Aharonov-Bohm Effect and Time-Dependent Inverse Scattering Theory *

Ricardo Weder †
Instituto Argentino de Matemática, CONICET
Saavedra 15, 1083-Buenos Aires, Argentina
E-Mail: weder@servidor.unam.mx

Abstract

We study the Aharonov-Bohm effect from the point of view of time-dependent inverse scattering theory. As this three dimensional problem is invariant under translations along the vertical axis, it reduces to a problem in $\mathbf{R}^2$. We first consider an unshielded magnetic field that has a singular part produced by a tiny solenoid and a regular part. The wave function is zero at the location of the solenoid. We then consider the case where the singular part of the magnetic field is shielded inside a cylinder whose transversal section is compact set K, and there is also a regular magnetic field. In this case the magnetic field inside K is quite general. Actually, the only condition is that the magnetic flux accross K has to be finite. Moreover, the wave function is defined in $\Omega := \mathbf{R}^2 \setminus K$ and it is zero on ∂K.
We prove that in the unshielded case the scattering operator determines uniquely the regular magnetic field and that in the shielded case it determines uniquely the magnetic field in Ω. Moreover, in the unshielded case the scattering operator determines the magnetic flux of the solenoid modulo 2 and in the shielded case it determines the magnetic flux across K modulo 2. Our results follow from a reconstruction formula with error term.

*2000 AMS classification 35P25, 35R30 and 81U40. Research partially supported by proyecto PAPIIT IN105799, DGAPA-UNAM.

†On leave of absence from Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, Apartado Postal 20-726, México D.F. 01000. Fellow, Sistema Nacional de Investigadores.

1 Introduction

In this paper we study the Aharonov-Bohm effect [1] from the point of view of time-dependent inverse scattering theory. Aharonov and Bohm considered the scattering of an electron off the magnetic field of a tiny solenoid, idealized as having infinite lenght and zero radius (scattering off a thread of magnetic flux). We assume that the solenoid is located on the vertical axis of coordinates, and in consequence it is enough to consider scattering in the plane orthogonal to the solenoid, as the problem is invariant under translation along the vertical direction. The magnetic field of an unshielded solenoid at zero is given by, $B_s\,\delta(x)$ with $x=(x_1,x_2)\in \mathbf{R}^2$ (this actually corresponds to a magnetic field in $\mathbf{R}^3$ with components $(0,0,B_s\,\delta(x))$. We also assume that there is a regular magnetic field $B_R \in C_0^1(\mathbf{R}^2)$ that is continuously differentiable and has compact support. In particular, this allows us to take into account that our idealized solenoid is only an approximation to a real solenoid. The total magnetic field is the sum of the two,

$$B := B_s\,\delta(x) + B_R, \tag{1.1}$$

where, of course, B_s is real and B_R is real valued. In order to define the Schrödinger operator for an electron in the presence of B we have to introduce a magnetic potential. The magnetic potential for $B_s\,\delta(x)$ in the Coulomb gauge is given by,

$$A_0 := \frac{\alpha_{\{0\}}}{|x|^2}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}, \tag{1.2}$$

where, we denote $\alpha_{\{0\}} := \frac{B_s}{2\pi}$. Observe that $\alpha_{\{0\}}$ is the flux (across zero) of the singular magnetic field normalized by 2π. It is easily checked that $\nabla\times A_s := \frac{\partial}{\partial x_1}A_{s,2} - \frac{\partial}{\partial x_2}A_{s,1} = B_s\,\delta(x)$ and that $\nabla\cdot A_s = 0$, with the derivatives taken in distribution sense in $\acute{\mathcal{D}}$. The magnetic potential in the Coulomb gauge for B_R is given by,

$$A_R = \frac{1}{2\pi}\int B_R(x-y)\begin{bmatrix} -\hat{y}_2 \\ \hat{y}_1 \end{bmatrix}\frac{dy}{|y|}, \tag{1.3}$$

where, $\hat{y} := \frac{y}{|y|}$. Clearly, $A_R \in C^1(\mathbf{R}^2, \mathbf{R}^2)$. The magnetic potential for B in the Coulomb gauge is given by,

$$A_c := A_s + A_R. \tag{1.4}$$

As is well known, the magnetic potential is not uniquely defined by the magnetic field, actually there is always the possibility of a gauge transformation. We introduce below a general class of magnetic potentials that is convenient for our purposes.

DEFINITION 1.1. *We denote by $\mathcal{A}_{\{0\}}(\alpha_{\{0\}}, B_R)$ the set of all real-valued $A \in C^1(\mathbf{R}^2 \setminus 0, \mathbf{R}^2) \cap L^1_{\text{loc}}(\mathbf{R}^2, \mathbf{R}^2)$, with $\nabla \times A = 2\pi\alpha_{\{0\}}\, \delta(x) + B_R$, in $\acute{\mathcal{D}}$. Moreover, we assume that $A(x) = O(|x|^{-1}), |x| \to \infty$ and that,*

$$a(r) := \sup_{|x| \geq r} |A(x) \cdot \hat{x}| \in L^1([0, \infty)). \tag{1.5}$$

We now consider the situation where a general singular magnetic field is contained inside an infinite cylinder, with axis along the vertical direction, and transversal section K, where K is a compact subset of $\mathbf{R}^2$. The purpose of the cylinder is to shield the singular magnetic field from the incoming electrons. As we will see below, we can not hope that the scattering operator determines uniquely the magnetic field inside K. In fact, we can only determine the (normalized) flux of the magnetic field across K modulo 2. This suggests that instead of specifying the magnetic field inside K we only fix the magnetic flux across K, normalized by 2π, that by Stoke's theorem is given by the circulation of the magnetic potential, A, along ∂K,

$$\alpha_K := \frac{1}{2\pi} \int_{\partial K} A(x) \cdot dx, \tag{1.6}$$

where we integrate in counter-clockwise sense. Of course, we also specify a regular magnetic field $B_R \in C_0^1\left(\overline{\Omega}\right)$ outside of K. The magnetic flux α_K could be produced, for example, by a finite number of solenoids inside K, and also by a regular magnetic field contained inside K, or by a combination of both. The considerations above suggest the definition of the following class of magnetic potentials. In what follows we will always denote, $\Omega := \mathbf{R}^2 \setminus K$.

DEFINITION 1.2. *Let K be a compact set such that $0 \in K$ and that its boundary, ∂K, is a simple, closed, C^1 curve. Then, for any $\alpha_K \in \mathbf{R}$ and any real-valued $B_R \in C_0^1\left(\overline{\Omega}\right)$ we denote by $\mathcal{A}_K(\alpha_K, B_R)$ the set of all real-valued $A \in C^1\left(\overline{\Omega}, \mathbf{R}^2\right)$ with $\nabla \times A = B_R$ and such that,*

$$\alpha_K = \frac{1}{2\pi} \int_{\partial K} A(x) \cdot dx, \tag{1.7}$$

where we integrate in counter-clockwise sense. Moreover, we assume that $A(x) = O(|x|^{-1}), |x| \to \infty$ and that,

$$a(r) := \sup_{x \in \Omega, |x| \geq r} |A(x) \cdot \hat{x}| \in L^1([0, \infty)). \tag{1.8}$$

As we will see in Appendix A, the magnetic potential in the Coulomb gauge belongs to $\mathcal{A}_K(\alpha_K, B_R)$.

In what follows we will denote by K either $K := \{0\}$ or a set as in Definition 1.2. It is an elementary fact that if $A^{(1)}, A^{(2)} \in \mathcal{A}_K(\alpha_K, B_R)$, then there is a real-valued function, λ, such that if $K = \{0\}$, $\lambda \in C^2\left(\mathbf{R}^2 \setminus 0\right) \cap L^\infty\left(\mathbf{R}^2\right)$ and $\lambda \in C^2\left(\overline{\Omega}\right) \cap L^\infty\left(\overline{\Omega}\right)$ if K is as in Definition 1.2, with the property that,

$$A^{(2)} = A^{(1)} + \nabla\lambda, \tag{1.9}$$

in $\acute{\mathcal{D}}$ if $K = \{0\}$ and in Ω otherwise. Moreover, for any $\hat{x} \in S^1, \lambda_\infty(\hat{x}) := \lim_{r\to\infty} \lambda(r\hat{x})$ exists and it is a continuous function on the circle. We also denote by λ_∞ the extension of this function to a continuous and homogeneous function of degree zero, defined in $\mathbf{R}^2 \setminus 0$, i.e., $\lambda_\infty(r\hat{x}) := \lambda_\infty(\hat{x})$, $r > 0$. For a proof of this fact in the case of regular fields defined in $\mathbf{R}^n$ see [12]. In Appendix A we give details of the proof for the reader's convenience.

