Finite gap potentials and WKB asymptotics for one-dimensional Schrödinger operators

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Abstract

Consider the Schrödinger operator $H = -d^2/dx^2 + V(x)$ with power-decaying potential $V(x) = O(x^{-\alpha})$. We prove that a previously obtained dimensional bound on exceptional sets of the WKB method is sharp in its whole range of validity. The construction relies on pointwise bounds on finite gap potentials. These bounds are obtained by an analysis of the Jacobi inversion problem on hyperelliptic Riemann surfaces.

1 Introduction

We are interested in one-dimensional Schrödinger equations,

$$-y''(x) + V(x)y(x) = Ey(x), \tag{1}$$

and the spectra of the corresponding self-adjoint operators $H_\beta = -d^2/dx^2 + V(x)$ on $L_2(0,\infty)$, say. The index $\beta \in [0,\pi)$ refers to the boundary condition $y(0)\cos\beta + y'(0)\sin\beta = 0$. The spectral properties of the operators H_β give information on the large time behavior of the quantum mechanical system described by (1).

In this paper, we will present an alternate approach to an earlier result of one of us [21]. With this new approach, we can remove a technical condition and thus prove that a previously obtained bound on the embedded singular spectrum of H_β is sharp in its whole range of validity. We will describe this result shortly; let us first point out that the new idea of this paper is to use finite gap potentials in the construction of [21]. The main difficulty is to obtain good pointwise bounds on these potentials. A substantial part of this paper is devoted to this problem. More specifically, we will have to study in some detail the Jacobi inversion problem on hyperelliptic Riemann surfaces. The result of this analysis is formulated as Theorem 3.1. Actually, our proof gives more than stated: We obtain a whole sequence of good pointwise approximations (where, very roughly, "good" means better than expected, due to cancellations) to finite gap potentials. While our motivation for proving Theorem 3.1 is to provide tools for the proof of Theorem 1.1 below, this discussion is perhaps of independent interest.

Let us now return to (1); suppose that the potential V is bounded by a decaying power, that is,

$$|V(x)| \leq \frac{C}{(1+x)^\alpha} \qquad (\alpha > 0). \tag{2}$$

Then, if $\alpha > 1/2$, the operators H_β have absolutely continuous spectrum essentially supported by $(0,\infty)$, as was first proved in [1, 19]. Embedded singular spectrum can occur if $\alpha \leq 1$ (see [18, 26, 30]), but there are restrictions on the dimension of the singular part of the spectral measure. This is intimately related to the problem of solving (1) asymptotically (for large x). We say that a solution $y(x,E)$ of the Schrödinger equation satisfies the WKB asymptotic formulae if

$$\begin{pmatrix} y(x,E) \\ y'(x,E) \end{pmatrix} = \begin{pmatrix} 1 \\ i\sqrt{E} \end{pmatrix} \exp\left(i \int_0^x \sqrt{E - V(t)}\, dt \right) + o(1) \qquad (x \to \infty). \tag{3}$$

It is well known that there exist solutions of (1) satisfying (3) for all $E > 0$ if the potential V decays and is slowly varying in a suitable sense (see, for instance, [7, Chapter 2]). Obviously, this latter assumption need not hold if V only satisfies (2). Nevertheless, recent work [1, 2, 3, 19, 20] has shown that (3) continues to hold off a small exceptional set of energies E as long as $\alpha > 1/2$. Call this exceptional set S; in other words,

$$S = \{E > 0 : \text{ No solution of (1) satisfies (3)}\}. \tag{4}$$

General criteria [11, 25, 29] show that if there is some embedded singular spectrum on $(0,\infty)$, then the corresponding parts of the spectral measures are supported by S. In other words, if $\rho^{(\beta)}$ denotes the spectral measure of H_β, then $\rho^{(\beta)}_{sing}((0,\infty) \setminus S) = 0$ for all β. Therefore, it is interesting to study S in detail. We know from [1, 19] that S is of Lebesgue measure zero if $\alpha > 1/2$; this was subsequently strengthened in [20] where it was proved that the Hausdorff dimension of S satisfies $\dim S \leq 2(1-\alpha)$. Formally, this result is valid for all $\alpha \in \mathbb{R}$ (if one defines $\dim \emptyset = -\infty$), but it gives nontrivial information only if $1/2 < \alpha \leq 1$. We will show that this bound is sharp and is even attained for suitable potentials:

Theorem 1.1 *For every* $\alpha \in (1/2, 1]$*, there exist potentials* $V(x)$ *satisfying* (2), *so that* $\dim S = 2(1-\alpha)$.

If $\alpha \notin (1/2, 1]$, the whole picture is different. More precisely, if $\alpha \leq 1/2$, then S can have full Lebesgue measure in $(0, \infty)$ [14, 15, 24] and the spectrum can be purely singular. On the other hand, it is easy to prove that $S = \emptyset$ if $\alpha > 1$ (see, e.g., [7]).

In [21], Theorem 1.1 was proved for $\alpha > 2/3$. Things get more difficult as α approaches $1/2$. In particular, we really need the full force of Theorem 3.1 in that the exponent N there gets larger and larger as α decreases to $1/2$.

Actually, here, too, we show more than stated: For any given function $\epsilon(x)$ with $\epsilon(x) \to 0$ as $x \to \infty$ (no matter how slowly), we can construct a potential $V(x) = O(x^{-\alpha-\epsilon(x)})$, so that $\dim S = 2(1-\alpha)$.

There are extensions of the results quoted above to far more general settings. Deift and Killip [5] have proved that there is absolutely continuous spectrum essentially supported by $(0, \infty)$ already if $V \in L_1 + L_2$; very recently, Killip has obtained even stronger results in this direction [13]. WKB asymptotics off exceptional sets have been established by Christ and Kiselev [2, 3] under very general conditions, including $V \in L_1 + L_p$ for some $p < 2$ (but not in the borderline case $p = 2$, which remains open).

A major open question in this context is Simon's problem no. 7 [27]: Are there potentials satisfying (2) with $\alpha > 1/2$, so that for some boundary condition β, the operator H_β has some singular *continuous* spectrum?

We organize this paper as follows. In Sect. 2, we discuss the construction of the so-called finite gap potentials, that is, of quasi-periodic potentials with finitely many prescribed gaps in the spectrum. Since this material is rather classical, we concentrate on those aspects of the theory that are needed later. The following section introduces the problem of obtaining pointwise bounds on finite gap potentials. We state our main result on finite gap potentials (Theorem 3.1) and discuss some general features of this result. The proof is given in Sect. 4, 5, 6; this analysis is perhaps the central part of this paper. It depends on a study of the Jacobi inversion problem in cases where a large number of small gaps is present. A major role will be played by a graphical representation of the terms of a perturbation series, which we introduce in Sect. 5. With Theorem 3.1 as new input, we can then obtain Theorem 1.1, relying mainly on the ideas already contained in [21]. This is done in Sect. 7. In fact, with our new approach, the treatment becomes more transparent.

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2 Finite gap potentials

In this section, we will briefly review the construction and some results on finite gap potentials. We will more or less follow the representation given in [16]. For further information on this many-faceted topic (for example, the connections to equations of the KdV hierarchy), see [9, 10]. The needed facts from the theory of compact Riemann surfaces can be found in [8, 28].

Let energies $E_0 < E_1 < \cdots < E_{2g}$ be given. The aim is to construct a family of (quasi-periodic) potentials $V \in C^\infty(\mathbb{R})$ so that the corresponding operators $H = -d^2/dx^2 + V(x)$ on $L_2(\mathbb{R})$ have purely absolutely continuous spectrum with precisely the prescribed gaps:

$$\sigma(H) = \mathbb{R} \setminus \bigcup_{n=0}^{g} (E_{2n-1}, E_{2n}) \qquad (E_{-1} := -\infty). \tag{5}$$

To this end, consider the Riemann surface S of

$$R(z) = \left(\prod_{n=0}^{2g} (E_n - z) \right)^{1/2}.$$

S is compact and hyperelliptic and its genus is equal to g. A topological model of S can be obtained by gluing together two copies of the extended complex plane cut along the gaps $(\infty, E_0), (E_1, E_2), \ldots , (E_{2g-1}, E_{2g})$.

The points of S may thus be viewed as pairs $\widehat{z} = (z, R(z))$. Here, $z \in \mathbb{C} \cup \{\infty\}$ is the canonical projection of $\widehat{z} \in S$ onto $\mathbb{C} \cup \{\infty\}$, and $R(z)$, which is of course already determined by z up to a sign, shows on which sheet of the surface $\widehat{z}$ lies. Put differently, the canonical projection $\widehat{z} \mapsto z$ gives a two-sheeted branched covering of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by S; the preimages of $\infty, E_0, \ldots , E_{2g}$ are branch points of order one.

Standard coordinates ζ on S can be defined as follows. If $\widehat{z}$ is not a branch point, put $\zeta = z$ in a neighborhood of $\widehat{z}$; near a finite branch point $\widehat{E}_j$, use $\zeta = (E_j - z)^{1/2}$, the sign being determined by the sign of R, and near infinity, use similarly $\zeta = (-z)^{-1/2}$.

There are precisely g linearly independent holomorphic differentials (also known as Abelian differentials of the first kind) on S. One can obtain unique basis elements by prescribing certain periods. We will work with the standard normalization which amounts to demanding that for $i, j = 1, \ldots , g$,

$$2 \int_{E_{2i-1}}^{E_{2i}} \omega_j = \delta_{ij}. \tag{6}$$

(The notation is a little sloppy: The path of integration projects onto $[E_{2i-1}, E_{2i}]$, and $(-1)^i R \leq 0$ on this path.) Note that the left-hand side is just the integral of ω_j over the cycle a_i of a standard homology basis (see, e.g., [10, p. 109ff]). One can then show that the ω_j thus defined are of the form

$$\omega_j = \frac{p_j(z)}{R(z)}\, dz, \quad p_j(z) = c_j \prod_{i \neq j} (\lambda_i^{(j)} - z), \tag{7}$$

with $\lambda_i^{(j)} \in (E_{2i-1}, E_{2i})$, $c_j > 0$. Of course, this representation refers to the coordinate maps $\widehat{z} \mapsto z$ discussed above.

The Abel-Jacobi map α sends positive divisors of degree g (that is, unordered collections of g points from S) to the Jacobi variety of S, which is the complex torus equal to $\mathbb{C}^g$ modulo the period lattice of the holomorphic differentials. This map is onto; in other words, the Jacobi inversion problem can be solved. We will need the Abel-Jacobi map only for divisors of the form $(\widehat{\mu}_1, \dots, \widehat{\mu}_g)$ with $\mu_i \in [E_{2i-1}, E_{2i}]$. The Abel-Jacobi map is then given by

$$\alpha_i(\widehat{\mu}_1, \dots, \widehat{\mu}_g) = 2\pi \sum_{j=1}^{g} \int_{\widehat{E}_{2j-1}}^{\widehat{\mu}_j} \omega_i \mod 2\pi; \tag{8}$$

here, we take paths of integration whose projections lie entirely in the corresponding gaps $[E_{2j-1}, E_{2j}]$. It follows from classical theorems of Abel and Jacobi [28, Chapter 10] that $\alpha = (\alpha_1, \dots, \alpha_g)$ is a bijection from the set of divisors specified above onto the real part of the Jacobi variety $\mathbb{T}^g = [0, 2\pi)^g$. Alternately, this fact may be verified directly, using a representation of the Abel-Jacobi map that will be derived below (see eq. (16)).

