A STRONG OPERATOR TOPOLOGY ADIABATIC THEOREM

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ABSTRACT. We prove an adiabatic theorem for the evolution of spectral data under a weak additive perturbation. For continuous functions of the unperturbed Hamiltonian the convergence is in norm while for a larger class functions, including the spectral projections associated to embedded eigenvalues, the convergence is in the strong operator topology.

1. INTRODUCTION

In this note, we discuss an adiabatic theorem for a certain class of quantum evolutions. Adiabatic theorems describe the limit $\tau \to \infty$ of a solution to an initial value problem of the form

$$\begin{cases} \mathrm{i}\dot{U}_\tau(t) &= H(t/\tau)U_\tau(t)\ , \quad t \in [0,\tau] \\ U_\tau(0) &= \mathbf{1} \end{cases} . \tag{1}$$

See, for example, [1, 3] for background and references therein. Here we examine a class of systems in which the time dependence is produced by a weak perturbation.

In the context of quantum mechanics the initial value problem is a Schrödinger equation where $U_\tau(\cdot)$ is a unitary evolution in Hilbert space and $H(\cdot)$ is a self adjoint operator with slow time dependence. One is often interested in the evolution of a spectral subspace associated to the instantaneous Hamiltonian $H(t)$. A prerequisite for an adiabatic theorem is the requirement that the spectral subspace be differentiable in time. In practice it is difficult to verify differentiability – unless the so called gap condition is satisfied. However for a family of Hamiltonians related by a unitary evolution – *i.e.*, $H(s) = V(s)H_o V^\dagger(s)$ – such smoothness is guaranteed by smoothness of V. Equivalently we may consider a family of the form

$$H_\tau(t/\tau) \;=\; H_o \;+\; \frac{1}{\tau}\Lambda(t/\tau) \tag{2}$$

where H_o and $\Lambda(s)$, for $s \in [0,1]$, are self adjoint operators. These two possibilities are connected via the interaction picture when Λ is the generator of $V(s)$:

$$\mathrm{i}\dot{V}(s) \;=\; \Lambda(s)V(s)\ , \quad V(0) \;=\; \mathbf{1}\ . \tag{3}$$

We discuss here the limit $\tau \to \infty$ of a solution, $A_\tau(t) = U_\tau(t)A(0)U_\tau^\dagger(t)$, to the associated Heisenberg equation

$$\mathrm{i}\dot{A}_\tau(t) \;=\; [H_\tau(t/\tau)\,,\,A_\tau(t)] \tag{4}$$

when the initial observable is a function of H_o,*i.e.*, $A(0) = f(H_o)$. Our main result, Theorem 1, states that

$$U_\tau(\tau)f(H_o)U_\tau^\dagger(\tau) \;\longrightarrow\; f(H_o) \tag{5}$$

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for a wide class of functions f.

The topology in which eq. 5 holds depends on the continuity of f relative to the spectral properties of H_o: for continuous functions we obtain norm convergence while for a class of discontinuous functions we obtain strong operator convergence. Let us recall that a family $\tau \mapsto A_\tau$ of operators converges to A in the *strong operator topology* (SOT) if

$$\lim_{\tau \to \infty} A_\tau \psi \ = \ A \psi \tag{6}$$

for every $\psi \in \mathcal{H}$ and converges in norm if

$$\lim_{\tau \to \infty} \|A_\tau - A\| \ = \ 0 \ . \tag{7}$$

We denote SOT convergence by "SOT-lim $A_\tau = A$".

Schödinger equations with a Hamiltonian of the form (2) find direct physical application in the description of the motion of a quantum particle in a time dependent potential energy. In this case, H_o describes the motion of the particle in the absence of time dependent terms and is generally the Laplacian or some perturbation thereof, possibly discretized, the underlying Hilbert space being $\ell^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$. The time dependent term $\Lambda(t)$ is the operator of multiplication by a bounded function $\Lambda(x,t)$. Theorem 1 is relevant to the adiabatic evolution of an ensemble of non-interacting particles with Fermi statistics. The observables, in this case, are the Fermi-Dirac distributions $F_{\mu,\beta}(H_o) = \frac{1}{1+e^{\beta(H_o-\mu)}}$ at positive temperatures and the spectral projections $\chi(H_o \le \mu)$ and/or $\chi(H_o < \mu)$ at zero temperature. We obtain an adiabatic evolution even if there is an eigenvalue at the chemical potential μ!

