What are the equations of motion of classical physics?

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Abstract

Action at a distance in Newtonian physics is replaced by finite propagation speeds in classical physics, the physics defined by the field theories of Maxwell and Einstein. As a result, the differential equations of motion in Newtonian physics are replaced in classical physics by functional differential equations, where the delay associated with the finite propagation speed (the speed of light) is taken into account. Newtonian equations of motion, with post-Newtonian corrections, are often used to approximate the functional differential equations of motion. Some mathematical issues related to the problem of extracting the "correct" approximate Newtonian equations of motion are discussed.

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1 Introduction

This paper—an expanded version of a lecture presented in the special session on Applied Dynamical Systems of the CAIMS 2001 meeting in Victoria,

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BC—is an invitation to explore some of the mathematical issues related to the foundations of post-Newtonian physics.

Let us recall that Newtonian forces (for example, the inverse square law for gravitation) imply "action at a distance." This absurd, but outstandingly successful, premise of Newtonian theory predicts that signals propagate instantaneously. In classical physics, relativity theory postulates that signals propagate with a velocity that does not exceed the velocity of light. Thus, the forces of Newtonian physics must be replaced by force laws that take into account the finite propagation speed of the classical fields—electromagnetic and gravitational—which determine the forces acting on a moving body. In turn, the ordinary (and partial) differential equations of Newtonian physics, which are derived from the second law of motion $d(mv)/dt = F$, must be replaced by corresponding functional differential equations where the force F is no longer a function of just position, time, and velocity; rather, the classical force law must take into account the time-delays due to the finite propagation speed of the classical fields.

The functional differential equations of motion for classical field theory are generally difficult, often impossible, to express in a form that is amenable to analysis. Thus, to obtain useful dynamical predictions from realistic models, it is natural to replace the functional differential equations of motion by approximations that are ordinary (or partial) differential equations (cf. [2]). Of course, Newton's equations are the premier choice for approximating the true equations of motion. Indeed, due to the overwhelming success of Newtonian models in applied mathematics—where in most cases characteristic velocities are so low that relativistic effects are negligible—the dynamics of the true equations of motion are often ignored. The purpose of this paper is to discuss some of the mathematical issues that must be addressed to obtain a rigorous foundation for post-Newtonian dynamics, that is, Newtonian dynamics with relativistic corrections taken into account.

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2 Newtonian versus relativistic physics

2.1 Newtonian physics

Let us consider the motion of a body p with mass m that is influenced by its *Newtonian* gravitational attraction to a second body P with mass M. In this case, the equation of motion for the body p is

$$m\ddot{x} = -\frac{GMm}{|x-y|^2}\frac{(x-y)}{|x-y|}, \tag{1}$$

where x denotes the position of p in space, y denotes the position of P, and G is the universal gravitational constant. Note that the force on the body p changes continuously with the position of P. By viewing changes in the gravitational force detected at p as a signal produced by manipulating the position of P, the (instantaneous) action at a distance of Newtonian gravity is seen to be equivalent to the infinite speed of propagation of gravity.

By coupling equation (1) with the equation

$$M\ddot{y} = -\frac{GMm}{|x-y|^2}\frac{(y-x)}{|x-y|},$$

which models the motion of P as it is influenced by its gravitational attraction to p, we obtain the prototypical model of Newtonian mechanics: a system of ordinary differential equations for the motion of two bodies influenced by their mutual gravitational attraction. The Newtonian model for two charged particles moving under the influence of the Coulomb force is essentially the same; the only change in the differential equations of motion is the replacement of the constant $-GMm$ by the product of the charges of the particles.

There are Newtonian models for systems of bodies, for fluids, elastic media, etc. All of these models are ordinary (or partial) differential equations.

2.2 Electrodynamics and gravitodynamics

In classical physics, the fundamental electromagnetic and gravitational forces are generally given by time-dependent fields that propagate at the speed of light.

Maxwell's field equations determine the properties of the electric and magnetic fields that influence the motion of charged particles through the

Lorentz force law. In fact, the motion of a relativistic charged particle p with mass m and velocity v, which is influenced by the electric field E and magnetic field B produced by a charged particle P, is given by

$$\frac{d}{dt}\Big(\frac{mv}{(1-|v|^2/c^2)^{1/2}}\Big) = q(E + \frac{1}{c}\, v \times B), \tag{2}$$

where c is the speed of light and q is the charge on the particle p.