Actually, (8) differs from the standard definition of the Abel-Jacobi map by an additive constant vector and the factor 2π; the choice (8) is more convenient here.

Now the stage has been set for the actual construction of the finite gap potentials. Consider the following linear flow on $\mathbb{T}^g$: $\phi_x\alpha_0 = \alpha_0 + \nu x$, where the frequency vector ν is given by

$$\nu_j = 2\pi \text{ res } \left((-z)^{1/2} \omega_j \right) \Big|_{z=\infty} .$$

Using the coordinate $\zeta = (-z)^{-1/2}$ at $z = \infty$, we can easily evaluate the residue to obtain $\nu_j = 4\pi c_j$, where c_j is the normalization constant of the polynomial p_j (see (7)). Now pull back the flow ϕ_x to the set of divisors $(\widehat{\mu}_1, \dots, \widehat{\mu}_g)$, using the Abel-Jacobi map. In other words, define the functions $\widehat{\mu}_j(x) \in S$ ($\mu_j(x) \in [E_{2j-1}, E_{2j}]$) by requiring that

$$\alpha(\widehat{\mu}_1(x), \dots, \widehat{\mu}_g(x)) = \alpha_0 + \nu x.$$

Next, introduce a potential V by the following trace formula:

$$V_{\alpha_0}(x) = E_0 + \sum_{n=1}^{g} (E_{2n-1} + E_{2n} - 2\mu_n(x)). \tag{9}$$

One can then show that this family of potentials solves the inverse problem stated at the beginning of this section. Namely, the operators $H = -d^2/dx^2 + V_{\alpha_0}(x)$ on $L_2(\mathbb{R})$ have purely absolutely continuous spectrum equal to the set given in (5). This follows from the following representation of the diagonal of the Green function of H:

$$G(x, x; z) = \frac{1}{2R(z)} \prod_{n=1}^{g} (\mu_n(x) - z) . \tag{10}$$

This important formula is derived with the aid of the so-called Baker-Akhieser function, which gives explicit expressions for the solutions y of the DE $Hy = zy$. See [16] for the details.

Actually, the right-hand side of (10) defines a meromorphic function on S (with simple poles precisely at the finite branch points) for every fixed $x \in \mathbb{R}$. The Green function, however, depends on $z \in \mathbb{C}$; therefore, we must complement (10) by recalling that Im $G(z)$ Im $z > 0$ for Im $z \neq 0$.

It is useful (and probably also more natural) to interpret the above recipe in a slightly different way. Namely, define $f : \mathbb{T}^g \to \mathbb{R}$ implicitly by the trace formula:

$$f(\beta) = E_0 + \sum_{n=1}^{g} (E_{2n-1} + E_{2n} - 2\mu_n), \tag{11}$$

where $\alpha(\widehat{\mu}_1, \ldots, \widehat{\mu}_g) = \beta$. Then the finite gap potential $V_\alpha(x)$ is obtained by evaluating f along the trajectory of α under the flow $\phi_x\alpha = \alpha + \nu x$.

So, if ν is known, then V_α is computed by inverting the Abel-Jacobi map. We remark parenthetically that there is an "explicit" solution to this problem which uses Riemann theta functions [9, 10, 17], but these formulae are not of much use here.

3 Pointwise estimates on finite gap potentials

Since $\mu_j \in [E_{2j-1}, E_{2j}]$, the trace formula (9) immediately implies the following bound on $V_\alpha(x)$:

$$\sup_{x\in\mathbb{R}} |V_\alpha(x) - E_0| \leq \sum_{n=1}^{g} (E_{2n} - E_{2n-1}). \tag{12}$$

In other words, $\|V_\alpha - E_0\|_\infty$ is bounded by the ℓ_1-norm of the sequence of the gap lengths $E_{2n} - E_{2n-1}$. It is also obvious that nothing more can be said in general: Indeed, if the components of the frequency vector ν are rationally independent, every trajectory $\{\phi_x\alpha : x \in \mathbb{R}\}$ is dense in the torus $\mathbb{T}^g$, and thus the L_∞-norm of $V_\alpha(x) - E_0$ is equal to the maximum of $f - E_0$ over the torus, so (12) holds with equality in this case. However, there still is hope that (12) *can* be improved if the supremum is only taken over a bounded (but large) interval $0 \leq x \leq L$. Then the problem is to choose a trajectory whose initial piece avoids those points of the torus where $|f|$ is large. Our next major goal is to confirm this hope. This will occupy us for the following four sections.

It will be convenient to use the centers and the half-lengths of the gaps as new parameters. So, define

$$m_n = \frac{E_{2n-1} + E_{2n}}{2}, \quad l_n = \frac{E_{2n} - E_{2n-1}}{2}.$$

The finite gap potentials that are needed in the construction underlying Theorem 1.1 have gaps which are small compared to the bands $[E_{2n}, E_{2n+1}]$. Therefore, we from now on concentrate on this situation. To make this condition precise, we introduce

$$l = \max_{n=1,\dots,g} l_n, \quad d = \min_{n=2,\dots,g} (m_n - m_{n-1});$$

our condition on the parameters of the construction will be that $(l/d)\ln g$ is sufficiently small.

The following theorem is our main result on finite gap potentials. It says that the family of finite gap potentials $\{V_\alpha : \alpha \in \mathbb{T}^g\}$ contains functions which are over long intervals "almost" bounded by the ℓ_2-norm of the gap lengths (rather than the ℓ_1-norm).

According to the above remarks, we now use the following parameters to describe finite gap potentials: $g \in \mathbb{N}$ $(g \geq 2)$ is the number of gaps; $E_0 < m_1 < \dots < m_g$ and $l_1, \dots, l_g > 0$ describe the locations and the lengths of the gaps, respectively. We require that the gaps do not touch or overlap. Clearly, this amounts to demanding that $E_0 < m_1 - l_1$ and $m_n + l_n < m_{n+1} - l_{n+1}$ for $n = 1, \dots, g-1$.

Theorem 3.1 *Let $C_1, C_2 > 0$ be constants so that $C_1 \leq m_1 - E_0$, $m_g - E_0 \leq C_2$, and let $N \in \mathbb{N}_0$. Then there exists a constant C, depending only on C_1, C_2, and N (but not on the parameters of the finite gap potentials), such that the following holds. For every $L \geq 1$, there exists an $\alpha \in \mathbb{T}^g$ so that*

$$\sup_{0\leq x\leq L} \left| f(\alpha+\nu x) - \widehat{f}_0 \right| \leq C \left[g^{1/2} l \left(\ln(gL)\right)^{1/2} + gl(ld^{-1}\ln g)^{N+1} \right],$$

where f was defined in (11) *and*

$$\widehat{f}_0 = \int_{\mathbb{T}^g} f(\beta)\, \frac{d\beta}{(2\pi)^g}.$$

Recall from Sect. 2 that $f(\alpha + \nu x)$ is just the finite gap potential $V_\alpha(x)$. In the proof of Theorem 3.1, we will in fact show that the assertion holds with large probability if $\alpha \in \mathbb{T}^g$ is chosen at random.

The first term of the bound is the ℓ_2-norm of the gap lengths (as promised), times a logarithmic factor. Of course, the point is that the increase in L is slow, so we can still take a relatively large L. Note, however, that we no longer get an improvement over the trivial bound gl if L is of the order e^g. In the application of Theorem 3.1 in this paper, we will have $L \leq g^\gamma$, and then Theorem 3.1 indeed gives a good bound.

From a theoretical point of view, a particularly neat situation arises when the flow ϕ_x and thus also the finite gap potentials are periodic with period p. In that case, one can take $L = p$ to obtain a bound which is valid for all $x \in \mathbb{R}$. This remark is not as academic as it may seem, because one can show, using topological arguments, that in situations with small gaps one can get a periodic ϕ_x by slightly moving the centers m_n. The period will be of the order $p \approx d^{-1}$. See also [6, Appendix C.2] for statements of this type.

The second term of the above bound contains the ℓ_1-norm gl, but multiplied by an arbitrarily high power of $ld^{-1}\ln g$. So Theorem 3.1 is interesting only if this combination is small, but this is the case in our construction for proving Theorem 1.1. What exactly "small" means obviously depends on C and thus on C_1, C_2, N, but on nothing else. This will be very important in the proof of Theorem 1.1, where we will apply Theorem 3.1 to a whole sequence of finite gap potentials.

We would like to emphasize the fact that we do not subtract E_0 from $V_\alpha(x)$ (which is perhaps the constant that comes to mind first), but rather the average of f over the real part of the Jacobi variety. This may be viewed as a renormalization, due to higher order terms. Indeed, E_0 is the limiting value of f at $l = 0$; now the Theorem says that the zeroth Fourier coefficient $\widehat{f}_0$ (which contains also terms which are of higher order in the small parameter l/d) gives a better constant approximation to $V_\alpha(x)$. This remark is actually true for approximations by trigonometric polynomials of arbitrarily high degree; we will comment on this point again after having discussed the proof of Theorem 3.1.

We will give this proof in the following three sections. This is the plan of attack: We will first solve the Jacobi inversion problem up to order N in the small parameter $ld^{-1} \ln g$. (As explained above, the problem of computing finite gap potentials basically is the Jacobi inversion problem, that is, the problem of inverting the Abel-Jacobi map.) This will be done by expanding in Fourier and Taylor series and solving the equations by iteration. The expressions obtained in this way rapidly get out of hand as N increases. However, things become surprisingly transparent if a graphical representation of the perturbation series is introduced. This will be developed in Sect. 5, after having discussed some preparatory material in Sect. 4.

Then, in Sect. 6, we extend classical methods, due to Salem and Zygmund [23], for bounding random trigonometric polynomials to finish the proof. In its original version, this argument shows, for example, that for a random choice of signs, $p(x) = \sum_{n=1}^{N} \pm a_n \cos nx$ is almost bounded by the ℓ_2-norm of its coefficients: $\|p\|_\infty \leq C \|a\|_2 (\ln N)^{1/2}$. Theorem 3.1 can perhaps be viewed as a nonlinear version of this result.

