A Survey on Bifurcations of Invariant Tori

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Abstract

Invariant tori of dynamical systems occur both in the dissipative and in the conservative context. I focus here on the latter, where the tori are intrinsically parametrised by the actions $y_1, \ldots, y_n$ conjugate to the angles $x_1, \ldots, x_n$ on the torus. The distribution of maximal tori in a nearly integrable Hamiltonian system is governed by the invariant tori of co–dimension one. The different Cantor families of maximal tori shrink down to normally elliptic tori and are separated by the web formed by stable and unstable manifolds of normally hyperbolic tori. The lower dimensional invariant tori form Cantor families themselves, and occurring bifurcations in turn organize the distribution of normally elliptic and hyperbolic tori.

1 Introduction

Invariant sets often provide a key for a better understanding of the behaviour of dynamical systems. Different types of motion lead to various invariant sets :

no motion	in	equilibria
periodic motion	in	periodic orbits
quasi–periodic motion	on	invariant tori
asymptotic motion	on	stable and unstable manifolds
chaotic motion	on	strange attractors (dissipative systems)
ergodic motion	on	energy shells (conservative systems).

In applications the equations of motion are known only to finite precision of the coefficients. Giving these coefficients the interpretation of parameters leads to a whole family of dynamical systems. Under variation of the parameters the invariant sets may then bifurcate.

Bifurcations of equilibria are fairly well understood, at least for low co–dimension, cf. [Guckenheimer,Holmes;86], [Kuznetsov;95] and references therein. Using a Poincaré map, periodic orbits can be studied as fixed points of a discrete dynamical system. In addition to the analogues of bifurcations of equilibria, periodic orbits may undergo period doubling bifurcations, cf. [Newhouse,Palis,Takens;83], [Broer,Roussarie,Simó;96].

It turns out that these bifurcations have quasi–periodic counterparts, see [Braaksma,Broer, Huitema;90]. In the *integrable* case where the system is equivariant under a torus action this is an immediate consequence of the behaviour of the reduced system obtained after reducing the torus symmetry. In the nearly integrable case where the torus symmetry is broken by a small perturbation one can use KAM–theory to show that the bifurcation persists on the Cantor sets defined by appropriate Diophantine conditions. Notably the bifurcating torus has to be in Floquet form. In the same way the higher topological complexity of periodic orbits leads to period doubling bifurcations (which have no counterpart for equilibria), tori that are not in Floquet form can bifurcate in a skew Hopf bifurcation, see [Wagener;98].

We focus here on Hamiltonian systems. One can ask similar questions for dissipative or for reversible systems — and obtain slightly different answers. An important feature of integrable Hamiltonian systems is that the torus symmetry yields conjugate actions by Noether's theorem. Accordingly, invariant n–tori in integrable Hamiltonian systems with d degrees of freedom, $d \geq n$, occur as "intrinsic" n–parameter families, without the need for external parameters. In particular, periodic orbits form 1–parameter families, while equilibria are in general isolated.

Thus, lower dimensional tori in (single) integrable Hamiltonian systems of d degrees of freedom may undergo bifurcations up to co–dimension $d-1$. When switching on the symmetry breaking perturbation, the (isolated) bifurcating tori with highest co–dimension may fall into a resonance hole. Using KAM–theory, we expect bifurcating n–tori of co–dimension k to be parametrised by Cantor sets of dimension $n-k$, *i.e.* the union of these persisting bifurcating tori is a $(2n-k)$–dimensional invariant set.

This paper is organized as follows. In the next two sections we review the Hamiltonian bifurcation theory for equilibria and periodic orbits, respectively. In Section 4 we then see how this carries over to invariant tori in Floquet form. In the final section we discuss additional complications for bifurcating tori that are not reducible to Floquet form.

2 Bifurcations of equilibria

The eigenvalues of a non–degenerate equilibrium of a Hamiltonian system are arranged in complex quartets $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$ and in pairs of purely imaginary or purely real eigenvalues. Bifurcations occur when there are zero eigenvalues or coinciding pairs of purely imaginary eigenvalues. Except for very degenerate situations (of infinite co–dimension), the equilibria of a single Hamiltonian system are all isolated. Thus, we introduce external parameters to study the bifurcations of equilibria in Hamiltonian systems.

2.1 Bifurcations at zero eigenvalues

We concentrate here on the least degenerate case where zero is a double eigenvalue and there are no further double eigenvalues on the imaginary axis. This still leaves two cases : if there is only one eigenvector to the zero eigenvalue the corresponding Jordan block takes the nilpotent form $\binom{0\,1}{0\,0}$, while it completely vanishes in the case of two linearly independent eigenvectors to the double zero eigenvalue.

