Ω) plane. For the non-rotating case, i.e. $\Omega = 0$, $E^{\rm GP}$ and $E^{\rm DM}$ are equal for all a. This remains true, if Ω is not too large.

PROPOSITION 2 (Equivalence of $\mathcal{E}^{\rm GP}$ and $\mathcal{E}^{\rm DM}$ for small Ω). *Assume that*

$$\Omega \leq \frac{1}{4}\frac{\tilde\Omega^2}{C_{\tilde\Omega} + \mu^{\rm DM}} \tag{5.7}$$

for some $\Omega < \tilde\Omega < \Omega_c$. Then $E^{\rm GP}(a,\Omega) = E^{\rm DM}(a,\Omega)$, and the minimizer of $\mathcal{E}^{\rm GP}$ has zero angular momentum.

Note that this Proposition implies a lower bound on a_Ω.

Proof. Assume that ψ is a ground state of $H^{\rm DM} \equiv H_0 + 2a\rho^{\rm DM}$ with angular momentum $L\psi = m\psi$. Then

$$\begin{aligned}
\mu^{\rm DM} &= \inf \operatorname{spec} H^{\rm DM} = \langle|\psi|| - \Delta - \Omega m + \frac{m^2}{r^2} + V(r) + 2a\rho^{\rm DM}||\psi|\rangle \\
&\geq \mu^{\rm DM} + \langle\psi|\frac{m^2}{r^2} - \Omega m|\psi\rangle,
\end{aligned} \tag{5.8}$$

implying that

$$m \leq \frac{\Omega}{\langle\psi|r^{-2}|\psi\rangle} \leq \Omega\langle\psi|r^2|\psi\rangle. \tag{5.9}$$

Choosing some $\tilde{\Omega}$ with $\Omega < \tilde{\Omega} < \Omega_c$ we have $r^2 \leq 4(C_{\tilde{\Omega}} + V(r))/\tilde{\Omega}^2$ by (1.4), and hence

$$\begin{aligned} \left(1 - \frac{\Omega^2}{\tilde{\Omega}^2}\right)\langle\psi|r^2|\psi\rangle &\leq \frac{4}{\tilde{\Omega}^2}\left(C_{\tilde{\Omega}} + \langle\psi|V(r) - \frac{\Omega^2}{4}r^2|\psi\rangle\right) \\ &\leq \frac{4}{\tilde{\Omega}^2}\left(C_{\tilde{\Omega}} + \mu^{\mathrm{DM}} - \Omega\right), \end{aligned} \tag{5.10}$$

where we have used that the ground state energy of $-\Delta - \Omega L + \Omega^2 r^2/4$ is Ω. Therefore

$$\Omega\langle\psi|r^2|\psi\rangle \leq \frac{4\Omega\left(C_{\tilde{\Omega}} + \mu^{\mathrm{DM}} - \Omega\right)}{\tilde{\Omega}^2 - \Omega^2} < 1 \tag{5.11}$$

if

$$\Omega \leq \frac{1}{4}\frac{\tilde{\Omega}^2}{C_{\tilde{\Omega}} + \mu^{\mathrm{DM}}}, \tag{5.12}$$

showing that any ground state of H^{DM} necessarily has angular momentum $m = 0$. □

6 The multi-component Bose gas

We now consider the Gross-Pitaevskii theory of a rotating Bose gas with n_c different components, or equivalently, a Bose gas consisting of particles with spin $(n_c - 1)/2$. The natural generalization of the Gross-Pitaevskii functional is

$$\mathcal{E}^{\mathrm{GP}}_{n_c}[\phi_1, \ldots, \phi_{n_c}] = \sum_{i=1}^{n_c}\langle\phi_i|H_0\phi_i\rangle + a\sum_{1\leq i,j\leq n_c}\int |\phi_i|^2|\phi_j|^2, \tag{6.1}$$

which is to minimized under the constraint $\sum_{i=1}^{n_c}\int|\phi_i|^2 = 1$. The corresponding ground state energy will be denoted by $E^{\mathrm{GP}}_{n_c}(a, \Omega)$.

Using standard methods, one can show that for all values of $n_c \in \mathbb{N}$, $0 \leq \Omega < \Omega_c$ and $a \geq 0$ there exist minimizing functions $\phi_1^{\rm GP}, \ldots, \phi_{n_c}^{\rm GP}$ for $\mathcal{E}_{n_c}^{\rm GP}$. The proof goes analogously to the proof of Thm. 5, noting that $\mathcal{E}_{n_c}^{\rm GP}$ can be considered as the restriction of $\mathcal{E}^{\rm DM}$ to density matrices of rank less than or equal to n_c. However, since this set is not convex, we can in general not conclude that the density of a minimizer, $\sum_{i=1}^{n_c} |\phi_i^{\rm GP}|^2$, is unique, as it was the case for the DM functional.

