No zero energy states for the supersymmetric x^2y^2 potential

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Abstract

We show that the positive supersymmetric matrix-valued differential operator $H = {p_x}^2 + {p_y}^2 + x^2y^2 + x\sigma_3 + y\sigma_1$ has no zero modes, i.e., $H\psi = 0$ implies $\psi = 0$.

1 Introduction

The Hamiltonian of the model is plainly given as

$$H = {p_x}^2 + {p_y}^2 + x^2y^2 + x\sigma_3 + y\sigma_1 \ ,$$

acting on the Hilbert space $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^2) \otimes \mathbb{C}^2$, where σ_i are the Pauli matrices. The supercharge is

$$Q = p_x\sigma_3 - p_y\sigma_1 - xy\sigma_2 \ . \tag{1}$$

Together with the reflection

$$(P\psi)(x, y) = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)\psi(y, x) \ ,$$

the system (H, P, Q) exhibits supersymmetry:

$$H = Q^2 \ , \qquad P^2 = 1 \ , \qquad QP + PQ = 0 \ .$$

It was shown in [1][1] that the spectrum of H is $\sigma(H) = [0, \infty)$. The question whether 0 is an eigenvalue of H has however so far eluded a definite answer — despite some efforts in this direction such as [2].

The precise definition of the model is as follows: Q is the closure of (1) on $C_0^\infty \otimes \mathbb{C}^2$, where $C_0^\infty \equiv C_0^\infty(\mathbb{R}^2)$, and H is the self-adjoint operator associated to the positive symmetric quadratic form $q(\varphi, \psi) = (Q\varphi, Q\psi)$ on $\mathcal{D}(Q) \times \mathcal{D}(Q)$. Hence $C_0^\infty \otimes \mathbb{C}^2$ is a form core for q. The main result is the following.

[1]The Pauli matrices used there differ from ours by an irrelevant unitary conjugation.

Theorem 1 *Let $\psi \in \mathcal{D}(H)$ with $H\psi = 0$. Then $\psi = 0$.*

The proof relies on a simple commutator argument. Let $f = f(x, y)$ be real-valued. Since $H\psi = 0$ implies $Q\psi = 0$ we then have

$$\begin{aligned}(\psi, i[Q, f]\psi) &= \mathrm{i}\{(Q\psi, f\psi) - (f\psi, Q\psi)\} = 0 \ ,\\ \mathrm{i}[Q, f] &= \sigma_3\partial_x f - \sigma_1\partial_y f\end{aligned}$$

for 'any' function f. In particular for $f = (x^2 - y^2)/2$ we get

$$(\psi, (x\sigma_3 + y\sigma_1)\psi) = 0 \ .$$

Subtracting this from $(\psi, H\psi) = 0$ we obtain

$$(\psi, (p_x^2 + p_y^2 + x^2y^2)\psi) = 0 \ ,$$

which is impossible unless $\psi = 0$, since $p_x^2 + p_y^2 + x^2y^2 > 0$ [3]. The trouble with this is that ψ may fail to be in the domain of $f = (x^2 - y^2)/2$ or, what matters more, in that of $\mathrm{i}[Q, f] = x\sigma_3 + y\sigma_1$. The proof given in the next section will circumvent this difficulty.

We stress that any natural realization of Q and H coincides with the one given above. This is the content of the following statement, which is however not needed for the main result.

Proposition 2

(a) *Q is self-adjoint, and its domain is*

$$\mathcal{D}(Q) = \{\psi \in \mathcal{H} \mid Q\psi \in \mathcal{H} \ \textit{(in the sense of distributions on } C_0^\infty \otimes \mathbb{C}^2\textit{)}\} \ .$$

(b) $\mathcal{D}(H) = \{\psi \in \mathcal{H} \mid H\psi \in \mathcal{H}$ *(in the sense of distributions on $C_0^\infty \otimes \mathbb{C}^2$)*$\}$.