4 Proof of Theorem 3.1: Basic estimates

We want to analyze the Abel-Jacobi map (8). We can parametrize the divisors $(\widehat{\mu}_1, \ldots, \widehat{\mu}_g)$ (as usual, $\mu_j \in [E_{2j-1}, E_{2j}]$) by the points $(\psi_1, \ldots, \psi_g)$ of another copy of the torus $\mathbb{T}^g = [0, 2\pi)^g$ (not to be confused with the real part of the Jacobi variety) as follows: Write

$$\mu_j = m_j - l_j \cos \psi_j, \tag{13}$$

$$R(\mu_j) = R_j(\mu_j) i l_j \sin \psi_j, \tag{14}$$

where $R_j(z) = R(z)/\sqrt{(E_{2j-1} - z)(E_{2j} - z)}$. This definition is not yet complete since the sign of $R_j(\mu_j)$ on the right-hand side of (14) also needs to be specified. Note that $iR_j(\mu)$ is real and non-zero for $E_{2j-1} < \mu < E_{2j}$. Therefore, it makes sense to require that $iR_j(\mu)$ be positive for odd j and negative for even j and

μ as above. So, for a given $\psi_j \in [0, 2\pi)$, eq. (13) tells us what the projection μ_j of $\widehat{\mu}_j$ is, while (14) determines the sheet on which $\widehat{\mu}_j$ lies. In particular, if $\psi_j + \psi_j' = 2\pi$, then the corresponding points $\widehat{\mu}_j, \widehat{\mu}_j' \in S$ have the same projections but lie on different sheets.

The substitution (13), (14) allows us to write integrals involving the ω_j in a particularly convenient way. Indeed, recalling (7), we see that the normalization condition (6) now takes the form

$$2 \int_0^{\pi} \left. \frac{p_j(\mu)}{iR_n(\mu)} \right|_{\mu = m_n - l_n \cos\psi} d\psi = \delta_{jn}. \tag{15}$$

Similarly, the Abel-Jacobi map, now viewed as a map from $\mathbb{T}^g$ to $\mathbb{T}^g$ (but still denoted by α), can be written as

$$\alpha_j(\psi_1, \ldots, \psi_g) = 2\pi \sum_{n=1}^{g} \int_0^{\psi_n} \left. \frac{p_j(\mu)}{iR_n(\mu)} \right|_{\mu = m_n - l_n \cos\psi} d\psi. \tag{16}$$

Finally, the function f from the trace formula for V (see (11)) takes the following form when expressed in terms of the new variables:

$$f = E_0 + 2 \sum_{n=1}^{g} l_n \cos\psi_n. \tag{17}$$

As already discussed, Theorem 3.1 is vacuous if $ld^{-1} \ln g$ is not small (if $ld^{-1} \ln g \geq \epsilon$, just take $C = 4\epsilon^{-N-1}$). Thus only the case where

$$ld^{-1} \ln g < \epsilon \tag{18}$$

needs proof; here, $\epsilon > 0$ can be chosen according to our needs and may depend on C_1, C_2, and N (but on nothing else). In addition to the hypotheses of Theorem 3.1, we will therefore assume (18) with a sufficiently small ϵ from now on. In particular, the reader should keep in mind that (18) with a suitable $\epsilon = \epsilon(C_1, C_2, N)$ as well as the hypotheses of Theorem 3.1 are (tacit) assumptions in all lemmas of Sect. 4–6.

Notational remark. In the sequel, we will use the following conventions. A "constant" (usually denoted by C) is a number that only depends on C_1, C_2, and N. In particular, the constants which are implicit in the Landau notation $O(\cdots)$ may only depend on C_1, C_2, and N. We will sometimes write $a \lesssim b$ instead of $a \leq Cb$ (or $a = O(b)$); here, C is a constant in the sense just explained. Similarly, $a \approx b$ is short-hand for two-sided estimates. Finally, the value of C may change from one formula to the next, so there is nothing wrong with an inequality like $C + 1 \leq C$ (to give a blatant example).

Assuming (18), we can analyze (15), (16) in some detail by using Taylor expansions. The following lemma will get us started.

Lemma 4.1 *For all $j,n = 1,\dots,g$, the function $p_j(z)/R_n(z)$ is holomorphic in a neighborhood of $[E_{2n-1},E_{2n}]$, and for all $s\in\mathbb{N}_0$,*

$$\max_{z\in[E_{2n-1},E_{2n}]}\left|\frac{d^s}{dz^s}\frac{p_j(z)}{R_n(z)}\right| \le \left(\frac{C}{d}\right)^s \frac{Cs!}{d^{-1}|m_j-m_n|+1}.$$

Moreover,

$$c_j = \frac{(m_j-E_0)^{1/2}}{2\pi}\left(1+O((l/d)^2\ln g)\right).$$

Proof. The first assertion is obvious; in fact, it holds on any simply connected neighborhood of $[E_{2n-1},E_{2n}]$ that avoids the other branch points. Thus, for $z\in[E_{2n-1},E_{2n}]$, we can use the Cauchy formula to represent the derivatives:

$$f^{(s)}(z) = \frac{s!}{2\pi i}\int_K \frac{f(\zeta)}{(\zeta-z)^{s+1}}\,d\zeta. \tag{19}$$

Here, we integrate over the contour $K=\{\zeta=m_n+(d/2)e^{i\varphi}:0\le\varphi\le 2\pi\}$ in counter-clockwise direction. Note that K is well separated from all gaps $[E_{2i-1},E_{2i}]$. In particular, if l/d is sufficiently small (for instance, $l/d\le 1/4$ will do), then $|\zeta-z|\gtrsim d$ for all $\zeta\in K$, $z\in[E_{2n-1},E_{2n}]$, so (19) implies that

$$\max_{z\in[E_{2n-1},E_{2n}]}\left|f^{(s)}(z)\right| \le C(C/d)^s s!\max_{\zeta\in K}|f(\zeta)|. \tag{20}$$

We want to apply this to $f=p_j/R_n$, so we need to estimate p_j and R_n: We have that for $\zeta\in K$,

$$\begin{aligned}|R_n(\zeta)| &= |\zeta-E_0|^{1/2}\left(\prod_{i\ne n}|m_i-l_i-\zeta|\,|m_i+l_i-\zeta|\right)^{1/2}\\ &\gtrsim \prod_{i\ne n}|m_i-\zeta|\left(1+O\left(\frac{l^2}{(m_i-m_n)^2}\right)\right).\end{aligned}$$

Here we used the fact that $m_n-E_0\approx 1$ by the hypotheses of Theorem 3.1.

Similarly, since the unknown zeros $\lambda_i^{(j)}$ of p_j satisfy $\lambda_i^{(j)}\in(E_{2i-1},E_{2i})$ and since $c_j>0$, we obtain

$$|p_j(\zeta)| = c_j\prod_{i\ne j}\left|\lambda_i^{(j)}-\zeta\right| = c_j\prod_{i\ne j}|m_i-\zeta|\left(1+O\left(\frac{l}{|m_i-m_n|+d}\right)\right).$$

Now $|m_i-m_j|\ge d|i-j|$, so taking logarithms, we see that for small $ld^{-1}\ln g$,

$$\begin{aligned}\prod_{i\ne j}\left(1+O\left(\frac{l}{|m_i-m_n|+d}\right)\right) &= 1+O(ld^{-1}\ln g),\\ \prod_{i\ne n}\left(1+O\left(\frac{l^2}{(m_i-m_n)^2}\right)\right) &= 1+O((l/d)^2).\end{aligned}$$

Estimates of this type will be used quite often in the sequel. Combining the bounds just proved, we get

$$\left|\frac{p_j}{R_n}(\zeta)\right| \leq c_j \, \frac{C}{d^{-1}|m_j - m_n| + 1},$$

and the claim on the derivatives of p_j/R_n would follow with (20) if we knew already the asserted formula for the c_j's.

So, it only remains to prove the estimate on c_j stated in Lemma 4.1. Taylor's theorem with remainder gives

$$\left.\frac{p_j(\mu)}{R_n(\mu)}\right|_{\mu=m_n-l_n\cos\psi} = \frac{p_j(m_n)}{R_n(m_n)} - \frac{d}{dz}\left(\frac{p_j}{R_n}\right)(m_n)\; l_n\cos\psi + O\left(c_j \, \frac{(l/d)^2}{d^{-1}|m_j - m_n| + 1}\right).$$

Plug this into (15). The first order term integrates to zero. Also,

$$R_n(m_n) = -i\sqrt{m_n - E_0}(1 + O((l/d)^2)) \prod_{i\neq n}(m_i - m_n),$$

so we obtain

$$\frac{2\pi c_j}{\sqrt{m_n - E_0}} \frac{\prod_{i\neq j}(\lambda_i^{(j)} - m_n)}{\prod_{i\neq n}(m_i - m_n)}(1 + O((l/d)^2)) + O\left(c_j \, \frac{(l/d)^2}{d^{-1}|m_j - m_n| + 1}\right) = \delta_{jn}. \quad (21)$$

For $j \neq n$, (21) leads to

$$\frac{\lambda_n^{(j)} - m_n}{m_j - m_n}(1 + O(ld^{-1}\ln g)) = O\left(\frac{l^2/d}{|m_j - m_n|}\right),$$

thus $\lambda_n^{(j)} = m_n + O(l^2/d)$. Using this in (21) with $j = n$, we finally obtain

$$\frac{2\pi c_n}{\sqrt{m_n - E_0}} \prod_{i\neq n}\left(1 + O\left(\frac{l^2 d^{-1}}{|m_i - m_n|}\right)\right) = 1 + O\left(c_n \, (l/d)^2\right),$$

and the lemma follows. □

We now expand the integrands of the Abel-Jacobi map (16) in a Fourier series. This, and not a Taylor series, is the appropriate choice here, because it gives the correct "renormalized" constant term immediately, without contributions from higher order terms. So write

$$\left.\frac{2\pi p_j(\mu)}{iR_n(\mu)}\right|_{\mu=m_n-l_n\cos\psi} = \sum_{m\in\mathbb{Z}} a_m(j,n)e^{im\psi}. \quad (22)$$

Since the left-hand side is in $C^\infty(\mathbb{T})$ as a function of ψ, this expansion converges uniformly. Moreover,

$$a_m(j,n) = \int_0^{2\pi} \frac{p_j(\mu)}{iR_n(\mu)}\bigg|_{\mu=m_n-l_n\cos\psi} e^{-im\psi}\, d\psi, \tag{23}$$

and, as a consequence, $a_0(j,n) = \delta_{jn}$ (by (15)).

