Resonance Free Domains for Non-Analytic Potentials

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Abstract

We study the resonances of the semiclassical Schrödinger operator $P = -h^2\Delta + V$ near a non-trapping energy level λ_0 in the case when the potential V is not necessarily analytic on all of $\mathbb{R}^n$ but only outside some compact set. Then we prove that for some $\delta > 0$ and for any $C > 0$, P admits no resonance in the domain $\Omega = [\lambda_0 - \delta, \lambda_0 + \delta] - i[0, Ch\log(h^{-1})]$ if V is C^∞, and $\Omega = [\lambda_0 - \delta, \lambda_0 + \delta] - i[0, \delta h^{1-\frac{1}{s}}]$ if V is Gevrey with index s. Here $\delta > 0$ does not depend on h and the results are uniform with respect to $h > 0$ small enough.

1 Introduction

The location of quantum resonances has been a very rich source of researchs during these last years, giving rise to a wide literature, including estimates on the number of resonances in various domains of the complex plane and relationship between resonances and the underlying classical geometry: see e.g. [GeSj, HeSj, SjZw] and references therein.

Concerning more specifically the resonances of the semiclassical Schrödinger opertator, one of the first results about their location is probably the one which states the absence of resonances near a non trapping energy level: see e.g. [BCD, GeSj]. However such a result has been proved under the assumption that the potential is analytic everywhere on $\mathbb{R}^n$, while the theory of resonances (and the notion of non trapping level as well) exists for potentials which may be analytic outside a compact set only. As far as we know, the only similar results concerning such potentials are those very recent of [Ro] in the Gevrey case, and

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[TaZw] in the compactly supported case (in the latter case a simpler proof has been given in [Bu] but leading to a less optimal result).

Here we plan to provide a rather simple proof which generalizes the result of [TaZw] to long range smooth potentials. Incidentally, our method also gives an alternative (and simpler) proof of the result of [Ro]. The idea is to use an almost analytic extension of the potential (as defined in [MeSj]) and to prove for it an equivalent of the exponential microlocal weighted estimates introduced in [BeMa]. Because of the non-analyticity, some extra terms will appear involving the $\overline{\partial}$ of the potential, but they are rather well controled if we stay sufficiently close to the real. These estimates together with a construction of a 'global' escape function permit to obtain the results

In the next section we give the precise statement of our main results. Then in Section 3 we introduce the a priori estimate that we use in Sections 4 and 5 to prove the main results. Section 6 is just a remark concerning the case of potentials which are not analytic at infinity. Finally, the appendix contains the proof of the a priori estimate.

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2 Notations and Results

We consider the n-dimensional semiclassical Schrödinger operator

$$P = P(h) = -h^2\Delta + V(x)$$

where $h > 0$ is small enough and the potential V satisfies the following hypothesis:

(H1) $V \in C^\infty(\mathbb{R}^n)$ is real valued and there exists some compact set $K \subset \mathbb{R}^n$ such that V is analytic on $K^C := \mathbb{R}^n \backslash K$ and can be extended as a holomorphic function on

$$D := \{x \in \mathbb{C}^n \ ; \ |\mathrm{Im}\, x| < \delta_0 \langle \mathrm{Re}\, x \rangle \ , \ \mathrm{Re}\, x \in K^C\}$$

for some constant $\delta_0 > 0$. Moreover,

$$V(x) \to 0 \ \text{ as } \ |\mathrm{Re}\, x| \to \infty \ , \ x \in D.$$

Then the continuous spectrum of P is $\mathbb{R}_+$, and one can define the *resonances* of P near the positive real axis as the complex eigenvalues of the so-called *distorded* operator P_θ ($\theta > 0$ small enough) defined in the following way (see e.g. [Hu]):

Let $F : I\!R^n \to I\!R^n$ be a smooth vector-field such that $F(x) = 0$ on the compact K given in (H1) and $F(x) = x$ for $|x|$ large enough, and for $\nu \in I\!R$ small enough consider the unitary operator U_ν on $L^2(I\!R^n)$ defined by:

$$U_\nu \varphi(x) = \det(1 + \nu dF(x))^{-1/2} \varphi(x + \nu F(x)).$$

Then we see by (H1) that the coefficients of the differential operator $\widetilde{P}_\nu := U_\nu P U_\nu^{-1}$ depend analytically on ν near 0, and can therefore be extended in a unique way to small complex values of ν. In particular for $\nu = i\theta$ ($\theta > 0$ small enough) we get a differential operator $P_\theta := \widetilde{P}_{i\theta}$ and by the Weyl perturbation theorem we see that the essential spectrum of P_θ is given by:

$$\sigma_{ess}(P_\theta) = e^{-2i\theta} I\!R_+.$$

In particular the spectrum $\sigma(P_\theta)$ of P_θ is discrete in the sector $S_\theta := \{\lambda \in \mathbb{C} \ ; \ \mathrm{Re}\,\lambda > 0 \ , \ -2\theta < \mathrm{Arg}\,\lambda \leq 0\}$ and standard arguments (see e.g. [AgCo, HeMa]) show that $\sigma(P_\theta) \cap S_\theta$ does not depend on the particular choice of the vector-field F and for any $0 < \theta < \theta'$ small enough one has:

$$\sigma(P_\theta) \cap S_\theta = \sigma(P_{\theta'}) \cap S_\theta.$$

Then the resonances of P in S_θ are by definition the eigenvalues of P_θ there, or equivalently the eigenvalues of $P_{\theta'}$ there, for any $\theta' > \theta$ small enough.

Now we fix some energy level $\lambda_0 > 0$ and we assume that it is non-trapping for the underlying classical dynamic, that is:

(H2) For all $(x, \xi) \in I\!R^{2n}$ such that $\xi^2 + V(x) = \lambda_0$, one has $|\exp tH_p(x,\xi)| \to \infty$ as $|t| \to \infty$.

Here $H_p := 2\xi\partial_\xi - \partial_x V(x)\partial_x$ denotes the Hamilton field of $p := \xi^2 + V(x)$.

Then we have:

Theorem 2.1 *Under assumtions (H1) and (H2), there exists $\delta > 0$ such that for any $C > 0$, $P(h)$ admits no resonance in the complex domain $\Omega(h) := [\lambda_0 - \delta, \lambda_0 + \delta] - i[0, Ch\log(h^{-1})]$ for all $h > 0$ small enough.*

Now let us assume more regularity on V in a neighborhood of K, namely that it is a Gevrey function of index $s \geq 1$ in the sense that:

(H3) There exists $C_1 > 0$ such that for all $\alpha \in I\!N^n$ the following estimate:

$$|\partial^\alpha V(x)| \leq C_1^{1+|\alpha|}(\alpha!)^s$$

holds uniformly in a neighborhood of K.

In this case we recover in a simpler way the result of [Ro], namely:

Theorem 2.2 *Under assumtions (H1), (H2) and (H3), there exists $\delta > 0$ such that $P(h)$ admits no resonance in the complex domain $\Omega_s(h) := [\lambda_0 - \delta, \lambda_0 + \delta] - i[0, \delta h^{1-\frac{1}{s}}]$ for all $h > 0$ small enough.*

Observe that when $s = 1$ one recovers a well-known result for analytic potentials.