The formal Hamiltonian is the operator,

$$h_A := \frac{(\mathbf{p} - A)^2}{2m}, \tag{1.10}$$

with domain $C_0^2(\Omega)$, and where and $\mathbf{p} := -i\nabla$ is the momentum operator,and $m > 0$ is the mass of the electron. We take Planck's constant and the speed of light all equal to one and the charge of the electron equal to minus one. The quadratic form associated to h is the following one,

$$q_A(\phi, \psi) := ((\mathbf{p} - A)\phi, (\mathbf{p} - A)\psi), \tag{1.11}$$

with domain $C_0^1(\Omega)$. The form q_A is non-negative and closable. The Hamiltonian, H_A, is the self-adjoint operator in $L^2(\Omega)$ associated to the closure of q_A (see [14]). H_A is the extension of h_A with Dirichlet boundary condition on $\partial\Omega$. Under the gauge transformation (1.9) the Hamiltonians are unitarily equivalent:

$$H_{A^{(2)}} = e^{i\lambda(x)}\, H_{A^{(1)}}\, e^{-i\lambda(x)}. \tag{1.12}$$

Let J be the identification operator from $L^2\left(\mathbf{R}^2\right)$ onto $L^2(\Omega)$ given by multiplication by the characteristic function of Ω. In the case $K = \{0\}$ we take $J = I$. The unperturbed Hamiltonian is given by $H_0 := \frac{\mathbf{p}^2}{2m}$, with domain the Sobolev space $W_{2,2}\left(\mathbf{R}^2\right)$. The wave operators are defined as,

$$W_\pm(A) := \mathrm{s} - \lim_{t\to\pm\infty} e^{itH_A}\, J\, e^{-itH_0}. \tag{1.13}$$

We prove in Section 2 that if $A \in \mathcal{A}_K(\alpha_K, B_R)$, the strong limits exist and are isometric and moreover, that the wave operators transform as follows under a change of gauge,

$$W_\pm\left(A^{(2)}\right) = e^{i\lambda(x)}\, W_\pm\left(A^{(1)}\right)\, e^{-i\lambda_\infty(\pm \mathbf{p})}. \tag{1.14}$$

The scattering operator is given by,

$$S(A) := W_+^*\,(A)\;\; W_-\,(A)\,. \tag{1.15}$$

Equation (1.14) implies that $S(A)$ transforms under a change of gauge according to,

$$S(A^{(2)}) = e^{i\lambda_\infty(\mathbf{p})} S(A^{(1)}) e^{-i\lambda_\infty(-\mathbf{p})}. \tag{1.16}$$

For a similar result for the Dirac equation in all space and with regular magnetic field see [12] and [13].Under the general class of gauge transformations that we consider the scattering operator is not gauge invariant. However, it is changed by unitary operators that are only a function of momentum. As we will see below, this leaves invariant relative phases in configuration space and hence, makes it possible to solve the inverse problem -in a gauge invariant way- taking the scattering operator corresponding to any magnetic potential in $\mathcal{A}_K(\alpha_K, B_R)$. Observe that to define H_A we only use the values of A in Ω. This means that as long as A is fixed in Ω, we can change the magnetic potential in the interior of K without changing H_A. Note however that as $A \in C^1\left(\overline{\Omega}\right)$ the flux α_K is uniquely defined by the values of A in Ω. This explains why we can not hope to uniquely reconstruct the magnetic field inside K from the scattering operator and makes it plausible that we can reconstruct α_K.

As we said above, the only purpose of the obstacle, K, is to shield the incoming electron from the magnetic field, and in order to separate the scattering effect of the magnetic potential from the one of the obstacle, we consider asymptotic configurations that have negligible interaction with K for all times in the high-velocity limit. For this purpose, given $\hat{\mathbf{v}} \in S^1$, let us denote,

$$\Omega_{\hat{\mathbf{v}}} := \{x \in \Omega : x + \hat{\mathbf{v}}\tau \in \Omega, \text{ for all } \tau \in \mathbf{R}\}\,. \tag{1.17}$$

Given $\mathbf{v} \in \mathbf{R}^2$ we take asymptotic configurations $\Phi \in C_0^\infty(\Omega_{\hat{\mathbf{v}}})$, where $\hat{\mathbf{v}} := \mathbf{v}/v$, with $v := |\mathbf{v}|$. The free evolution boosted by $\mathbf{v}$ is given by $e^{-im\mathbf{v}\cdot x}\, e^{-itH_0}\, e^{im\mathbf{v}\cdot x} = e^{-imv^2t/2} e^{-i\mathbf{p}\cdot\mathbf{v}t} e^{-itH_0}$, and -to a good approximation-in the limit when $v \to \infty$ with $\hat{\mathbf{v}}$ fixed this can be replaced (modulo an unimportant phase) by the classical translation $e^{-it\mathbf{p}\cdot\mathbf{v}}$. Then, in the high-velocity limit it is a good approximation to assume that the free evolution of our asymptotic configuration is given by $e^{-it\mathbf{p}\cdot\mathbf{v}}\Phi_0 = \Phi_0(x - \mathbf{v}t)$, and as $\Phi_0 \in C_0^\infty(\Omega_{\hat{\mathbf{v}}})$, it has negligible interaction with K for all times. In the following theorem we evaluate the high-velocity asymptotics of the scattering operator.

THEOREM 1.3. *Suppose that $A \in \mathcal{A}_K(\alpha_K, B_R)$ and that $\Phi_0, \Psi_0 \in C_0^\infty(\Omega_{\hat{\mathbf{v}}})$. Let $\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}$ be the boosted asymptotic configurations,*

$$\Phi_{\mathbf{v}} := e^{im\mathbf{v}\cdot x}\Phi_0, \;\; \Psi_{\mathbf{v}} := e^{im\mathbf{v}\cdot x}\Psi_0. \tag{1.18}$$

Then,

$$(S(A)\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) = \left(e^{i\int_{-\infty}^{\infty} \hat{\mathbf{v}}\cdot A(x+\hat{\mathbf{v}}\tau)\, d\tau}\Phi_0, \Psi_0\right) + O\left(\frac{1}{|v|}\right), \; |v| \to \infty. \tag{1.19}$$

By Theorem 1.3, from the high-velocity limit of the scattering operator we uniquely reconstruct the phase,

$$e^{i\int_{-\infty}^{\infty} \hat{\mathbf{v}}\cdot A(x+\hat{\mathbf{v}}\tau)\, d\tau}, \; x \in \Omega_{\hat{\mathbf{v}}}. \tag{1.20}$$

Equation (1.9) implies that if $A^{(1)}, A^{(2)} \in \mathcal{A}_K(\alpha_K, B_R)$, then

$$\int_{-\infty}^{\infty} \hat{\mathbf{v}} \cdot A^{(2)}(x + \hat{\mathbf{v}}\tau)\, d\tau = \int_{-\infty}^{\infty} \hat{\mathbf{v}} \cdot A^{(1)}(x + \hat{\mathbf{v}}\tau)\, d\tau + \lambda_\infty(\hat{\mathbf{v}}) - \lambda_\infty(-\hat{\mathbf{v}}). \tag{1.21}$$

In consequence, (1.20) is not gauge invariant. However, the function

$$\mathcal{R}(x, y, \hat{\mathbf{v}}) := e^{i\int_{-\infty}^{\infty} \hat{\mathbf{v}}\cdot A(x+\hat{\mathbf{v}}\tau)\, d\tau - i\int_{-\infty}^{\infty} \hat{\mathbf{v}}\cdot A(y+\hat{\mathbf{v}}\tau)\, d\tau}, \;\; x, y \in \Omega_{\hat{\mathbf{v}}}, \tag{1.22}$$

is gauge invariant and this allows us to solve the inverse scattering problem in a gauge invariant way. We prove in Section 2 that (1.22) determines uniquely the Radon transforms,

$$\int_{-\infty}^{\infty} B_R(x + \hat{\mathbf{v}}\tau)\, d\tau, \; x \in \Omega_{\hat{v}}. \tag{1.23}$$

By the support theorem for the Radon transform [10] [16] B_R is uniquely determined in Ω, provided that K is convex. The fact that the magnetic flux, α_K, is determined modulo 2 follows from an explicit calculation. This gives us the following theorem.

THEOREM 1.4. *Suppose that $A^{(j)} \in \mathcal{A}_K \left(\alpha_K^{(j)}, B_R^{(j)}\right), j = 1,2$ and that K is convex. Then, if $S\left(A^{(1)}\right) = S\left(A^{(2)}\right)$, we have that, $\alpha_K^{(1)} = \alpha_K^{(2)}$ modulo 2 and that $B_R^{(1)}(x) = B_R^{(2)}(x), x \in \Omega$.*

Observe that since (1.22) is obtained from (1.19) we actually only need to know the high-velocity limit of the scattering operator. Moreover, our proof gives a method for the reconstruction of B_R in Ω. Note that in spite of the fact that the scattering of the electron takes place outside of K, we determine the magnetic flux in K- modulo 2- from the scattering operator. This is the Aharonov-Bohm effect [1] that shows that in quantum mechanics the magnetic field acts on a charged particle -by means of the magnetic potential- even in regions where it is identically zero. Nicoleau has proven in [17] the following result using stationary methods. Suppose that the magnetic field, B, is infinitely differentiable in $\mathbf{R}^2$ and has compact support, and that K is compact, convex, $0 \in K$ and ∂K is smooth. Then, if $S\left(A^{(1)}\right) = S\left(A^{(2)}\right)$, with $A^{(j)}, j = 1,2$, the Coulomb gauge potentials, then the magnetic fluxs across $\mathbf{R}^2$ are equal modulo 2 and $B_R^{(1)}(x) = B_R^{(2)}(x), x \in \Omega$. The case of an unshielded solenoid with $K = \{0\}$ is not covered by the result of [17]. Note that this is actually the problem considered by Aharonov and Bohm [1]. In the case where the interior of K is non-empty [17] considers the situation where the magnetic flux across K is produced by a magnetic field in $C_0^\infty(\mathbf{R}^2)$. Our result is considerably more general in the sense that we study the case where the magnetic flux across K is produced by any magnetic field inside K. The only restriction is that the magnetic flux across K has to be finite. We could as well consider the case where there is also a scalar potential-as is done in [17]- however, we do not pursue this direction here.

We prove Theorems 1.3 and 1.4 by extending to this case the time-dependent method for inverse scattering theory [7], [8]. This method was previously extended to Schrödinger operators with magnetic fields in [2], [3] and [4] and to Dirac operators in [12] and [13].These papers study the problem $\mathbf{R}^n$ (i.e. when $K = \emptyset$) and with regular magnetic fields.

Since the asymptotic configurations that have negligible interaction with the obstacle K have compact support in configuration space we have to introduce an energy cut-off and this requires the extension of the propagation estimate of [6] -that was used in [7], [8]- to the case where the momentum cut-off has support in a ball whose radious increases as the velocity tends to infinity (see Lemma 2.1).

The Aharonov-Bohm effect has been extensively studied. In [22] Ruijsenaars analyses in great detail the direct scattering problem -with time-dependent and stationary methods- in the case where $B_R \equiv 0$

and K is either $\{0\}$ or a circle. This problem has an explicit solution. The paper [22] also contains an extensive review of the literature up to 1982. For reviews of the physics literature see [18] and [19], and for related results see [11]. In the mathematics literature the Aharonov-Bohm effect was studied from the point of view of semiclassical analysis in [15] and [5]. In [9] this effect was studied in eigenvalues.