(c) *$C_0^\infty \otimes \mathbb{C}^2$ is an operator core for H.*

(d) *$H = Q^2$, i.e., $\psi \in \mathcal{D}(H)$ iff $\psi \in \mathcal{D}(Q)$ and $Q\psi \in \mathcal{D}(Q)$, in which case $H\psi = Q^2\psi$.*

Before turning to the proofs we give a simple argument showing that a possible zero mode of H cannot be unique, though this statement is superseded by Theorem 1. The operators on $\mathcal{H}$

$$(P_1\psi)(x, y) = \sigma_1\psi(-x, y) \ , \ (P_2\psi)(x, y) = \sigma_2\psi(-x, -y) \ , \ (P_3\psi)(x, y) = \sigma_3\psi(x, -y)$$

satisfy $P_i^2 = 1$, $[P_i, P_j] = 2\mathrm{i}\varepsilon_{ijk}P_k$ and $[Q, P_i] = 0$. Thus $Q\psi = 0$ implies $QP_i\psi = 0$ and, if uniqueness is assumed, $P_i\psi = s_i\psi$ with $s_i = \pm 1$. But this contradicts the commutation relations of the P_i.

A related argument shows that the index $\mathrm{tr}(P\Pi)$, where Π is the ground state projection of H, vanishes. This is seen from $PP_2 = -P_2P$ and

$$\mathrm{tr}(P\Pi) = \mathrm{tr}(P\Pi P_2^2) = -\mathrm{tr}(P\Pi P_2^2) \ ,$$

where one power of P_2 has been turned around the trace.

Finally we remark that absence of zero energy states had been suggested by the asymptotic analysis of [4] and, for a slightly less elementary model, been proven in [5] by different means.

2 Proofs

Proof of Theorem 1. Let

$$h(x) = \begin{cases} -Mx - \frac{1}{2}M^2 & (x \le -M) \\ \frac{1}{2}x^2 & (-M \le x \le M) \\ Mx - \frac{1}{2}M^2 & (x \ge M) \end{cases} ,$$

whence

$$\begin{aligned} h'(x) &= \begin{cases} -M & (x \le -M) \\ x & (-M \le x \le M) \\ M & (x \ge M) \end{cases} \\ &= x - g(x) , \end{aligned}$$

where we have defined g as

$$g(x) = \begin{cases} x + M & (x \le -M) \\ 0 & (-M \le x \le M) \\ x - M & (x \ge M) \end{cases} .$$

Let furthermore $h_\epsilon(x) = h(x)\mathrm{e}^{-\epsilon\sqrt{1+x^2}}$, $f_\epsilon(x,y) = h_\epsilon(x) - h_\epsilon(y)$. For $\varphi, \psi \in \mathcal{D}(Q)$ we have

$$\mathrm{i}[(Q\varphi, f_\epsilon\psi) - (f_\epsilon\varphi, Q\psi)] = (\varphi, (\sigma_3\partial_x f_\epsilon - \sigma_1\partial_y f_\epsilon)\psi) . \tag{2}$$

This equality is straightforward for $\varphi, \psi \in \mathcal{C}_0^\infty \otimes \mathbb{C}^2$ and extends to $\mathcal{D}(Q)$, since the operator on the right side is bounded and $\mathcal{C}_0^\infty \otimes \mathbb{C}^2$ is an operator core for Q. By dominated convergence,

$$\text{s-}\lim_{\epsilon\downarrow 0}(\sigma_3\partial_x f_\epsilon - \sigma_1\partial_y f_\epsilon) = h'(x)\sigma_3 + h'(y)\sigma_1 . \tag{3}$$

Subtracting the r.h.s. from H, we obtain

$$\begin{aligned} H - h'(x)\sigma_3 - h'(y)\sigma_1 &= p_x^2 + p_y^2 + x^2y^2 + g(x)\sigma_3 + g(y)\sigma_1 \\ &\ge p_x^2 + p_y^2 + x^2y^2 - |g(x)| - |g(y)| \equiv H_M . \end{aligned} \tag{4}$$