3 Microlocal Exponential Weighted Estimates for Operators with Non Analytic Symbol

The purpose of this section is to generalize a result of [BeMa] to the case of operators with non analytic symbol. In the next sections these estimates, together with the construction of a global escape function, will permit to define a h-dependent norm on $L^2(\mathbb{R}^n)$ for which P_θ ($\theta = \theta(h)$ convenient) will satisfy a global ellipticity-type energy inequality.

For the sake of simplicity, and since we do not need more in this paper, here we consider *differential* operators only, with smooth bounded coefficients. More precisely we consider the operator A defined by the oscillatory integral:

$$Au(x) = \mathrm{Op}_h^W(a)u(x) := \frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h} a\left(\frac{x+y}{2}, \xi; h\right) u(y)dyd\xi$$

($u \in \mathcal{S}'(\mathbb{R}^n)$) where a is a polynome of degree k with respect to ξ, whose coefficients are smooth functions of x on $\mathbb{R}^n$, uniformly bounded together with all their derivatives.

Let $G \in C_0^\infty(\mathbb{R}^{2n})$ be a compactly supported smooth real-valued function on $\mathbb{R}^{2n}$. We denote

$$H(x,\xi) = G(x,\xi) - \xi \cdot \partial_\xi G(x,\xi)$$

and, for $t \in \mathbb{R}$ small enough:

$$\Lambda_{tG} = \{(x + it\partial_\xi G(x,\xi), \xi - it\partial_x G(x,\xi)) \ ; \ (x,\xi) \in \mathbb{R}^{2n}\}$$

which is a I-Lagrangian submanifold of $\mathbb{C}^{2n}$ (that is Lagrangian with respect to $\mathrm{Im}\, d\zeta \wedge dz$) and $\mathbb{R}$-symplectic if t is small enough. In particular one can define $L^2(\Lambda_{tG})$ e.g. by using the measure $d\mathrm{Re}\,(z,\zeta)$ which is equivalent to $(\mathrm{Re}\, d\zeta \wedge dz)^{\wedge n}$ on Λ_{tG}.

We also introduce the F.B.I. transform T defined by:

$$Tu(x,\xi) = c\int e^{i(x-y)\xi/h - (x-y)^2/2h} u(y)dy$$

where $c = 2^{-\frac{n}{2}}(\pi h)^{-\frac{3n}{4}}$. Then T maps $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^{2n}) \cap e^{-\xi^2}\mathcal{H}(\mathbb{C}^n_{x-i\xi})$ where $\mathcal{H}(\mathbb{C}^n_{x-i\xi})$ denotes the space of entire function with respect to $x - i\xi$ on $\mathbb{C}^n$, and it is also an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ (see e.g.[Ma1]).

In order to state the main result of this section, we also need to recall the notion of *almost analytic extension* introduced by A. Melin and J. Sjöstrand in [MeSj] and subsequently widely used in the literature. Let $f \in C_b^\infty(\mathbb{R}^n)$, the space of smooth functions on $\mathbb{R}^n$ which are uniformly bounded together with all their derivatives. For $(x,y) \in \mathbb{R}^{2n}$, one considers the formal series:

$$F(x,y) = \sum_{\alpha \in \mathbb{N}^n} \frac{(iy)^\alpha}{\alpha!}\partial^\alpha f(x).$$

Then, considering y as a small vector-parameter, this series can be resummed up to $\mathcal{O}(|y|^\infty)$ in a standard way (see e.g. [Ho, Ma1]), and gives rise to a smooth function $\widetilde{f}(x,y)$ on $\mathbb{R}^{2n}$ of the type:

$$\widetilde{f}(x,y) = \sum_{\alpha\in\mathbb{N}^n} \frac{(iy)^\alpha}{\alpha!}\partial^\alpha f(x)\left(1-\chi(\varepsilon_{|\alpha|}/|y|)\right) \tag{3.1}$$

where $\chi \in C_0^\infty(\mathbb{R})$ is a fixed cut-off function near 0, and $(\varepsilon_k)_{k\in\mathbb{N}}$ is a decreasing sequence of positive numbers converging to 0 sufficiently rapidly (depending only on the L^∞-norms of $\partial^\alpha f$, $\alpha \in \mathbb{N}^n$). Then, $\widetilde{f}$ satisfies:

$$\widetilde{f}(x,0) = f(x) \tag{3.2}$$

and

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\widetilde{f}(x,y) = \mathcal{O}(|y|^\infty) \tag{3.3}$$

as $|y| \to 0$, uniformly with respect to $x \in \mathbb{R}^n$. Because of the latter property, $\widetilde{f}$ is called an *almost analytic extension* of f. Moreover, in the case when f is globally Gevrey with index $s > 1$ one can take $\varepsilon_k = \dfrac{1}{Ck^{s-1}}$ with $C > 0$ a large enough constant, and then (3.3) is improved into:

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$\widetilde{f}(x,y) = \mathcal{O}\left(e^{-\varepsilon/|y|^{1/(s-1)}}\right)$ (3.4)for some positive constant ε.

The previous discussion permits in particular to consider an almost analytic extension of the coefficients of a, giving rise to a function $\widetilde{a} = \widetilde{a}(x,\xi)$ polynomial with respect to ξ with coefficients which are smooth functions of x on a complex strip around $\mathbb{R}^n$ of the type

$$\Gamma_\varepsilon := \{x \in \mathbb{C}^n \; ; \; |\mathrm{Im}\, x| < \varepsilon\}$$

($\varepsilon > 0$ arbitrary), and satisfying:

$$\widetilde{a}\,|_{\mathbb{R}^{2n}} = a$$
$$\frac{\partial}{\partial \overline{x}}\widetilde{a} = \mathcal{O}(|\mathrm{Im}\, x|^\infty)\langle\xi\rangle^k$$

uniformly on $\Gamma_\varepsilon \times \mathbb{R}^n$. In particular, since we have taken a function G compactly supported, $\widetilde{a}$ is well defined on Λ_{tG}. Then, the main technical tool to prove our theorems is the following estimate:

Proposition 3.1 *For any $u \in C_0^\infty(\mathbb{R}^n)$ one has:*

$$\begin{aligned}\|e^{-tH/h}TAu\|^2_{L^2(\Lambda_{tG})} &= \|\widetilde{a}e^{-tH/h}Tu\|^2_{L^2(\Lambda_{tG})} \\ &\quad +\mathcal{O}\left(h + h^{-3n}|t|^\infty e^{J_0|t|/h}\right)\|\langle\xi\rangle^k e^{-tH/h}Tu\|^2_{L^2(\Lambda_{tG})}\end{aligned}$$

uniformly with respect to t and h both small enough. Here $J_0 := 2 \operatorname{Sup}_{\mathbb{R}^{2n}}(|G| + |\nabla G|^2)$.
Moreover, if the coefficients of A are globally Gevrey with index $s > 1$, the previous estimate can be improved into:

$$\begin{aligned}\|e^{-tH/h}TAu\|^2_{L^2(\Lambda_{tG})} &= \|\widetilde{a}e^{-tH/h}Tu\|^2_{L^2(\Lambda_{tG})} \\ &+\mathcal{O}\left(h + h^{-3n}e^{-\varepsilon/t^{1/(s-1)}}e^{J_0|t|/h}\right)\|\langle\xi\rangle^k e^{-tH/h}Tu\|^2_{L^2(\Lambda_{tG})}\end{aligned}$$

for some positive constant ε.