2 The Proofs

We first prove a propagation estimate that is an extension of the estimate of [6]-that was used in [7], [8]- to the case where the momentum cut-off has support in a ball with center $m\mathbf{v}$ and whose radius increases as $v^\rho, \rho > 0$, $v \to \infty$. We denote by $F(x \in \mathcal{M})$ the operator of multiplication by the characteristic function of $\mathcal{M}$. For any Borel function f let us define the operator $f(\mathbf{p})$ by functional calculus, or equivalently, as $f(\mathbf{p}) := \mathcal{F}^{-1} f(\cdot) \mathcal{F}$, where $\mathcal{F}$ denotes the Fourier transform. Remark that under translation in configuration or momentum space generated, respectively, by $\mathbf{p}$ and x we obtain,

$$e^{i\mathbf{p}\cdot\mathbf{v}t} f(x)\, e^{-i\mathbf{p}\cdot\mathbf{v}t} = f(x + \mathbf{v}t), \tag{2.1}$$

$$e^{-im\mathbf{v}\cdot x} f(\mathbf{p})\, e^{im\mathbf{v}\cdot x} = f(\mathbf{p} + m\mathbf{v}), \tag{2.2}$$

and in particular,

$$e^{-im\mathbf{v}\cdot x}\, e^{-itH_0}\, e^{im\mathbf{v}\cdot x} = e^{-imv^2t/2} e^{-i\mathbf{p}\cdot\mathbf{v}t} e^{-itH_0}. \tag{2.3}$$

For any $\mathbf{v} \in \mathbf{R}^n, n = 1, 2, \cdots$, let us denote by $B_{m\eta}(m\mathbf{v})$ the open ball in $\mathbf{R}^n$, of center $m\mathbf{v}$ and radius $m\eta$. In the case where $\mathbf{v} = 0$ we use the notation $B_{m\eta}$. We designate by $\hat{f}$ the Fourier transform of f, $\hat{f}(p) := 1/(2\pi)^{n/2} \int e^{-ip\cdot x} f(x)\, dx$.

LEMMA 2.1. *For any $f \in C_0^\infty(B_{m\eta})$, for some $\eta > 0$, and any $l = 1, 2, \cdots$, there is a constant C_l such that the following estimate holds:*

$$\left\| F(x \in \tilde{\mathcal{M}})\, e^{-itH_0}\, f\left(\frac{\mathbf{p} - m\mathbf{v}}{v^\rho}\right) F(x \in \mathcal{M}) \right\| \leq C_l (1 + r v^\rho + \eta\, v^{2\rho}|t|)^{-l}, \tag{2.4}$$

for every $\mathbf{v} \in \mathbf{R}^n$, $t \in \mathbf{R}$ and any measurable sets $\mathcal{M}$ and $\tilde{\mathcal{M}}$ in $\mathbf{R}^n$ such that $r := \mathrm{dist}\left(\tilde{\mathcal{M}}, \mathcal{M} + \mathbf{v}t\right) - \eta v^\rho |t| \geq 0$.

Proof: by (2.1)-(2.3) it is enough to prove the estimate,

$$\left\| F(x \in \tilde{\mathcal{M}})\, e^{-itH_0}\, f\left(\frac{\mathbf{p}}{v^\rho}\right) F(x \in \mathcal{M}) \right\| \leq C_l\, (1 + r v^\rho + \eta\, v^{2\rho} |t|)^{-l}, \tag{2.5}$$

provided that, $r := \text{dist}\left(\tilde{\mathcal{M}}, \mathcal{M}\right) - \eta v^\rho |t| \geq 0$. We have that,

$$\left\| F(x \in \tilde{\mathcal{M}})\, e^{-itH_0}\, f\left(\frac{\mathbf{p}}{v^\rho}\right)\, F(x \in \mathcal{M})\phi \right\|^2 \leq \frac{1}{(2\pi)^n} \int_{\tilde{\mathcal{M}}} dx \int_{\mathcal{M}} dy \int_{\mathcal{M}} dz\, \tilde{f}_t(x-y)\, \phi(y)\, \overline{\tilde{f}_t(x-z)\, \phi(z)}, \tag{2.6}$$

where,

$$\tilde{f}_t(x) := \frac{1}{(2\pi)^{n/2}} \int e^{ip\cdot x} e^{-ip^2 t/2m}\, f(p/v^\rho)\, dp. \tag{2.7}$$

But, as $2|\phi(y)|\, |\phi(z)| \leq |\phi(y)|^2 + |\phi(z)|^2$,

$$\left\| F(x \in \tilde{\mathcal{M}})\, e^{-itH_0}\, f\left(\frac{\mathbf{p}}{v^\rho}\right)\, F(x \in \mathcal{M})\phi \right\|^2 \leq \frac{1}{(2\pi)^n} \left[\int_{|x| \geq r + \eta v^\rho |t|} |\tilde{f}_t(x)|\, dx \right]^2 \|\phi\|^2. \tag{2.8}$$

Finally, by the Corollary to Theorem XI.14 in [21], for any $N = 1, 2, \cdots$, there is a constant C_N such that,

$$|\tilde{f}_t(x)| \leq C_N\, v^{n\rho}\, (1 + |x| v^\rho + v^{2\rho} |t|)^{-N}, \tag{2.9}$$

for $x \geq \eta v^\rho |t|$. The lemma follows inserting (2.9) in (2.8).

COROLLARY 2.2. *For any $Q \in \mathbf{R}^2$, $f \in C_0^\infty(B_{m\eta})$, $0 \leq \rho < 1$ and for any $l = 1, 2, \cdots$, there is a constant C_l such that,*

$$\left\| F(|x - Q - \mathbf{v}t| > |\mathbf{v}t|/4)\, e^{-itH_0}\, f\left(\frac{\mathbf{p} - m\mathbf{v}}{v^\rho}\right) F(|x - Q| \leq |\mathbf{v}t|/8) \right\| \leq C_l (1 + |\mathbf{v}t|)^{-l}, \tag{2.10}$$

for $v > (8\eta)^{1/(1-\rho)}$.

Proof : the corollary follows from Lemma 2.1 with $\tilde{\mathcal{M}} = \{|x - Q - \mathbf{v}t| > |\mathbf{v}t|/4\}$, $\mathcal{M} = \{|x - Q| \leq |\mathbf{v}t|/8\}$. Observe that $r := \text{dist}\left(\tilde{\mathcal{M}}, \mathcal{M} + \mathbf{v}t\right) - \eta v^\rho |t| \geq |\mathbf{v}t|/8 - \eta v^\rho |t| \geq 0$ if $v > (8\eta)^{1/(1-\rho)}$.

$\blacksquare$

In the proofs of Lemmas 2.3 and 2.4 below we first use a special shifted transversal gauge and we then transfer the result to a general gauge. Similar ideas were used previously in [4] and [12], respectively, in the cases of the Schrödinger and the Dirac equation in all space with regular magnetic field.

LEMMA 2.3. *The wave operators $W_\pm(H_A)$ exist and are isometric for all $A \in \mathcal{A}_K(\alpha_K, B_R)$. Moreover, if $A^{(1)}, A^{(2)} \in \mathcal{A}_K(\alpha_K, B_R)$ with $A^{(2)} = A^{(1)} + \nabla\lambda$, then,*

$$W_\pm\left(A^{(2)}\right) = e^{i\lambda(x)}\, W_\pm\left(A^{(1)}\right)\, e^{-i\lambda_\infty(\pm\mathbf{p})}. \tag{2.11}$$

Proof : Let $\chi \in C^\infty(\mathbf{R}^2)$ satisfy $\chi(x) = 0$ for x in a neigborhood of K and $\chi(x) = 1$ for $|x| \geq M$, with M large enough. Then as $(1-\chi(x))(H_0+1)^{-1}$ is a compact operator,

$$W_\pm(A)\phi = \mathrm{s} - \lim_{t\to\pm\infty} e^{itH(A)}\chi(x)e^{-itH_0}\phi, \tag{2.12}$$

for all $\phi \in D(H_0)$, and by density for all $\phi \in L^2$. We prove the existence of $W_+(A)$. The case of $W_-(A)$ follows in the same way. It is enough to prove that the strong limits (2.12) exist for all $\Phi_{\mathbf{v}}$ with $\hat{\Phi}_{\mathbf{v}} \in C_0^\infty(B_{m\eta}(m\mathbf{v}))$, for all $\mathbf{v}, \eta$ with $v > 8\eta$, since the set of all these functions is dense in $L^2(\mathbf{R}^2)$. Note that $\Phi_{\mathbf{v}} = e^{im\mathbf{v}\cdot x}\Phi_0$ with $\hat{\Phi}_0 \in C_0^\infty(B_{m\eta})$. Let us take a gauge $A^{(f,Q)}$, as in (3.14) in Appendix A, with f, Q such that,

$$A^{(f,Q)}(x)\, F(|x - Q - \hat{\mathbf{v}}\tau| \leq |\tau|/4) = 0,\, \tau \in \mathbf{R}. \tag{2.13}$$

By Duhamel's formula,

$$W_+(A^{(f,Q)})\Phi_{\mathbf{v}} = \chi(x)\Phi_{\mathbf{v}} + i\int_0^\infty dt\, e^{itH_{A^{(f,Q)}}}\, V(x,\mathbf{p})\, e^{-itH_0}\,\Phi_{\mathbf{v}}, \tag{2.14}$$

where,

$$V(x,\mathbf{p}) := \chi(x)\left[-\frac{A^{(f,Q)}\cdot\mathbf{p}}{m} + \frac{\left(-\mathbf{p}\cdot A^{(f,Q)}\right) + \left(A^{(f,Q)}\right)^2}{2m}\right] - \frac{1}{2m}\left((\Delta\chi) + 2i(\nabla\chi)\cdot\mathbf{p} + 2A^{(f,Q)}\cdot(\mathbf{p}\chi)\right). \tag{2.15}$$