The electric field E that affects p at position x at time t is produced by P at y at the retarded time $t-\tau$ where

$$|x(t) - y(t-\tau)| = c\tau$$

(distance=speed$\times$time). Hence, the electric field at (x,t) is given by

$$E(x,t) = f(t, x, \tau, y(t-\tau), \dot{y}(t-\tau), \ddot{y}(t-\tau)),$$

where f is some smooth function. The magnetic field has a similar form. It follows that the Lorentz force acting in spacetime at (x,t) is delayed by time τ (itself an implicitly defined function of space and time) as p and P move, and therefore the equations of motion are a coupled system of retarded functional differential equations.

Explicit representations of E and B are obtained from Maxwell's field equations

$$\begin{aligned} \operatorname{div} E &= 4\pi\rho, & \operatorname{curl} E &= -\tfrac{1}{c}\tfrac{\partial B}{\partial t}, \\ \operatorname{div} B &= 0, & c\operatorname{curl} B &= 4\pi j + \tfrac{\partial E}{\partial t}, \end{aligned}$$

where ρ is the charge density, j is the current density, and c is the speed of light. In addition, we have the conservation of charge: $\operatorname{div} j = -\partial\rho/\partial t$.

A standard computation using vector calculus shows that the fields E and B are given by

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \operatorname{grad}\phi, \qquad B = \operatorname{curl} A, \tag{3}$$

where, once the Lorentz gauge condition

$$\operatorname{div} A + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$$

is imposed, the scalar potential ϕ and the vector potential A satisfy the following wave equations with sources:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \Delta\phi = 4\pi\rho,$$
$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \Delta A = \frac{4\pi}{c} j.$$

Using classical potential theory, the wave equation, and variation of parameters, the retarded-time potentials are given by

$$\phi(x,t) = \int \frac{\rho(y, t-|x-y|/c)}{|x-y|}\, dy,$$
$$A(x,t) = \frac{1}{c}\int \frac{j(y, t-|x-y|/c)}{|x-y|}\, dy,$$

where the integrals are over all of space. The retarded-time electric and magnetic fields are then computed as in display (3). In practice, the current density is given by $j = \rho v$. Hence, by the conservation of charge, the charge density satisfies the continuity equation

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho v) = 0.$$

The electric and magnetic fields produced by an arbitrary charge density are complicated. The potentials, however, can be computed explicitly for a point-charge moving in spacetime. In this ideal case, the charge density is a Dirac-type measure associated with the moving point-charge, and the resulting retarded-time potentials—called the Liénard-Wiechert potentials—are given by

$$\phi(x,t) = \Big(\frac{q}{|x-y| - \dot{y}\cdot(x-y)/c}\Big)_{\text{ret}},$$
$$A(x,t) = \frac{1}{c}\phi(x,t)\,\dot{y}_{\text{ret}},$$

where the subscript ret indicates that y and $\dot{y}$ are evaluated at the retarded time $t-\tau$ and τ is given implicitly by the equation $c\tau = |x(t) - y(t-\tau)|$. The associated electric and magnetic fields are computed using the equations in display (3), and the corresponding equation of motion (2) for a point particle influenced by these fields is relatively simple. For example, the motion of two charged particles confined to move on a line, with only the Lorentz force taken into account, is modeled by the Driver-Travis system (see [10, 18])

$$\frac{\ddot{x}_i}{(1-\dot{x}_i^2/c^2)^{3/2}} = \frac{(-1)^i q_1 q_2 (c+(-1)^i \dot{x}_j(t-\tau_{ij}))}{c^2 m_i \tau_{ij}^2 (c-(-1)^i \dot{x}_j(t-\tau_{ij}))},$$
$$c\tau_{ij} = |x_i - x_j(t-\tau_{ij})|,$$

where (i,j) is in the set $\{(1,2),\,(2,1)\}$.