Proof - Essentially, the proof proceeds as in [BeMa] Section 3, with the difference that the various changes of contour give rise, by Stokes formula, to extra terms involving the $\overline{\partial}$ of the integrands. The details can be found in the appendix. ◇

4 Proof of Theorem 2.1

We start by constructing a kind of 'global escape function' in the same spirit as in [GeMa]. First we fix $\delta > 0$ and $R > 1$ such that for $|x| \geq R$ and $(x,\xi) \in p^{-1}([\lambda_0 - \delta, \lambda_0 + \delta])$ one has:

$$H_p(x \cdot \xi) = 2\xi^2 - x\nabla_x V(x) \geq \lambda_0. \tag{4.1}$$

We can also assume that R is sufficiently large so that $K \subset \{|x| < R-1\}$. Then one can construct as in [GeMa] a real-valued function $f \in C_b^\infty(\mathbb{R}^{2n})$ such that $H_p f \geq 0$ everywhere, $H_p f = 1$ for $|x| \leq R$ and $H_p f = 0$ for $|x| \geq R+1$.

We also specify our choice of the vector-field F by taking it as: $F(x) = \chi_1(x)x$ where $\chi_1 \in C^\infty(\mathbb{R}^n)$, $0 \leq \chi_1 \leq 1$, $\chi_1(x) = 1$ for $|x| \geq R$. If we denote by p_θ the Weyl-symbol of the distorded Hamiltonian P_θ defined in Section 2, we see that it is a second order polynome with respect to ξ with coefficients in $C_b^\infty(\mathbb{R}_x^n)$ and thus we can consider an almost analytic extension $\widetilde{p}_\theta$ of p_θ as defined in Section 3.

Then we set

$$G(x,\xi) := \chi_2(x)\chi_3(\xi)f(x,\xi) \tag{4.2}$$

where $\chi_2, \chi_3 \in C_0^\infty(\mathbb{R}^n)$, $\chi_2(x) = 1$ for $|x| \leq R+1$, $\chi_3(\xi) = 1$ for $\xi^2 \leq \lambda_0 + \operatorname{Sup}|V| + \delta$. In particular we have $\chi_2(x)H_p f(x,\xi) = H_p f(x,\xi)$. Moreover χ_2 can be chosen arbitrarily flat and we denote:

$$\mu := \operatorname{Sup}|fH_p\chi_2| \tag{4.3}$$

which will be fixed small enough later. The following lemma will allow us to obtain an ellipticity estimate of $\widetilde{p}_\theta$ on Λ_{tG} for a suitable choice of t, θ:

Lemma 4.1 *For any $(x,\xi) \in \mathbb{R}^{2n}$ one has:*

$$\widetilde{p}_\theta(x + it\partial_\xi G, \xi - it\partial_x G) = p(x,\xi) - iH_p(\theta F(x)\xi + tG) + \mathcal{O}\left((t^2 + \theta^2 + h)\langle\xi\rangle^2\right) \tag{4.4}$$

uniformly with respect to h, t and θ (all of them small enough). Moreover, for $(x,\xi)\in p^{-1}([\lambda_0-\delta,\lambda_0+\delta])$ one has:

$$H_p(\theta F(x)\xi + tG) \geq \theta\lambda_0\chi_1(x) + (t - C_2\theta)H_p f(x,\xi) - \mu t \tag{4.5}$$

where $C_2 := \underset{p^{-1}([\lambda_0-\delta,\lambda_0+\delta])}{\mathrm{Sup}} |(x\cdot\xi)H_p\chi_2|$ *$(<\infty)$.*

Proof - By construction we have:

$$p_\theta(x,\xi) = p_\theta^0(x,\xi) + \mathcal{O}(h\langle\xi\rangle^2)$$

with

$$p_\theta^0(x,\xi) = p\left(x + i\theta F(x), (1+i\theta dF(x))^{-1}\xi\right).$$

Considering almost analytic extensions of all the involved smooth function of x, this permits to obtain by a Taylor expansion:

$$\begin{aligned}\widetilde{p}_\theta(x+it\partial_\xi G, \xi - it\partial_x G) &= \widetilde{p}(x+i\theta F(x) + it\partial_\xi G, \xi - i\theta dF(x)\xi - it\partial_x G)\\ &\quad + \mathcal{O}\left((t\theta + \theta^2 + h)\langle\xi\rangle^2\right)\end{aligned}$$

and formula (4.4) follows easily by using the definition of the almost analytic extension $\widetilde{p}$ of p and by observing that $(\theta F(x) + t\partial_\xi G, \theta dF(x)\xi + t\partial_x G) = (\partial_\xi G_1, \partial_x G_1)$ with $G_1 := \theta F(x)\xi + tG$.

Moreover, on $p^{-1}([\lambda_0-\delta,\lambda_0+\delta])$ we have:

$$H_p(\theta F(x)\xi + tG) = \theta\chi_2 H_p(x\cdot\xi) + t\chi_1 H_p f + \theta x\cdot\xi H_p\chi_2 + tfH_p\chi_1$$

and since $H_p\chi_2$ is supported in $\{\chi_1 H_p f = 1\}$, the estimate (4.5) follows by using (4.1)-(4.3). $\diamond$

Now we take

$$t = 2C_2\theta$$

and we choose χ_2 in such a way that

$$\mu \leq \frac{1}{2}\,\mathrm{Min}\{C_2,\lambda_0\}.$$

In this case, since $\chi_1 + H_p f \geq 1$ everywhere, the estimate (4.5) gives:

$$H_p(\theta F(x)\xi + tG) \geq \frac{\theta}{2}\,\mathrm{Min}\{C_2,\lambda_0\}$$

on $p^{-1}([\lambda_0-\delta,\lambda_0+\delta])$. As a consequence,

(4.4) allows to deduce the existence of a positive constant C_3 such that if θ and h are small enough, then:

$$|\widetilde{p}_\theta(x+it\partial_\xi G, \xi - it\partial_x G) - \rho| \geq \frac{\theta}{C_3}\langle\xi\rangle^2 \tag{4.6}$$

for all $(x,\xi) \in I\!R^{2n}$ and $\rho \in [\lambda_0 - \delta', \lambda_0 + \delta'] - i[0, \delta'\theta]$ with $\delta' > 0$ small enough.