Let $g \in C_0^\infty(B_{m\eta})$ satisfy, $g \equiv 1$ on the support of $\hat{\Phi}_0$. Then, By (2.13)

$$\left\|\chi A^{(f,Q)}e^{-itH_0}\left(\mathbf{p}-m\mathbf{v}\right)\Phi_{\mathbf{v}}\right\| \leq C\left[\left\|\chi A^{(f,Q)}F(|x-Q-\mathbf{v}t| > |\mathbf{v}t|/4)e^{-itH_0}\, g(\mathbf{p}-m\mathbf{v})\right.\right.$$

$$(\mathbf{p}-m\mathbf{v})F(|x-Q|\leq|\mathbf{v}t|/8)\| \, \|\Phi_0\| + \|F(|x-Q|>|\mathbf{v}t|/8) \ \Phi_0\|] \, . \tag{2.16}$$

By (2.10) with $\rho=0$,

$$\left\|\chi A^{(f,Q)}F(|x-Q-\mathbf{v}t|>|\mathbf{v}t|/4)e^{-itH_0}\, g(\mathbf{p}-m\mathbf{v})(\mathbf{p}-m\mathbf{v})F(|x-Q|\leq|\mathbf{v}t|/8)\right\|\leq C_l(1+|\mathbf{v}t|)^{-l}. \tag{2.17}$$

As Φ_0 belongs to the Schwartz class,

$$\|F(|x-Q|>|\mathbf{v}t|/8)\,\Phi_0\|\leq C_l(1+|\mathbf{v}t|)^{-l},\ l=1,2,\cdots, \tag{2.18}$$

and it follows that,

$$\left\|\chi A^{(f,Q)}e^{-itH_0}\,(\mathbf{p}-m\mathbf{v})\,\Phi_{\mathbf{v}}\right\|\leq C_l(1+|\mathbf{v}t|)^{-l}. \tag{2.19}$$

In a similar way we prove that,

$$\left\|\chi A^{(f,Q)}e^{-itH_0}\,m\mathbf{v}\,\Phi_{\mathbf{v}}\right\|\leq C_l\,v\,(1+|\mathbf{v}t|)^{-l}, \tag{2.20}$$

and that,

$$\left\|\chi\left((\nabla\cdot A^{(f,Q)})+\left(A^{(f,Q)}\right)^2\right)\,e^{-itH_0}\,\Phi_{\mathbf{v}}\right\|\leq C_l(1+|\mathbf{v}t|)^{-l}. \tag{2.21}$$

Moreover,

$$\begin{aligned}\left\|(\Delta\chi(x))\,e^{-itH_0}\,\Phi_{\mathbf{v}}\right\|&\leq C\left[\left\|F(|x-\mathbf{v}t|>|\mathbf{v}t|/4)\,e^{-itH_0}\,g(\mathbf{p}-m\mathbf{v})F(|x|\leq|\mathbf{v}t|/8)\right\|\,\|\Phi_0\|\right.\\ &\left.+\left\|(\Delta\chi(x))\,F(|x-\mathbf{v}t|\leq|\mathbf{v}t|/4)\,e^{-itH_0}\,g(\mathbf{p}-m\mathbf{v})F(|x|\leq|\mathbf{v}t|/8)\right\|\,\|\Phi_0\|+\|F(|x|>|\mathbf{v}t|/8)\,\Phi_0\|\right].\end{aligned} \tag{2.22}$$

Since $\Delta\chi$ has compact support,

$$\begin{aligned}&\left\|(\Delta\chi(x))\,F(|x-\mathbf{v}t|\leq|\mathbf{v}t|/4)e^{-itH_0}\,g(\mathbf{p}-m\mathbf{v})F(|x|\leq|\mathbf{v}t|/8)\right\|\leq\\ &C\,\|(\Delta\chi(x))\,F(|x-\mathbf{v}t|\leq|\mathbf{v}t|/4)\|\leq C\,\|(\Delta\chi(x))\,F(|x|\geq 3|\mathbf{v}t|/4)\|\leq C_l(1+|\mathbf{v}t|)^{-l},\end{aligned} \tag{2.23}$$

and we prove as above that,

$$\left\|(\Delta\chi(x))\,e^{-itH_0}\,\Phi_{\mathbf{v}}\right\|\leq C_l(1+|\mathbf{v}t|)^{-l}. \tag{2.24}$$

In the same way we obtain that

$$\left\|(\nabla\chi)\cdot\mathbf{p}\, e^{-itH_0}\,\Phi_{\mathbf{v}}\right\| \leq C_l\, v\,(1+|\mathbf{v}t|)^{-l}, \tag{2.25}$$

and that,

$$\left\|A^{(f,Q)}\cdot(\mathbf{p}\chi)\, e^{-itH_0}\,\Phi_{\mathbf{v}}\right\| \leq C_l\,(1+|\mathbf{v}t|)^{-l}, \tag{2.26}$$

By (2.15),(2.19),(2.20),(2.21),(2.24), (2.25)and (2.26),

$$\left\|V(x,\mathbf{p})e^{-itH_0}\,\Phi_{\mathbf{v}}\right\| \leq C_l\, v\,(1+|\mathbf{v}t|)^{-l},\ l=1,2,\cdots, \tag{2.27}$$

and this implies that the integral in the right-hand side of (2.14) is absolutely convergent. In particular, $W_+(A^{(f,Q)})\Phi_{\mathbf{v}}$ exists. Let λ satisfy (see (1.9) and Remark 3.1 in Appendix A), $A=A^{(f,Q)}+\nabla\lambda$. Since, $H_A=e^{i\lambda(x)}\,H_{A^{(f,Q)}}\,e^{-i\lambda(x)}$, we have that for $\phi\in D(H_0)$,

$$W_+(A)\phi_{\mathbf{v}}=e^{i\lambda(x)}\,\mathrm{s}-\lim_{t\to\infty}e^{itH_{A^{(f,Q)}}}e^{-i\lambda(x)}\,J\,e^{-itH_0}\phi_{\mathbf{v}}=e^{i\lambda(x)}\,\mathrm{s}-\lim_{t\to\infty}e^{itH_{A^{(f,Q)}}}\,J\,e^{-i\lambda_\infty(x)}e^{-itH_0}\phi_{\mathbf{v}}, \tag{2.28}$$

where we used the fact that by (3.11) if K is contained inside the open ball of center zero and radius M, then $|\lambda(x)-\lambda_\infty(x)|\leq\int_{|x|}^\infty a(r)\,dr$, for $|x|\geq M$, and then by Rellich's selection theorem the multiplication operator $e^{-i\lambda(x)}-e^{-i\lambda_\infty(x)}$ is compact from $D(H_0)$ into L^2. As, $e^{itH_0}\,x\,e^{-itH_0}=x+\mathbf{p}t/m$, it follows by functional calculus that, $e^{itH_0}\,\lambda_\infty(x)\,e^{-itH_0}=\lambda_\infty(x+\mathbf{p}t/m)$, and as λ_∞ is homogeneous of degree zero, $e^{itH_0}\lambda_\infty(x)e^{-itH_0}=\lambda_\infty(\frac{mx}{t}+\mathbf{p}), t>0$. Finally, as $\mathrm{s}-\lim_{t\to\infty}\lambda_\infty(mx/t+\mathbf{p})=\lambda_\infty(\mathbf{p})$, and by (2.28),

$$W_+(A)\phi_{\mathbf{v}}=e^{i\lambda(x)}\,\mathrm{s}-\lim_{t\to\infty}e^{itH_{A^{(f,Q)}}}\,Je^{-itH_0}\;e^{itH_0}e^{-i\lambda_\infty(x)}e^{-itH_0}\phi_{\mathbf{v}}=e^{i\lambda(x)}\,W_+\left(A^{(f,Q)}\right)\,e^{-i\lambda_\infty(\mathbf{p})}\phi_{\mathbf{v}}. \tag{2.29}$$

This proves the existence of $W_+(A)$. Note that the same argument proves (2.11) for $W_+(A)$. In the case of $W_-(A)$ we proceed as above using that for $t<0$, $\lambda_\infty(x+\mathbf{p}t/m)=\lambda_\infty(-\frac{mx}{t}-\mathbf{p})$. The fact that the wave operators are isometric is immediate because $e^{itH(A)}$ and e^{-itH_0} are unitary and $(1-\chi_\Omega)(H_0+1)^{-1}$ is a compact operator.

LEMMA 2.4. *For any $A\in\mathcal{A}_K(\alpha_K,B_R)$, and every $\Phi_0\in C_0^\infty(\Omega_{\hat{\mathbf{v}}})$,*

$$\left\|\left(e^{-im\mathbf{v}\cdot x}W_\pm(A)\,e^{im\mathbf{v}\cdot x}-e^{-i\int_0^{\pm\infty}\hat{\mathbf{v}}\cdot A(x+\hat{\mathbf{v}}\tau)\,d\tau}\right)\Phi_0\right\|=O\left(1/v\right),\ v\to\infty. \tag{2.30}$$

Proof: We give the proof for $W_+(A)$. The case of $W_-(A)$ follows in the same way. let $A^{(f,Q)}$ be the gauge chosen in the proof of Lemma 2.3 . We first prove that,

$$\left\| \left(e^{-im\mathbf{v}\cdot x}\, W_+ \left(A^{(f,Q)} \right) \, e^{im\mathbf{v}\cdot x} - e^{-i\int_0^\infty \hat{\mathbf{v}}\cdot A^{(f,Q)}(x+\hat{\mathbf{v}}\tau)\, d\tau} \right) \Phi_0 \right\| = O\left(1/v\right), \, v \to \infty. \tag{2.31}$$

As $\Phi_0 \in C_0^\infty\left(\Omega_{\hat{\mathbf{v}}}\right)$, distance $(K, \{x : x = y + \hat{\mathbf{v}}\tau, y \in \text{support}\, \Phi_0, \tau \in \mathbf{R}\}) > 0$. Hence, there is a function $\chi \in C^\infty(\mathbf{R}^2)$ such that $\chi \equiv 0$ for x in a neighborhood of K and $\chi \equiv 1$, for $x \in \{x : x = y + \hat{\mathbf{v}}\tau, y \in \text{support}\, \Phi_0, \tau \in \mathbf{R}\} \cup \{x : |x| \geq M\}$, for some M large enough. We designate, $\tilde{A}(x) := \chi(x) A^{(f,Q)}(x)$. Let us denote,

$$H_1 := \frac{1}{v} e^{-im\mathbf{v}\cdot x}\, H_0\, e^{im\mathbf{v}\cdot x}, \text{ and } H_2 := \frac{1}{v} e^{-im\mathbf{v}\cdot x}\, H \left(A^{(f,Q)} \right) \, e^{im\mathbf{v}\cdot x}. \tag{2.32}$$