We may discard the hyperbolic eigenvalues by passing to a centre manifold. Each remaining pair of non–zero eigenvalues generates an S^1–action, and in case of non–resonance reducing the combined torus action leads to a one–degree–of–freedom problem.
In the nilpotent case the resulting Hamiltonian

$$H(p,q) \;=\; \frac{1}{2}p^2 \;+\; V(q)$$

describes a one–dimensional particle with potential energy V with $V(0) = V'(0) = 0$. While elliptic and hyperbolic equilibria satisfy $V''(0) \neq 0$, the co–dimension k of a bifurcating parabolic equilibrium is determined by the first non–vanishing derivative of V'' at the origin. The versal unfolding is then given by

$$H_\mu(p,q) \;=\; \frac{a}{2}p^2 \;\pm\; \frac{b}{(k+2)!}q^{(k+2)} \;+\; \sum_{j=1}^{k}\frac{\mu_j}{j!}q^j \tag{1}$$

where $\mu_1,\ldots,\mu_k$ are the unfolding parameters while $a,b>0$ are of order one. Thus, the potential energy function V_μ unfolds the singularity $A^\pm_{k+1}$. For odd k the sign $\pm$ in front of b can be transformed away, but for even k these constitute two dynamically different cases.

In the $\binom{0\,0}{0\,0}$ case the pertinent *simple* singularities, *i.e.* those with modality 0 , are of types $D^\pm_{k+1}$, $k\geq 3$, and E_6, E_7, E_8 , with versal unfoldings

$$H_\mu(p,q) \;=\; \frac{a}{2}p^2q \;\pm\; \frac{b}{k!}q^k \;+\; \sum_{j=1}^{k-1}\frac{\mu_j}{j!}q^j \;+\; \mu_k p \tag{2}$$

and

$$\begin{aligned}
H_\mu(p,q) \;&=\; \frac{a}{6}p^3 \;+\; \frac{b}{24}q^4 \;+\; \sum_{j=1}^{2}\frac{\mu_j}{j!}q^j \;+\; \sum_{j=0}^{2}\frac{\mu_{j+3}}{j!}pq^j \\
H_\mu(p,q) \;&=\; \frac{a}{6}p^3 \;+\; \frac{b}{6}pq^3 \;+\; \sum_{j=1}^{4}\frac{\mu_j}{j!}q^j \;+\; \sum_{j=0}^{1}\frac{\mu_{j+5}}{j!}pq^j \\
H_\mu(p,q) \;&=\; \frac{a}{6}p^3 \;+\; \frac{b}{5!}q^5 \;+\; \sum_{j=1}^{3}\frac{\mu_j}{j!}q^j \;+\; \sum_{j=0}^{3}\frac{\mu_{j+4}}{j!}pq^j
\end{aligned} \tag{3}$$

where the sign $\pm$ of the $D^\pm_{k+1}$ constitutes two dynamically different cases for odd k and can be transformed away for even k .

For further details on these bifurcations the reader is referred to [Broer,Chow,Kim,Vegter;92,95], where in particular the reconstruction of an S^1–action generated by an additional pair of imaginary eigenvalues is carried out.

2.2 The Hamiltonian Hopf bifurcation

In a Hamiltonian Hopf bifurcation two pairs of purely imaginary eigenvalues meet and split off from the imaginary axis to form a complex quartet. This requires two degrees of freedom. At the bifurcation the double pair $\pm \mathrm{i}\omega_0$ of complex eigenvalues generates an S^1–action. Normalization with respect to the semi–simple part $\omega_0 S$ of the linearization at the bifurcating equilibrium pushes this symmetry through the Taylor series, and expressing the normalized part in terms of the invariants N, M, P effectively performs the reduction to one degree of freedom. Here N denotes the nilpotent part of the linearization at the bifurcating equilibrium, M its adjoint and $P := \{N, M\}$. These co–ordinates are constrained by the relations

$$2NM \;=\; \frac{1}{2}P^2 \;+\; \frac{1}{2}S^2 \;, \quad N \geq 0 \,,\; M \geq 0$$

and the conic singularity at the origin of the reduced phase space with $S = 0$ corresponds to the equilibrium in two degrees of freedom. A further normalization with respect to the nilpotent part makes the terms of order two and higher independent of N and P , and the decisive 1–parameter unfolding reads

$$H_\mu(S, N, M, P) \;=\; \omega_0 S \;+\; aN \;\pm\; bM^2 \;+ \mu_1 M \tag{4}$$

where the $\pm$ sign in front of $b > 0$ distinguishes between the supercritical case and the subcritical case.