To end the argument we apply Proposition 3.1 with $A = P_\theta - \rho$, $t = 2C_2\theta$ and $\theta = Ch\log(h^{-1})$ where $C >> 1$ is arbitrary. Then for any $N >> 1$ one has $h^{-3n}|t|^{2N}e^{J_0|t|/h} = \mathcal{O}(h^{N-CJ_0-3n}) = \mathcal{O}(h)$ and thus by (4.6) Proposition 3.1 gives in this case:

$$\begin{aligned} \|\langle\xi\rangle^2 e^{-tH/h}Tu\|^2_{L^2(\Lambda_{tG})} &\leq \frac{C_4}{h\log(h^{-1})}\|e^{-tH/h}T(P_\theta-\rho)u\|^2_{L^2(\Lambda_{tG})} \\ &\quad + \frac{C_4}{\log(h^{-1})}\|\langle\xi\rangle^2 e^{-tH/h}Tu\|^2_{L^2(\Lambda_{tG})} \end{aligned}$$

for some positive constant C_4 and for any $u \in C_0^\infty$. Taking h small enough, we obtain

$$\|\langle\xi\rangle^2 e^{-tH/h}Tu\|^2_{L^2(\Lambda_{tG})} \leq \frac{2C_4}{h\log(h^{-1})}\|e^{-tH/h}T(P_\theta-\rho)u\|^2_{L^2(\Lambda_{tG})}$$

and by density this estimate extends for all $u \in H^2(I\!R^n)$. In particular the equation $(P_\theta - \rho)u = 0$ does not admit a non trivial solution in $L^2(I\!R^n)$ and Theorem 2.1 follows. $\diamond$

5 Proof of Theorem 2.2

We can assume $s > 1$ since the result for $s = 1$ is well known. The arguments of the previous section are unchanged until (4.6), then the difference consists in the choice of θ only: here we take $\theta = \delta h^{1-\frac{1}{s}}$ and we observe that if δ is chosen small enough, then for $t = 2C_2\theta = 2C_2\delta h^{1-\frac{1}{s}}$ we have:

$$h^{-3n}e^{-\varepsilon/t^{1/(s-1)}}e^{J_0|t|/h} = \mathcal{O}\left(e^{-\varepsilon/(2h^{1/s})}\right). \tag{5.1}$$

Moreover, the cut-off function χ_1 used in the definition of $F(x)$ can be taken Gevrey of index s, and this permits to apply the Gevrey version of Proposition 3.1 with $A = P_\theta - \rho$. By (4.6) and (5.1) it gives in this case:

$$\begin{aligned} \|\langle\xi\rangle^2 e^{-tH/h}Tu\|^2_{L^2(\Lambda_{tG})} &\leq \frac{C_5}{h^{1-1/s}}\|e^{-tH/h}T(P_\theta-\rho)u\|^2_{L^2(\Lambda_{tG})} \\ &\quad + C_5 h^{1/s}\|\langle\xi\rangle^2 e^{-tH/h}Tu\|^2_{L^2(\Lambda_{tG})} \end{aligned}$$

for some positive constant C_5. Taking again h small enough, we obtain:

$$\|\langle\xi\rangle^2 e^{-tH/h}Tu\|^2_{L^2(\Lambda_{tG})} \leq \frac{2C_5}{h^{1-1/s}}\|e^{-tH/h}T(P_\theta-\rho)u\|^2_{L^2(\Lambda_{tG})}$$

and we conclude again that the equation $(P_\theta - \rho)u = 0$ admits no non-trivial solution in $L^2(I\!R^n)$ when $\theta = \delta h^{1-\frac{1}{s}}$ and $\rho \in [\lambda_0 - \delta', \lambda_0 + \delta'] - i[0, \delta'\theta]$ ($\delta, \delta' > 0$ small enough). This completes the proof of Theorem 2.2. $\diamond$

6 Remark: Nowhere Analytic Potentials

If one just assume that V is C^∞ on $\mathbb{R}^n$, then it is not possible in general to define the resonances of P. However, if one adds the following assumption:

(H4) For all $\alpha \in \mathbb{N}^n$, $|x^\alpha \partial^\alpha V(x)|$ tends to 0 as $|x|$ tends to infinity;

then the previous constructions of an analytic extension for V can be performed globally and permit to define $\widetilde{V}((1+it)x)$ up to an error term $\mathcal{O}(t^\infty)$ uniformly with respect to $x \in \mathbb{R}^n$. As a consequence, it becomes possible to define up to $\mathcal{O}(t^\infty)$ the resonances of P as the eigenvalues of $\widetilde{P} := -(1+it)^{-2}h^2\Delta + \widetilde{V}((1+it)x)$ (the spectrum of which is discrete outside $(1+it)^{-2}\mathbb{R}_+$) and, taking $t = Ch\log(h^{-1})$, the previous arguments apply and permit to generalize Theorem 2.1 to the resonances of P which are now defined up to $\mathcal{O}(h^\infty)$.

Similarly, if in addition to (H4) we also assume:

(H5) There exists $C_1 > 0$ such that for all $\alpha \in \mathbb{N}^n$, $|x^\alpha \partial^\alpha V(x)| \leq C_1^{1+|\alpha|}(\alpha!)^s$ uniformly on $\mathbb{R}^n$;

(for some $s > 1$) then the global construction of $\widetilde{V}((1+it)x)$ can be performed up to an error term $\mathcal{O}(e^{-1/Ct^{1/(s-1)}})$ uniformly with respect to $x \in \mathbb{R}^n$, and the resonances of P can be defined up to the same quantity. Taking $t = \delta h^{1-1/s}$, this quantity becomes $\mathcal{O}(e^{-1/(Ch^{1/s})})$ and again the previous arguments permit to generalize Theorem 2.2 to this situation.

7 Appendix: Proof of Proposition 3.1

We start by proving:

Proposition 7.1 *Let $m \in C^\infty(\Lambda_{tG})$ such that $\partial^\gamma m = \mathcal{O}(\langle\xi\rangle^\ell)$ for some $\ell \geq 0$ and for all $\gamma \in \mathbb{N}^{2n}$. Then there exists an operator $R(t,h)$ such that for all $u, v \in C_0^\infty(\mathbb{R}^n)$ one has:*

$$\langle me^{-tH/h}TAu, e^{-tH/h}Tv\rangle_{L^2(\Lambda_{tG})} = \langle (m\widetilde{a}+R(t,h))e^{-tH/h}Tu, e^{-tH/h}Tv\rangle_{L^2(\Lambda_{tG})}$$

with

$$\left\| \langle\xi\rangle^\sigma R(t,h)\langle\xi\rangle^{-(k+\ell+\sigma)} \right\|_{\mathcal{L}(L^2(\Lambda_{tG}))} = \mathcal{O}\left(h + h^{-3n/2}|t|^\infty e^{J_0|t|/h}\right)$$

for any $\sigma \in \mathbb{R}$ and uniformly with respect to t and h both small enough. Here $J_0 := 2\operatorname{Sup}_{\mathbb{R}^{2n}} |H|$.