We see that for all values of a, Ω and n_c we always have $E^{\rm DM} \leq E_{n_c}^{\rm GP} \leq E^{\rm GP}$. Denoting

$$n^{\rm DM}(a,\Omega) = \operatorname{rank} \gamma^{\rm DM}, \tag{6.2}$$

which we showed to be finite in Thm. 6, we can distinguish the following cases.

THEOREM 7 (Minimizers of the multi-component GP functional). *Let $\phi_1^{\rm GP}, \ldots, \phi_{n_c}^{\rm GP}$ be minimizers of $\mathcal{E}_{n_c}^{\rm GP}$.*

(i) If $n_c \geq n^{\rm DM}$, then $E^{\rm DM}(a,\Omega) = E_{n_c}^{\rm GP}(a,\Omega)$. Moreover,

$$\sum_{i=1}^{n_c} |\phi_i^{\rm GP}\rangle\langle\phi_i^{\rm GP}| = \gamma^{\rm DM}, \tag{6.3}$$

implying that the $\phi_i^{\rm GP}$'s can be written as $\phi_i^{\rm GP} = \sum_{j=1}^{n^{\rm DM}} A_{ij}\psi_j$, where $\gamma^{\rm DM} = \sum_{i=1}^{n^{\rm DM}} |\psi_i\rangle\langle\psi_i|$, ψ_i orthogonal, and A is an $n_c \times n^{\rm DM}$-matrix with $A^\dagger A = 1$.

(ii) If $n_c < n^{\rm DM}$, then $E^{\rm DM}(a,\Omega) < E_{n_c}^{\rm GP}(a,\Omega)$.

(iii) If $n_c \geq 2$, $a \geq a_\Omega$, then $E_{n_c}^{\rm GP}(a,\Omega) < E^{\rm GP}(a,\Omega)$, and the minimizers $\phi_i^{\rm GP}$ are not *all equal; i.e., $\sum_{i=1}^{n_c} |\phi_i^{\rm GP}\rangle\langle\phi_i^{\rm GP}|$ has at least rank 2.*

Proof. As remarked earlier, $E^{\rm DM} \leq E_{n_c}^{\rm GP}$. To prove (i), we write $\gamma^{\rm DM} = \sum_{i=1}^{n^{\rm DM}} |\psi_i\rangle\langle\psi_i|$, where the ψ_i are orthogonal (but not necessarily normalized). Using $\phi_i = \psi_i$ for $i \leq n^{\rm DM}$ and $\phi_i = 0$ for $n^{\rm DM} < i \leq n_c$ as trial functions, we obtain $E^{\rm DM} \geq E_{n_c}^{\rm GP}$. Since $\gamma^{\rm DM}$ is unique, (6.3) follows. (ii) is a trivial consequence of the uniqueness of $\gamma^{\rm DM}$.

For $a \geq a_\Omega$, we know from Corollary 1 that there are at least two different minimizers, $\phi^{(1)}$ and $\phi^{(2)}$, for $\mathcal{E}^{\rm GP}$, whose absolute values are not the same.

For $n_c \geq 2$ we use as trial functions $\phi_1 = \phi^{(1)}$, $\phi_i = \phi^{(2)}$ for $i \geq 2$ to obtain $E^{\rm GP}_{n_c} < E^{\rm GP}$. Therefore the minimizers for $\mathcal{E}^{\rm GP}_{n_c}$ cannot be all equal, and (iii) is proved. □

An important consequence of part (iii) of the theorem above is that the GP ground state energy depends non-trivially on the number of spin-components, at least in the symmetry breaking regime $a \geq a_\Omega$. Moreover, there is a clear separation between different spin-components, their individual densities $|\phi_i^{\rm GP}|^2$ can never be all equal.

7 The special case $V(r) = r^2$

In the case of a harmonic potential $V(r) = r^2$ the theorems above are more explicit, setting $\Omega_c = 2$ and $C_\Omega = 0$. Moreover, the special case $\Omega = \Omega_c = 2$ is easy to treat: only for $a = 0$ there is a minimizer for $\mathcal{E}^{\rm GP}$ (in fact there are infinitely many), whereas for $a > 0$ there is no minimizer, and also all n-vortices are unstable.