There is a similar equation for the gravitational two-body problem (restricted to a line), but the equations are more complicated. The basic reason is that Einstein's field equation ($R_{\mu\nu} - (1/2)g_{\mu\nu}R = 8\pi\kappa T_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci tensor, R is the scalar curvature, $\kappa = G/c^4$, $T_{\mu\nu}$ is the stress-energy tensor, $g_{\mu\nu}$ is the metric tensor, and the indices μ and ν run over the integers from zero to four) is *nonlinear*. While the motion of a single test particle follows a geodesic in spacetime, the general relativistic two-body system seems to be too difficult to write down explicitly. Models of this type are the subject of current research. For example, relativistic effects, including radiation damping, are important in the dynamics of binary neutron stars (see [3] and the references therein). In modern physics, the theoretical study of gravitational dynamics is generally more important than classical electrodynamics. The reason is that quantum mechanics has superseded classical electrodynamics, but there is yet no quantum theory of gravity. At any rate, the true equations of motion of the classical field theories are functional differential equations.

3 Post-Newtonian approximation

The basic idea of post-Newtonian approximation, from a mathematical point of view, is the expansion of model equations in powers of $1/c$. From a physical point of view, the idea is to consider low velocity (compared with the speed of light) weak field limits. Note, for example, that the relativistic form of Newton's second law, where the rate of change of the momentum is given by $\frac{d}{dt}(mv(1-|v|^2/c^2)^{-1/2})$, reverts to Newton's law in the low-velocity limit.

3.1 Radiation damping

> Classical mechanics is a mathematically consistent theory; it just doesn't agree with experience. It is interesting, though, that the

> classical theory of electromagnetism is an unsatisfactory theory all by itself. There are difficulties associated with the *ideas* of Maxwell's theory which are not solved by and not directly associated with quantum mechanics—R. Feynman [12, p. 28-1].

According to Maxwell's field equations, a charged particle produces electromagnetic fields as it moves. Since, in this case, a particle radiates energy, it must slow down. This basic intuition has led to some thorny issues in physics. To describe one of them briefly, let us consider a sphere consisting of identical charged particles. As this body accelerates, the various charges produce fields that affect the motion of the body through the Lorentz force.

By considering motion along a line, Dirac reasoned that there are retarded and advanced self-forces that have (on average over the body) the post-Newtonian expansions

$$\begin{aligned} F_{\text{ret}} &= \frac{\alpha q^2}{ac^2}\ddot{x} - \frac{2q^2}{3c^3}\dddot{x} + O(\frac{a}{c^4}), \\ F_{\text{adv}} &= \frac{\alpha q^2}{ac^2}\ddot{x} + \frac{2q^2}{3c^3}\dddot{x} + O(\frac{a}{c^4}), \end{aligned}$$

where x is the position of the centroid of the sphere, α depends on the charge distribution ($\alpha = 2/3$ for a round sphere), q is the charge on one of the particles, and a is the radius of the sphere (see [12, Ch. 28]). These forces are derived by expanding the Liénard-Wiechert potentials (see [15, Ch. 9]).

The existence of advanced forces (preaccelerations) would imply that the motion of a body is influenced by forces that are produced in its future. While this notion is surely problematic, it does lead to an interesting result.

The ideal situation—Dirac's original motivation—is the theory of the electron, an elementary particle that seems to have no internal structure; it is supposed to be a point charge, that is, a sphere with radius zero. One manifestation of the internal inconsistencies mentioned in the quote by Feynman is the blowup of the coefficients of the first terms in these expansions as $a \to 0$. From Newton's second law, this coefficient, $m_{\text{e}} := \alpha q^2/(ac^2)$, represents a mass, called the electromagnetic mass. Thus, the difficulty can be stated as follows: the electromagnetic mass blows up as the radius of the sphere shrinks to zero.

The radius of the sphere is given by $a = \alpha(q^2/(m_{\text{e}}c^2))$ and the number $q^2/(m_{\text{e}}c^2)$ is called the classical electron radius (α is scaled out, because this coefficient depends on the shape of the original charge distribution).

This value for the electron radius does not agree with the value that can be computed directly from relativity theory, a fact that led to a crisis in classical physics that is still not completely resolved (see [12, Ch. 28] for an extended discussion).