Moreover, if the coefficients of A are globally Gevrey with index $s > 1$, the estimate on $R(t,h)$ can be improved into:

$$\left\| \langle\xi\rangle^\sigma R(t,h)\langle\xi\rangle^{-(k+\ell+\sigma)} \right\|_{\mathcal{L}(L^2(\Lambda_{tG}))} = \mathcal{O}\left(h + h^{-3n/2}e^{-\varepsilon/t^{1/(s-1)}}e^{J_0|t|/h}\right)$$

for some positive constant ε.

Denote

$$I := \langle m e^{-tH/h} T A u, e^{-tH/h} T v\rangle_{L^2(\Lambda_{tG})}.$$

Using the definitions of A and T we obtain:

$$I = \frac{c^2}{(2\pi h)^n} \int e^{\Phi/h} m(\alpha) a\left(\frac{y+x'}{2}, \eta\right) u(x')\overline{v(y')} d(x', \eta, y, y', \mathrm{Re}\,\alpha) \qquad (7.1)$$

where the integral runs over $\{(\alpha, y, y', x', \eta) \in \Lambda_{tG} \times I\!R^{4n}\}$, and where (denoting $\alpha = (\alpha_x, \alpha_\xi) \in \mathbb{C}^n \times \mathbb{C}^n$ for α in Λ_{tG}):

$$\Phi = -2tH(\mathrm{Re}\,\alpha) + i(\alpha_x - y)\alpha_\xi - i(\overline{\alpha_x} - y')\overline{\alpha_\xi} - \frac{1}{2}(\alpha_x - y)^2 - \frac{1}{2}(\overline{\alpha_x} - y')^2 + i(y - x')\eta. \qquad (7.2)$$

Then we observe that by construction of $\widetilde{a}$ we have for all $Y = (y, \eta) \in I\!R^{2n}$ and $\alpha = (\alpha_x, \alpha_\xi) \in \Lambda_{tG}$ (and denoting $X_s := sY + (1-s)\alpha$, $0 \le s \le 1$):

$$\begin{aligned} a(Y) - \widetilde{a}(\alpha) &= \int_0^1 \left((Y - \mathrm{Re}\,\alpha) \frac{\partial \widetilde{a}}{\partial \mathrm{Re}\,\alpha}(X_s) - \mathrm{Im}\,\alpha \frac{\partial \widetilde{a}}{\partial \mathrm{Im}\,\alpha}(X_s) \right) ds \\ &= \int_0^1 \left((Y - \alpha) \frac{\partial \widetilde{a}}{\partial \mathrm{Re}\,\alpha}(X_s) + 2i \mathrm{Im}\,\alpha \frac{\partial \widetilde{a}}{\partial \overline{\alpha}}(X_s) \right) ds \\ &= (\alpha - Y) \cdot b(\alpha, Y) + r(\alpha, Y) \end{aligned} \qquad (7.3)$$

where b and r are C^∞ on $\Gamma_\varepsilon \times I\!R^n \times I\!R^{2n}$ (polynomial with respect to α_ξ and η) and satisfy the following estimates for all $\gamma \in I\!N^{4n}$:

$$\begin{aligned} \partial^\gamma b &= \mathcal{O}(\langle(\alpha_\xi, \eta)\rangle^k) \\ \partial^\gamma r &= \mathcal{O}(\mu(\mathrm{Im}\,\alpha_x)\langle(\alpha_\xi, \eta)\rangle^k) \end{aligned} \qquad (7.4)$$

uniformly on $\Gamma_\varepsilon \times I\!R^n \times I\!R^{2n}$. Here we have used following the notation:

$$\begin{aligned} \mu(t) &= |t|^\infty \text{ in the } C^\infty \text{ case;} \\ \mu(t) &= e^{-\varepsilon/|t|^{1/(s-1)}} \ (\text{some } \varepsilon > 0) \text{ in the Gevrey case,} \end{aligned}$$

and we shall continue to use it all along the proof.

Inserting (7.3) into (7.1) we obtain:

$$I = \langle m\widetilde{a} e^{-tH/h} T u, e^{-tH/h} T v\rangle_{L^2(\Lambda_{tG})} + R_1 + R_2$$

with

$$R_1 = \frac{c^2}{(2\pi h)^n} \int e^{\Phi/h} \left(\alpha_x - \frac{y+x'}{2}, \alpha_\xi - \eta\right) \cdot b_1\, m(\alpha) u(x')\overline{v(y')} d(x', \eta, y, y', \mathrm{Re}\,\alpha) \qquad (7.5)$$

$$R_2 = \frac{c^2}{(2\pi h)^n} \int e^{\Phi/h} r\left(\alpha, \frac{y+x'}{2}, \eta\right) m(\alpha) u(x')\overline{v(y')} d(x', \eta, y, y', \mathrm{Re}\,\alpha) \qquad (7.6)$$

where we have denoted $b_1 = b\left(\alpha, \frac{y+x'}{2}, \eta\right)$. Now the arguments of [BeMa] Section 3 show the existence of a smooth vector-field of the type:

$$L := L\left(\alpha, \frac{y+x'}{2}, \eta, \frac{\partial}{\partial \mathrm{Re}\,\alpha}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \eta}\right)$$

with coefficients $\mathcal{O}(\langle(\alpha_\xi, \eta)\rangle^k)$ together with all their derivatives (actually polynomial with respect to η), such that:

$$\left(\alpha_x - \frac{y+x'}{2}, \alpha_\xi - \eta\right) \cdot b_1 = hL(e^{\Phi/h}).$$

Inserting this inside (7.5), making integrations by parts and denoting:

$$f\left(\alpha, \frac{y+x'}{2}, \eta\right) = {}^tL[m] \tag{7.7}$$

(where ${}^tL$ denotes the transposed of L), we obtain

$$R_1 = h\langle e^{-tH/h}T_f u, e^{-tH/h}Tv\rangle_{L^2(\Lambda_{tG})}$$

with

$$T_f u(\alpha) := c\int e^{i(\alpha_x - y)\alpha_\xi/h - (\alpha_x - y)^2/2h}\mathrm{Op}_h^W(f(\alpha, \cdot))u(y)dy.$$