The following bound on the energies $E_n(a)$ can be easily obtained, and will be used several times in the considerations below.

LEMMA 6 (Upper bound on $E_n(a)$).

$$E_n(a) \leq 2(n+1)\sqrt{1 + \frac{a}{b_n(n+1)}} \tag{7.1}$$

with $b_n = 2\pi 4^n (n!)^2/(2n)!$.

Proof. This follows from a trial function of the form $Cr^n \exp(-cr^2)$, where C is a normalization constant and c is to be optimized. □

The estimate above implies that, for $\tilde\mu$ the chemical potential corresponding to the minimizer of $\mathcal{E}_n$,

$$\tilde\mu - 2n \leq 2(E_n - 2n) - 2 \leq 2\left(1 + 2\sqrt{a\frac{(n+1)}{b_n}}\right). \tag{7.2}$$

This will be useful for an upper bound on f, since in the case of $V(r) = r^2$ we can improve the estimate (2.11) by

$$\|f\|_\infty^2 \leq \frac{1}{2a}(\tilde\mu - 2n). \tag{7.3}$$

The proof is analogous to the one of (2.11), using in addition that $n^2/r^2 + r^2 \geq 2n$.

We now consider the stability of n-vortices ϕ, i.e., solutions to (2.1) of the form (2.5). In addition to the results of Section 2 we can state another proposition.

PROPOSITION 3 (Instability for small a). *Assume that $a < \pi n(2 - \Omega)$. Then ϕ is unstable.*

Proof. Let $n \geq 1$, and let $w(\mathbf{x}) = \sqrt{1/\pi}\exp(-r^2/2)$ be the ground state of H_0. Then

$$Q(w) = 2 - \mu + 4a \int w^2 |\phi|^2 \leq n(\Omega - 2) + \frac{a}{\pi}, \tag{7.4}$$

where we used Cauchy-Schwarz and the fact that $\mu \geq 2 - n(\Omega - 2) + 2a \int |\phi|^4$. □

Moreover, we can improve Theorem 1 in the following way.

THEOREM 8 (Instability for large n, harmonic potential). *Assume that $n \geq 10$ and*

$$\tilde{\mu} \geq \frac{n\Omega}{1 - d_n}, \tag{7.5}$$

where d_n is a monotone decreasing function of n, with $d_n < 1$ for $n \geq 10$, namely

$$d_n = \min\left\{ \frac{2}{e^2} + \frac{2\pi\Gamma(n + \frac{1}{2})^2}{n^2\Gamma(n)^2} + 2^{1-n}\left(n! - \Gamma(n+1, 2)\right), \frac{19}{n} \right\}. \tag{7.6}$$

Then ϕ is unstable.

Note that since $\tilde{\mu} > 2n$, (7.5) is in particular fulfilled if $d_n \leq 1 - \Omega/2$.

Proof. Let $n \geq 1$, and let $w_1 \in H^1(\mathbb{R}^2)$ be radial and normalized. Let $T = \langle w_1 | -\Delta w_1 \rangle$ and $X = \langle w_1 | r^2 w_1 \rangle$, and define w by

$$w(\mathbf{x}) = c_n \tilde{\mu}^{1/2} w_1(c_n \tilde{\mu}^{1/2} r), \tag{7.7}$$

with c_n given in (2.14). Using (7.3) and (2.13), we can estimate

$$\begin{aligned} Q(w) &\leq n\Omega + \tilde{\mu}\left(c_n^2 T + \frac{1}{c_n^2\tilde{\mu}^2}X - 1\right) \\ &\quad +2(\tilde{\mu} - 2n)\int |w(\mathbf{x})|^2 \min\{1, (r^2\tilde{\mu}c_n^2)^n\} d^2\mathbf{x}. \end{aligned} \tag{7.8}$$

With $N = \int_{r\geq 1} |w_1|^2$ and $M = \int_{r\leq 1} |w_1|^2 r^{2n}$ this gives, using $\tilde{\mu} \geq 2(n+1)$ in front of X,

$$Q(w) \leq n\Omega + \tilde{\mu}\left(c_n^2 T - 1 + 2\left(N + M\right)\right) + \left(\frac{1}{c_n^2 2(n+1)}X - 4n(N+M)\right). \tag{7.9}$$

Now if we choose $w_1(r) = (2/\pi)^{1/2}\exp(-r^2)$ the last term in (7.9) is negative, and the computation of T, N and M yields

$$Q(w) < n\Omega - \tilde{\mu}(1 - d_n), \tag{7.10}$$

where d_n is the first part in the parenthesis in (7.6). For large n, this can be improved by choosing $w_1(r) = (35/9\pi)^{1/2}[1 - r^{3/2}]_+$, which gives

$$Q(w) < n\Omega - \tilde{\mu}\left(1 - \frac{19}{n}\right). \tag{7.11}$$

□

In [6] the authors used a similar method to the one of Thm. 8 and a particular assumption on the form of the vortex state ϕ to obtain a d_n in (7.5) that is less than 1 if $n \geq 2$.