Observe that,

$$e^{-im\mathbf{v}\cdot x}\, W_+ \left(A^{(f,Q)} \right) \, e^{im\mathbf{v}\cdot x} = \text{s} - \lim_{t\to\infty} e^{itH_2}\, \chi(x)\, e^{-itH_1}. \tag{2.33}$$

Moreover, as $\tilde{A}(x + \hat{\mathbf{v}}\tau) = A^{(f,Q)}(x + \hat{\mathbf{v}}\tau), x \in \text{support}\, \Phi_0$,

$$\left(e^{-im\mathbf{v}\cdot x}\, W_+ \left(A^{(f,Q)} \right) \, e^{im\mathbf{v}\cdot x} - e^{-i\int_0^\infty \hat{\mathbf{v}}\cdot A^{(f,Q)}(x+\hat{\mathbf{v}}\tau)\, d\tau} \right) \Phi_0 = \text{s-} \lim_{t\to\infty} \left[e^{itH_2}\, \chi(x)\, e^{-itH_1} - e^{-i\int_0^t \hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\tau)\, d\tau} \right] \Phi_0. \tag{2.34}$$

Let $g \in C_0^\infty(\mathbf{R}^2)$ satisfy, $g(k) = 1, |k| \leq 1$ and $g(k) = 0, |k| \geq 2$. Then, as $\hat{\Phi}_0$ belongs to the Schwartz space, we have that for every integer N,

$$\left\| \left[g\left(\frac{\mathbf{p}}{v^\rho} \right) - 1 \right] \, \Phi_0 \right\|_{L^2(\mathbf{R}^2)} = O\left(1/\, v^{\rho N} \right), v \to \infty, \tag{2.35}$$

where $\rho > 0$. Then, it is enough to prove that,

$$\left\| \lim_{t\to\infty} \left[e^{itH_2}\, \chi(x)\, e^{-itH_1} - e^{-i\int_0^t \hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\tau)\, d\tau} \right] \tilde{\Phi} \right\| = O\left(1/v\right), \, v \to \infty, \tag{2.36}$$

with $\tilde{\Phi} := g\left(\frac{\mathbf{p}}{v^\rho}\right) \Phi_0$, and $0 < \rho < 1$.

By Duhamel's formula,

$$\left(e^{itH_2}\, \chi(x)\, e^{-itH_1} - \chi(x) e^{-i\int_0^t \hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\tau)\, d\tau} \right) \tilde{\Phi} = \int_0^t d\tau e^{i\tau H_2} i \left[H_2 e^{-i\int_0^{t-\tau} \hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\mu) d\mu} \chi(x) \right.$$

$$-e^{-i\int_0^{t-\tau}\hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\mu)d\mu}\chi(x)(H_1-\hat{\mathbf{v}}\cdot\tilde{A}(x+(t-\tau)\hat{\mathbf{v}}))\Big]\, e^{-i\tau H_1}\,\tilde{\Phi}. \tag{2.37}$$

It follows from the identity [2], [3],

$$(\mathbf{p}-\tilde{A}(x))e^{-i\int_0^t\hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\tau)d\tau}=e^{-i\int_0^t\hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\tau)d\tau}\,(\mathbf{p}-b(x,t)),\ \ b(x,t):=\tilde{A}(x+\hat{\mathbf{v}}t)+\int_0^t(\hat{\mathbf{v}}\times\tilde{B})(x+\hat{\mathbf{v}}\mu)\,d\mu, \tag{2.38}$$

with $(\hat{\mathbf{v}}\times\tilde{B})_i:=\sum_{j=1}^2(\tilde{F}_{ij})\hat{\mathbf{v}}_j$, $\tilde{F}_{ij}:=\frac{\partial}{\partial x_i}\tilde{A}_j-\frac{\partial}{\partial x_j}\tilde{A}_i$, and a simple computation that,

$$\Big(H_2\,e^{-i\int_0^{t-\tau}\hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\mu)d\mu}\chi(x)-e^{-i\int_0^{t-\tau}\hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\mu)d\mu}\chi(x)[H_1-\hat{\mathbf{v}}\cdot\tilde{A}(x+(t-\tau)\hat{\mathbf{v}})]\Big)=T_1+T_2+T_3, \tag{2.39}$$

where,

$$T_1:=\frac{1}{v}\,e^{-i\int_0^{t-\tau}\hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\mu)d\mu}\frac{-1}{2m}\Big\{\chi(x)\left[\mathbf{p}\ b(x,t-\tau)+b(x,t-\tau)\,\mathbf{p}-(b(x,t-\tau))^2\right]$$
$$+(\Delta\chi(x))-2(\mathbf{p}\chi(x))\cdot\mathbf{p}+2\tilde{A}(x)\cdot(\mathbf{p}\chi(x))\Big\}, \tag{2.40}$$

$$T_2:=\frac{1}{v}\,e^{-i\int_0^{t-\tau}\hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\mu)d\mu}\left\{\frac{\chi(x)}{2m}\left[-\Big(\mathbf{p}\cdot(A^{(f,Q)}(x)-\tilde{A}(x))\Big)+|A^{(f,Q)}(x)|^2-|\tilde{A}(x)|^2\right]\right.$$
$$\left.-\frac{\chi(x)}{m}(A^{(f,Q)}(x)-\tilde{A}(x))\cdot(\tilde{A}(x)+\mathbf{p}-b(x,t-\tau))-\frac{1}{m}(\mathbf{p}\chi(x))\cdot(A^{(f,Q)}(x)-\tilde{A}(x))\right\}, \tag{2.41}$$

$$T_3:=e^{-i\int_0^{t-\tau}\hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\mu)d\mu}\left[(\mathbf{p}\chi(x))\cdot\hat{\mathbf{v}}-\chi(x)(\ A^{(f,Q)}(x)-\tilde{A}(x))\cdot\hat{\mathbf{v}}\right]. \tag{2.42}$$

Equation (2.2) implies that,

$$T_1\,e^{-i\tau H_1}\,\tilde{\Phi}=\frac{1}{v}\,e^{-im\mathbf{v}\cdot x}\,e^{-i\int_0^{t-\tau}\hat{\mathbf{v}}\cdot\tilde{A}(x+\hat{\mathbf{v}}\mu)d\mu}\frac{-1}{2m}\left\{\chi(x)\left[(\mathbf{p}-m\mathbf{v})\ b(x,t-\tau)+b(x,t-\tau)\,(\mathbf{p}-m\mathbf{v})\right.\right.$$
$$\left.\left.-\,(b(x,t-\tau))^2\right]+(\Delta\chi(x))-2(\mathbf{p}\chi(x))\cdot(\mathbf{p}-m\mathbf{v})+2\tilde{A}(x)\cdot(\mathbf{p}\chi(x))\right\}\,e^{-iH_0\tau/v}e^{im\mathbf{v}\cdot x}\tilde{\Phi}. \tag{2.43}$$

Furthermore, by (2.1), for $0\le\tau\le t$,

$$\left\|\tilde{A}(x+\mathbf{v}(t-\tau))F\,(|x-Q-\hat{\mathbf{v}}\tau|\le\tau/4)\right\|=\left\|e^{i\mathbf{p}\cdot\hat{\mathbf{v}}(t-\tau)}\tilde{A}(x)F\,(|x-Q-\hat{\mathbf{v}}t|\le\tau/4)\,e^{-i\mathbf{p}\cdot\hat{\mathbf{v}}(t-\tau)}\right\|$$
$$\le\left\|\tilde{A}(x)F\,(|x-Q-\hat{\mathbf{v}}t|\le t/4)\right\|=0, \tag{2.44}$$

where in the last equality we used (2.13). We prove in the same way that,

$$\tilde{B}(x+\hat{\mathbf{v}}\mu)F\,(|x-Q-\hat{\mathbf{v}}\tau|\le\tau/4)=0, \tag{2.45}$$

for $\mu, \tau \geq 0$ and then,

$$b(x,t-\tau)\,F\left(|x-Q-\hat{\mathbf{v}}\tau| \leq \tau/4\right) = 0, \text{for}\, \tau \in (0,t). \tag{2.46}$$

We prove as in Lemma 2.3, using (2.2),(2.10), (2.32), (2.40) and (2.46) that, for each $l = 1, 2, \cdots$, there are constants C_l such that,

$$\left\| T_1\, e^{-i\tau H_1}\,\tilde{\Phi}\right\| \leq \frac{C_l}{v}\,(1+|\tau|)^{-l}. \tag{2.47}$$

For this purpose observe that,

$$e^{im\mathbf{v}\cdot x}\,\tilde{\Phi} = g\left(\frac{\mathbf{p}-m\mathbf{v}}{v^{\rho}}\right)\,e^{im\mathbf{v}\cdot x}\,\Phi_0. \tag{2.48}$$

We prove in the same way that,

$$\left\| T_2\, e^{-i\tau H_1}\,\tilde{\Phi}\right\| \leq \frac{C_l}{v}\,(1+|\tau|)^{-l}. \tag{2.49}$$

Let us denote,

$$a(x) := |(\mathbf{p}\chi(x))| + |\chi(x)|\left|A^{(f,Q)}(x) - \tilde{A}(x)\right|. \tag{2.50}$$

Then,

$$\left\| T_3\, e^{-i\tau H_1}\,\tilde{\Phi}\right\| \leq \left\| a(x)\, e^{-i\tau H_1}\,\tilde{\Phi}\right\|. \tag{2.51}$$

We prove as above that,

$$\left\| a(x)\, e^{-i\tau H_1}\,\tilde{\Phi}\right\| \leq C_l\,(1+|\tau|)^{-l}. \tag{2.52}$$