With the same notation we also have:

$$R_2 = \langle e^{-tH/h}T_{mr}u, e^{-tH/h}Tv\rangle_{L^2(\Lambda_{tG})}.$$

Then we introduce the formal adjoint of T as:

$$T^*w(x) = c\int e^{-i(x-y)\xi/h - (x-y)^2/2h}w(x,\xi)dxd\xi \tag{7.8}$$

where the (oscillatory) integral runs over $\mathbb{R}^{2n}$ in the general case when $w \in \mathcal{S}'(\mathbb{R}^{2n})$. In the case when $w = Tu$ with $u \in C_0^\infty(\mathbb{R}^n)$, one can make a change of contour of integration in (7.8) so that the new contour becomes Λ_{tG}, and using that $u = T^*Tu$ we see that R_1 and R_2 can be rewritten as:

$$\begin{aligned} R_1 &= h\left\langle \langle\alpha_\xi\rangle^{-\sigma}K_f\langle\alpha_\xi\rangle^{k+\ell+\sigma}e^{-tH/h}Tu, e^{-tH/h}Tv\right\rangle_{L^2(\Lambda_{tG})} \\ R_2 &= \left\langle \langle\alpha_\xi\rangle^{-\sigma}K_{mr}\langle\alpha_\xi\rangle^{k+\ell+\sigma}e^{-tH/h}Tu, e^{-tH/h}Tv\right\rangle_{L^2(\Lambda_{tG})} \end{aligned} \tag{7.9}$$

where we have used the following notation:

$$K_g = \langle\alpha_\xi\rangle^\sigma e^{-tH/h}T_g T^* e^{tH/h}\langle\alpha_\xi\rangle^{-(k+\ell+\sigma)}$$

for $g = f, mr$. Now, in view of (7.9), it is clear that Proposition 7.1 is a consequence of the following lemma:

Lemma 7.2 *the operators K_f and K_{mr} are bounded on $L^2(\Lambda_{tG})$, and their norms satisfy:*

$$\begin{aligned}
\|K_f\|_{\mathcal{L}(L^2(\Lambda_{tG}))} &= \mathcal{O}\left(1 + h^{-3n/2}\mu(t)e^{J|t|/h}\right) \\
\|K_{mr}\|_{\mathcal{L}(L^2(\Lambda_{tG}))} &= \mathcal{O}\left(h^{-3n/2}\mu(t)e^{J|t|/h} + h^{\infty}\right).
\end{aligned}$$

Proof - The distribution kernel of K_g is given by the oscillatory integral:

$$K_g(\alpha,\beta) = \frac{c^2\langle\alpha_\xi\rangle^\sigma}{(2\pi h)^n\langle\beta_\xi\rangle^{k+\ell+\sigma}} \int e^{\Phi_1/h} g\left(\alpha, \frac{y+x'}{2}, \eta\right) d(x', y, \eta) \tag{7.10}$$

with

$$\Phi_1 = t[H(\mathrm{Re}\beta) - H(\mathrm{Re}\alpha)] + \Phi_2 - \frac{1}{2}(\alpha_x - y)^2 - \frac{1}{2}(\beta_x - x')^2$$

where $\Phi_2 := i\alpha_x\alpha_\xi - i\beta_x\beta_\xi + i(\eta - \alpha_\xi)y + i(\beta_\xi - \eta)x'$ $(\alpha, \beta \in \Lambda_{tG})$. At first we perform repeated integrations by parts in (7.10) by using the operator:

$$\begin{aligned}
L_1 := &\left(1 + |\eta-\alpha_\xi|^2 + |\eta - \beta_\xi|^2 + |y - x'|^2\right)^{-1} \times \\
&\times (1 + (\eta - \alpha_\xi)\cdot hD_y + (\beta_\xi - \eta)\cdot hD_{x'} + (y - x')\cdot hD_\eta)
\end{aligned}$$

which satisfies : $L_1(e^{\Phi_2/h}) = e^{\Phi_2/h}$. This permits to obtain for any $N \geq 1$:

$$K_g(\alpha,\beta) = \frac{c^2\langle\alpha_\xi\rangle^\sigma}{(2\pi h)^n\langle\beta_\xi\rangle^{k+\ell+\sigma}} \int e^{\Phi_1/h} g_N(\alpha, \beta, x', y, \eta) d(x', y, \eta) \tag{7.11}$$

with

$$g_N(\alpha, \beta, x', y, \eta) = ({}^tL_2)^N\left(g\left(\alpha, \frac{y+x'}{2}, \eta\right)\right) \tag{7.12}$$

$$L_2 = e^{-[(\alpha_x - y)^2 + (\beta_x - x')^2]/2h} L_1 e^{[(\alpha_x-y)^2 + (\beta_x - x')^2]/2h}$$

In particular, g_N satisfies to the estimates:

$$g_N = \mathcal{O}\left(\frac{\langle(\alpha_x - y, \beta_x - x')\rangle^N}{\langle(\alpha_\xi - \eta, \beta_\xi - \eta, y - x')\rangle^N}\right) \sum_{|\gamma|\leq N} \left|\partial^{\gamma'} g\left(\alpha, \frac{y+x'}{2}, \eta\right)\right|. \tag{7.13}$$

We denote by $\widetilde{g}_N$ an almost analytic extension of g_N constructed from an almost analytic extension $\widetilde{g}$ of g *via* (7.12), and, for $C > 0$ large enough (to be specified later on) we consider the following family of contours in $\mathbb{C}^{3n}$:

$$\mathcal{C}_s \;:\; I\!R^{3n} \ni \begin{pmatrix} y \\ x' \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} y \\ x' \\ \eta \end{pmatrix} + iCs \cdot \begin{pmatrix} (\eta - \overline{\alpha_\xi})/\langle\eta - \alpha_\xi\rangle \\ (\overline{\beta_\xi} - \eta)/\langle\beta_\xi - \eta\rangle \\ (y - x')/\langle y - x'\rangle \end{pmatrix}$$

for $s = (s_1, ..., s_{3n}) \in [0, |t|]^{3n}$. Then, along $\mathcal{C}_s$ we have:

$$\begin{aligned}
\mathrm{Re}\,\phi_1 \leq\; &|t|F_t(\alpha,\beta) - (\operatorname*{Min}_j s_j)\left[\frac{|y-x'|^2}{\langle y - x'\rangle} + \frac{|\eta - \alpha_\xi|^2}{\langle\eta - \alpha_\xi\rangle} + \frac{|\eta - \beta_\xi|^2}{\langle\eta - \beta_\xi\rangle}\right] \\
&-\frac{1}{4}\left[|\alpha_x - y|^2 + |\mathrm{Re}\beta_x - x'|^2\right]
\end{aligned}$$

with

$$F_t(\alpha,\beta) = |G(\mathrm{Re}\beta) - G(\mathrm{Re}\alpha)| + 4|t\nabla G(\mathrm{Re}\alpha)|^2 + 4|t\nabla G(\mathrm{Re}\beta)|^2.$$