We conjecture that in the case of an harmonic potential an n-vortex with $n \geq 2$ is unstable, for all values of $\Omega \geq 0$ and $a > 0$. However, we can prove this only for $\Omega \leq 1$. Namely, if we insert $V(r) = r^2$ in (2.32), set $d = 2$, use the improved bound (7.3) and $\mu \geq 2 - n(\Omega - 2) + 2a\int |\phi|^4$ this shows the negativity of $Q(w)$ as long as $n \geq 2$ and

$$\Omega < 1 + \frac{1}{2n}\left(1 + a\int |\phi|^4\right) \tag{7.12}$$

is satisfied. This implies that all vortices with $n \geq 2$ and $a \geq 0$ are unstable as long as $\Omega \leq 1$.

As a consequence of the considerations above we can state an explicit condition on a where an n-vortex is necessarily unstable, using the general lower bound

$$\int |\phi(\mathbf{x})|^4 d^2\mathbf{x} \geq \frac{4}{9\pi} \frac{\left(\int |\phi(\mathbf{x})|^2 d^2\mathbf{x}\right)^3}{\int |\phi(\mathbf{x})|^2 |\mathbf{x}|^2 d^2\mathbf{x}}, \tag{7.13}$$

which can easily be proved using elementary calculus of variations. Moreover, since in two dimensions $\int |\phi|^4$ scales as $1/(\text{length})^2$, the virial theorem implies for the minimizer f of $\mathcal{E}_n$

$$\int f(r)^2 r^2 d^2\mathbf{x} = \frac{1}{2} E_n(a). \tag{7.14}$$

Hence we only need an upper bound on $E_n(a)$, which is given in Lemma 6, to obtain a condition on a for validity of (7.12).

The critical frequencies $\Omega_n(a)$, defined in (3.1), have the following properties.

LEMMA 7 (Properties of critical frequencies). *For all* $n \in \mathbb{N}_0$ *we have* $\Omega_n(0) = 2$ *and for all* $a \geq 0$ $\lim_{n\to\infty} \Omega_n(a) = 2$. *Moreover,*

$$\Omega_n'(0) = -\frac{1}{4^{n+1}\pi} \frac{(2n)!}{n!(n+1)!} < \Omega_{n+1}'(0) < 0. \tag{7.15}$$

Proof. The first assertion follows from $E_n(0) = 2(n+1)$. Using the harmonic oscillator eigenstates $\chi_n(r) = \sqrt{1/\pi n!} r^n \exp(-r^2/2)$ as trial functions the second assertion is proved by

$$\Omega_n \leq 2 + a \int |\chi_{n+1}|^4 \qquad \text{and} \quad \Omega_n \geq 2 - a \int |\chi_n|^4, \tag{7.16}$$

noting that $\int |\chi_n|^4 = O(n^{-1/2})$ as $n \to \infty$. To prove (7.15) we use the Feynman-Hellmann principle to calculate

$$\Omega_n'(a) = \int |f_{n+1}|^4 - \int |f_n|^4, \tag{7.17}$$

where f_n is the minimizer of $\mathcal{E}_n$. For $a = 0$ we have $f_n = \chi_n$, yielding (7.15). □

In the special case of a harmonic potential, the results of Thm. 3 can be improved. We get the following bounds on the critical frequencies.

THEOREM 9 (Decrease of Ω_n, harmonic potential). *For all $n \in \mathbb{N}_0$*

$$\Omega_n(a) \leq (2n+1)\frac{2\pi e}{a}\left(1+\sqrt{\frac{2a}{\pi}}\right)\left(3+\left[\ln\left(\frac{a}{2\pi e^2}\right)\right]_+\right), \quad (7.18)$$

$$\Omega_n(a) \geq \frac{2n+1}{(n+2)\sqrt{1+\frac{a}{b_{n+1}(n+2)}}}, \quad (7.19)$$

with b_n given in Lemma 6.