Moreover, by (2.1), (2.3) and as $a(x+\hat{\mathbf{v}}\tau)\,\Phi_0(x) = 0$,

$$a(x)\,e^{-i\tau H_1}\,\tilde{\Phi} = a(x)\,e^{-i\tau H_1}\,(\tilde{\Phi}-\Phi_0) + e^{-i(\mathbf{p}\cdot\hat{\mathbf{v}}\tau + mv\tau/2)}a(x+\hat{\mathbf{v}}\tau)\left(e^{-iH_0\tau/v} - I\right)\,\Phi_0. \tag{2.53}$$

Hence,

$$\left\| a(x)\, e^{-i\tau H_1}\,\tilde{\Phi}\right\| \leq C\frac{(1+|\tau|)}{v}, \tag{2.54}$$

where we used (2.35). By (2.52) and (2.54),

$$\left\| a(x)\, e^{-i\tau H_1}\,\tilde{\Phi}\right\| \leq C_{\delta,l}\,\frac{1}{v^{\delta}}(1+|\tau|)^{-l}, l = 1, 2, \cdots, 0 \leq \delta < 1. \tag{2.55}$$

Let us denote,

$$I(\mathbf{v}) := \int \gamma(\mathbf{v},\tau)\,d\tau, \tag{2.56}$$

where,

$$\gamma(\mathbf{v},\tau) := \left[\left\| a(x)\, e^{-iH_1\tau} \tilde{\Phi} \right\|^2 + \epsilon\, v^{-4} (1+|\tau|)^{-4} \right]^{1/2}, \tag{2.57}$$

with $\epsilon > 0$. It follows from (2.55) that $I(\mathbf{v}) \ < \infty$ and that $\lim_{v\to\infty} I(\mathbf{v}) = 0$. Equations (2.1) and (2.3) imply that,

$$\left\| a(x)\, e^{-iH_1\tau} \tilde{\Phi} \right\|^2 = \left\| a(x+\hat{\mathbf{v}}\tau))\, e^{-iH_0\tau/v} \tilde{\Phi} \right\|^2. \tag{2.58}$$

Hence,

$$\left| \frac{\partial}{\partial v} \gamma(\mathbf{v},\tau) \right| \leq C \left[\frac{|\tau|}{v^2} \left\| a(x+\hat{\mathbf{v}}\tau)\, e^{-iH_0\tau/v} H_0\, \tilde{\Phi} \right\| + v^{-3}\, (1+|\tau|)^{-2} \right]. \tag{2.59}$$

But, as

$$\left\| a(x+\hat{\mathbf{v}}\tau)\, e^{-iH_0\tau/v} H_0\, \tilde{\Phi} \right\| = \left\| a(x)\, e^{-iH_1\tau} H_0\, \tilde{\Phi} \right\|, \tag{2.60}$$

we prove as above that,

$$\left\| a(x+\hat{\mathbf{v}}\tau)\, e^{-iH_0\tau/v} H_0\, \tilde{\Phi} \right\| \leq C_l (1+|\tau|)^{-l}, \tag{2.61}$$

and we obtain,

$$\left| \frac{\partial}{\partial v} \gamma(\mathbf{v},\tau) \right| \leq\leq C\, v^{-2}\, (1+|\tau|)^{-2}, v \geq 1. \tag{2.62}$$

This proves the following estimate,

$$\left| \frac{\partial}{\partial v} I(\mathbf{v}) \right| \leq C\, v^{-2}, \tag{2.63}$$

and then,

$$I(\mathbf{v}) = \left| \int_v^\infty \frac{\partial}{\partial s} I(s\hat{\mathbf{v}})\, ds \right| \leq C v^{-1}. \tag{2.64}$$

By (2.51), (2.56), (2.57) and (2.64),

$$\int \left\| T_3\, e^{-i\tau H_1}\, \tilde{\Phi} \right\| d\tau \leq I(\mathbf{v}) \leq C v^{-1}, v \geq 1. \tag{2.65}$$

Equation (2.36) follows from (2.37),(2.39), (2.47), (2.49) and (2.65). Equation (2.31) is a consequence of (2.34), (2.35)and (2.36). By (2.2), (2.11) and as λ_∞ is homogeneous of degree zero,

$$\left(e^{-im\mathbf{v}\cdot x} W_+(A)\, e^{im\mathbf{v}\cdot x} - e^{-i\int_0^\infty \hat{\mathbf{v}}\cdot A(x+\hat{\mathbf{v}}\tau)\, d\tau} \right) \Phi_0 = e^{i\lambda(x)} \left(e^{-im\mathbf{v}\cdot x}\, W_+ \left(A^{(f,Q)} \right)\, e^{im\mathbf{v}\cdot x} e^{-i\lambda_\infty(\hat{\mathbf{v}}+\mathbf{p}/mv)} \right.$$

$$-e^{-i\lambda_\infty(\hat{\mathbf{v}})}e^{-i\int_0^\infty \hat{\mathbf{v}}\cdot A^{(f,Q)}(x+\hat{\mathbf{v}}\tau)\,d\tau}\Big)\,\Phi_0, \tag{2.66}$$

where we used that $A = A^{(f,Q)} + \nabla\lambda$. As $|\lambda_\infty(\hat{\mathbf{v}}+p/mv)-\lambda_\infty(\hat{\mathbf{v}})| \leq C\,|p|/v$ for $|p|/mv \leq 1/2$ and since $\hat{\Phi}_0$ belongs to the Schwartz space,

$$\left\|\left(e^{-i\lambda_\infty(\hat{\mathbf{v}}+\mathbf{p}/mv)} - e^{-i\lambda_\infty(\hat{\mathbf{v}})}\right)\Phi_0\right\| \leq \frac{C}{v}. \tag{2.67}$$

Equation (2.30) follows from (2.31), (2.66) and (2.67).

Proof of Theorem 1.2: The theorem follows from (1.15) and Lemma 2.4.

Proof of Theorem 1.3 : By (1.3) and (3.1)

$$A_R(x) = \frac{\alpha_\Omega}{|x|^2}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + \tilde{A}_R(x), \tag{2.68}$$

where, $\alpha_\Omega := \frac{1}{2\pi}\int_\Omega B_R(y)\,dy$, and with, $\tilde{A}_R(x) = O\left(\frac{1}{|x|^2}\right), |x| \to \infty$.

We designate, $\Omega^{\pm}_{\hat{\mathbf{v}}} := \{x \in \Omega_{\hat{\mathbf{v}}} : \pm x\cdot\tilde{\mathbf{v}} > 0\}$, where $\tilde{\mathbf{v}} := (\hat{\mathbf{v}}_2, -\hat{\mathbf{v}}_1)$. By explicit calculation (see (1.2),(1.4), (2.68) and (3.7) we see that,

$$L_c(x,\hat{\mathbf{v}}) := \int_{-\infty}^{\infty}\hat{\mathbf{v}}\cdot A_c(x+\hat{\mathbf{v}}\tau)\,d\tau = \pm\alpha\pi + \int_{-\infty}^{\infty}\hat{\mathbf{v}}\cdot\tilde{A}_R(x+\hat{\mathbf{v}}\tau)\,d\tau,\ x\in\Omega^{\pm}_{\hat{\mathbf{v}}}, \tag{2.69}$$

with $\alpha := \alpha_K - \alpha_A + \alpha_\Omega$, where in the case $K = \{0\}$ we take $\alpha_A = 0$. Moreover,

$$\int_{-\infty}^{\infty}\hat{\mathbf{v}}\cdot\tilde{A}_R(x+\hat{\mathbf{v}}\tau)\,d\tau = O\left(\frac{1}{|x_\perp|}\right),\ |x_\perp| \to \infty, \tag{2.70}$$

where, $x_\perp := x - (x\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$. In consequence, for any $A \in \mathcal{A}_K(\alpha_K, B_R)$,

$$L(x,\hat{\mathbf{v}}) := \int_{-\infty}^{\infty}\hat{\mathbf{v}}\cdot A(x+\hat{\mathbf{v}}\tau)\,d\tau = \pm\alpha\pi + \int_{-\infty}^{\infty}\hat{\mathbf{v}}\cdot\tilde{A}_R(x+\hat{\mathbf{v}}\tau)\,d\tau + \lambda_\infty(\hat{\mathbf{v}}) - \lambda_\infty(-\hat{\mathbf{v}}),\ x\in\Omega^{\pm}_{\hat{\mathbf{v}}}, \tag{2.71}$$

where we used (1.21) with $A^{(1)} = A_c$. Then, by (2.70),

$$\lim_{|x_\perp|\to\infty} L(x,\hat{\mathbf{v}}) = \pm\alpha\pi + \lambda_\infty(\hat{\mathbf{v}}) - \lambda_\infty(-\hat{\mathbf{v}}). \tag{2.72}$$

In the limit above, $x \in \Omega^{\pm}_{\hat{\mathbf{v}}}$. By (1.19), from $S(A)$ we uniquely reconstruct $e^{iL(x,\hat{\mathbf{v}})}$, $x \in \Omega_{\hat{\mathbf{v}}}$ and it follows from (2.72) that if $S(A^{(1)}) = S(A^{(2)})$,

$$\alpha^{(1)}\pi + \lambda^{(1)}_\infty(\hat{\mathbf{v}}) - \lambda^{(1)}_\infty(-\hat{\mathbf{v}}) = \alpha^{(2)}\pi + \lambda^{(2)}_\infty(\hat{\mathbf{v}}) - \lambda^{(2)}_\infty(-\hat{\mathbf{v}}) + 2\pi k(\hat{\mathbf{v}}), \tag{2.73}$$

where the function $k(\hat{\mathbf{v}})$ takes integers values. But as $\lambda_\infty^{(j)}(\hat{\mathbf{v}}), j=1,2$ are continuous for $\hat{\mathbf{v}}$ on the unit circle, the function k is a constant independent of $\hat{\mathbf{v}}$. Hence, (2.73) taken at $\hat{\mathbf{v}}$ and at $-\hat{\mathbf{v}}$ implies that $\lambda_\infty^{(1)}(\hat{\mathbf{v}}) - \lambda_\infty^{(1)}(-\hat{\mathbf{v}}) = \lambda_\infty^{(2)}(\hat{\mathbf{v}}) - \lambda_\infty^{(2)}(-\hat{\mathbf{v}})$ and we have that $\alpha^{(1)} = \alpha^{(2)}$ modulo 2.