Therefore, since $\overline{\partial}\widetilde{g}_N = ({}^tL_2)^N\overline{\partial}\widetilde{g}$, we obtain from (7.11) by Stokes' formula:

$$\begin{aligned}
K_g(\alpha,\beta) &= \mathcal{O}(h^{-5n/2}e^{|t|F_t(\alpha,\beta)/h})\int_{\mathcal{C}_{|t|}} e^{-\frac{C|t|}{h}\left[\frac{(y-x')^2}{\langle y-x'\rangle}+\frac{|\eta-\alpha_\xi|^2}{\langle \eta-\alpha_\xi\rangle}+\frac{|\eta-\beta_\xi|^2}{\langle \eta-\beta_\xi\rangle}\right]} \qquad (7.14)\\
&\qquad \times e^{-\left[|\alpha_x-y|^2+|\beta_x-x'|^2\right]/4h}\frac{\langle\alpha_\xi\rangle^\sigma}{\langle\beta_\xi\rangle^{k+\ell+\sigma}}\widetilde{g}_N d(x',y,\eta)\\
&\quad +\mathcal{O}(h^{-5n/2}e^{|t|F_t(\alpha,\beta)/h})\,\mathrm{Sup}_{\Omega_t}\left(\sum_{|\gamma|\le N}|\langle(\alpha_\xi,\eta)\rangle^{-(k+\ell)}\partial^\gamma\overline{\partial}\widetilde{g}|\right)\\
&\qquad \times\int_{\Omega_t}\frac{e^{-\left[|\alpha_x-y|^2+|\beta_x-x'|^2\right]/4h}\langle(\alpha_\xi,\eta)\rangle^{k+\ell+\sigma}}{\langle\beta_\xi\rangle^{k+\ell+\sigma}\langle(\alpha_\xi-\eta,\beta_\xi-\eta,y-x')\rangle^N}\mathrm{L}(d(x',y,\eta))
\end{aligned}$$

where $\mathcal{C}_{|t|}$ stands for $\mathcal{C}_{(|t|,\dots,|t|)}$, $\Omega_t := \bigcup_{s\in[0,|t|]^{3n}} \mathcal{C}_s$ and $\mathrm{L}(d(x',y,\eta))$ denotes the Lebesgue measure in $\mathbb{C}^{3n}$.

Let $C'>0$ be such that $\mathrm{Supp}\,G \subset \{|(x,\xi)|\le C'\}$. Then, as in [BeMa], we split $\Lambda_{tG}\times\Lambda_{tG}$ in 4 regions:

1st region: $|\mathrm{Re}\,\alpha|\ge C'$ and $|\mathrm{Re}\,\beta|\ge C'$

In this region $F_t(\alpha,\beta)=0$ and thus, if we take N large enough (7.14) gives there (using also (7.13):

$$\begin{aligned}
K_g(\alpha,\beta) &= \mathcal{O}(h^{-n})e^{-\frac{|t|}{h}\frac{\alpha_\xi-\beta_\xi|^2}{\langle\alpha_\xi-\beta_\xi\rangle}-\nu|\alpha_x-\beta_x|^2/h}\,\mathrm{Sup}_{\Omega_t}\left(\sum_{|\gamma|\le N}|\langle(\alpha_\xi,\eta)\rangle^{-(k+\ell)}\partial^\gamma\widetilde{g}|\right)\\
&\quad +\mathcal{O}(h^{-3n/2})\frac{e^{-\nu|\alpha_x-\beta_x|^2/h}}{\langle\alpha_\xi-\beta_\xi\rangle^{n+1}}\,\mathrm{Sup}_{\Omega_t}\left(\sum_{|\gamma|\le N}|\langle(\alpha_\xi,\eta)\rangle^{-(k+\ell)}\partial^\gamma\overline{\partial}\widetilde{g}|\right)
\end{aligned}$$

for some $\nu>0$ and any fixed $N\ge1$ large enough. In particular, if we use (7.4) and the properties of the almost analytic extentions:

$$\begin{aligned}
K_f(\alpha,\beta) &= \mathcal{O}\left(h^{-n}e^{-\frac{|t|}{h}\frac{\alpha_\xi-\beta_\xi|^2}{\langle\alpha_\xi-\beta_\xi\rangle}-\nu|\alpha_x-\beta_x|^2/h}+h^{-3n/2}\mu(t)\frac{e^{-\nu|\alpha_x-\beta_x|^2/h}}{\langle\alpha_\xi-\beta_\xi\rangle^{n+1}}\right)\\
K_{mr}(\alpha,\beta) &= \mathcal{O}\left(h^{-3n/2}\mu(t)\frac{e^{-\nu|\alpha_x-\beta_x|^2/h}}{\langle\alpha_\xi-\beta_\xi\rangle^{n+1}}\right) \qquad (7.15)
\end{aligned}$$

2nd region: $|\mathrm{Re}\,\alpha|\ge 2C'$ and $|\mathrm{Re}\,\beta|\le C'$

In this region we have

$$|\alpha-\beta|\ge\frac{1}{2}|\mathrm{Re}\,\alpha|\ge\frac{C'}{2}+\frac{1}{4}|\mathrm{Re}\,\alpha|. \qquad (7.16)$$

Therefore, denoting

$$J := \operatorname*{Sup}_{\mathbb{R}^{2n}}(|G| + |\nabla G|^2)$$

we see on (7.14) that if we have chosen C and N sufficiently large we have in this region:

$$\begin{aligned} K_g(\alpha,\beta) &= \mathcal{O}(e^{-|t\mathrm{Re}\,\alpha|/h}) \operatorname*{Sup}_{\mathcal{C}_{|t|}} \left(\sum_{|\gamma|\leq N} |\langle(\alpha_\xi,\eta)\rangle^{-(k+\ell)}\partial^\gamma g| \right) \\ &\quad + \mathcal{O}(h^{-3n/2}e^{J|t|/h})\langle \mathrm{Re}\,\alpha\rangle^{-(2n+1)} \operatorname*{Sup}_{\Omega_t} \left(\sum_{|\gamma|\leq N} |\langle(\alpha_\xi,\eta)\rangle^{-(k+\ell)}\partial^\gamma \overline{\partial} g| \right). \end{aligned}$$

In particular:

$$\begin{aligned} K_f(\alpha,\beta) &= \mathcal{O}(e^{-|t\mathrm{Re}\,\alpha|/h} + h^{-3n/2}\mu(t)e^{J|t|/h}\langle \mathrm{Re}\,\alpha\rangle^{-(2n+1)}) \\ K_{mr}(\alpha,\beta) &= \mathcal{O}(h^{-3n/2}\mu(t)e^{J|t|/h}\langle \mathrm{Re}\,\alpha\rangle^{-(2n+1)}) \end{aligned} \tag{7.17}$$

3rd region: $|\mathrm{Re}\,\alpha| \leq C'$ and $|\mathrm{Re}\,\beta| \geq 2C'$

We obtain in the same way:

$$\begin{aligned} K_f(\alpha,\beta) &= \mathcal{O}(e^{-|t\mathrm{Re}\,\beta|/h} + h^{-3n/2}\mu(t)e^{J|t|/h}\langle \mathrm{Re}\,\beta\rangle^{-(2n+1)}) \\ K_{mr}(\alpha,\beta) &= \mathcal{O}(h^{-3n/2}\mu(t)e^{J|t|/h}\langle \mathrm{Re}\,\beta\rangle^{-(2n+1)}) \end{aligned} \tag{7.18}$$