Dirac proposed a way to remove the apparent blowup of the electromagnetic mass. He theorized that the true self-force (on a moving electron) is half the difference of the retarded and advanced forces; that is,

$$F_{\text{self}} = \lim_{a \to 0} \frac{1}{2}(F_{\text{adv}} - F_{\text{ret}}).$$

Therefore, the self-force (the radiation reaction force) is

$$F_{\text{self}} = \frac{2q^2}{3c^3} \dddot{x} .$$

Using Dirac's theory, a post-Newtonian model for the motion of an electron, confined to move on a line and with radiation reaction taken into account, is given by the Abraham-Lorentz equation

$$m\ddot{x} = \frac{2q^2}{3c^3} \dddot{x} + F,$$

where F is some external force. In the presence of an electromagnetic field, the equation of motion in space would be

$$m\ddot{x} = q(E + \frac{1}{c} v \times B) + \frac{2q^2}{3c^3}\ddot{v} + F.$$

Since the particle radiates (produces fields that carry energy) the self force should cause the particle to lose energy and slow down. For this reason, the presence of the third-order time-derivative term in the first differential equation is called radiation damping. Is this intuition correct; that is, does the presence of this term cause damping?

As a concrete example, consider two identical (isolated) charged particles each with unit mass that are one unit apart at rest. Imagine that these bodies are elastically connected by a force that can be modeled by Hooke's law with unit spring constant. Their relative motion, ignoring their mutual Coulomb and gravitational interactions, is modeled by the differential equation

$$\ddot{x} + x - 1 = \epsilon \dddot{x},$$

where $\epsilon \sim c^{-3}$. A similar scenario for two bodies with gravitational radiation damping taken into account, leads to a *nonlinear* oscillator of the form

$$\epsilon z \frac{d^5 z^2}{dt^5} + \ddot{z} + z = 1,$$

where $\epsilon \sim c^{-5}$ (see [4]). In both cases, these post-Newtonian models are not Newtonian—for example, the differential equations are not second-order.

Physical intuition suggests that the postulated two-body systems are damped oscillators. This leaves open the mathematical question: Is damped oscillatory motion predicted by these post-Newtonian models?

The electrodynamic model is linear. Its characteristic equation

$$-\epsilon r^3 + r^2 + 1 = 0$$

has the roots

$$i - \epsilon + O(\epsilon^2), \qquad -i - \epsilon + O(\epsilon^2), \qquad 1/\epsilon + O(1);$$

therefore, the first-order approximation of the general homogeneous solution, which is given by

$$ae^{t/\epsilon} + be^{-\epsilon t} \cos t + ce^{-\epsilon t} \sin t,$$

has a "runaway" mode. A similar result is true, but more difficult to prove, for the *nonlinear* oscillator

$$\epsilon z \frac{d^5 z^2}{dt^5} + \ddot{z} + z = 1.$$

Runaway solutions are clearly not physical. What do they represent? How should they be eliminated? More precisely, we may ask: What is the correct *Newtonian* equation of motion with the radiation damping taken into account?

There is an obvious *mathematical* answer to our question once our post-Newtonian models are recognized as singularly perturbed Newtonian equations. By the definition of ϵ as the reciprocal of a power of the speed of light, it is reasonable to assume that ϵ is a small parameter, at least in the low-velocity regime. To recover the correct Newtonian model, we can apply Fenichel's geometric singular perturbation theory (see [16, 17]).

For the electrodynamic oscillator, we have the equivalent (singularly perturbed) first-order system given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \epsilon \dot{z} &= z + x - 1. \end{aligned} \tag{4}$$

For the gravitodynamic oscillator, the appropriate first-order system is given by

$$\begin{aligned} \dot{z} &= u, \\ \dot{u} &= v, \\ \epsilon^{1/3} \dot{v} &= w, \\ \epsilon^{1/3} \dot{w} &= x, \\ \epsilon^{1/3} \dot{x} &= \frac{1 - z - v}{2z^2} - \epsilon^{1/3} \frac{5ux}{z} - \epsilon^{2/3} \frac{10vw}{z}. \end{aligned}$$

These systems are converted to regular perturbation problems by rescaling time. The corresponding fast-time electrodynamic first-order system is obtained by the change of variables $s = t/\epsilon$ ($s = t/\epsilon^{1/3}$ for the gravitodynamic oscillator). These fast-time systems are equivalent to their original slow-time counterparts for $\epsilon \neq 0$. In effect, the rescaling of time produces a new family of systems, still parametrized by ϵ, but with a different limit as $\epsilon \to 0$.