Lemma 4.2

$$|a_m(j,n)| \le \frac{(Cl/d)^{|m|}}{d^{-1}|m_j - m_n| + 1}$$

Proof. This is trivially satisfied if $m = 0$, so we suppose that $m \neq 0$. Then, by Taylor's theorem and Lemma 4.1,

$$\frac{p_j}{R_n}(m_n - l_n\cos\psi) = \sum_{k=0}^{|m|-1} b_k(j,n)(-l_n\cos\psi)^k + \rho_{|m|}(\psi),$$

where the remainder satisfies the estimate

$$\left|\rho_{|m|}(\psi)\right| \le \frac{(Cl/d)^{|m|}}{d^{-1}|m_j - m_n| + 1}.$$

Since $\int_0^{2\pi} \cos^k\psi\, e^{-im\psi}\, d\psi = 0$ for $|m| > k$, the claim now follows from (23). □

Using $a_0(j,n) = \delta_{jn}$, we can now plug (22) into (16) to write the Abel-Jacobi map in the form

$$\alpha_j = \psi_j + \sum_{m\in\mathbb{Z}}{}' \sum_{n=1}^{g} \frac{a_m(j,n)}{im}(e^{im\psi_n} - 1). \tag{24}$$

Here and in the sequel, the prime at the sum sign indicates omission of the term with $m = 0$. To obtain (24), we have integrated (22) term by term, which is allowed because of the uniform convergence. We want to solve the system of equations (24) for $\psi_1, \ldots, \psi_g$. It is useful to separate the leading term, which, due to the smallness of the $a_m(j,n)$'s expressed by Lemma 4.2, is α_j. So, introduce θ_j by writing $\psi_j = \alpha_j + \theta_j$; then (24) becomes

$$\theta_j + \sum_m{}' \sum_{n=1}^{g} \frac{a_m(j,n)}{im}(e^{im\alpha_n} e^{im\theta_n} - 1) = 0, \tag{25}$$

and these equations must now be solved for the θ_j's. Actually, we will compute the θ_j's only up to an error of order $O((ld^{-1}\ln g)^{N+1})$. Note that by Lemma 4.2 and (25),

$$|\theta_j| \le 2\sum_m{}' (Cl/d)^{|m|} \sum_{n=1}^{g} \frac{1}{d^{-1}|m_j - m_n| + 1} \lesssim ld^{-1}\ln g, \tag{26}$$

since $d^{-1}|m_j - m_n| \geq |j-n|$. Now we keep only those terms of (25) which are of order $\leq N$ in the small parameter $ld^{-1}\ln g$, and we iterate these new equations N times. The following lemma justifies this procedure; we get indeed a good approximation to θ_j.

Lemma 4.3 *Define* $\theta_j^{(0)} = 0$,

$$\theta_j^{(s+1)} = -\sum_{|m|\leq N}{}' \sum_{n=1}^{g} \frac{a_m(j,n)}{im} \left(e^{im\alpha_n} - 1\right) - \sum_{|m|\leq N}{}' \sum_{n=1}^{g} \frac{a_m(j,n)}{im} e^{im\alpha_n} \sum_{t=1}^{N-|m|} \frac{(im\theta_n^{(s)})^t}{t!}, \tag{27}$$

$s = 0, 1, \ldots, N-1$. *Then* $\left|\theta_j^{(N)} - \theta_j\right| \leq C(ld^{-1}\ln g)^{N+1}$.

Proof. We will prove by induction that

$$\left|\theta_j^{(s)} - \theta_j\right| \leq C(ld^{-1}\ln g)^{s+1} \tag{28}$$

for $s = 0, 1, \ldots, N$. For $s = 0$, this is just (26). Now assume (28) holds for some $s \geq 0$. We claim that then

$$\theta_j^{(s+1)} = -\sum_{m\in\mathbb{Z}}{}' \sum_{n=1}^{g} \frac{a_m(j,n)}{im} \left(e^{im\alpha_n} e^{im\theta_n^{(s)}} - 1\right) + O((ld^{-1}\ln g)^{N+1}). \tag{29}$$

Indeed, comparison with (27) shows that the error from (29), which we want to bound by $C(ld^{-1}\ln g)^{N+1}$, is equal to

$$\sum_{|m|\leq N}{}' \sum_{n=1}^{g} \frac{a_m(j,n)}{im} e^{im\alpha_n} \sum_{t=N+1-|m|}^{\infty} \frac{(im\theta_n^{(s)})^t}{t!} + \sum_{|m|>N} \sum_{n=1}^{g} \frac{a_m(j,n)}{im} \left(e^{im\alpha_n} e^{im\theta_n^{(s)}} - 1\right).$$

The induction hypothesis implies that

$$\left|\theta_n^{(s)}\right| \leq \left|\theta_n^{(s)} - \theta_n\right| + |\theta_n| = O(ld^{-1}\ln g),$$

so, by Lemma 4.2, the first contribution to the error is bounded by a constant times

$$\sum_{|m|\leq N}{}' \sum_{n=1}^{g} \frac{(l/d)^{|m|}}{d^{-1}|m_j - m_n| + 1} (ld^{-1}\ln g)^{N+1-|m|} \lesssim \sum_{|m|\leq N}{}' (l/d)^{N+1} (\ln g)^{N+2-|m|} \lesssim (ld^{-1}\ln g)^{N+1},$$

as desired. Similarly, the second contribution to the error term can be estimated by

$$\sum_{|m|>N} \sum_{n=1}^{g} \frac{(Cl/d)^{|m|}}{d^{-1}|m_j - m_n| + 1} \lesssim \ln g \sum_{|m|>N} (Cl/d)^{|m|} \lesssim (l/d)^{N+1} \ln g.$$

This concludes the proof of (29). Adding (29) and (25), we obtain

$$\theta_j^{(s+1)} = \theta_j - {\sum_{m\in\mathbb{Z}}}' \sum_{n=1}^{g} \frac{a_m(j,n)}{im} e^{im\alpha_n} \left(e^{im\theta_n^{(s)}} - e^{im\theta_n} \right) + O((ld^{-1} \ln g)^{N+1}).$$

Lemma 4.2 together with

$$\left| e^{im\theta_n^{(s)}} - e^{im\theta_n} \right| \lesssim |m| (ld^{-1} \ln g)^{s+1},$$

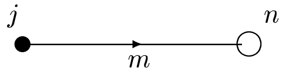
which follows from the induction hypothesis, now yield the induction statement (28) for $s+1$. $\square$

5 The Feynman rules

We now introduce, as announced above, a graphical representation of the terms obtained from the recursion (27). We have $\theta_j^{(0)} = 0$ and

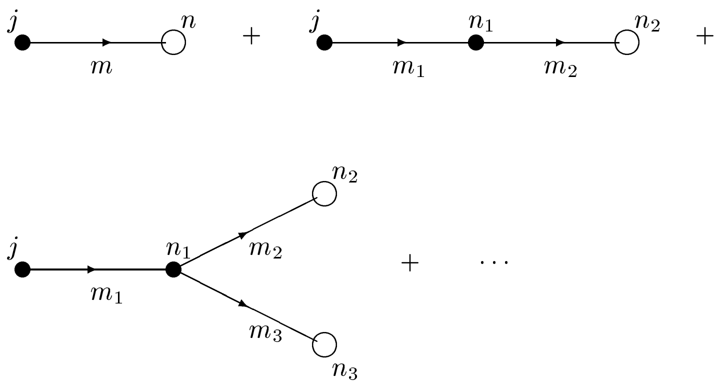
$$\theta_j^{(1)} = - {\sum_{|m|\le N}}' \sum_{n=1}^{g} \frac{a_m(j,n)}{im} (e^{im\alpha_n} - 1).$$

This latter expression can be represented by the following graph:



Here is the recipe to recover $\theta_j^{(1)}$ from this graph: Associate the factor $a_m(j,n)$ with the edge m with vertices j and n. The circled vertex n contributes a factor $e^{im\alpha_n} - 1$, where m is the parameter of the incoming edge. Finally, multiply by i/m and sum over $m = \pm 1, \ldots, \pm N$ and $n = 1, \ldots, g$.

These rules, suitably generalized, also work for larger values of s. At first sight, the formula for $\theta_j^{(2)}$ looks considerably more complicated than that for $\theta_j^{(1)}$ because now the second line of (27) also contributes. However, it is not hard to convince oneself that $\theta_j^{(2)}$ can actually be computed by evaluating the following graphs.



More precisely, there are N such graphs; they have the common property that every edge except the first one emanates from the second vertex. Again, edges contribute factors of the form $a_m(n,n')$, and for each vertex $\neq j$, there is a factor $e^{im\alpha_n}$ ($e^{im\alpha_n}-1$ if the vertex is marked by a circle). Then one has to multiply by a factor that depends on the edge indices m_i and also on the graph and finally sum over all parameters except j. (Explicitly, this factor is

$$i(-1)^{E+1}\frac{m_1^{E-2}}{(E-1)!}\prod_{k=2}^{E} m_k^{-1},$$

where E is the number of edges.)

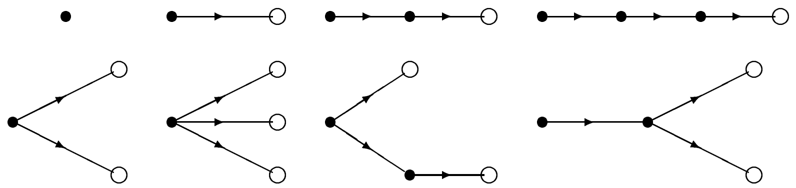
We are now ready to formulate the rules for computing $e^{i(\alpha_j+\theta_j)}$ from graphs of this type. The quantity $e^{i(\alpha_j+\theta_j)}=e^{i\psi_j}$ is of especial interest here because the function f from (17) depends on exactly this combination.

Feynman rules for $e^{i(\alpha_j+\theta_j)}$

1. Draw all directed trees with at most N edges. By a "directed tree", we mean a connected graph with the property that there is precisely one vertex with only outgoing edges, while for every other vertex, there is exactly one incoming edge. The vertices without outgoing edges are called final vertices (the trivial graph consisting of just one vertex is excluded in this definition); they are marked by circles.

   Formally, such a graph (with E edges, say) may be represented by $E+1$ symbols $V_1,\ldots,V_{E+1}$ ("vertices") and a collection of E ordered pairs (V_i,V_j) with $i\neq j$ ("edges"). Two graphs are equal if there is a bijection from one set of vertices to the other which preserves the edges.

   The figure below illustrates the case $N=3$.



2. For every graph, label the (unique) vertex without incoming edge j. Then, attach the indices $n_1, \dots, n_E$ to the remaining vertices, and label the edges $m_1, \dots, m_E$. It is of no significance how the indices $n_1, \dots, n_E$ and $m_1, \dots, m_E$ are assigned to the vertices and edges, respectively, but once a graph has been labeled, this particular labeling is fixed once and for all.

3. These labeled graphs are translated into formulae as follows. An edge labeled m pointing from vertex n to vertex n' stands for a factor $a_m(n, n')$. A non-final vertex with index $n \neq j$ contributes $e^{im\alpha_n}$, where m is the (unique) incoming edge. In case n is a final vertex, the rule is similar except that the factor now is $e^{im\alpha_n} - 1$. The vertex j always carries the factor $e^{i\alpha_j}$. Finally, the result is multiplied by a number $c_G(m_1, \dots, m_E)$ which depends on the graph and the edge indices (and N, but this is fixed throughout). In principle, c_G can be computed, as the discussion below will show, but we do not need to know the precise values of the c_G's here.