The remainder of this note is organized as follows. We state our main result in section 2 and there prove those parts of it which follow from norm resolvent convergence. This section of the proof is very simple but is unrelated to the arguments in the subsequent sections. Lemma 1, presented at the start of section 3, states that the portion of Theorem 1 which relates to functions of bounded variation (BV) may be reduced to a statement about spectral projections. A proof of this statement, based on ideas that go back to Kato [5], is presented in section 3. In section 4 we prove Lemma 1. In section 5, we describe an example, due to Michael Aizenman, which shows that the norm topology is inadequate when we consider discontinuous functions of H_o. Finally, in section 6 we describe a stronger result which holds when H_o has purely discrete spectrum and motivate a related conjecture regarding the Schrödinger evolution.

2. The theorem and all we can show with the resolvent

Before we state Theorem 1, let us recall the definition of certain classes of functions $f : \mathbb{R} \to \mathbb{C}$:

(1) Let C_b denote the bounded continuous functions.
(2) Let C_o denote those functions in C_b which vanish at $\pm\infty$.
(3) Let BV denote the functions of *bounded variation*, *i.e.*, functions f for which

$$\mathrm{Var}(f) \ := \ \sup_{n \ge 1} \ \sup_{x_o < \cdots < x_n \in \mathbb{R}} \ \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \ < \ \infty \ . \tag{8}$$

A function in BV can have only countably many points of discontinuity. See [4, Ch. 3] for further discussion.

Theorem 1. *Let H_o be a self adjoint operator and suppose that the time evolution U_τ satisfies the initial value problem* (1) *with $H_\tau(t/\tau) = H_0 + (1/\tau)\,\Lambda(t/\tau)$ where $\Lambda(\cdot)$ is a uniformly bounded continuously differentiable self adjoint family. Given a measurable function f, consider the statement*

$$\lim_{\tau\to\infty} W_\tau(s) f(H_0) W_\tau^\dagger(s) \;=\; f(H_o)\ , \ \textit{uniformly for } s \in [0,1]\ , \tag{9}$$

where W_τ is the evolution at scaled time, $W_\tau(s) = U_\tau(\tau\cdot s)$.

(1) *If $f \in C_o$ then eq.* (9) *is true in the operator norm topology .*
(2) *If $f = g + h$ with $g \in C_b$ and $h \in BV$ then eq.* (9) *is true in the strong operator topology.*

Remarks:

(1) Operators $A_\tau(s)$ are said to converge uniformly to A in the strong operator topology if

$$\lim_{\tau\to\infty} \sup_s \|A_\tau(s)\psi - A\psi\| \;=\; 0 \tag{10}$$

for every $\psi \in \mathcal{H}$. Uniform norm convergence is defined similarly.
(2) The strong operator topology is the strongest topology in which we can expect an adiabatic limit for discontinuous functions of H_o. In section 5 we describe an elementary example of a system for which $W_\tau(s)f(H_o)W_\tau^\dagger(s)$ fails to converge in the norm topology.
(3) With some extra work one can prove the conclusions of the theorem assuming about $\Lambda(\cdot)$ only that it is self adjoint and essentially bounded. However, we restrict our attention to differentiable perturbations to streamline the presentation.
(4) If the operator H_o is unbounded, the distinction between C_o and C_b is meaningful. Functions in C_b may be "discontinuous at infinity" which explains the loss of norm convergence.
(5) Among the functions of bounded variation are the *Krönecker* delta functions: $\delta_E(x) = 1$ if $x = E$ and 0 otherwise. Thus we obtain an adiabatic evolution for the spectral projection associated to any eigenvalue – even if it has infinite degeneracy and is embedded in the essential spectrum![a]
(6) The standard adiabatic theorems describe the limiting behavior of the Schödinger evolution for a system having a gap in its spectrum with initial data being a spectral projection onto an energy band.[b] A projection onto a spectral band is a *continuous* function of H_o, thus the convergence occurs in the norm topology. In such a setting it is possible to find an explicit bound on the rate of convergence in eq. 9 (see, for example, eq. (14) and Lemma 3).

[a] The general adiabatic theorem for an embedded eigenvalue with *finite* degeneracy was proved previously [1]. That work was motivated by consideration of the behavior of an atom in a radiation field.

[b] We are aware of one example of an adiabatic theorem without a gap condition in this context. This is a result for finite rank perturbations of dense point spectrum [2].

