ADDENDUM TO: NORMAL FORMS AND QUANTIZATION FORMULAE

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Abstract

The main result of [BGP] is reformulated in terms of the standard quantization formula constructed quantizing the Lie transformation.

Theorem 1.1 of [BGP] yields a quantization formula modulo exponentially small terms in $\hbar$ for the semi-excited eigenvalues of the Schrödinger operator $Q(\hbar) := -\hbar^2\Delta/2 + V$ on $L^2(\mathbb{R}^n), n \geq 1$ whose symbol $q(x,\xi) := \xi^2/2 + V(x)$ is a non-integrable classical Hamiltonian. It extends a formula obtained by Sjöstrand [Sj] quantizing the classical normal form of q modulo terms of order $\hbar^\infty$; however it has an *a priori* different form since it is generatated by an *a priori* different normal form for the symbol q. Here we solve the question left open in [BGP] of the relation between these two normal forms by proving their coincidence. The main result of [BGP] can thus be reformulated, within the same assumptions and notations (in particular, ℓ is the Gevrey order of $V \in C^\infty(\mathbb{R}^n)$, τ the power in the diophantine condition fulfilled by the frequencies $\omega_i := \partial_i^2 V(0)$) as follows:

Theorem 1 *Assume (A1-A3), and let once more $b := [(\tau+2)(2\ell+3)]^{-1}$. Then:*

There exist $h_* > 0, \epsilon_* > 0, A > 0, B > 0$ (independent of $\hbar$ and ϵ) and a smooth real valued function $\mathcal{W}(I_1,\dots,I_n;\hbar,\epsilon)$ with asymptotic expansion in $\hbar$:

$$\mathcal{W}(I_1,\dots,I_n;\hbar,\epsilon) \sim \mathcal{W}_0(I_1,\dots,I_n;\epsilon) + \hbar\mathcal{W}_1(I_1,\dots,I_n;\epsilon) + \dots \tag{1}$$

such that $\forall \epsilon \leq \epsilon_*, \hbar \leq \hbar_*\epsilon^2$ the eigenvalues $\lambda(k,\hbar,\epsilon)$ of $Q(\hbar)$ in $[0,\epsilon^2]$ have the representation

$$\lambda(k,\hbar,\epsilon) = \langle (k+1/2),\omega\rangle\hbar + \mathcal{W}\left((k+1/2)\hbar,\hbar,\epsilon\right) + \epsilon^2 O\left(e^{-(B\epsilon^2/\hbar)^{1/(\ell+1)}}\right) + O\left(\epsilon^3 e^{-A/\epsilon^b}\right) \tag{2}$$

Here $\mathcal{W}_0$ is the classical Birkhoff normal form of q truncated at a suitable order.

Remark.

The main difference with the original formulation of [BGP], Theorem 1.1, is that the quantization formula (2) has the same form as that of [Sj], Theorem 0.1, even in the

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most general case. We will indeed explicitly reabsorb the rescaling of $\hbar$ introduced in [BGP] which is responsible for the possible noncoincidence of the two normal forms for q. Thus (2) extends the validity range of the quantization formula of [Sj] to a larger class of eigenvalues (Corollary 1.1), providing in addition an explicit error estimate. Choosing $\epsilon^2 = \varphi(\hbar)$ as in [BGP] one actually obtains that the eigenvalues of $Q(\hbar)$ in $[0, \varphi(\hbar)]$ are given by

$$\lambda(k,\hbar) = \langle (k+1/2), \omega \rangle \hbar + \mathcal{W}\left((k+1/2)\hbar, \hbar, \varphi(\hbar)\right) + O\left(e^{-A/\varphi(\hbar)^{b/2}}\right) \tag{3}$$

and the choice of $\varphi(\hbar)$ can be optimized as in Corollary 1.1.

Proof. We outline the additions and modifications with respect to the arguments of [BGP], whose formulas and statements are directly referred to here.

1. Consider once more the (Weyl) pseudodifferential operator $P_R(\hbar', \epsilon)$ of Gevrey symbol $p_{\epsilon,R}(x,\xi) := p_0(x,\xi) + a_{\epsilon,R}(x,\xi)$, $\hbar' = \hbar/\epsilon^2$. Perform the inverse unitary rescaling (1.3). The result is the pseudodifferential operator $P'(\hbar,\epsilon)$ of Weyl symbol $p'_\epsilon(x,\xi) := p_0(x,\xi) + \epsilon^3 a_{\epsilon,R}(x/\epsilon, \xi/\epsilon)$. We omit the easy verification.