Let us denote,

$$I(x,y,\hat{\mathbf{v}}) := \int_{-\infty}^{\infty} \left(\hat{\mathbf{v}} \cdot (A(x+\hat{\mathbf{v}}\tau) - \hat{\mathbf{v}} \cdot A(y+\hat{\mathbf{v}}\tau)\right) d\tau, \ x,y \in \Omega_{\hat{\mathbf{v}}}. \tag{2.74}$$

Observe that,

$$\mathcal{R}(x,y,\hat{\mathbf{v}}) = e^{iI(x,y,\hat{\mathbf{v}})}. \tag{2.75}$$

Using the fact that $I(x,y,\hat{\mathbf{v}})$ is gauge invariant and that $B_R \in C_0^1\left(\overline{\Omega}\right)$, we observe from (1.3), (3.1) and (2.69) that $I(x,y,\hat{\mathbf{v}}) \in C^1\left(\Omega_{\hat{\mathbf{v}}} \times \Omega_{\hat{\mathbf{v}}}\right)$. Then, $\mathcal{R}(x,y,\hat{\mathbf{v}}) \in C^1\left(\Omega^{\pm}_{\hat{\mathbf{v}}} \times \Omega^{\pm}_{\hat{\mathbf{v}}}\right)$ and

$$\frac{\partial}{\partial x_\perp}\mathcal{R}(x,y,\hat{\mathbf{v}}) = i\mathcal{R}(x,y,\hat{\mathbf{v}})\frac{\partial}{\partial x_\perp}I(x,y,\hat{\mathbf{v}}), \ x,y \in \Omega_{\hat{\mathbf{v}}}. \tag{2.76}$$

This implies that from the high-velocity limit of the scattering operator we uniquely reconstruct $\frac{\partial}{\partial x_\perp}I(x,y,\hat{\mathbf{v}}), x,y \in \Omega_{\hat{\mathbf{v}}}$. But it follows from the dominated convergence theorem, taking derivatives under the integral sign, using (1.21) and the gauge (3.14), in Appendix A, with appropriate f and Q,(see [12]) that,

$$\frac{\partial}{\partial x_\perp}I(x,y,\hat{\mathbf{v}}) = \frac{\partial}{\partial x_\perp}\int_{-\infty}^{\infty} \left(\hat{\mathbf{v}} \cdot (A(x+\hat{\mathbf{v}}\tau)\right) d\tau = \int_{-\infty}^{\infty} B_R(x+\hat{\mathbf{v}}\tau)\, d\tau, \ x,y \in \Omega_{\hat{\mathbf{v}}}. \tag{2.77}$$

By the support theorem for the Radon transform [10], [16], if K is convex, B_R is uniquely determined in Ω by the Radon transforms,

$$\int_{-\infty}^{\infty} B_R(x+\hat{\mathbf{v}}\mu)\, d\mu, \ x \in \Omega_{\hat{\mathbf{v}}}. \tag{2.78}$$

In consequence, if $S\left(A^{(1)}\right) = S\left(A^{(2)}\right), A^{(j)} \in \mathcal{A}_K\left(\alpha_K^{(j)}, B_R^{(j)}\right), j=1,2$, we have that, $B_R^{(1)}(x) = B_R^{(2)}(x), x \in \Omega$. Once B_R is uniquely reconstructed, α_A and α_Ω are uniquely known, and then α_K is determined modulo 2.

$\blacksquare$

Note that in the case where $K = \{0\}$ we have that, $\Omega_{\hat{\mathbf{v}}} = \{x \in \mathbf{R}^2 : x_\perp \neq 0\}$. Hence, in this case we know the Radon transform of B_R for all lines in $\mathbf{R}^2$, because the integrals in (2.78) are continuous functions of $x_\perp$.

We can reconstruct α from the gauge invariant quantity $\mathcal{R}(x, y, \hat{\mathbf{v}})$ as follows. By (2.72),

$$\lim_{|x_\perp|\to\infty, |y_\perp|\to\infty} \mathcal{R}(x, y, \hat{\mathbf{v}}) = e^{2i\alpha\pi}. \tag{2.79}$$

In the limit above, $x \in \Omega^+_{\hat{\mathbf{v}}}, y \in \Omega^-_{\hat{\mathbf{v}}}$. From (2.79)we reconstruct α modulo 1.

Lemma 2.4 implies -by a density argument- that, for every $\Phi \in L^2(\mathbf{R}^2)$ with support$\Phi \in \Omega_{\hat{\mathbf{v}}}$,

$$\text{s-}\lim_{v\to\infty} e^{-im\mathbf{v}\cdot x}\, W_\pm(A)\, e^{im\mathbf{v}\cdot x}\Phi = e^{-i\int_0^{\pm\infty} \hat{\mathbf{v}}\cdot A(x+\hat{\mathbf{v}}\tau)\, d\tau}\Phi. \tag{2.80}$$

Moreover, as the wave operators are isometric, and the limits in (2.80) are unitary, we also have strong convergence of the adjoint of the wave operators, as the following argument shows. Let us denote, $W_{\pm,\mathbf{v}} := e^{-im\mathbf{v}\cdot x}\, W_\pm(A)\, e^{im\mathbf{v}\cdot x}$, $W^*_{\pm,\mathbf{v}} := e^{-im\mathbf{v}\cdot x}\, W^*_\pm(A)\, e^{im\mathbf{v}\cdot x}$ and $L_\pm := \int_0^{\pm\infty} \hat{\mathbf{v}}\cdot A(x+\hat{\mathbf{v}}\tau)\, d\tau$. Then,

$$\left\|\left(W^*_{\pm,\mathbf{v}} - e^{iL_\pm}\right)\Phi\right\| = \left\|W^*_{\pm,\mathbf{v}}\left(e^{-iL_\pm} - W_{\pm,\mathbf{v}}\right)e^{iL_\pm}\Phi\right\| \leq \left\|\left(e^{-iL_\pm} - W_{\pm,\mathbf{v}}\right)e^{iL_\pm}\Phi\right\| \to 0, \text{ as } v\to\infty, \tag{2.81}$$

and it follows that,

$$\text{s-}\lim_{v\to\infty} e^{-im\mathbf{v}\cdot x}\, W^*_\pm(A)\, e^{im\mathbf{v}\cdot x}\,\Phi = e^{i\int_0^{\pm\infty} \hat{\mathbf{v}}\cdot A(x+\hat{\mathbf{v}}\tau)\, d\tau}\,\Phi. \tag{2.82}$$

Hence, for all $\Phi \in L^2(\mathbf{R}^2)$ with support$\Phi \in \Omega_{\hat{\mathbf{v}}}$,

$$\text{s-}\lim_{v\to\infty} e^{-im\mathbf{v}\cdot x}\, S(A)\, e^{im\mathbf{v}\cdot x}\,\Phi = e^{i\int_{-\infty}^{\infty} \hat{\mathbf{v}}\cdot A(x+\hat{\mathbf{v}}\tau)\, d\tau}\,\Phi. \tag{2.83}$$

3 Appendix A

In this Appendix we prove, for the reader's convenience, some elementary results on gauge transformations, and on special gauges with support in cones. See also [4] and [12] on these points. We recall that the classes of allowed magnetic potentials were given in Definitions 1.1 and 1.2. In the case were the obstacle

K is as in Definition 1.2 we define,

$$A_R(x) := \frac{1}{2\pi} \int \tilde{B}_R(x-y) \begin{bmatrix} -\hat{y}_2 \\ \hat{y}_1 \end{bmatrix} \frac{dy}{|y|} =$$

$$\frac{1}{2\pi} \int \tilde{B}_R(y) \frac{1}{|x-y|^2} \begin{bmatrix} -(x-y)_2 \\ (x-y)_1 \end{bmatrix} dy, \tag{3.1}$$

where, $\tilde{B}_R(x) = B_R(x), x \in \Omega$, and $\tilde{B}_R(x) = 0$ if $x \in K$. For any $x \in \Omega$, let us denote $\delta := dist(x, K) > 0$. Let $\phi \in C_0^\infty(\mathbf{R}^2)$ satisfy $\phi(x) = 1, |x| \leq \delta/3$ and $\phi(x) = 0$, for $|x| \geq \delta/2$. We define,

$$A_R^{(1)}(x) := \frac{1}{2\pi} \int \tilde{B}_R(y)\, \phi(|x-y|) \frac{1}{|x-y|^2} \begin{bmatrix} -(x-y)_2 \\ (x-y)_1 \end{bmatrix} dy, \tag{3.2}$$

and

$$A_R^{(2)}(x) := \frac{1}{2\pi} \int \tilde{B}_R(y)\, (1-\phi(|x-y|)) \frac{1}{|x-y|^2} \begin{bmatrix} -(x-y)_2 \\ (x-y)_1 \end{bmatrix} dy. \tag{3.3}$$

Since for all $\acute{x}$ in a small neighborhood of x and for all $y \in \mathbf{R}^2$, with $|y| \leq \delta/2$ we have that, $\acute{x} - y \in \Omega$, it follows that,

$$\frac{\partial}{\partial x_j} A_R^{(1)}(x) = \frac{1}{2\pi} \int_\Omega \left(\frac{\partial}{\partial x_j} B_R(x-y) \right) \phi(y) \begin{bmatrix} -\hat{y}_2 \\ \hat{y}_1 \end{bmatrix} \frac{dy}{|y|}. \tag{3.4}$$

Moreover, as $|z| \geq \delta/3$ for all z in the support of $(1-\phi(z))$,

$$\frac{\partial}{\partial x_j} A_R^{(2)}(x) = \frac{1}{2\pi} \int \tilde{B}_R(y) \left(\frac{\partial}{\partial x_j} (1-\phi(|x-y|)) \frac{1}{|x-y|^2} \begin{bmatrix} -(x-y)_2 \\ (x-y)_1 \end{bmatrix} \right) dy. \tag{3.5}$$

By (3.4) and (3.5), $A_R \in C^1\left(\overline{\Omega}\right)$. It follows from Gauss theorem that for any $\phi \in C_0^\infty(\Omega)$

$$-\int_\Omega dx \left[A_{R,2} \frac{\partial}{\partial x_1} \phi(x) - A_{R,1} \frac{\partial}{\partial x_2} \phi(x) \right] = \int_\Omega dx\, B_R(x) \phi(x)\, dx. \tag{3.6}$$

This implies that, $\nabla \times A_R(x) = B_R(x), x \in \Omega$.