The fast-time electrodynamic system is given by

$$\begin{aligned} x' &= \epsilon y, \\ y' &= \epsilon z, \\ z' &= z + x - 1; \end{aligned} \tag{5}$$

it is a regularly perturbed first-order system. The corresponding unperturbed system ($\epsilon = 0$) has a two-dimensional invariant manifold, $\{(x, y, z) : z + x - 1 = 0\}$, consisting entirely of rest points. Moreover, this manifold is normally hyperbolic. In our special case, where the invariant manifold consists entirely of rest points, the requirement for normal hyperbolicity is that nearby trajectories are attracted to (or repelled from) the vicinity of the invariant manifold exponentially fast; or, in other words, the system matrix of the linearized system at each rest point has a zero eigenvalue with a two-dimensional spectral subspace (corresponding to the tangent directions on the invariant manifold) and a nonzero eigenvalue with a one-dimensional spectral subspace (corresponding to the normal direction to the invariant manifold). For the unperturbed system (5), the nonzero eigenvalue λ is the same at each rest point. In fact, $\lambda = 1$ and all nearby solutions are repelled from the unperturbed invariant manifold at this exponential rate.

A fundamental result of Fenichel's theory states that a normally hyperbolic invariant manifold persists. Hence, for each sufficiently small $\epsilon \neq 0$, the corresponding system (5) has a two-dimensional normally hyperbolic invariant manifold (not necessarily a linear subspace), called the slow-manifold. Moreover, the corresponding family of slow-manifolds depends smoothly on the parameter ϵ and each slow-manifold is invariant, normally hyperbolic, and repelling.

The original singularly perturbed slow-time family (4) has a corresponding family of normally hyperbolic invariant manifolds. Each of them repels nearby solutions at an exponential rate. This accounts for the existence of runaway solutions.

Using the invariance of the perturbed slow-manifolds, it is easy to obtain (in local coordinates) the equations of motion restricted to these two-dimensional manifolds. The perturbed invariant manifold, for the fast-time system, is the graph of a function of the form

$$z = h(x, y) = 1 - x + \epsilon h_1(x, y) + \epsilon^2 h_2(x, y) + O(\epsilon^3),$$

where the h_i are functions to be determined. Moreover, the dynamical system on this invariant manifold is given by

$$x' = \epsilon y, \quad y' = \epsilon z = \epsilon(1 - x + \epsilon h_1(x, y) + O(\epsilon^2)).$$

For $\epsilon \neq 0$, the dynamical system on the corresponding slow-manifold for the original system is thus given by

$$\dot{x} = y, \quad \dot{y} = 1 - x + \epsilon h_1(x, y) + O(\epsilon^2).$$

To solve for the functions h_i, we use the invariance of the perturbed manifold. In effect, at each point on the manifold, each tangent vector $X = (\epsilon y, \epsilon z, z + x - 1)$ corresponding to the system of differential equations (5), is a linear combination of the basis vectors $b_x = (1, 0, h_x(x, y))$ and $b_y = (0, 1, h_y(x, y))$ for the corresponding tangent space on this manifold. The functions h_i are obtained by equating coefficients of powers of ϵ in the vector identity

$$X = \alpha b_x + \beta b_y,$$

where α and β are scalar variables (which are determined by the first two components of this identity). To first-order in ϵ and in the original slow-time, the equation of motion on the invariant manifold is equivalent to the (Newtonian) second-order system

$$\ddot{x} + \epsilon\dot{x} + x = 1,$$

a dynamical equation for a damped harmonic oscillator—just as it should be.

For the gravitodynamic two-body system, a similar analysis results in the Newtonian dynamical equation

$$\ddot{z} + 32\epsilon\dot{z}z(z - \frac{15}{16}) + z = 1.$$

Again, this is a (nonlinear) under-damped oscillator, at least if the initial separation between the bodies is near $z = 1$.