The inequality is understood in the sense of forms [6, 7], where H_M is the self-adjoint operator associated to the corresponding form with form core $\mathcal{C}_0^\infty \otimes \mathbb{C}^2$. We claim that for M large enough H_M is positive, i.e., $H_M > 0$. Now let $\psi \in \mathcal{D}(H)$ with $H\psi = 0$. This implies $Q\psi = 0$ and, by (2) and (3),

$$(\psi, (H - h'(x)\sigma_3 - h'(y)\sigma_1)\psi) = 0 .$$

By (4) and the claim this is possible only if $\psi = 0$ and the theorem follows. To prove the claim, we consider the partition of $\mathbb{R}^2$ as in the figure and introduce Neumann conditions along the boundaries. Then ([6], XIII.15, Proposition 4)

$$H_M \ge H_I + H_{II} + H_{III} + H_{IV} ,$$

where the forms

$$\begin{aligned} H_I &= p_x^2 + p_y^2 + x^2y^2 \\ H_{II} &= p_x^2 + p_y^2 + x^2y^2 - |y| + M \\ H_{III} &= p_x^2 + p_y^2 + x^2y^2 - |x| + M \\ H_{IV} &= p_x^2 + p_y^2 + x^2y^2 - |x| - |y| + 2M \end{aligned}$$

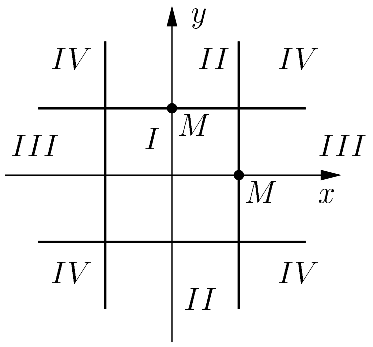

Figure 1: Partition of $\mathbb{R}^2$

act on the corresponding regions. We show that $H_a > 0$ for $a = I, \dots IV$ and M large enough.

I. We have $H_I \geq 0$, and $(\psi, H_I \psi) = 0$ implies $(\psi, x^2 y^2 \psi) = 0$ and hence $\psi = 0$.

II. The operator $p_x^2 + x^2$ on $\mathrm{L}^2(-a, a)$ with Neumann boundary conditions at $x = \pm a$ satisfies

$$p_x^2 + x^2 \geq 1 - Ca^{-2} \ ,$$

where C denotes a generic constant. This can e.g. be seen by means of a partition of unity $j_1^2 + j_2^2 = 1$ with $j_1 = j_1(x/a)$ equal to 1 near $x/a = 0$ and to 0 near $x/a = \pm 1$. Hence, by scaling, $p_x^2 + x^2 y^2$ on $\mathrm{L}^2(-M, M)$ is estimated from below as

$$p_x^2 + x^2 y^2 \geq |y|(1 - C(|y|^{1/2} M)^{-2}) = |y| - CM^{-2} \ .$$

As a result, $H_{II} \geq M - CM^{-2}$, which is positive for M large enough.

III. Is analogous to case II.

IV. There we have $x^2 y^2 \geq M^3 |x|$, $M^3 |y|$ and hence

$$x^2 y^2 - |x| - |y| \geq \Big(\frac{1}{2} M^3 - 1\Big)(|x| + |y|) \geq \Big(\frac{1}{2} M^3 - 1\Big) 2M \ ,$$

which is again positive for M large enough.

Proof of Proposition 2. (a) Let $\mathcal{D}(Q_{\min}) = \mathcal{C}_0^\infty \otimes \mathbb{C}^2$, then $\overline{Q}_{\min} = Q$ by definition of Q and

$$\mathcal{D}(Q_{\min}^*) = \{\psi \in \mathcal{H} \mid Q\psi \in \mathcal{H} \text{ (in the sense of distributions on } \mathcal{C}_0^\infty \otimes \mathbb{C}^2)\} \ .$$