For a detailed study of these two 1–parameter families and the effect of the neglected higher order terms the reader is referred to [van der Meer;85], while a first step towards higher co–dimensions has been done in [van der Meer;96].

3 Bifurcations of periodic orbits

According to Floquet's theorem there are co–ordinates $x \in S^1$, $y \in \mathsf{Y} \subseteq \mathbb{R}^n$ conjugate to x , and $z \in \mathsf{S} \subseteq \mathbb{R}^{2d-2}$ such that the linearized equations of motion read

$$\begin{aligned} \dot{x} &= \omega \\ \dot{y} &= 0 \\ \dot{z} &= \Omega z \quad . \end{aligned}$$

The Floquet multipliers are given by $\mathrm{e}^{\lambda/\omega}$, where λ runs through the eigenvalues of the constant matrix Ω . Thus, these Floquet exponents determine a whole lattice

$$(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}) \;+\; 2\pi \mathrm{i}\omega \mathbb{Z}$$

(with the special case $\mathrm{Im}(\lambda) \in 2\pi \mathrm{i}\omega\mathbb{Z}$ of those intersecting the real line), and a purely imaginary lattice

$$(\lambda, \bar{\lambda}) \;+\; 2\pi \mathrm{i}\omega \mathbb{Z}$$

for $\mathrm{Re}(\lambda) = 0$. Bifurcations occur for multiple Floquet multipliers on the unit circle, *i.e.* when $\lambda \in \pi \mathrm{i}\omega\mathbb{Z}$ or when there are two Floquet exponents with coinciding purely imaginary lattices.

3.1 Bifurcations inherited from equilibria

Simply adding $\omega \cdot y$ to (1)–(4) yields standard forms of periodic bifurcations, the equilibria get simply replaced by the superposed periodic motion. The corresponding lattices of Floquet exponents are $2\pi \mathrm{i} \omega \mathbb{Z}$ for (1)–(3) and a purely imaginary double lattice $(\lambda, \bar{\lambda}) + 2\pi \mathrm{i} \omega \mathbb{Z}$ for the periodic Hamiltonian Hopf bifurcation.

Given a bifurcating periodic orbit, we put it in Floquet form

$$H(x, y, z) \;=\; \omega \cdot y \;+\; (z \mid \Omega z) \;+\; \textit{higher order terms}$$

and then normalize with respect to y , *i.e.* push the x–independence through the *higher order terms*. For a single Hamiltonian system the unfolding parameters thus become functions $\mu_j = \mu_j(y)$ and in case of co–dimension 1 we require the non–degeneracy condition $\mu_1'(0) \neq 0$ to hold true. For higher co–dimensions we still need external parameters.

From Section 2 we already know that x–independent higher order terms have no qualitative impact. But x–dependent terms, however small, may break up resonant tori. For instance, there is a quasi–periodic centre–saddle bifurcation subordinate to the subcritical periodic Hamiltonian Hopf bifurcation.

3.2 The Hamiltonian flip bifurcation

We expect a periodic orbit with Floquet multiplier -1 to undergo a period–doubling bifurcation. Therefore, we normalize with respect to a periodic flow with twice the period. Reducing the S^1–symmetry of the two–degrees–of–freedom normal form then leads to a one–degree–of–freedom system with $\mathbb{Z}_2$–symmetry

$$(p, q) \mapsto (-p, -q) \quad . \tag{5}$$

The $\mathbb{Z}_2$–isotropy of the origin under this action reflects that the periodic orbit we start with has half the period of the normalizing flow. While the singularities of types $D^{\pm}_{k+1}$ and E_6, E_7, E_8 are not invariant under the π–rotation (5), the singularities $A^{\pm}_{k+1}$ with even k lead, following [Poénaru;75], to the standard forms

$$H_\mu(x, y, p, q) \;=\; \omega \cdot y \;+\; \frac{a}{2} p^2 \;\pm\; \frac{b}{(k+2)!} q^{(k+2)} \;+\; \sum_{i=1}^{k/2} \frac{\mu_i}{(2i)!} q^{2i} \tag{6}$$

of the Hamiltonian flip bifurcations. For $k = 2$ the rôle of the sole parameter μ_1 may be played by the action y conjugate to the angle x .

To factor out the discrete symmetry (5) one can pass to the invariants

$$u = \frac{1}{2} q^2 \; , \quad v = \frac{1}{2} p^2 \quad \text{and} \quad w = pq$$

which are constrained by the relations

$$2uv \;=\; \frac{1}{2} w^2 \; , \quad u, v \geq 0 \quad .$$

Regular equilibria on this reduced phase space correspond to periodic orbits, and the conic singularity at the origin corresponds to the family of periodic orbits of half the period. In particular the bifurcating periodic orbit does not vanish, but merely changes its normal behaviour.