3.2 Synthesis

We have just seen that geometric singular perturbation theory (in particular, reduction to the slow-manifold) produces Newtonian model equations with post-Newtonian corrections that give physically reasonable dynamics; in particular, the runaway solutions are eliminated. How can we justify using these models? Note, for instance, that the slow-manifolds in our models are *unstable*; nearby solutions run away. In applied mathematics, we usually justify approximations by their *stability*. To validate the slow-manifold reductions, we must show that the resulting Newtonian model equations are "stable" with respect to the dynamics of the original functional differential equations, the true equations of motion in classical physics.

Conjecture 3.1. *In the low-velocity regime, a functional differential equation of motion derived using the forces in classical field theory (the Lorentz force or the relativistic gravitational force) has an inertial manifold $\mathcal{I}$ (a finite-dimensional, invariant, exponentially attracting, and smooth manifold)* *such that the restriction of the motion to this manifold is a Newtonian dynamical system. The Nth-order singular perturbation problem, obtained by (post-Newtonian) expansion in powers of $1/c$ and truncation at order N, has an equivalent first-order system with a normally hyperbolic slow-manifold $\mathcal{S}$. The corresponding vector fields that generate the dynamical systems on $\mathcal{I}$ and $\mathcal{S}$ agree to order $N-1$ in $1/c$.*

While the post-Newtonian expansion (and truncation) results in a system that might (and usually does) have runaway modes, these are simply artifacts of the singular nature of the expansion. The long-term dynamics of the functional differential equations obtained by a direct application of classical field theory is given by a Newtonian equation on a finite-dimensional inertial manifold, and the same dynamical system is obtained by appropriate reduction to a slow-manifold for the singular high-order post-Newtonian equation. Runaway modes generally have no physical significance; equivalently, the slow-manifold is generally *not* an attractor.

The conjecture states a rigorous justification for the validity of the post-Newtonian model that agrees with the dynamics on an inertial manifold which is supposed to exist in the low-velocity regime. But, what is a low velocity? A deeper question addresses this issue directly. For what range of parameter values does the inertial manifold persist?

In the mathematical analysis, $1/c$, or (even better) a characteristic velocity divided by c, is to be viewed as a small parameter. If the conjecture is valid, then there is a lower bound for ϵ such that the corresponding functional differential equation has an inertial manifold. A theorem meant to validate the conjecture should include an estimate for this bound.

What accounts for the transition from the low velocity to the high velocity regime? Answer: As a characteristic velocity increases, a bifurcation will occur that destroys an inertial manifold and, therefore, the validity of post-Newtonian approximation. How can such bifurcations be detected?

4 Delay equations

To test the conjecture stated in the last section, let us replace the functional differential equations of mathematical physics (with space-dependent delays) with families of delay differential equations of the form

$$\dot{x}(t) = f(x(t), x(t-\tau)), \quad x \in \mathbf{R}^n, \quad \tau \in \mathbf{R}, \tag{6}$$

where the delay τ replaces the small parameter corresponding to $1/c$ and f is a smooth function.

The usual state space for the delay equation (6) is $C([-\tau,0])$, the space of continuous functions that map the interval $[-\tau,0]$ into $\mathbf{R}^n$. A basic result in the well-developed theory of delay equations (see [6, 9, 11, 14]) states that the initial value problem consisting of equation (6) and an initial function in $C([-\tau,0])$ has a unique solution $t \mapsto x(t)$ for t in some interval $[0,\beta)$, where $\beta > 0$ or $\beta = \infty$. For such a solution, the state of the system at time $t > 0$ is given by $x_t \in C([-\tau,0])$, where $x_t(\theta) = x(t+\theta)$. This assignment defines a semi-flow in the state space given by $(T^t\psi)(\theta) = x(t+\theta)$, where $t \mapsto x(t)$ is the solution with initial condition ψ.