For a great many functions, eq. (9) follows from well known convergence theorems and a simple formula – eq. (12) – which shows that

$$\sup_{s\in[0,1]} \left\| W_\tau(s)(H_o - z)^{-1} W_\tau^\dagger(s) - (H_o - z)^{-1}\right\| \longrightarrow 0 \tag{11}$$

for every $z \notin \mathbb{R}$, which is to say that $W_\tau(s) H_o W_\tau^\dagger(s) \to H_o$ uniformly in s in the "norm resolvent sense". The implications of norm resolvent convergence for Theorem 1 are that

(1) Eq. 9 holds in the norm topology for $f \in C_o$ [6, Thm. VIII.20].
(2) Eq. 9 holds in the strong operator topology for $f \in C_b$ or when f is the characteristic function of an open interval (a, b) provided that a and b are not eigenvalues of H_o. This follows from [6, Thm. VIII.20 and VIII.24] since uniform "strong resolvent convergence" is implied by eq. 11.

What is remarkable is that with some additional work we can prove that eq. 9 holds, for example, when we take f to be the characteristic function of an open interval (a, b) and *one or both of a, b is an eigenvalue with arbitrary degeneracy.*

To verify eq. (11), we use the identity

$$\begin{aligned}(H_\tau(s) - z)^{-1} \;&-\; W_\tau(s)\, (H_\tau(0) - z)^{-1}\, W_\tau(s) \\ &= W_\tau(s) \int_0^s W_\tau(t)^\dagger \left(\frac{\mathrm{d}}{\mathrm{d}t} (H_\tau(t) - z)^{-1} \right) W_\tau(t)\, \mathrm{d}t\, W_\tau(s)^\dagger , \end{aligned} \tag{12}$$

where $H_\tau(s) = H_o + \frac{1}{\tau}\Lambda(s)$. Eq. (12) follows from the fundamental theorem of calculus and the observation that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(W_\tau(t)^\dagger\, (H_\tau(t) - z)^{-1}\, W_\tau(t) \Big) = W_\tau(t)^\dagger \left(\frac{\mathrm{d}}{\mathrm{d}t} (H_\tau(t) - z)^{-1} \right) W_\tau(t) . \tag{13}$$

Now, eq. (11) follows from eq. (12) because the latter implies that

$$\left\| W_\tau(s)(H_o - z)^{-1} W_\tau(s) - (H_o - z)^{-1} \right\| \le \frac{C}{(\mathrm{Im} z)^2} \frac{1}{\tau} , \tag{14}$$

since

$$\frac{\mathrm{d}}{\mathrm{d}t} (H_\tau(t) - z)^{-1} = \frac{1}{\tau} (H_\tau(t) - z)^{-1}\, \dot{\Lambda}(t)\, (H_\tau(t) - z)^{-1} , \tag{15}$$

and

$$(H_\tau(s) - z)^{-1} = (H_o - z)^{-1} - \frac{1}{\tau} (H_\tau(s) - z)^{-1}\, \Lambda(s)\, (H_o - z)^{-1} . \tag{16}$$

Before we proceed, let us describe an example which demonstrates that we cannot really go any further using only norm resolvent convergence. Consider the self adjoint operator on $\ell^2(\mathbb{Z})$ given in Dirac notation by $H_o = \sum_{m\neq 0} \frac{1}{m} |m\rangle\langle m|$. For each n let V_n be the unitary on $\ell^2(\mathbb{Z})$ which "swaps 0 and n", *i.e.*,

$$(V_n \psi)(m) = \begin{cases} \psi(m) & \text{if } m \neq 0, n \\ \psi(0) & \text{if } m = n \\ \psi(n) & \text{if } m = 0 \end{cases} . \tag{17}$$

Then $V_n H_o V_n^\dagger = H_o + \frac{1}{n}(|0\rangle\langle 0| - |n\rangle\langle n|)$. Thus $V_n H_o V_n^\dagger \to H_o$ in norm, and therefore in norm resolvent sense, as $n \to \infty$. Yet, if $P_0 = |0\rangle\langle 0|$ – the spectral projection of H_o associated to eigenvalue 0 – then

$$V_n P_0 V_n = |n\rangle\langle n| \xrightarrow{\text{SOT}} 0 \quad n \to \infty . \tag{18}$$

3. SOT convergence for spectral projections

The claim that eq. (9) holds whenever $f \in BV$ is, at heart, a statement about spectral projections as is indicated by the following lemma:

Lemma 1. *Let H_o and $\Lambda(t)$ be as in Theorem 1. Then eq. (9) holds in the SOT for every $f \in BV$ if and only if it holds whenever $f(x) = \chi(x \leq E)$ or $f(x) = \chi(x \geq E)$.*

We postpone the proof of Lemma 1 to section 4 and focus here on proving eq. 9 with $f(x) = \chi(x \geq E)$ and $f(x) = \chi(x \leq E)$ for every E in $\mathbb{R}$.