For more details on bifurcations of periodic orbits the reader is referred to [Meyer,Hall;92] and references therein, for the periodic Hamiltonian Hopf bifurcation see also [Ollé,Villanueva,Pacha;02].

4 Bifurcations of Floquet–tori

An invariant n–torus of a d–degrees–of–freedom Hamiltonian system can be put in Floquet form if there are co–ordinates $x \in \mathbb{T}^n$, $y \in \mathsf{Y} \subseteq \mathbb{R}^n$, $z \in \mathsf{S} \subseteq \mathbb{R}^{2(d-n)}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;=\; \begin{pmatrix} \omega \\ 0 \\ \Omega z \end{pmatrix} + \textit{higher order terms}(y, z) \quad ,$$

i.e. with a matrix Ω independent of the angles x . While this is always possible in the periodic case $n = 1$, there do exist invariant tori that cannot be transformed into Floquet form. The eigenvalues of Ω now determine lattices

$$(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}) \;+\; 2\pi\mathrm{i}\omega_1\mathbb{Z} \;+\; \ldots \;+\; 2\pi\mathrm{i}\omega_n\mathbb{Z}$$

with simplifications for real and for purely imaginary λ as in the periodic case. If all eigenvalues are hyperbolic the n–tori will persist the replacement of the *higher order terms* by any small perturbation provided that the usual Diophantine conditions

$$|k_1\omega_1 + \ldots + k_n\omega_n| \;\geq\; \frac{\gamma}{|k|^\tau} \quad \forall_{k \in \mathbb{Z}^n \setminus \{0\}} \tag{7}$$

with $\gamma > 0$ and $\tau > n - 1$ on the internal frequencies $\omega_1, \ldots, \omega_n$ are met. For purely imaginary λ we call $\alpha := |\frac{\lambda}{2\pi\mathrm{i}}|$ the corresponding normal frequency. The Diophantine conditions for non–hyperbolic tori involve these normal frequencies and read

$$|k_1\omega_1 + \ldots + k_n\omega_n \;+\; \ell_1\alpha_1 + \ldots + \ell_m\alpha_m| \;\geq\; \frac{\gamma}{|k|^\tau} \qquad \forall_{k \in \mathbb{Z}^n \setminus \{0\}} \quad \forall_{\ell \in \mathbb{Z}^m \,,\, |\ell| \leq 2} \tag{8}$$

where $2m \leq 2(d - n)$ is the number of purely imaginary eigenvalues. The Diophantine conditions (7) on the internal frequencies are included through $\ell = 0$.

For $\ell = (1, 0, \ldots, 0)$ we have to stay away from the lattice $0 + 2\pi\mathrm{i}\omega_1\mathbf{Z} + \ldots + 2\pi\mathrm{i}\omega_n\mathbf{Z}$, and correspondingly a bifurcation occurs when

$$k_1\omega_1 \;+\; \ldots \;+\; k_n\omega_n \;+\; \alpha_1 \;=\; 0 \quad .$$

Standard (integrable) forms are obtained by adding $\omega_1 y_1 + \ldots + \omega_n y_n$ to the one–degree–of–freedom Hamiltonians (1)–(3).

Similarly $\ell = (2, 0, \ldots, 0)$ is related to the lattice $\pi\mathrm{i} + 2\pi\mathrm{i}\omega_1\mathbb{Z} + \ldots + 2\pi\mathrm{i}\omega_n\mathbb{Z}$ and a quasi–periodic Hamiltonian flip bifurcation occurs when

$$k_1\omega_1 + \ldots + k_n\omega_n + 2\alpha_1 = 0$$

with at least one of the k_i odd. Instead of period doubling it seems more appropriate to speak of frequency halving in this case.

Finally $\ell = (1, -1, 0, \ldots, 0)$ corresponds to a purely imaginary double lattice and a quasi–periodic Hamiltonian Hopf bifurcation occurs when

$$k_1\omega_1 + \ldots + k_n\omega_n + \alpha_1 - \alpha_2 = 0 \quad .$$

As shown in [de Jong;99], tori with (8) violated by $\ell = (1, 1, 0, \ldots, 0)$ persist and do not bifurcate.