4. Sum over $m_i \neq 0$, $\sum_{i=1}^{E} |m_i| \leq N$, and $n_i = 1, \dots, g$. Finally, sum over all graphs.

Carrying out these instructions produces a (complicated) function of the α_n's. The claim is that up to an error $O((ld^{-1} \ln g)^{N+1})$, this function coincides with $e^{i(\alpha_j + \theta_j)}$. We will now prove this assertion, which is the central result of this section.

This proof, though not really difficult, is not easy to formulate; to get a feeling for the underlying principles, it is advisable to try things out by iterating (27) a few times and drawing some pictures. Our verbal description will thus be somewhat sketchy.

The strategy of the proof, however, is straightforward. First of all, we show that the approximations $\theta_j^{(s)}$ from (27) admit a representation by diagrams. We have demonstrated this already for $s = 1, 2$, and the general case is hardly more difficult. Then, we use this knowledge to formulate similar rules for $e^{i(\alpha_j + \theta_j)}$. Finally, terms of order $(ld^{-1} \ln g)^{N+1}$ or higher can be dropped on the way.

So, our first claim is the following statement: The $\theta_j^{(s)}$ from Lemma 4.3 can be calculated by evaluating certain graphs according to similar rules like the ones given above. There are a number of differences: All graphs have exactly one edge emanating from j, there is no factor $e^{i\alpha_j}$ attached to j, the factors c_G are different, the m_i's are summed over the range $|m_i| \leq N$, $m_i \neq 0$, and there may be graphs with more than N edges.

In fact, we know this already for $s = 1, 2$, and the proof of the general case is by induction on s. By its definition, $\theta_j^{(s+1)}$ is obtained by inserting $\theta_n^{(s)}$ on the right-hand side of (27). By induction hypothesis, $\theta_n^{(s)}$ is a sum of many terms each of which corresponds to a graph with certain parameters. We now multiply out the right-hand side of (27) and only then take the various sums. To prove our claim, it suffices to make the following observations: graphs are multiplied together by attaching them to one another at the "initial" vertex j. Similarly, multiplying a graph by $a_m(j,n)e^{im\alpha_n}$, as in the second line of (27), amounts to attaching this graph to the single-edge graph in such a way that the final vertex of this single-edge graph and the initial vertex of the other graph combine to one new vertex.

Also, we may restrict ourselves to graphs with at most N edges and to parameters m_i with $\sum_{i=1}^E |m_i| \leq N$. Indeed, since each edge carries a factor $a_m(n,n')$, Lemma 4.2 implies that the omitted contributions are $O((ld^{-1}\ln g)^{N+1})$. Here, the logarithmic factors come from the denominators of the bound of Lemma 4.2, when the vertex indices are summed over. Note also in this context that summing over the m_i's is never dangerous because the restrictions $|m_i| \leq N$ imply that there is an a priori bound (depending on N only) on the number of summands.

Next, we have that

$$e^{i(\alpha_j+\theta_j)} = e^{i\alpha_j} \sum_{t=0}^{N} \frac{(i\tilde{\theta}_j^{(N)})^t}{t!} + O((ld^{-1}\ln g)^{N+1}).$$

The tilde on the right-hand side indicates the omission of higher order terms, as discussed in the preceding paragraph. Again, the task is to multiply this out. The $\tilde{\theta}_j^{(N)}$ have graphical representations, as we have just seen, and the above remarks about multiplying together different graphs are still relevant here. The asserted rules follow from this. The additional factor $e^{i\alpha_j}$ has simply been attached to the vertex j.

Note also that the same graph may arise many times when the process of multiplying out is performed, but then we can simply combine these contributions to a single one. This will only affect the numbers $c_G(m_1, \dots, m_E)$.

6 Bounds along a random trajectory

This last part of the proof of Theorem 3.1 deals with the problem of bounding $f(\alpha) - \widehat{f}_0$ along trajectories $\alpha = \alpha_0 + \nu x$, given the information obtained in the preceding section. First of all, recall from (17) that

$$f(\alpha) = E_0 + 2\sum_{j=1}^{g} l_j \cos\psi_j = E_0 + 2 \text{ Re } \sum_{j=1}^{g} l_j e^{i(\alpha_j+\theta_j)}. \tag{30}$$

We will first convince ourselves that f is of the form

$$f(\alpha) = \widehat{f}_0 + {\sum}'_{|m|_1 \leq N+1} b(m) \sin(m\cdot\alpha + \varphi_m) + O(lg(ld^{-1}\ln g)^{N+1}). \tag{31}$$

We use a slightly different notation in this section in that now $m = (m_1, \ldots, m_g)$ with $m_i \in \mathbb{Z}$. Also, $|m|_1 = \sum |m_i|$ and $m \cdot \alpha = \sum m_i \alpha_i$; finally, the prime at the sum sign now means omission of the summand with $m = (0, \ldots, 0)$.

To prove (31), use (30) and think of the exponentials $e^{i(\alpha_j+\theta_j)}$ as being evaluated according to the Feynman rules. Then α-dependent factors come in only through the vertices of the graphs; more precisely, vertices contribute factors of the form $e^{im\alpha_n}$ (or $e^{im\alpha_n} - 1$ if the vertex is final), where m is the index of the incoming edge. The vertex j always contributes a factor $e^{i\alpha_j}$, so each graph is a sum of α-independent factors times an exponential of the form $\exp(i(\alpha_j + \sum m_i \alpha_{n_i}))$. Since rule 4 imposes the restriction $\sum |m_i| \leq N$, a rearrangement of terms gives (31), as asserted.

Clearly, this argument has not only established (31), but it has also indicated how the coefficients $b(m)$ can be computed, at least in principle, using the graphs introduced in Sect. 5. This will become very important in a moment. (Just proving (31) is easy and does not require the Feynman rules.)

To prove Theorem 3.1, we need to estimate the second term on the right-hand side of (31). Call this sum $f_N(\alpha)$. The main step will be to prove the following estimate. Given Lemma 6.1, we will then be able to apply the methods of [23].

Lemma 6.1 *There is a constant C, so that for every $\lambda \in \mathbb{R}$,*

$$\int_{\mathbb{T}^g} e^{\lambda f_N(\alpha)} \frac{d\alpha}{(2\pi)^g} \leq e^{C\lambda^2 l^2 g}.$$

Remark. Our "definition" of f_N is not quite complete, since (31) does *not* uniquely determine f_N, given f. Lemma 6.1 really asserts that for some fixed choice of f_N, consistent with (31), the stated estimate holds. More precisely, f_N is obtained by going from (30) to (31) in exactly the way described above. The following proof will also clarify this.

Proof. We will further decompose f_N and then analyze the individual terms separately. To this end, we first introduce equivalence classes of indices m. Namely, we say that m and m' are equivalent if they have the same non-zero entries, taking the order into account. To put this into more formal language, write

$$m = (0, \ldots, 0, k_1, 0, \ldots, 0, k_2, 0, \ldots, 0, k_r, 0, \ldots, 0),$$

with $r \in \mathbb{N}$ and $k_i \neq 0$ for all $i = 1, \ldots, r$. Then m and m' are equivalent precisely if $r = r'$ and $k_i = k_i'$ for all i.

This definition may not look very useful at first sight, but recall that N (which bounds the ℓ_1-norm of m) is fixed while g (which is the length of the vectors m) is typically large, so the vectors m indeed have only relatively few non-zero entries.

The number of equivalence classes in the set of indices $\{m \in \mathbb{Z}^g : |m|_1 \leq N+1\}$ only depends on N, but not on g. (Note, however, that the cardinality of the equivalence classes themselves does go to infinity as g increases.) Now fix an equivalence class (m_0) and consider

$$\sum_{m \in (m_0)} b(m) \sin(m \cdot \alpha + \varphi_m).$$

Denote the positions of the non-zero entries k_i of $m \in (m_0)$ by n_i. Then, if we vary the n_i's (respecting the obvious restrictions $1 \leq n_1 < n_2 < \cdots < n_r \leq g$), but keep r and the k_i's fixed, we get exactly all elements of the equivalence class under consideration. Thus the above sum is equal to

$$\sum_{1 \leq n_1 < \cdots < n_r \leq g} b(m) \sin\left(k_1 \alpha_{n_1} + \cdots + k_r \alpha_{n_r} + \varphi_m\right). \tag{32}$$

In this formula, m is defined by $m_{n_i} = k_i$ and $m_i = 0$ otherwise.

Using the addition laws for sine and cosine $r-1$ times, we can write (32) as a sum of 2^{r-1} terms of the form

$$\sum_{1 \leq n_1 < \cdots < n_r \leq g} b(m) \sin(k_1 \alpha_{n_1} + \gamma_1) \cdots \sin(k_r \alpha_{n_r} + \gamma_r), \tag{33}$$

Here, the dependence of the phases γ_i on the index m has not been made explicit because the precise values of the γ_i's will not matter anyway. (On top of that, we of course have a lot of freedom in the choice of the γ_i's, given φ_m from (32).)

Since $r \leq N+1$, we still have a universal bound (depending on N only) on the number of different sums of the form (33) that arise in the decomposition of f_N just performed. Fix such a sum and call it $F = F(\alpha_1, \ldots, \alpha_g)$ for easier reference.

It suffices to establish the lemma with F in place of f_N. Indeed, if this is proved, then, since $f_N = \sum F$ with an a priori bound on the number of summands F, the claimed estimate follows from Hölder's inequality.