In what follows we fix E and take $P = \chi(H_o \leq E)$. The other case – $\chi(H_o \geq E)$ – is handled in exactly the same way by changing $\leq$ to $\geq$ in the appropriate places. We must show that for any $\psi \in \mathcal{H}$

$$\lim_{\tau\to\infty} \sup_{s\in[0,1]} \left\| \left(W_\tau(s)\, P\, W_\tau(s)^\dagger \;-\; P \right) \psi \right\| \;=\; 0\,. \tag{19}$$

Our argument is stated most readily with the semigroup $W_\tau(t,s) = W_\tau(t)W_\tau^\dagger(s)$ – note that $W_\tau(s)PW_\tau^\dagger(s) = W_\tau(s,0)PW_\tau(0,s)$ and $P = W_\tau(s,s)PW_\tau(s,s)$. We would like to compare $W_\tau(s,t)$ with the semigroup associated to H_o, so we define

$$\Omega_\tau(t,s) \;:=\; \mathrm{e}^{\mathrm{i}\tau(t-s)H_o} W_\tau(t) W_\tau(s)^\dagger\,. \tag{20}$$

Since the exponential of H_o commutes with P and $\Omega_\tau(t,s)$ is unitary

$$\begin{aligned} \left\| \left(W_\tau(s) P W_\tau(s)^\dagger - P \right) \psi \right\| \;&=\; \left\| \left(\Omega_\tau(0,s)^\dagger P \Omega_\tau(0,s) \;-\; P \right) \psi \right\| \\ &= \left\| [P, \Omega_\tau(0,s)] \psi \right\|\,. \end{aligned} \tag{21}$$

Finally, because P is a projection

$$[P, \Omega_\tau(t,s)] \;=\; P\,\Omega_\tau(t,s)\bar{P} - \bar{P}\,\Omega_\tau(t,s)P\,, \tag{22}$$

where $\bar{P} = \mathbf{1} - P$. Therefore, eq. 19 will follow if we can verify that both terms on the right side of eq. (22) uniformly converge to zero in the SOT.

Consider the first term. Let $P_\Delta := \chi(E < H_0 < E + \Delta)$, then

$$P\,\Omega_\tau(t,s)\bar{P} \;=\; P\,\Omega_\tau(t,s)(\bar{P} - P_\Delta) \;+\; P\,\Omega_\tau(t,s)P_\Delta. \tag{23}$$

We will see below (Lemma 3) that the operator norm of $P\Omega(P - P_\Delta)$ is uniformly bounded by $1/\tau\Delta$. Thus given $\psi \in \mathcal{H}$

$$\left\| P\,\Omega_\tau(t,s)\bar{P}\psi \right\| \;\leq\; \frac{C}{\tau\Delta}\,\|\psi\| \;+\; \|P_\Delta \psi\|\,. \tag{24}$$

If, for instance, $\Delta = 1/\sqrt{\tau}$ then both terms converges to zero since SOT-lim $P_\Delta = 0$ – whether or not there is an eigenvalue at E.

The second term of (22) requires a little more care. Because E may be an eigenvalue, we need to isolate the contribution from the associated projection $P_E = \chi(H_o = E)$. Let $P'_\Delta = \chi(E - \Delta < H_o < E)$ and consider

$$\begin{aligned} \bar{P}\,\Omega_\tau(t,s)P \;=\; \bar{P}\,\Omega_\tau(t,s)(P - P'_\Delta - P_E) \;+\; \bar{P}\,\Omega_\tau(t,s)P'_\Delta \\ +\; \bar{P}\,\Omega_\tau(t,s)P_E\,. \end{aligned} \tag{25}$$

As above, if we take $\Delta = 1/\sqrt{\tau}$ then the first and second terms tend uniformly to zero. That the third term also converges to zero is the content of the following lemma:

Lemma 2. *Let $P_E := \chi(H_o = E)$. Then $(\mathbf{1} - P_E)\,\Omega_\tau(t,s)P_E$ uniformly tends to zero in the strong operator topology.*

Proof. The operator $\Omega_\tau(t,s)$ satisfies a Volterra equation

$$\Omega_\tau(t,s) \ = \mathbf{1} + \int_s^t \mathrm{d}r \ K_\tau(r,s)\Omega_\tau(r,s) \ , \tag{26}$$

with

$$K_\tau(r,s) \ = \ -\mathrm{i}\mathrm{e}^{\mathrm{i}\tau(r-s)H_o}\Lambda(r)\mathrm{e}^{\mathrm{i}\tau(s-r)H_o} \ . \tag{27}$$

By iterating eq. (26) we obtain a norm convergent series

$$\Omega_\tau(t,s) \ = \ \sum_{n=0}^{\infty} A_\tau^n(t,s) \tag{28}$$

where

$$A_\tau^n(t,s) \ = \ \int\cdots\int_{s\le r_n\le\ldots\le r_1\le t} \mathrm{d}r_1\ldots\mathrm{d}r_n K_\tau(r_1,s)\ldots K_\tau(r_n,s) \ . \tag{29}$$

Since $A_\tau^n(t,s)$ is obtained by integrating a product of n factors of K over a simplex of volume $(t-s)^n/n!$ we have the elementary norm bound

$$\|A_\tau^n(t,s)\| \ \le \ \frac{1}{n!}\kappa^n(t-s)^n \ , \tag{30}$$

where $\kappa = \sup_r \|\Lambda(r)\|$.

We see from (28), (30), and dominated convergence that it suffices to show for each n that $\bar{P}_E A_\tau^n(t,s)P_E \to 0$ uniformly in the SOT. This may be proved as follows. First note that

$$P_E K_\tau(r,s)P_E \ = \ -\,\mathrm{i}P_E\Lambda(r)P_E \ , \tag{31}$$

$$\bar{P}_E K_\tau(r,s)P_E \ = \ -\,\mathrm{i}\bar{P}_E\mathrm{e}^{\mathrm{i}\tau(r-s)(H_o-E)}\Lambda(r)P_E \ . \tag{32}$$

Next observe that

$$\int_s^r \mathrm{d}r'\,\bar{P}_E\mathrm{e}^{\mathrm{i}\tau(r'-s)(H_o-E)} \ \longrightarrow \ 0 \tag{33}$$

uniformly in the strong operator topology from which it follows via integration by parts that

$$\int_s^r \mathrm{d}r'\,\bar{P}_E\mathrm{e}^{\mathrm{i}\tau(r'-s)(H_o-E)}B(r') \ \longrightarrow \ 0 \tag{34}$$

for any differentiable family of operators $B(r)$ which does not depend on τ.

Now consider the expression for A_τ^n obtained by inserting $\mathbf{1} = \bar{P}_E + P_E$ between the two right most factors of K_τ in the integral which appears in eq. (29). Proceed with the term obtained from P_E by inserting $\bar{P}_E + P_E$ between the next two factors of K. Continue from right to left in this way, expanding only the terms obtained from P_E. We obtain an expression for $\bar{P}_E A_\tau^n(t,s)P_E$ as of sum of n terms, the jth term being

$$(-\mathrm{i})^j \int\cdots\int_{s\le r_n\le\ldots\le r_1\le t} \mathrm{d}r_1\ldots\mathrm{d}r_n\bar{P}_E K_\tau(r_1,s)\ldots K_\tau(r_{n-j},s)$$
$$\times\ \bar{P}_E\mathrm{e}^{\mathrm{i}\tau(r_{n-j+1}-s)(H_o-E)}\Lambda(r_{n-j+1})P_E\ldots\Lambda(r_n)P_E \ , \tag{35}$$

which uniformly converges to zero by virtue of eq. (34). Since A^n_τ is a finite linear combination of terms which uniformly tend to zero it does so as well. □

It remains to show that $\left\|P\Omega_\tau(t,s)(\bar{P}-P_\Delta)\right\|$ is bounded by $1/\tau\Delta$.