We prove in a similar way that, $\nabla \cdot A_R(x) = 0, x \in \Omega$. The magnetic potential in the Coulomb gauge, in the case were K is as in Definition 1.2, is defined as follows,

$$A_c := \frac{(\alpha_K - \alpha_R)}{|x|^2} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + A_R(x), \tag{3.7}$$

with, $\alpha_R := \frac{1}{2\pi} \int_{\partial K} A_R(x) \cdot dx$, where we integrate in counter-clockwise sense. It is easyly checked that, $A_c \in \mathcal{A}_K(\alpha_K, B_R)$.

REMARK 3.1. *Suppose that* $A^{(1)}$ *and* $A^2 \in \mathcal{A}_K(\alpha_K, B_R)$. *Then, if* $K = \{0\}$, *there is a real-valued function,* $\lambda \in C^2(\mathbf{R}^2 \setminus 0) \cap L^\infty(\mathbf{R}^2)$, *such that* $A^{(1)} = A^{(2)} + \nabla\lambda$, *in* $\acute{\mathcal{D}}(\mathbf{R}^2)$. *Furtheremore, if* K *is as in Definition 1.2 there is a real-valued function,* λ, *in* $C^2(\overline{\Omega}) \cap L^\infty\left(\overline{\Omega}\right)$ *such that,* $A^{(1)} = A^{(2)} + \nabla\lambda$, *in* Ω. *Moreover, in both cases,* $\lambda_\infty(\hat{x}) := \lim_{r\to\infty} \lambda(rx)$ *exists and it is a continuous function on the circle,* S_1. *We also denote by* λ_∞ *the extension of this function to a continuous and homogeneous function of degree zero, defined in* $\mathbf{R}^2 \setminus 0$, *i.e.,* $\lambda_\infty(r\hat{x}) := \lambda_\infty(\hat{x})$, $r > 0$.

Proof : We denote, $A := A^{(1)} - A^{(2)}$. Then $\nabla \times A = 0$. Suppose first that $K = \{0\}$. Let $\phi \in C_0^\infty(\mathbf{R}^2)$ satisfy $\int \phi(x)dx = 1$, and let us denote $\phi_n := \phi(x/n)$. We define, $A_n := \phi_n * A$. Then, $A_n \in C^\infty$, $\nabla\times A_n = 0$, and

$$\lim_{n\to\infty} \|A - A_n\|_{L^\infty(D)} = 0, \tag{3.8}$$

for any compact set $D \in \mathbf{R}^2 \setminus 0$. It follows from Stoke's theorem that for any simple closed C^1 curve, $\mathcal{C}$, that does not go through zero,

$$\int_{\mathcal{C}} A \cdot dl = \lim_{n\to\infty} \int_{\mathcal{C}} A_n \cdot dl = \lim_{n\to\infty} \int_{\mathcal{C}_{int}} \nabla \times A_n \, dx = 0. \tag{3.9}$$

When K is as in Definition 1.2, Stoke's theorem implies that (3.9) holds for any simple closed C^1 curve in Ω. For this purpose observe that,

$$\int_{\partial K} A(x) \cdot dx = 0. \tag{3.10}$$

For any fixed $x_0 \in \Omega$, we denote,

$$\lambda(x) := \int_{\mathcal{C}_{x_0}^x} A \cdot dl, \tag{3.11}$$

where $\mathcal{C}_{x_0}^x$ is any simple, C^1 curve in Ω that goes from x_0 to x. Then, when $K = \{0\}$, $\lambda \in C^2(\mathbf{R}^2 \setminus 0)$ and when K is as in Definition 1.2, $\lambda \in C^2\left(\overline{\Omega}\right)$. Furthermore,$\nabla\lambda = A$. Since, $a(r) := \sup_{|x|\geq r} |A(x) \cdot \hat{x}| \in L^1((0,\infty))$, we have that, $\lambda \in L^\infty(\overline{\Omega})$. Moreover, $\lambda_\infty(\hat{x}) := \lim_{r\to\infty} \lambda(r\hat{x})$ exists, and it is a continuous function on S_1.

■

We state some special gauges with support in a cone. Let us take polar coordinates, $(r, \theta), r > 0, \theta_0 \leq \theta < \theta_0 + 2\pi$ for any $0 \leq \theta_0 < 2\pi$. Let $f(\theta) \in C^1(S^1)$ satisfy, $f(\theta) = 0$ for $\theta \leq \theta_0 + \epsilon/2$, and for $\theta \geq \theta_0 + 2\pi - \epsilon/2$, and moreover, $f(\theta) = \theta$ for $\theta_0 + \epsilon \leq \theta \leq \theta_0 + 2\pi - \epsilon$, for some $\epsilon > 0$. We define,

$$A^{(f)} := \frac{(\alpha_K - \alpha_R)}{|x|^2} \left[\begin{array}{c} -x_2 \\ x_1 \end{array} \right] - (\alpha_K - \alpha_R) \nabla f(\theta). \tag{3.12}$$

Then, $A^{(f)}$ has support in a cone with center zero, axis $e^{i\theta_0}$, and opening angle ϵ.

For any $Q = (q_1, q_2) \in \mathbf{R}^2$ let $A^{(Q)}$ be the following shifted transversal magnetic potential for B_R,

$$A^{(Q)} := \left[\begin{array}{c} q_2 - x_2 \\ x_1 - q_1 \end{array} \right] \int_0^1 \tilde{B}_R(x\mu + (1-\mu)Q)\, \mu\, d\mu, \tag{3.13}$$

where if $K = \{0\}$, $\tilde{B}_R = B_R$ and if K is as in Definition 1.2, $\tilde{B}_R(x) = B_R(x), x \in \Omega$, and $\tilde{B}_R(x) = 0, x \in K$. If we take Q far enough from the support of B_R the support of $A^{(Q)}$ will be contained in a cone with vertex Q and with a small opening angle. Furthermore,

$$A^{(f,Q)} := A^{(f)} + A^{(Q)} \in \mathcal{A}_K\left(\alpha_K, B_R\right), \tag{3.14}$$

where if $K = \{0\}$ we take $\alpha_R = 0$, and we can always choose θ_0 and Q in such a way that $A^{(f)} + A^{(Q)}$ has support in a cone with small opening angle and vertex Q.

Acknowledgement

I thank G. Lozano, F. Nicoleau and F. Schaposnik for useful remarks.

References

[1] Aharonov Y., Bohm D., "Significance of electromagnetic potentials in the quantum theory", Phys. Rev. **115** ,485-491 (1959).

[2] Arians S., "Geometric approach to inverse scattering for the Schrödinger equation with magnetic and electric potential", J. Math. Phys. **38**, 2761-2773 (1997).

[3] Arians S., "Geometric approach to inverse scattering for hydrogen-like systems in a homogeneous magnetic field", J. Math. Phys. **39**, 1730-1743 (1998).

[4] Arians S., *"Inverse Streutheorie für die Schrödingergleigung mit Magnetfeld"*, Dissertation RWTH-Aachen. Berlin: Logos Verlag, 1997.

[5] Bily J.-M. and Robert D., "The semi-classical Van-Vleck formula. Applications to the Aharonov-Bohm effect", Preprint 00/03-1, Université de Nantes, 2001.

[6] Enss V., "Propagation properties of quantum scattering states", J. Funct. Anal. **52**, 219-251 (1983).

[7] Enss V., Weder R., "Inverse potential scattering: A geometrical approach". In Feldman J., Froese R. and Rosen L. (eds.) *Mathematical Quantum Theory II: Schrödinger operators. CRM Proceedings and Lecture Notes* 8, pp. 151–162. Providence: American Mathematical Society,1995.

[8] Enss V., Weder R., "The geometrical approach to multidimensional inverse scattering", J. Math. Phys. **36**, 3902-3921 (1995).

[9] Helffer B., "Effect d'Aharonov-Bohm sur un état borné de l'équation de Schrödinger", Commun. Math. Phys. **119**, 315-329 (1988).

[10] Helgason S., *"The Radon Transform, Second Edition"*. Progress in Mathematics vol. 5. Berlin: Birkhäuser, 1999.

[11] Jackiv R., "Dynamical symmetry of the magnetic vortex", Annals of Physics **201**, 83-116 (1990).

[12] Jung W., "Der geometrische Ansatz zur inversen Streutheorie bei der Dirac-Gleichung", Diplomarbeit, RWTH-Aachen, 1996.

[13] Jung W., "Geometrical approach to inverse scattering for the Dirac equation", J. Math. Phys. **38**, 39-48 (1997).

[14] Kato T., *"Perturbation Theory of Linear Operators, Second Edition"*. Berlin: Springer-Verlag, 1978.

[15] Laptev A., Sigal I.M., "Global Fourier integral operators and semiclassical asymptotics", Rev. Math. Phys. **12**, 749-766 (2000).

[16] Natterer F., *"The Mathematics of Computerized Tomography"*. Stuttgart: B. Teubner, Chichester: J. Wiley & Sons, 1989.

[17] Nicoleau F., "An inverse scattering problem with the Aharonov-Bohm effect", J. Math. Phys. **41**, 5223-5237 (2000).

[18] Olariu S.,Popescu I.I., "The quantum effects of electromagnetic fluxes", Reviews of Modern Physics **57**, 339-436 (1985).

[19] Peskin M, Tanamura A., *"The Aharonov-Bohm Effect"*. Lecture Notes in Physics **340**. Berlin: Springer-Verlag, 1989.

[20] Reed M., Simon B., *"Methods of Modern Mathematical Physics I. Functional Analysis"*. New York: Academic Press, 1972.

[21] Reed M., Simon B., *"Methods of Modern Mathematical Physics III. Scattering Theory"*. New York: Academic Press, 1979.

[22] Ruijsenaars S.N.M., "The Aharonov-Bohm effect and scattering theory", Annals of Physics **146**, 1-34 (1983).