Now, to bound $\int e^{\lambda F}$, we first do the integration with respect to the last variable α_g. The sum defining F contains many terms that do not depend on α_g. More precisely, we have that

$$F(\alpha_1, \ldots, \alpha_g) = F_1(\alpha_1, \ldots, \alpha_{g-1}) \sin(k_r \alpha_g + \gamma_r) + F_2(\alpha_1, \ldots, \alpha_{g-1}),$$

with

$$F_1 = \sum_{\substack{1 \leq n_1 < \ldots < n_{r-1} \leq g-1 \\ n_r = g}} b(m) \sin(k_1 \alpha_{n_1} + \gamma_1) \cdots \sin(k_{r-1} \alpha_{n_{r-1}} + \gamma_{r-1}),$$

$$F_2 = \sum_{1 \leq n_1 < \ldots < n_r \leq g-1} b(m) \sin(k_1 \alpha_{n_1} + \gamma_1) \cdots \sin(k_r \alpha_{n_r} + \gamma_r).$$

So, we now have to evaluate $(2\pi)^{-1} \int_0^{2\pi} \exp[\lambda F_1 \sin(k_r \alpha + \gamma_r)]\, d\alpha$. The substitution $\beta = k_r \alpha + \gamma_r$ together with the computation

$$\int_0^{2\pi} e^{c \sin\beta} \frac{d\beta}{2\pi} = \sum_{n=0}^{\infty} \frac{c^n}{n!} \int_0^{2\pi} \sin^n \beta \, \frac{d\beta}{2\pi} = \sum_{n=0}^{\infty} \frac{(c/2)^{2n}}{(2n)!} \binom{2n}{n} \leq e^{c^2/4}$$

show that

$$\int_0^{2\pi} e^{\lambda F}\,\frac{d\alpha_g}{2\pi} = e^{\lambda F_2}\int_0^{2\pi} e^{\lambda F_1 \sin(k_r\alpha_g+\gamma_r)}\,\frac{d\alpha_g}{2\pi}$$
$$\leq e^{\lambda F_2}\exp\left(\frac{\lambda^2}{4}\left(\sum |b(m)|\right)^2\right), \tag{34}$$

where the sum is over $1 \leq n_1 < \ldots < n_{r-1} \leq g-1$, and $n_r = g$. We now estimate this sum $\sum |b(m)|$. By the discussion following (31), the coefficients $b(m)$ can be obtained with the help of the Feynman rules of the preceding section by collecting those contributions which depend on α in exactly the way described by m. (More precisely, collect everything that comes with a factor $e^{im\cdot\alpha}$, multiply by the corresponding l_j's, and then take the real part and read off $b(m)$.) By the triangle inequality, we can estimate $\sum |b(m)|$ by bounding the individual contributions associated with certain fixed graphs with fixed labelings and then taking the various sums at the very end.

So, fix a graph that contributes to some $b(m)$ occuring in the sum $\sum |b(m)|$. To avoid confusion with the numbers m_i, n_i introduced in this section, the parameters labeling the edges and vertices of the graph will now be called m_i' and n_i', respectively. Since the factors $e^{im'\alpha_{n'}}$ are attached to the vertices of the graph and since the m's under consideration have r non-zero entries, the graph fixed above must have at least r vertices and hence at least $E \geq r-1$ edges. Moreover, since n_r is set equal to g in the sum $\sum |b(m)|$ we are trying to estimate, at least one vertex of the graph must have its parameter equal to g. In other words, $j = g$ or $n_i' = g$ for some i. (There is the additional restriction that $\sum |m_i'| \geq \sum_{i=1}^r |k_i| - 1$, but this will not be used.)

Armed with these observations, we are now ready to do the estimates. By the Feynman rules, the contribution coming from the fixed graph admits a bound of the form

$$Cl\left|a_{m_1'}(\ldots)\cdots a_{m_E'}(\ldots)\right| \tag{35}$$

if the parameters $j, n_1', \ldots, n_E', m_1', \ldots, m_E'$ are all kept fixed. The factor l allows for the fact that the l_j's from (30) have been absorbed by the $b(m)$'s when passing to (31). The arguments of the factors $a_{m_i'}$ depend on the particular form of the graph and also on the way the graph was labeled. Note that the (unknown) factors c_G from rule 3 can be absorbed by C because there are only finitely many different values of $c_G(m_1', \ldots, m_E')$ and we can thus simply estimate these numbers by their maximum.

We now use Lemma 4.2 to estimate (35), and we want to sum these bounds over those values of the parameters j, n_i', m_i' which satisfy the restrictions obtained above. (In contrast to the rules from Sect. 5, there is now a sum over $j = 1, \ldots, g$ also; this is simply the sum from (30).) In particular, j or one of the n_i''s is held fixed (equal to g). This implies that we can sum over the indices of the remaining vertices in the following way: First of all, delete the vertex corresponding to the fixed index. From the remaining graph, pick a vertex which is connected to just one edge and perform the corresponding sum $n_i' = 1, \ldots, g$.

By the choice of the vertex, n_i' appears in precisely one of the factors $a_{m_i'}$ as the argument. The denominator of the bound of Lemma 4.2 thus yields a factor $\lesssim \ln g$ when summed in the way just described. As a reminder that the corresponding sum has been performed, delete the chosen vertex together with the edge connected to it. Then repeat the whole procedure with the modified graph to determine the next index to be summed over. Again, just one $a_{m_i'}$ is involved in the sum, and thus another factor $\lesssim \ln g$ results. Since at each stage, there are equally many vertices and edges, this process can only stop after the whole graph has disappeared. There are only $E+1 \leq N+1$ possible choices for the vertex whose parameter is set equal to g, so the net result is that after summation over the vertices, (35) can be estimated by

$$Cl(l/d)^{|m_1'|+\cdots+|m_E'|}(\ln g)^E. \tag{36}$$

Indeed, the numerators from Lemma 4.2 contribute the factor $(l/d)^{|m_1'|+\cdots+|m_E'|}$, and by the argument just given, each of the E sums over the vertices accounts for a factor $\ln g$.

We can further estimate (36). By rule 4, $m_i' \neq 0$ for all $i = 1, \ldots, E$, hence $\sum |m_i'| \geq E$; since $E \geq 0$, we can thus can bound (36) simply by Cl.

Now the rest is easy. First of all, each of the at most N parameters m_i' has values in $\pm 1, \ldots, \pm N$, so summing over the m_i''s just increases the constant C (by at most a factor $(2N)^N$). Then, the total number of graphs also depends on N only, so we can finally sum over those graphs which could in principle contribute to $\sum |b(m)|$, and we still have the bound Cl (again, with a possibly larger constant C).

Returning to (34) now, we have thus proved that

$$\int_{\mathbb{T}^g} e^{\lambda F(\alpha)} \frac{d\alpha}{(2\pi)^g} \leq e^{C\lambda^2 l^2} \int_{\mathbb{T}^{g-1}} e^{\lambda F_2(\alpha')} \frac{d\alpha'}{(2\pi)^{g-1}},$$

where $\alpha = (\alpha_1, \ldots, \alpha_g)$ and $\alpha' = (\alpha_1, \ldots, \alpha_{g-1})$. The integral on the right-hand side has the same structure as the original one, except that g has been replaced by $g-1$. We can therefore repeat the whole argument; the second step would be to carry out the integration with respect to α_{g-1} in the same way as discussed above. We need at most g steps to do the integral completely, and at each step, we get a factor $\exp(C\lambda^2 l^2)$. As a result, we obtain the estimate $(2\pi)^{-g}\int e^{\lambda F} \leq e^{C\lambda^2 l^2 g}$, and, as already explained, the lemma follows. □

Remarks. 1. The key point of this proof was the observation that the summation over the vertices only gives logarithmic factors $(\ln g)^E$, but no powers of g. Note that to establish this, in turn, we only used some structural information contained in the Feynman rules and Lemma 4.2; the precise form of the underlying iteration was largely irrelevant.

2. The estimate Cl on (36) is of course crude unless we are in the extreme case $E = 0$ (which, it turns out now, gives the dominant contributions to f_N). Indeed, to prove the extension of Theorem 3.1 mentioned in the beginning of the Introduction and in Sect. 3, one has to keep (36) as it stands.

We are now ready to finish the proof of Theorem 3.1. This final part of the argument uses methods developed in [23]. We will follow the presentation given in [12, Chapter 6]. We want to bound f_N along a trajectory $\phi_x\alpha = \alpha + \nu x$. So, let

$$M(\alpha) = \max_{0\le x\le L} |f_N(\alpha + \nu x)| .$$

To run the argument from [12, 23], we need a bound on $(d/dx)f_N(\phi_x\alpha)$. Write f_N as a sum of terms of the form (33), as in the proof of Lemma 6.1, and fix again one of the summands F. Then, by the argument presented in that proof, the sum over the corresponding coefficients $b(m)$ satisfies

$$\sum_{1\le n_1<\ldots<n_r\le g} |b(m)| \le Cgl$$

Indeed, to show this, it just remains to sum the bound Cl also over the vertex index that had been fixed, and this gives an additional factor g. Since

$$\frac{d}{dx}F(\alpha+\nu x) = \sum b(m) \sum_{j=1}^{r} k_j\nu_{n_j} \cos(k_j(\alpha_{n_j} + \nu_{n_j}x) + \gamma_j) \prod\nolimits_j ,$$

where $\prod_j$ is short-hand for the product of sines with the jth factor omitted, it now follows that

$$\left| \frac{d}{dx}F(\alpha+\nu x)\right| \le C\sum |b(m)| \le Cgl.$$

Here we also used the fact the ν_n's are bounded; this follows from Lemma 4.1 since $\nu_n = 4\pi c_n$. Summing the above bounds, we see that also $|(d/dx)f_N(\phi_x\alpha)| \le Cgl$.

The maximum $M(\alpha)$ is attained at some point x_0, and thus by the mean value theorem, there is a constant C_0 (as usual, depending only on C_1, C_2, and N) together with an interval $I = I(\alpha) \subset [0, L]$ of length $|I| = \min\{C_0 M(\alpha)/(gl), L\}$, so that $|f_N(\alpha+\nu x)| \ge M(\alpha)/2$ for all $x \in I$. We may assume that $M(\alpha) \ge C_0^{-1}l$ for all $\alpha \in \mathbb{T}^g$, since in the opposite case we have for free a better bound than the one we are trying to prove. We then have that $g|I| \ge 1$, and it follows that

$$\begin{aligned}
\int_{\mathbb{T}^g} e^{\lambda M(\alpha)/2}\,\frac{d\alpha}{(2\pi)^g} &\le g\int_{\mathbb{T}^g} |I(\alpha)| e^{\lambda M(\alpha)/2}\,\frac{d\alpha}{(2\pi)^g}\\
&\le g\int_{\mathbb{T}^g}\frac{d\alpha}{(2\pi)^g}\int_{I(\alpha)} dx\,\left(e^{\lambda f_N(\alpha+\nu x)} + e^{-\lambda f_N(\alpha+\nu x)}\right)\\
&\le g\int_0^L dx\int_{\mathbb{T}^g}\frac{d\alpha}{(2\pi)^g}\,\left(e^{\lambda f_N(\alpha+\nu x)} + e^{-\lambda f_N(\alpha+\nu x)}\right)\\
&= gL\int_{\mathbb{T}^g}\frac{d\alpha}{(2\pi)^g}\,\left(e^{\lambda f_N(\alpha)} + e^{-\lambda f_N(\alpha)}\right)\\
&\le 2gLe^{C\lambda^2 l^2 g}.
\end{aligned}$$

The last step is by Lemma 6.1. For $\lambda > 0$, we can write this inequality in the form

$$E\left(\exp\left(\frac{\lambda}{2}\left[M - 2C\lambda l^2 g - \frac{2}{\lambda}\ln(4gL)\right]\right)\right) \leq \frac{1}{2},$$

with $E(\cdots)$ denoting the expectation taken with respect to the probability measure $(2\pi)^{-g} d\alpha$ on the torus $\mathbb{T}^g$. By a Chebyshev estimate, the inequality

$$M(\alpha) \leq 2C\lambda g l^2 + \frac{2}{\lambda}\ln(4gL)$$

holds with probability $\geq 1/2$. The parameter $\lambda > 0$ is still at our disposal, the optimal choice being

$$\lambda = \left(\frac{\ln(4gL)}{Cgl^2}\right)^{1/2}.$$

Then the bound becomes

$$M(\alpha) \leq 4C^{1/2} g^{1/2} l \left(\ln(4gL)\right)^{1/2},$$

and this holds for α's from a set of $(2\pi)^{-g} d\alpha$ measure at least $1/2$. The proof of Theorem 3.1 is complete. $\square$

Moreover, by re-examining the reasoning of this section, we see that we can also prove the more general result already mentioned. We can obtain a whole series of pointwise approximations to $V_\alpha(x)$. More specifically, the difference between $f(\phi_x \alpha)$ and those terms of $\sum b(m)\sin(m\cdot\phi_x\alpha + \varphi_m)$ for which $|m|_1 \leq M$ is

$$\lesssim g^{1/2} l (l d^{-1} \ln g)^M \left(\ln(gL)\right)^{1/2} + gl(ld^{-1}\ln g)^{N+1}$$

for suitable α. In other words, the Fourier series of f (viewed as a function on the Jacobi variety), up to some order, gives a very good pointwise approximation to $V_\alpha(x)$ with positive probability (in fact, with as large probability as we please) if α is chosen at random.