Lemma 3. *Let* $P_1 := \chi(H_o \le E_1)$ *and* $P_2 := \chi(H_o \ge E_2)$ *with* $E_2 > E_1$. *Then*

$$\|P_1\Omega_\tau(t,s)P_2\| \;\le\; \frac{C}{\Delta\tau}\,, \tag{36}$$

where $\Delta = E_2 - E_1$ *and* C *is a constant which does not depend on* E_1 *or* E_2. *The same inequality holds with* $P_1,\ P_2$ *interchanged.*

Proof. As in the proof of Lemma 2 the idea is to prove a bound on each term A^n_τ in the expansion for Ω_τ. In this case, we will show that

$$\|P_1A^n_\tau(s,t)P_2\| \;\le\; \frac{n}{\tau\Delta}\frac{\alpha^n}{(n-1)!} \tag{37}$$

where α is a constant independent of s,t. Summing these bounds clearly implies eq. (36) – see eq. (28).

The main step is to show that

$$\|P_1\int_t^s drK_\tau(r,s)\;P_2\| \;\le\; \frac{C}{\Delta\tau}\,, \tag{38}$$

and the same with P_1 and P_2 interchanged. The idea is that, since $K_\tau(r,s) = \mathrm{e}^{\mathrm{i}\tau(r-s)H_o}\Lambda(r)\mathrm{e}^{\mathrm{i}\tau(s-t)H_o}$ and the spectral supports of P_1 and P_2 are distance Δ apart, the integral over r has a highly oscillating phase of order $\tau\Delta$. For a rigorous argument, however, it is convenient to use a commutator equation and integration by parts to extract eq (38). This method goes back to Kato [5].

The commutator $[H_o, X]$ might be ill defined if H_o is unbounded. Thus we introduce a cutoff and work instead with $[H_o, P_MXP_M]$ where $P_M = \chi(-M < H_o < M)$ and $M \in (0,\infty)$. At the end of the argument we take $M \to \infty$. The X we have in mind is

$$X(r) \;:=\; \frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathrm{d}z\, P_1\, R(z)\Lambda(r)R(z)\, P_2\,. \tag{39}$$

where $R(z) := (H_0 - z)^{-1}$ and the contour Γ is the line $\{E' + \mathrm{i}\eta \,:\, \eta \in \mathbb{R}\}$ with $E' = (E_2 + E_1)/2$. A simple calculation yields

$$[H_o, P_MX(r)P_M] \;=\; P_MP_1\Lambda(r)P_2P_M\,. \tag{40}$$

Therefore

$$\begin{aligned} P_MP_1K_\tau(r,s)P_2P_M &= [H_o, \mathrm{e}^{\mathrm{i}\tau(r-s)H_o}P_MX(r)P_M\mathrm{e}^{\mathrm{i}\tau(s-r)H_o}] \\ &= \frac{1}{\mathrm{i}\tau}P_M\left(\frac{\mathrm{d}}{\mathrm{d}r}(\mathrm{e}^{\mathrm{i}\tau(r-s)H_o}X(r)\mathrm{e}^{\mathrm{i}\tau(s-r)H_o})\right. \\ &\qquad \left. -\;\mathrm{e}^{\mathrm{i}\tau(r-s)H_o}\dot{X}(r)\mathrm{e}^{\mathrm{i}\tau(s-r)H_o}\right)P_M\,. \end{aligned} \tag{41}$$

However $X(r)$ and $\dot{X}(r)$ are uniformly bounded, $\|X(r)\|\,,\|\dot{X}(r)\| \le C/\Delta$, so integrating (41) yields

$$\|P_M\,P_1\int_t^s drK_\tau(r,s)\;P_2\,P_M\| \;\le\; \frac{C}{\Delta\tau}\,. \tag{42}$$

In the limit $M \to \infty$ this implies eq. (38) by lower semi-continuity of the norm. The second case with P_1 and P_2 interchanged follows with an obvious modification of X.