4th region: $|\mathrm{Re}\,\alpha| \leq 2C'$ and $|\mathrm{Re}\,\beta| \leq 2C'$

In this last region we try to apply the stationary phase theorem to (7.10). The unique stationary point (given by $\partial_y\Phi_1 = \partial_{x'}\Phi_1 = \partial_\eta\Phi_1 = 0$) is:

$$\begin{cases} y = x' = \frac{\alpha_x+\beta_x}{2} - \frac{i}{2}(\alpha_\xi - \beta_\xi) \\ \eta = \frac{\alpha_\xi+\beta_\xi}{2} + \frac{i}{2}(\alpha_x - \beta_x) \end{cases}$$

In order to be able to apply the stationary phase theorem with complex-valued phase functin (see [MeSj] p.145), we need that the critical point remains close enough to the real (it already remains bounded because so are α and β in this region). That is, we need that $|\alpha-\beta|$ is small enough. But if it is not the case (that is if $|\alpha-\beta| \geq \delta'$ for some $\delta' > 0$), then (7.14) gives as for the 2nd and 3rd regions (possibly by increasing the value of the constant C again):

$$\begin{aligned} K_f(\alpha,\beta) &= \mathcal{O}(e^{-C''|t|/h} + h^{-3n/2}\mu(t)e^{J|t|/h}) \\ K_{mr}(\alpha,\beta) &= \mathcal{O}(h^{-3n/2}\mu(t)e^{J|t|/h}) \end{aligned} \tag{7.19}$$

for some arbitrary large constant $C'' > 0$. Otherwise, if $|\alpha-\beta| \leq \delta'$ with δ' small enough, then Theorem 2.3 of [MeSj] can be applied and gives for any $N \geq 1$:

$$K_g(\alpha,\beta) = h^{-n}\widetilde{b}_N(\alpha,\beta,h)e^{\varphi(\alpha,\beta)/h} + \mathcal{O}(h^N) \tag{7.20}$$

where $\varphi(\alpha,\beta)$ is the critical value of Φ_1, that is:

$$\varphi(\alpha,\beta)=t(H(\beta)-H(\alpha))+\frac{i}{2}(\alpha_x-\beta_x)(\alpha_\xi+\beta_\xi)-\frac{1}{4}(\alpha_x-\beta_x)^2-\frac{1}{4}(\alpha_\xi-\beta_\xi)^2$$

and $\widetilde{b}_N$ is a bounded symbol which can be estimated by the derivatives of g at the critical point, up to order $2(N+n)$. Moreover, using the definition of H we can see that

$$\mathrm{Re}\,\varphi(\alpha,\beta)=-\frac{1}{4}\mathrm{Re}\,(\alpha_x-\beta_x)^2-\frac{1}{4}\mathrm{Re}\,(\alpha_\xi-\beta_\xi)^2+t\psi(\alpha,\beta) \tag{7.21}$$

with

$$\begin{aligned}\psi(\alpha,\beta) &= G(\beta)-G(\alpha)-\frac{1}{2}\mathrm{Re}\,(\beta_x-\alpha_x)(\partial_x G(\alpha)+\partial_x G(\beta))\\ &\qquad -\frac{1}{2}\mathrm{Re}\,(\beta_\xi-\alpha_\xi)(\partial_\xi G(\alpha)+\partial_\xi G(\beta))\\ &= \mathcal{O}(|\alpha-\beta|^2).\end{aligned}$$

As a consequence, if we take t small enough we deduce from (7.4), (7.20) and (7.21):

$$\begin{aligned}K_f(\alpha,\beta) &= \mathcal{O}\left(h^{-n}e^{-\nu|\alpha-\beta|^2/h}+h^N\right)\\ K_{mr}(\alpha,\beta) &= \mathcal{O}\left(h^{-n}(\mu(t)+\mu(|\alpha-\beta|)(e^{-\nu|\alpha-\beta|^2/h}+h^N\right)\end{aligned} \tag{7.22}$$

with $\nu>0$ constant and $N\geq 1$ arbitrary. (Here we have also used that $\mu(t+s)=\mathcal{O}(\mu(t/2)+\mu(s/2))$ for $s,t\geq 0$.)

Now, gathering the estimates (7.15), (7.17), (7.18), (7.19) and (7.22), we obtain uniform bounds for both $\int_{\Lambda_{tG}}|K_g(\alpha,\beta)|d\mathrm{Re}\,\alpha$ and $\int_{\Lambda_{tG}}|K_g(\alpha,\beta)|d\mathrm{Re}\,\beta$ $(g=f,mr)$ which, by the Schur lemma, give the following estimates on the operators K_f K_{mr}:

$$\begin{aligned}\|K_f\|_{\mathcal{L}(L^2(\Lambda_{tG}))} &= \mathcal{O}\left(1+h^{-3n/2}\mu(t)e^{J|t|/h}\right)\\ \|K_{mr}\|_{\mathcal{L}(L^2(\Lambda_{tG}))} &= \mathcal{O}\left(h^{-3n/2}\mu(t)e^{J|t|/h}+h^\infty\right).\end{aligned}$$

this completes the proof of Lemma 7.2 and of Proposition 7.1. ◇

To complete the proof of Theorem 3.1, we first apply Proposition 7.1 with $m=1$ and $v=Au$. We obtain:

$$\|e^{-tH/h}TAu\|^2_{L^2(\Lambda_{tG})}=\langle\widetilde{a}e^{-tH/h}Tu,e^{-tH/h}TAu\rangle+R_3$$

with

$$R_3=\mathcal{O}\left(h+h^{-3n/2}\mu(t)e^{J|t|/h}\right)\|\langle\xi\rangle^k e^{-tH/h}Tu\|\cdot\|e^{-tH/h}TAu\|.$$

Then we apply again Proposition 7.1 but this time with $v = u$ and $m = \overline{\widetilde{a}}$. This gives:

$$\|e^{-tH/h}TAu\|^2_{L^2(\Lambda_{tG})} = \|\widetilde{a}e^{-tH/h}Tu\|^2_{L^2(\Lambda_{tG})} + R_4 \tag{7.23}$$

with

$$\begin{aligned} R_4 &= \mathcal{O}\left(h + h^{-3n/2}\mu(t)e^{J|t|/h}\right) \times \\ &\quad \times \left(\|\langle\xi\rangle^k e^{-tH/h}Tu\|^2 + \|\langle\xi\rangle^k e^{-tH/h}Tu\| \cdot \|e^{-tH/h}TAu\|\right). \end{aligned} \tag{7.24}$$

In particular, we deduce from these two last inequalities (and the fact that $\|\widetilde{a}e^{-tH/h}Tu\| = \mathcal{O}(\|\langle\xi\rangle^k e^{-tH/h}Tu\|)$):

$$\|e^{-tH/h}TAu\| = \mathcal{O}\left(1 + h^{-3n/2}\mu(t)e^{J|t|/h}\right)\|\langle\xi\rangle^k e^{-tH/h}Tu\|$$

which, inserted into (7.24) gives

$$R_4 = \mathcal{O}\left(h + h^{-3n}\mu(t)e^{2J|t|/h}\right)\|\langle\xi\rangle^k e^{-tH/h}Tu\|^2. \tag{7.25}$$

Then Proposition 3.1 follows from (7.23) and (7.25). $\diamond$

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