4.1 Inertial manifold reduction

A heuristic argument indicates that the delay equation (6) has an inertial manifold for small $|\tau|$. Let us first introduce a new variable $t = s\tau$ so that $y(s) = x(s\tau)$ is a solution of the delay equation

$$\dot{y}(s) = \tau f(y(s), y(s-1)) \tag{7}$$

whenever x is a solution of the delay equation (6). The state space for the family (7) is $C([-1,0])$. Also, the unperturbed system ($\tau = 0$)

$$\dot{y}(s) = 0$$

generates the semi-flow given by

$$T^t\psi = \begin{cases} \psi(t+\theta), & 0 \le t < 1 \text{ and } t+\theta < 0, \\ \psi(0), & \text{otherwise.} \end{cases}$$

The n-dimensional submanifold

$$\mathcal{I} := \{\psi \in C([-1,0]) : \psi(\theta) \equiv a \text{ for some } a \in \mathbf{R}^n\}$$

consists entirely of rest points and is normally hyperbolic. In fact, every solution reaches $\mathcal{I}$ in time $t = 1$, a rate of normal contraction that is faster than any exponential decay. Thus, it is reasonable to expect that $\mathcal{I}$ persists as an inertial manifold for sufficiently small $|\tau|$.

We have not proved the persistence of the invariant manifold. Because the state space is infinite-dimensional, there are some delicate issues involved in proving the existence and smoothness of an inertial manifold. Recent results (see, for example, [1]) on the persistence of infinite-dimensional normally hyperbolic invariant manifolds may apply. But, at the first level, a theorem in this direction would state the existence of an inertial manifold for sufficiently small values of the parameter. This would not be completely satisfactory for future applications to physics where some estimate of the relevant parameter values would be required. The foundation for a direct proof of the existence of an inertial manifold, with explicit bounds but without a proof of the required smoothness, is contained in the work of Yu. A. Ryabov and R. D. Driver (see [7, 8]). The smoothness of the inertial manifold is proved in [5].

Having indicated that an inertial manifold exists, let us reduce the dynamical system to the inertial manifold and thus obtain the "Newtonian system" corresponding to the delay equation. To do this, suppose that ξ is a coordinate on $\mathbf{R}^n$ and $y(t, \xi, \tau)$ is the flow on the inertial manifold; that is; $t \mapsto y(t, \xi, \tau)$ is the solution of the delay equation such that $y(0, \xi, \tau) = \xi$. More precisely, $\psi(\theta) \equiv \xi$ is the initial condition for the solution y of the delay equation. With this notation, we have that

$$\dot{y}(t, \xi, \tau) = f(y(t, \xi, \tau), y(t - \tau, \xi, \tau));$$

hence, the vector field that generates the flow Y on the inertial manifold is given by

$$X(\xi, \tau) := \dot{y}(0, \xi, \tau) = f(\xi, y(-\tau, \xi, \tau)).$$

Its expansion at $\tau = 0$ is

$$\begin{aligned} X(\xi, \tau) = f(\xi, \xi) - \tau D_2 f(\xi, \xi) f(\xi, \xi) + \frac{\tau^2}{2!} \{ & D_2^2 f(\xi, \xi)(f(\xi, \xi), f(\xi, \xi)) \\ & + D_2 f(\xi, \xi)(D_1 f(\xi, \xi) + 3 D_2 f(\xi, \xi)) f(\xi, \xi))\} + O(\tau^3), \end{aligned} \tag{8}$$

where the operator D_1, respectively D_2, denotes differentiation with respect to the first, respectively the second, argument of f.

It is interesting to note that the presence of a small delay in a conservative system often results in damped long-term dynamics on an associated inertial manifold. For example, the Duffing-type model equation

$$\ddot{x} + \omega^2 x = -ax(t - \tau) + bx^3(t - \tau)$$

with small delay τ in the restoring force, reduces (by a formal computation to first-order in τ) to the van der Pol-type model equation

$$\ddot{x} + \tau(3bx^2 - a)\dot{x} + (a + \omega^2)x - bx^3 = 0$$

on its inertial manifold. This example illustrates a phenomenon that is reminiscent of quantization: while most periodic solutions in one-parameter families of periodic solutions in a conservative system disappear in the presence of a small delay, some persist as limit cycles. Does this observation have physical significance?