The rest of the argument is similar to the proof of Lemma 2. We insert a decomposition of the identity $\mathbf{1} = Q + \bar{Q}$ between the factors of K in the integral expression for A^n_τ, eq. (29). To apply (38), we should maintain a spectral gap between the projections which sits to the left and right of K. Therefore we define $Q_j := \chi(H_0 \le Q_1 + j/n\Delta)$ for $j = 0, ..., n$ and insert $1 = Q_j + \bar{Q}_j$ between the jth and $(j+1)$th factors of K. With these insertions, $P_1 A^n_\tau P_2$ breaks into 2^n terms and each term includes at least one factor of the type $Q_j K_\tau(r_{j+1}, s)\bar{Q}_{j+1}$ or $\bar{Q}_j K_\tau(r_{j+1}, s) Q_{j+1}$ where there is a gap of size Δ/n between the spectral supports of the two projections. We apply integration by parts to the integral over r_{j+1} to obtain a factor which may be bounded by eq. (38):

$$\int_0^{r_j} \mathrm{d}r_{j+1}\, Q_j K_\tau(r_{j+1}, s)\bar{Q}_{j+1} B(r_{j+1}) = \int_0^{r_j} \mathrm{d}r_{j+1}\, Q_j \int_{r'}^{r_j} \mathrm{d}r'\, K_\tau(r', s)\bar{Q}_{j+1}\dot{B}(r_{j+1}) \,. \tag{43}$$

Elementary norm estimates and eq (38) now show that each of the 2^n terms is bounded by $n\beta^n/(\Delta\tau(n-1)!)$ for some β which implies eq. (37) with $\alpha = 2\beta$. ☐

4. Integration by parts and the proof of lemma 1

Turning to the proof of Lemma 1, we note that the spectral theorem provides the representation

$$f(H_o) \;=\; \int f(E)\mathrm{d}P_o(E) \tag{44}$$

valid for bounded measurable f. The goal is to integrate this expression by parts thereby obtaining an expression involving $\mathrm{d}f$ and $P_o(E) = \chi(H_o \le E)$. This argument works precisely when $f \in BV$ as we shall now explain.

The projection valued measure $\mathrm{d}P_o(E)$ is the differential of $P_o(E) = \chi(H_o \le E)$ which is of bounded variation *in the strong operator topology.* That is, for any $\psi \in \mathcal{H}$,

$$\sup_{n\ge1} \sup_{E_0<\cdots<E_n\in\mathbb{R}} \sum_{j=1}^{n} \|P_o(E_j)\psi - P_o(E_{j-1})\psi\| \;<\; \infty \,. \tag{45}$$

We could equally well work with $P_o(E) = \chi(H_o < E)$ or a number of other choices – the distinction being meaningful only if H_o has point spectrum.

Since the function P_o is SOT-continuous from the left at every E, *i.e.* $P_o(E-0) = P_o(E)$, we may integrate (44) by parts whenever $f \in BV$ and everywhere continuous from the right:[c]

$$f(H_o) \;=\; f(\infty)\mathbf{1} - \int \mathrm{d}f(E)P_o(E) \,, \qquad f \in BV \text{ and continuous from the right.} \tag{46}$$

[c]The extension of integration by parts to functions in BV is a standard part of real analysis – we direct the reader to [4, Ch. 3] for details.

For general $f \in BV$ this formula is replaced by

$$
\begin{aligned}
f(H_o) \; &= \; f(\infty)\mathbf{1} - \int \mathrm{d}f(E)\,\chi(H_o \leq E) \\
&\quad + \sum_{E\in\mathbb{R}} (f(E) - f(E+0))\,\chi(H_o = E)\;.
\end{aligned} \tag{47}
$$

Note that $\sum_{E\in\mathbb{R}} |f(E) - f(E+0)| \leq \mathrm{Var}(f) < \infty$. In particular, there can be only countably many $E \in \mathbb{R}$ for which $f(E) \neq f(E+0)$.

Now suppose that

$$\text{SOT-lim}\; W_\tau(s)\, A\, W_\tau(s)^\dagger \;=\; A \tag{48}$$

uniformly in s whenever $A = \chi(H_o \leq E)$ or $A = \chi(H_o \geq E)$ with $E \in \mathbb{R}$. Since

$$\chi(H_o = E) \;=\; \chi(H_o \leq E) + \chi(H_o \geq E) - \mathbf{1}\;, \tag{49}$$

eq. (48) also holds with $A = \chi(H_o = E)$. Now given $f \in BV$, use (47) to express $f(H_o)$ and find that

$$\text{SOT-lim}\; W_\tau(s) f(H_o) W_\tau(s)^\dagger \;=\; f(H_o) \tag{50}$$

uniformly in s by dominated convergence.

5. Why the norm topology is inadequate – an example

The following example is due to Michael Aizenman and is motivated by the consideration of systems with dense point spectrum.