With $M = 0$, Theorem 3.1 is recovered. We do not need these more refined statements to prove Theorem 1.1.

7 Proof of Theorem 1.1

The basic idea of the construction of [21] was to glue together suitably chosen periodic potentials. In this paper, we will instead use finite gap potentials with gaps of equal length. Roughly speaking, the construction runs as follows. We will choose the first finite gap potential V_1 so that all gaps lie in, let us say, $[1, 2]$. V_2 will have much smaller gaps; also, these new gaps will be contained in the gaps of V_1. If we continue in this way, the intersection over all n of the unions of the gaps of V_n will be a Cantor type set whose dimension is easily controlled, provided there is an appropriate scaling. Moreover, the set S defined in (4) will contain this Cantor type set because if the energy E is in a gap of V_n, the solutions to the Schrödinger equation are on average exponentially increasing or decreasing and hence do not satisfy (3). Of course, we must also take care of the required decay of $V(x)$, that is, V_n must be sufficiently small for large n. The bounds on V_n will be established with the aid of Theorem 3.1.

We start by investigating the solutions of (1) for finite gap potentials V and energies E which lie in some gap of V.

Lemma 7.1 *Let $V(x)$ be a finite gap potential whose parameters satisfy the assumptions of Theorem 3.1. Then there exists an $\epsilon = \epsilon(C_1, C_2, N) > 0$ and a constant $C = C(C_1, C_2, N)$, such that for $ld^{-1} \ln g < \epsilon$, the following holds. If $|E - m_n| \leq l_n/2$ for some $n \in \{1, \ldots, g\}$, then there is a solution $y(x)$ of the Schrödinger equation (1) with $y(x_0) = 1$ for some $x_0 \in [0,1]$ and*

$$\int_{x_0}^{\infty} |y(x)|^2 \, dx \leq C/l_n.$$

Remark. This statement cannot, in general, hold with a fixed, prescribed x_0 because the decaying solution has zeros. Roughly speaking, the lemma says that there is a solution which has some decay over intervals of length $\gg l_n^{-1}$.

Proof. Our starting point is the following formula (see, for example, [4, Chapter 9]):

$$\int_x^{\infty} |f(t,z)|^2 \, dt = \frac{\text{Im } m_x(z)}{\text{Im } z}. \tag{37}$$

Here, m_x is the m-function of $-d^2/dt^2 + V(t)$ on $[x, \infty)$ with Dirichlet boundary conditions at $t = x$. More specifically, let u, v be the solutions of $-y'' + Vy = zy$ with the initial values $u(x,z) = v'(x,z) = 1$, $u'(x,z) = v(x,z) = 0$ and write

$$f(t,z) = u(t,z) + m_x(z)v(t,z);$$

then $m_x(z)$ is defined by requiring that $f \in L_2(x, \infty)$.

The Green function of the whole line problem is related to the m-function by $G(x,x;z) = (m_x^-(z) - m_x(z))^{-1}$, where m_x^- is the m-function of the operator on $L_2(-\infty, x)$ (see again [4]). Since the imaginary parts of m_x^- and m_x have opposite signs, the right-hand side of (37) is less than $-\text{Im } G(x,x;z)^{-1}/\text{Im } z$. So, if we use (10) and abbreviate $\prod(\mu_j(x) - z) = U_x(z)$, then (37) becomes

$$\int_x^{\infty} |f(t,z)|^2 \, dt < -\frac{2}{\text{Im } z} \text{Im } \frac{R(z)}{U_x(z)}. \tag{38}$$

Here, the sign of $R(z)$ is determined by the fact that Im $G(x,x;z)$ has the same sign as Im z for Im $z \neq 0$ (compare the discussion following (10)).

Now let E be as in the hypothesis, and put $z = E + i\delta$ with $\delta > 0$. By slightly changing x if necessary, we may assume that $\mu_n(x) \neq E$. Then E is not in the spectrum of the operator on $L_2(x, \infty)$ with Dirichlet boundary conditions. This is so simply because $\mu_n(x)$ is the only eigenvalue in the gap (E_{2n-1}, E_{2n}). Thus $m_x(z)$ and $R(z)/U_x(z)$ are holomorphic in a neighborhood of $z = E$. For this latter function, this may of course be seen by direct inspection. Moreover, $R(E)/U_x(E)$ is real. Therefore, the right-hand side of (38) converges to $-2(R/U_x)'(E)$ as $\delta \to 0+$, while the function $f(t, E + i\delta)$ tends to $f(t, E) = u(t, E) + m_x(E)v(t, E)$. Fatou's Lemma together with (38) imply

$$\int_x^\infty |f(t,E)|^2\, dt \le -2\,\frac{d}{dz}\left(\frac{R}{U_x}\right)(E).$$

We have that $f(x, E) = 1$, so it remains to evaluate $(R/U_x)'(E)$. To this end, note that

$$\left(\ln \frac{R}{U_x}\right)'(E) = \frac{1}{2(E-E_0)} + \sum_{j=1}^{g}\left(\frac{1}{\mu_j(x)-E} - \frac{1}{2(E_{2j-1}-E)} - \frac{1}{2(E_{2j}-E)}\right).$$

Estimating as in the proof of Lemma 4.1, we see that

$$\sum_{j\neq n}\left|\frac{1}{\mu_j(x)-E} - \frac{1}{2(E_{2j-1}-E)} - \frac{1}{2(E_{2j}-E)}\right| \lesssim l d^{-2}.$$

Furthermore, the term with $j = n$ can be bounded by $C/|\mu_n(x) - E|$. Since this is $\gtrsim l_n^{-1}$ which is much larger than ld^{-2}, we can in fact estimate the whole logarithmic derivative by $C/|\mu_n(x) - E|$. Finally, similar arguments show that $|(R/U_x)(E)| \lesssim l_n/|\mu_n(x) - E|$, so we conclude that

$$\int_x^\infty |f(t,E)|^2\, dt \le \frac{C l_n}{(\mu_n(x)-E)^2}.$$

The proof is finished by observing that $\mu_n(x)$ moves by an amount $\gtrsim l_n$ if x varies over an interval of length one. Indeed, if we again use the variables ψ_j (see (13), (14)), then the ψ_j's evolve according to the differential equations

$$\frac{d\psi_n}{dx} = \frac{2iR_n(m_n - l_n\cos\psi_n)}{\prod_{j\neq n}(m_j - m_n - l_j\cos\psi_j + l_n\cos\psi_n)},$$

and the right-hand sides are ≈ 1, independently of the positions the $\psi_j(x)$'s. $\square$

Now let $a_1 = 0$, $a_{n+1} = a_n + L_n$, where $L_n > 0$ will be chosen later. Then V will be of the form

$$V(x) = \sum_{n=1}^{\infty} \chi_{(a_n, a_{n+1})}(x) V_n(x - a_n);$$

the building blocks V_n are finite gap potentials.

We now pick these V_n's. We basically keep the notation of the preceding sections, except that there is now an additional index n. The gaps of V_n are taken to be of equal length l_n, and g_n denotes the number of gaps of V_n. Let

$\alpha \in (1/2, 1)$ be the exponent from (2) (if $\alpha = 1$, there is nothing to prove). We abbreviate $2(1-\alpha) = D$, so $D \in (0,1)$, and D is the dimension the set S from (4) must have. Fix a number $a > (1-D)^{-1}$ and put

$$l_n = \exp(-a^n).$$

A Cantor type set with g_n intervals of length l_n as its nth approximation has dimension D if there is a scaling of the type $g_n l_n^D \sim 1$. This suggests to take $g_n \sim \exp(Da^n)$, but for technical reasons, the actual definition is slightly different. First of all, choose a sequence $\epsilon_n > 0$ which tends to zero, but so slowly that $\epsilon_n a^n - \epsilon_{n-1} a^{n-1} \to \infty$ and $a^n \exp(-\epsilon_n a^n) \to 0$. (In fact, we could take $\epsilon_n = q^n$ with $a^{-1} < q < 1$ right away, but if we leave ϵ_n unspecified, we can improve the decay rate of V.) Now put $g_{n_0 - 1} = 1$, where $n_0 \in \mathbb{N}$ must be sufficiently large (this will be made precise below), and then define inductively for $n \geq n_0$

$$g_n = \exp\left((D - \epsilon_n)a^n\right) + \theta_n g_{n-1}.$$

Here, we require that $g_n / g_{n-1} \in \mathbb{N}$, and this determines $\theta_n \in [0,1)$ uniquely.

The parameter d_n describes the spacing between adjacent gaps of V_n. Since we want these gaps to lie in some gap of V_{n-1}, we get $g_n \lesssim g_{n-1} l_{n-1} / d_n$. If we disregard the ϵ_n's and θ_n's for the moment and take d_n about as large as possible, this motivates the choice

$$d_n = \exp\left((-D + (D-1)a^{-1})a^n\right).$$

A computation then shows that $l_n d_n^{-1} \ln g_n \to 0$, so Theorem 3.1 can indeed be used to bound V_n for sufficiently large n. Finally, Lemma 7.1 suggests that we put $L_n = A/l_n$, with a large constant A (how large A has to be will become clear later on).

It remains to choose the locations of the gaps, that is, we must pick, for each n, the parameters $E_0^{(n)} < m_n(1) < \cdots < m_n(g_n)$. Let $G_{n_0 - 1} = [1, 2]$, say (any compact subinterval of $(0, \infty)$ will do). For the time being, set $E_0^{(n_0)} = 0$.