4.2 The analog of post-Newtonian expansion

For the delay equation (6), the analog of post-Newtonian expansion is the expansion of the function $\tau \mapsto f(x(t), x(t-\tau))$ at $\tau = 0$, where the first few terms of the series are given by

$$\begin{aligned} f(x(t), x(t-\tau)) &= f(x(t), x(t)) - \tau D_2 f(x(t), x(t))\dot{x}(t) \\ &\quad + \frac{\tau^2}{2!}(D_2 f(x(t), x(t))\ddot{x}(t) + D_2^2 f(x(t), x(t))(\dot{x}(t), \dot{x}(t))) \\ &\quad + O(\tau^3). \end{aligned}$$

Truncation of the expansion at order N in τ produces an Nth order ordinary differential equation of the form

$$(-1)^N \frac{\tau^N}{N!} D_2 f(x, x) x^{(N)} = F(x, \dot{x}, \ldots, x^{(N-1)}, \tau), \tag{9}$$

the desired analog of a post-Newtonian expansion in classical physics. Moreover, by setting $\mu := \tau^{1/(N-1)}$ and treating μ as a small parameter, we obtain a singularly perturbed first-order system of the form

$$\begin{aligned} \dot{x} &= y_1, \\ \mu^N \dot{y}_1 &= y_2, \\ &\vdots \\ \mu^N \dot{y}_{N-2} &= y_{N-1}, \\ \mu^N \dot{y}_{N-1} &= F(x, y_1, \ldots, y_{N-1}, \mu^{N-1}). \end{aligned}$$

Under the assumption that $D_2 f(x, x)$ is invertible, the geometric singular perturbation theory can be applied to prove that this system has an n-dimensional slow-manifold for sufficiently small $\tau \neq 0$.

4.3 Synthesis

For delay equations, the family of flows restricted to the family of slow-manifolds (parametrized by τ) is generated by the family of vector fields $Y(\xi, \tau)$. A result in [5] states that this family of vector fields agrees with $X(\xi, \tau)$, the family that generates the flows on the inertial manifolds, to order τ^2.

For applications to physics, the result that X and Y agree to order-two in τ is sufficient for many applications. Of course, it is not difficult to show that these vector fields agree to order three, four, etc. On the other hand, the complexity of the terms that appear in the expansion increases rapidly with the order. While the equality of the low-order coefficients of τ in the two expansions can be proved by direct computation, there does not seem to be an easy abstract proof for the equality of all coefficients. Thus, the conjecture that X and Y agree to order $N-1$ (one order less than the order of the "post-Newtonian" truncation) is open. The hypothesis that $D_2 f(x, x)$ is invertible should be sufficient for the conjecture, at least in the case where N is sufficiently large. Under the stronger hypothesis that $D_2 f(x, x)$ is infinitesimally hyperbolic (no eigenvalues on the imaginary axis), no such restriction on N should be necessary.

A apparent difficulty to be overcome in the proof of the conjecture for delay equations is encountered in the proof of the conjecture for the special case of linear delay equations (see [5]).

Theorem 4.1. *If A is invertible, $|\tau| \|A\| e < 1$, and*

$$\dot{x}(t) = Ax(t - \tau),$$

then the family of vector fields that generates the flow of this system on its family of inertial manifolds (parametrized by τ) agrees to order $N-1$ with the family of vector fields on the corresponding family of slow-manifolds for the corresponding family of Nth order systems (9). Moreover, the family of vector fields on the inertial manifolds is given by

$$X(x, \tau) = \sum_{j=0}^{\infty} (-1)^j \frac{(1+j)^j}{(1+j)!} \tau^j A^{1+j} x.$$

The proof in [5] requires the combinatorial identity

$$\sum_{i=0}^{m} \binom{m}{i} (\ell - 1 + i)^{i-1} (m + 1 - i)^{m-i-1} = \frac{\ell}{\ell - 1} (m + \ell)^{m-1}.$$

This nontrivial identity can be proved using Abel's generalization of the binomial theorem; namely, the identity

$$\alpha\beta \sum_{i=0}^{m} \binom{m}{i} (\alpha+i)^{i-1}(\beta+m-i)^{m-i-1} = (\alpha+\beta)(\alpha+\beta+m)^{m-1}$$

(see [13, p. 19]). It seems that a "nonlinear" replacement for this combinatorial identity will be required to prove the analog of Theorem 4.1 for the delay equation (6).

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