Consider a countable collection of non interacting two-level systems:

$$H_\tau(t/\tau) \;=\; \sum_{k=0}^{\infty} \oplus H_k(t/\tau)\;, \quad H_k(t/\tau) \;=\; m_k\sigma_z + \epsilon\frac{t}{\tau}\sigma_x\;, \tag{51}$$

with $m_k := 1/k$ and σ_z, σ_x the Pauli spin matrices. Theorem 1 certainly applies to the unitary evolution U_τ associated with Hamiltonian $H_\tau(t/\tau)$. The unitaries, of course, decompose into a direct sum of two by two matrices U_τ^k generated by H_k.

Let us choose $f(H(0)) = P_o$ to be the spectral projection onto negative energies: $P_o = \chi(H(0) < 0)$. We will show that, although in accordance with Theorem 1

$$\text{SOT-lim}\, U_\tau(\tau s) P_o U_\tau^\dagger(\tau s) \;=\; P_o\;, \tag{52}$$

we nonetheless have

$$\limsup_{\tau\to\infty} \left\| U_\tau(\tau s) P_o U_\tau^\dagger(\tau s) \;-\; P_o \right\| \;\geq\; \alpha_\epsilon(s) \tag{53}$$

where, for sufficiently small ϵ, $\alpha_\epsilon(s) > 0$ for every s in $[0,1]$.

Indeed, choose the particular sequence $\tau_n := n$ of integer values for τ. For each n, we consider the two-level system with $m_n := 1/\tau_n$. For this system

$$\mathrm{i}\dot{U}_\tau^n(t) \;=\; H_n U_\tau^n(t) \;=\; \frac{1}{\tau_n}(\sigma_z + t\sigma_x) U_\tau^n(t)\,, \tag{54}$$

so that $U_{\tau_n}^n(\tau_n s) =: V(s)$ is independent of n. The matrix $V(s)$ is obtained by integrating

$$\mathrm{i}\dot{V}(s) \;=\; (\sigma_z + \epsilon s\sigma_x) V(s) \tag{55}$$

with initial condition $V(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly, with ϵ small enough

$$\|V(s)\sigma_z V(s)^\dagger - \sigma_z\| \;>\; 0\,, \tag{56}$$

for all $s \in [0,1]$. Since, as mentioned above, U_τ is the direct sum of the two by two matrices U_τ^k we have

$$\left\|U_{\tau_n}(\tau_n s)P_o U_{\tau_n}^\dagger(\tau_n s)) - P_o\right\| \;\geq\; \left\|V(s)\theta(-\sigma_z)V(s)^\dagger - \theta(-\sigma_z)\right\| \;>\; 0\,. \tag{57}$$

6. The Schrödinger picture – a conjecture

Theorem 1 describes the adiabatic limit of the *Heisenberg picture* of quantum dynamics. As for the *Schrödinger picture*, in general it is not true that $U_\tau(\tau s) \to \mathbf{1}$; nor is there reason to expect $U_\tau(\tau s)$ to converge to anything at all. In fact, in case the spectrum of H_o consists of countably many isolated eigenvalues, a simple modification of our proof shows that

$$\operatorname*{SOT-lim}_{\tau\to\infty} \mathrm{e}^{\mathrm{i}\tau s H_o} U_\tau(\tau s) \;=\; \Phi(s) \tag{58}$$

where $\Phi(s)$ is a unitary operator which commutes with H_o. If there is a uniform lower bound on the spacing between neighboring eigenvalues then the convergence is in *norm*. The unitary $\Phi(s)$ is related to $\Lambda(\cdot)$ and the spectral projections $P_E = \chi(H_o = E)$ via the evolution equation:

$$\mathrm{i}\dot{\Phi}(s) \;=\; \left(\sum_{E\in\sigma(H_o)} P_E \Lambda(s) P_E\right) \Phi(s)\,. \tag{59}$$

For general H_o it is not clear from our proof that $\mathrm{e}^{\mathrm{i}\tau s H_o} U_\tau(\tau s)$ even converges. However if it does converge, it is clear from Theorem 1 that the limiting operator commutes with H_o which motivates the following conjecture:

Conjecture 1 (Schödinger Adiabatic Theorem)**.** *Under the assumptions of Theorem 1,* $\mathrm{e}^{\mathrm{i}\tau s H_o} U_\tau(\tau s) \xrightarrow{SOT} \Phi(s)$ *where* $\Phi(s)$ *is a unitrary operator which commutes with* H_o.

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