Pick g_{n_0} different centers $m_{n_0}(1) < \cdots < m_{n_0}(g_{n_0})$ that lie well inside $[1, 2]$ and satisfy

$$\min_k (m_{n_0}(k) - m_{n_0}(k-1)) \geq d_{n_0}.$$

By the first requirement we mean that $|m_{n_0}(k) - 3/2| \leq 1/4$ for all $k = 1, \ldots, g_{n_0}$. Since $d_{n_0} g_{n_0} \to 0$ as $n_0 \to \infty$, this can certainly be done, provided n_0 is large enough.

Let G_{n_0} be the union of (the middle halves of) the corresponding gaps:

$$G_{n_0} = \bigcup_{k=1}^{g_{n_0}} \left[m_{n_0}(k) - l_{n_0}/2, m_{n_0}(k) + l_{n_0}/2\right]. \tag{39}$$

The general step is similar. Suppose G_{n-1} has been constructed for some $n \geq n_0 + 1$. More specifically, G_{n-1} is then a union of (half the) gaps of V_{n-1}.

For every such interval $[m_{n-1}(j) - l_{n-1}/2, m_{n-1}(j) + l_{n-1}/2] \subset G_{n-1}$, pick g_n/g_{n-1} centers $m_n(k)$ with $|m_n(k) - m_{n-1}(j)| \leq l_{n-1}/4$, and also so that the $m_n(k)$ are separated from one another by a distance $\geq d_n$, Again, $m_n(k)$'s with these properties exist in sufficiently large supply: Indeed, we can find $\approx l_{n-1}/d_n$ centers $m_n(k)$ with the required properties in each subinterval of G_{n-1}. On the other hand, by our assumptions on the sequence ϵ_n, the desired number of centers $g_n/g_{n-1} = (l_{n-1}/d_n)e^{\epsilon_{n-1}a^{n-1} - \epsilon_n a^n} + \theta_n$ is much smaller than l_{n-1}/d_n for all $n \geq n_0 + 1$, provided n_0 is large enough. So the construction is possible. We then get a total of $(g_n/g_{n-1})g_{n-1} = g_n$ new gaps, and we would like to define G_n as the union of these gaps, as in (39).

Unfortunately, there is an additional complication: If we proceed as above, we have no control on $\widehat{f}_0$, so the bound of Theorem 3.1 is useless. The problem is that we have not renormalized correctly. So it is necessary to also adjust $E_0^{(n)}$. We will add $E_0^{(n)}$ to all the previously chosen centers $m_n(k)$ ($k = 1, \ldots, g_n$), while keeping the gap length l_n fixed. This just amounts to adding $E_0^{(n)}$ to the function $f(\alpha)$. In particular, the original mean value $\widehat{f}_0$ will be replaced by $\widehat{f}_0 + E_0^{(n)}$.

Now (30) implies that (for $E_0^{(n)} = 0$) $\widehat{f}_0 = O(g_n l_n)$, so for a suitable $E_0^{(n)} = O(g_n l_n)$, we have $\widehat{f}_0 = 0$, as desired. On the other hand, since we were cautious enough to take the gaps of V_n well inside the gaps of V_{n-1}, the new centers $E_0^{(n)} + m_n(k)$ will still satisfy, let us say, $|E_0^{(n)} + m_n(k) - m_{n-1}(j)| \leq l_{n-1}/3$. This follows from the fact that the shift $E_0^{(n)} = O(g_n l_n)$ is much smaller than l_{n-1} for large n (here we use the inequality $a > (1-D)^{-1}$).

Once $E_0^{(n)}$ has been picked, G_n can again be defined as in (39), but with the shifted centers $E_0^{(n)} + m_n(k)$ taking the role of $m_n(k)$.

This two step procedure (first choose $m_n(k)$'s, then shift by an appropriate $E_0^{(n)}$ to make $\widehat{f}_0 = 0$) can now be used to pick the $m_n(k)$'s and $E_0^{(n)}$ (inductively) for all $n \geq n_0$. Note that the construction ensures that $G_n \subset G_{n-1}$.

We must still choose, for every $n \geq n_0$, a particular potential from the corresponding family V_{α_0} of finite gap potentials. Fortunately, this choice is easy: We fix once and for all a sufficiently large $N \in \mathbb{N}$ (where "sufficiently large" will be made precise at the end of the proof) and then simply take a V_n that satisfies the conclusion of Theorem 3.1 for $L = L_n$. Note also that the assumptions of Theorem 3.1 on the location of the gaps (that is, $C_1 \leq m_n(k) - E_0^{(n)} \leq C_2$ for all $k = 1, \ldots, g_n$) hold with n-independent constants $C_1, C_2 > 0$ because $E_0^{(n)}$ lies in a small interval centered at zero while all gaps are in $[1, 2]$. Therefore, the constant C from the statement of Theorem 3.1 is also independent of n.

Finally, for $n < n_0$, we can take an arbitrary bounded (and measurable) function as V_n; for instance, we can put $V_n \equiv 0$ for $n < n_0$.

Let $T = \bigcap_n G_n$. We will now show that $S \supset T$, where S is the set defined in (4). Suppose $E \in T$. Then E satisfies the assumptions of Lemma 7.1 for the potentials $V(x) = V_n(x)$ for all sufficiently large n. Assume, to obtain a contradiction, that $E \notin S$. Then there is a solution $y(x)$ of the form (3), and $y, \overline{y}$ span the space of solutions of (1). In particular, the solution f_n from Lemma 7.1 must be a linear combination of y and $\overline{y}$, that is,

$$f_n(x) = A_n e^{i\omega(x)} + B_n e^{-i\omega(x)} + r_n(x), \tag{40}$$

with $\omega(x) = \int_0^x \sqrt{E - V(t)}\, dt$ and $|r_n(x)| \le (|A_n| + |B_n|)\rho(x)$, where $\rho(x) \to 0$. Of course, since $V = V_n$ only on the interval (a_n, a_{n+1}), eq. (40) holds for $a_n \le x \le a_{n+1}$. Lemma 7.1 shows that $f_n(x_0^{(n)}) = 1$ for some $x_0^{(n)} \in [a_n, a_n + 1]$ and

$$\int_{x_0^{(n)}}^{a_n+1} |f_n(x)|^2 \, dx \le C/l_n \tag{41}$$

for large values of n. We also take n so large that $\rho(x) < \delta$ for $x \ge a_n$ and $\|V_n\|_\infty < \delta E$, which clearly is possible since $g_n l_n \to 0$ (of course, Theorem 3.1 would give a better bound on V_n, but this is not needed here). Then, since $f_n(x_0^{(n)}) = 1$, we must have that $|A_n| + |B_n| > (1+\delta)^{-1}$. Now routine estimates show that if $\delta > 0$ was chosen sufficiently small, then (40) implies that

$$\int_{x_0^{(n)}}^{a_n+1} |f_n(x)|^2 \, dx \ge C_0 L_n = AC_0/l_n.$$

This inequality contradicts (41) if A is sufficienctly large, so $T \subset S$, as claimed.

The next step is to prove that $\dim T = D$. To this end, we introduce a Borel measure μ that reflects the self-similar scaling structure of T. More specifically, μ gives equal weight to the intervals of G_n for every n: $\mu(I_n) = g_n^{-1}$ if I_n is one of the intervals $[m_n(k) - l_n/2, m_n(k) + l_n/2]$. Moreover, we also demand that μ be supported by T: $\mu(\mathbb{R} \setminus T) = 0$. It is not hard to show (for instance, by considering approximations μ_n supported by G_n) that there indeed exists a unique Borel (probability) measure μ satisfying these requirements.

We will now establish the following property of the generalized derivatives of μ: For every fixed $\gamma < D$, we have that

$$\lim_{\delta \to 0+} \sup_{|I| \le \delta} \frac{\mu(I)}{|I|^\gamma} = 0. \tag{42}$$

The supremum is over all intervals $I \subset \mathbb{R}$ of length at most δ.

If (42) holds, then, by general facts on Hausdorff measures [22, Section 3.4, Theorem 67], μ gives zero weight to sets of dimension strictly less than D, and therefore $\dim T \ge D$, as desired. The converse inequality $\dim T \le D$ does not need explicit proof (although that would actually be easy to do) because we know that always $\dim S \le 2(1-\alpha)$ (this is the result whose optimality we are about to prove), and thus $\dim T \le \dim S \le D$ will follow automatically once we have established that $V(x) = O(x^{-\alpha})$.

So let us prove (42): Fix γ, and let I be an interval with $|I| \le \delta$, where $\delta > 0$ is small. Then, define $n \in \mathbb{N}$ by requiring that $l_n < |I| \le l_{n-1}$. Clearly,

n is large if δ is small. We first treat the case when $|I| \leq d_n$. Recall that d_n is the minimal distance between adjacent gaps of V_n. So the above assumption implies that I intersects at most two of the intervals that build up G_n. Each of these intervals has measure g_n^{-1}, hence

$$\frac{\mu(I)}{|I|^\gamma} \leq \frac{2}{g_n |I|^\gamma} \leq \frac{2}{g_n l_n^\gamma}.$$

On the other hand, if $|I| > d_n$, then the number of subintervals of G_n intersecting I is $\leq 3|I|/d_n$, thus in this case,

$$\frac{\mu(I)}{|I|^\gamma} \leq \frac{3|I|^{1-\gamma}}{d_n g_n} \leq 3\, \frac{g_{n-1} l_{n-1}}{d_n g_n}\, \frac{1}{g_{n-1} l_{n-1}^\gamma}.$$

Now $g_n l_n^\gamma \gtrsim \exp(\sigma a^n)$, where $\sigma > 0$ depends on γ, and $g_{n-1} l_{n-1}/(d_n g_n) \lesssim \exp(\epsilon_n a^n)$; indeed, relations of this type motivated our definition of g_n and d_n. Since, as noted above, $n \to \infty$ as $\delta \to 0+$, (42) now follows.

It remains to show that V satisfies the bound (2). So, let $x \in (a_n, a_{n+1})$ with large n. Recall that $\widehat{f}_0 = 0$, where f is the function from the trace formula for V_n. Theorem 3.1 therefore implies that

$$|V(x)| \leq C \left[g_n^{1/2} l_n \left(\ln(g_n L_n)\right)^{1/2} + g_n l_n (l_n d_n^{-1} \ln g_n)^{N+1} \right]$$

for these x. On the other hand,

$$x \leq a_{n+1} = \sum_{m=1}^{n} L_m \lesssim \sum_{m=1}^{n} l_m^{-1} \lesssim l_n^{-1},$$

and (2) indeed follows, provided we took

$$N + 1 \geq \frac{1-\alpha}{2\alpha - 1}\, \frac{a}{a-1}.$$

(As expected, $N \to \infty$ as $\alpha \to 1/2+$.) $\square$

Actually, doing these final estimates carefully, we obtain a stronger bound of the form $x^{-\alpha-\epsilon_n/2}(\ln x)^{1/2}$. The strengthening of Theorem 1.1 mentioned in Sect. 1 follows from this by taking a sequence ϵ_n that tends to zero sufficiently slowly.

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