Proof. We proceed as in Thm. 3, but now use the improved estimate (7.3) to replace (3.12) by

$$\frac{\partial E_n}{\partial n} \leq \frac{\pi e}{a}(\tilde{\mu}-2n)\max\left\{2n+1, n\ln\left(\frac{c_n^2 a}{2\pi}\right)\right\}. \quad (7.20)$$

Inserting (7.2) and using $(n+1)/b_n \leq (2\pi)^{-1}$ for $0 \leq n \leq 1$ and $c_n^{2n} \leq e$ this gives for Ω_0

$$\Omega_0(a) \leq \frac{2\pi e}{a}\left(1+\sqrt{\frac{2a}{\pi}}\right)\max\left\{3, 1+\ln\left(\frac{a}{2\pi}\right)\right\}, \quad (7.21)$$

and using Lemma 3 we obtain (7.18).

For the lower bound we proceed as in (3.14) and note that $\int f_{n+1}^2 r^2 = E_{n+1}(a)/2$ by the virial theorem. (7.19) is obtained by inserting the bound (7.1) for $E_{n+1}(a)$. □

Numerical investigations in [11] indicate that for all $n \geq 0$, Ω_n is strictly monotone decreasing in a, and, for $a > 0$, $\Omega_n < \Omega_{n+1}$, i.e., E_n is convex in n. Note that the theorem above states that Ω_n behaves at most as $a^{-1/2}\ln a$ for large a, in accordance with previous considerations (see [6] and references therein).

We can now use the results above to derive explicit upper and lower bounds on a_Ω. Denote

$$\Xi(a) = \frac{2\pi e}{a}\left(1+\sqrt{\frac{2a}{\pi}}\right)\left(3+\left[\ln\left(\frac{a}{2\pi e^2}\right)\right]_+\right), \quad (7.22)$$

which is a strictly monotone decreasing function of a. We know from Thm. 8 and (7.12) that all n-vortices are unstable for $n \geq N_\Omega$, where

$$N_\Omega = \begin{cases} 2 & \text{for } \Omega \leq 1 \\ \frac{38}{2-\Omega} & \text{for } \Omega > 1. \end{cases} \tag{7.23}$$

Using the bound on the critical frequencies (7.18), we see by analogous considerations as in the proof of Thm. 4 that symmetry breaking occurs if

$$\Xi(a) \leq \frac{\Omega}{2N_\Omega - 1}. \tag{7.24}$$

From Thm. 2 we see that $E^{\rm DM}(a,\Omega) = E^{\rm GP}(a,\Omega)$ if $\Omega \leq 1/\mu^{\rm DM}$. Moreover, it is easy to see that the same holds for $\Omega \leq 2 - a/\pi$. Namely, using Cauchy-Schwarz and the fact that $\int (\rho^{\rm DM})^2$ is monotone decreasing in a (because of concavity of $E^{\rm DM}$ in a) we have, for $H^{\rm DM} = H_0 + 2a\rho^{\rm DM}$,

$$\inf \operatorname{spec} H^{\rm DM} \restriction_{L=0} \leq 2 + 2a \int \rho^{\rm DM} \chi^2 < 2 + 2a \int \chi^4 = 2 + \frac{a}{\pi}, \tag{7.25}$$

where $\chi(\mathbf{x}) = \sqrt{1/\pi} \exp(-r^2/2)$ is the ground state of H_0. Moreover,

$$H^{\rm DM} \restriction_{|L|\geq 1} > 4 - \Omega, \tag{7.26}$$

showing that, for $a \leq \pi(2-\Omega)$, $H^{\rm DM}$ has a unique ground state with zero angular momentum which necessarily also minimizes the GP functional.

Since the minimizer of the GP functional is unique and therefore an angular momentum eigenfunction as long as $E^{\rm DM} = E^{\rm GP}$, we obtain as a consequence a lower bound on a_Ω, using $\mu^{\rm DM} \leq 2E^{\rm DM}(a,\Omega) \leq 2E_0(a)$ and the bound on $E_0(a)$ given in (7.1). Thus we have proved the following Theorem:

THEOREM 10 (Bounds on a_Ω). *In the case of a harmonic potential, the critical parameter for symmetry breaking fulfills the bounds*

$$a_\Omega \leq \Xi^{-1}\left(\frac{\Omega}{2N_\Omega - 1}\right), \tag{7.27}$$

with Ξ defined in (7.22) and N_Ω given in (7.23), and

$$a_\Omega \geq \pi \max\left\{2 - \Omega, \frac{1}{8\Omega^2} - 2\right\}. \tag{7.28}$$

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