Since $Q_{\min}$ is symmetric, $\overline{Q}_{\min} \subset Q_{\min}^*$. We will show that

$$Q_{\min}^* \subset \overline{Q}_{\min} \ . \tag{5}$$

This implies $\overline{Q}_{\min} = Q_{\min}^* = \overline{Q}_{\min}^*$. It remains to prove (5). We pick $f \in \mathcal{C}_0^\infty$ with $f(0) = 1$, such that $\text{s}-\lim_{n\to\infty} f_n = 1$, $\|\nabla f_n\|_\infty \to 0$ for $f_n(\vec{x}) = f(\vec{x}/n)$, and set $\tilde{f}_n(\vec{p}) = f(\vec{p}/n^2)$. We approximate a given $\psi \in \mathcal{D}(Q_{\min})$ by $\psi_n = f_n \tilde{f}_n f_n \psi \in \mathcal{C}_0^\infty \otimes \mathbb{C}^2$. with $n \to \infty$. We have

$$Q\psi_n = f_n \tilde{f}_n f_n Q\psi + [Q, f_n \tilde{f}_n f_n]\psi \ ,$$

as one checks by taking inner products with $\varphi \in \mathcal{C}_0^\infty \otimes \mathbb{C}^2$. Here

$$\begin{aligned} [Q, f_n\tilde{f}_n f_n] &= [Q, f_n]\tilde{f}_n f_n + f_n[Q, \tilde{f}_n]f_n + f_n\tilde{f}_n[Q, f_n] \,, \\ [Q, f_n] &= -\mathrm{i}(\partial_x f_n)\sigma_3 + \mathrm{i}(\partial_y f_n)\sigma_1 \,, \\ [Q, \tilde{f}_n] &= [xy, \tilde{f}_n] = \mathrm{i}\big(y(\partial_{p_x}\tilde{f}_n) + x(\partial_{p_y}\tilde{f}_n)\big) + \partial^2_{p_x p_y}\tilde{f}_n \end{aligned}$$

are bounded operators with $\|[Q, f_n]\| \to 0$ and, due to $|x|$, $|y| \le Cn$ on $\mathrm{supp} f_n$,

$$\|f_n[Q, \tilde{f}_n]\| \le Cn \cdot \frac{1}{n^2} + C\frac{1}{n^4} \to 0 \ .$$

Hence $\psi_n \to \psi$ and $Q\psi_n \to Q\psi$, i.e., $\psi \in \mathcal{D}(\overline{Q}_{\min})$.

(d) By definition of the operator H associated to the form q we have: $\psi \in \mathcal{D}(H)$ iff

$$\psi \in \mathcal{D}(Q) \quad \text{and} \quad \exists \varphi \in \mathcal{H}\ \forall \eta \in \mathcal{D}(Q) : \ (Q\psi, Q\eta) = (\varphi, \eta) \ , \tag{6}$$

in which case $H\psi = \varphi$. This condition is also equivalent to $Q\psi \in \mathcal{D}(Q^*)$, with $Q^*Q\psi = \varphi$ in case of validity. That proves (d).

(b) In (6) one can replace $\mathcal{D}(Q) \ni \eta$ by the core $\mathcal{C}_0^\infty \otimes \mathbb{C}^2$, which proves (b) with $\mathcal{H}$ replaced by $\mathcal{D}(Q)$. Left to show is that if $\psi \in \mathcal{H}$ with $H\psi\,(= Q^2\psi) \in \mathcal{H}$ (in the sense of distributions), then $Q\psi \in \mathcal{H}$ (in the same sense). By elliptic regularity, $f_n^2 Q\psi \in \mathcal{D}(Q)$ and

$$Q f_n^2 Q\psi = f_n^2 Q^2\psi + 2[Q, f_n]f_n Q\psi \ .$$

This implies $\|f_n Q\psi\|^2 \le C\|\psi\|(\|H\psi\| + \|f_n Q\psi\|)$ and hence that $\|f_n Q\psi\|$ is bounded. We conclude $Q\psi \in \mathcal{H}$ by monotone convergence.

(c) Let $\mathcal{D}(H_{\min}) = \mathcal{C}_0^\infty \otimes \mathbb{C}^2$. By (b), $H^*_{\min} = H$ and thus $\overline{H}_{\min} = H^{**}_{\min} = H^* = H$.

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