2. As in Section 3, we approximate p'_ϵ with an analytic symbol fulfilling the conditions of Proposition 3.1. The modifications of the argument leading from Proposition 3.1 to Theorem 3.1 are as follows: define $a'(x,\xi) := \epsilon^3 a_{\epsilon,R}(x/\epsilon, \xi/\epsilon)$, and recalling formulas (3.7-11) define also the analogous of the norm $\langle \|a_\epsilon\| \rangle$ of (3.33)

$$\langle \|a'\| \rangle_{\mu,\mu\epsilon^\beta} := \sum_{k\in\mathbb{Z}^n} e^{\mu|k|^\beta} \int_{\mathbb{R}^{2n}} |\widehat{a}'_k(s)| e^{\mu(|s|\epsilon)^\beta} ds \ , \tag{4}$$

Performing the change of variables $\epsilon s = s'$ in (3.33) we immediately get $\langle \|a'\| \rangle_{\mu,\mu\epsilon^\beta} = \epsilon^3 \langle \|a_\epsilon\| \rangle_{\mu,\mu}$ whence

$$\langle \|a'\| \rangle_{\mu,\mu\epsilon^\beta} = \epsilon^3 \langle \|a_{\epsilon,R}\| \rangle \le C_{11}\epsilon^3 \ ,$$

where C_{11} is the constant whose existence is ensured by Corollary 3.1.

3. Define now $a'_{\Lambda,K}(z)$ by (3.38) (up to the obvious modifications induced by the replacement of a_ϵ by a'). Then we repeat the argument in the proof of Theorem 3.1 to get

$$\left\| \mathrm{Op}^W(a') - \mathrm{Op}^W\left(a'_{\Lambda,K}\right) \right\|_{L^2\to L^2} \le \epsilon^3 C_{11}\left[e^{-\mu K^\beta} + e^{-\mu(\Lambda\epsilon)^\beta}\right] \ .$$

4. Now we approximate p'_ϵ by $p_0 + a'_{\Lambda,K}$, and we apply to it Proposition 3.1. Before doing that we reformulate Proposition 3.1 in a more convenient way (the proof does not change):

Proposition 3.1 *Consider the symbol $p_0 + f$, with $f \in \mathcal{A}_{\rho,\sigma}$ for some $\rho > 0, \sigma > 0$. Denote $P_f := P_0(\hbar) + F$, $F := Op^W(f)$ the corresponding operator. Let (A3) hold. Set:*

$$E := \|f\|_{\rho,\sigma} \ , \epsilon_0 := \frac{2^{\tau+5}e^2\tau^\tau}{\gamma}\frac{E}{\rho^\tau\sigma^2}$$

and assume $\epsilon_0 < 1$ *. Then for* $0 \leq \epsilon \leq \epsilon_0$ *there exists a unitary transformation* $T_\epsilon : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *such that* $S_f(\hbar) := T_\epsilon P_f(\hbar) T_\epsilon^{-1}$ *is an* $\hbar-$ *pseudodifferential operator family with symbol*

$$\sigma_f(x,\xi,\hbar,\epsilon) = p_0(x,\xi) + \mathcal{Z}(I(x,\xi);\hbar,\epsilon) + \mathcal{R}(x,\xi,\hbar,\epsilon). \tag{5}$$

Here $\mathcal{Z} \in A_{\rho/2,\sigma/2}$ *and fulfills the estimate*

$$\|\mathcal{Z}\|_{\rho/2,\sigma/2} \leq 2E, \tag{6}$$

The principal symbol of $\mathcal{Z}$ *is the normal form of the Hamiltonian* $p_0(x,\xi) + f(x,\xi)$ *computed up to order* $(\epsilon_0)^{-\frac{1}{\tau+2}}$*, and* $\mathcal{R} \in \mathcal{A}_{\rho/2,\sigma/2}$ *is exponentially small, namely*

$$\|\mathcal{R}\|_{\rho/2,\sigma/2} \leq eE \exp\left[-(\tau+2)(1/\epsilon_0)^{\frac{1}{\tau+2}}\right] \tag{7}$$

5. Apply now Propostion 3.1 to the regularized symbol. A simple computation gives

$$E = e^{\rho K + \sigma \Lambda} \left\langle \left\| a'_{\Lambda,K} \right\| \right\rangle_{0,0} \leq e^{\rho K + \sigma\Lambda} C_{11} \epsilon^3 \ .$$

Choosing $K = \epsilon^{-\alpha}$, $\Lambda = \epsilon^{-(\alpha+1)}$, $\rho = \epsilon^\alpha$, $\sigma = \epsilon^{\alpha+1}$ we obtain the normal form result for the regularized system. Finally, choose $\alpha := [(2\ell+1)/2(\tau+2)(\ell+1)]$ (exactly as in the proof of Theorem 3.1). This yields the result for the symbol p'_ϵ and therefore for the corresponding operator.

References

[BGP] D.Bambusi, S.Graffi,T.Paul, *Normal forms and quantization formulae.* Commun. Math. Phys., **207**, 173-195, (1999).

[Sj] J.Siöstrand, *Semi-excited levels in non-degenerate potential wells*, Asymptotic Analysis **6** (1992), 29-43