4.1 Bifurcations of co–dimension one

In an integrable Hamiltonian system with d degrees of freedom the isotropic invariant tori of dimension $n(\leq d)$ form smooth families parametrised by the n actions $y_1, \ldots, y_n$ conjugate to the toral angles $x_1, \ldots, x_n$. When using one of these actions to unfold the bifurcation we still have $n - 1$ (internal) parameters left to control the n frequencies. Therefore we do not need external parameters. In the absence of additional symmetries the three types of resonances lead to the quasi–periodic centre–saddle, Hamiltonian flip and Hamiltonian Hopf bifurcations.

A quasi–periodic centre–saddle bifurcation requires no more than $d = n+1$ degrees of freedom and a standard (integrable) form is obtained by superposing the quasi–periodic flow

$$\begin{aligned} \dot{x}_i &= \omega_i \\ \dot{y}_i &= 0 \end{aligned}$$

with the planar flow

$$\begin{aligned} \dot{q} &= ap \\ \dot{p} &= -\frac{b}{2}q^2 - \mu \end{aligned}$$

of the unfolding (1) of A_2 . As μ passes through zero, two n–parameter families of normally hyperbolic and elliptic invariant n–tori meet in an $(n-1)$–parameter family of normally parabolic n–tori and vanish. Only tori with Diophantine frequency ratios $[\omega_1 : \omega_2 : \ldots : \omega_n]$ may be expected to survive small perturbations. As shown in [Hanßmann;98], the bifurcation scenario persists on the Cantor sets parametrising the Diophantine tori provided that the mapping

$$[\mu : \omega_1 : \ldots : \omega_n] : \ \mathsf{Y} \longrightarrow \mathbb{RP}^n$$

is locally invertible, *i.e.* its derivative has non–zero determinant.

For a quasi–periodic Hamiltonian flip bifurcation we again pass to a 2 : 1 covering, where a standard form has the equations of motion

$$\begin{aligned} \dot{x}_i &= \omega_i \\ \dot{y}_i &= 0 \\ \dot{q} &= ap \\ \dot{p} &= \mp\frac{b}{6}q^3 - \mu q \end{aligned}$$

of the $\mathbb{Z}_2$–symmetric unfolding (6) derived from $A_3^{\pm}$. As μ passes through zero, the n-torus at the origin changes its normal behaviour and two further tori branch off from the origin. Passing down from the 2 : 1 covering, *i.e.* dividing out the $\mathbb{Z}_2$–symmetry (5), these two additional tori project to a single invariant torus with one frequency halved. Again we need the mapping

$$[\mu : \omega_1 : \ldots : \omega_n] : \quad \mathsf{Y} \longrightarrow \mathbb{RP}^n \tag{9}$$

to be locally invertible for the bifurcation scenario to persist (on Cantor sets), see Section 4.2 below.

For a quasi–periodic Hamiltonian Hopf bifurcation we need at least $d = n+2$ degrees of freedom and the standard form obtained from (4) has the equations of motion

$$\begin{aligned} \dot{x}_i &= \omega_i \\ \dot{y}_i &= 0 \\ \dot{q}_1 &= -\omega_0 q_2 \pm b(p_1^2+p_2^2)p_1 + \mu p_1 \\ \dot{p}_1 &= -\omega_0 p_2 - a q_1 \\ \dot{q}_2 &= \omega_0 q_1 \pm b(p_1^2+p_2^2)p_2 + \mu p_2 \\ \dot{p}_2 &= \omega_0 p_1 - a q_2 \quad . \end{aligned}$$

Here the bifurcation scenario not only involves the n–torus at the origin changing its normal behaviour, but also invariant tori of dimension $n+1$. Correspondingly, next to

$$[\mu : \omega_1 : \ldots : \omega_n] : \quad \mathsf{Y} \longrightarrow \mathbb{RP}^n$$

to control the torus at the origin, we also need to control the frequencies $\omega_0, \omega_1, \ldots, \omega_n$ of the invariant $(n+1)$–tori, where we may use $S = q_1 p_2 - q_2 p_1$ as an additional (internal) parameter. The necessary conditions have yet to be formulated, they should in particular imply that the subordinate quasi–periodic centre–saddle bifurcation of $(n+1)$–tori in the subcritical case satisfies "its" non–degeneracy condition.

4.2 Bifurcations of higher co–dimension

Invariant n–tori that undergo a bifurcation of co–dimension n are typically isolated and we do not expect them to persist — they may fall into "resonance holes". This can only be remedied by allowing the Hamiltonian system to depend on external parameters.

Even for co–dimensions $k < n$ we run into the "lack of parameter" problem that k actions $y_1, \ldots, y_n$ conjugate to the toral angles $x_1, \ldots, x_n$ are necessary to unfold the bifurcation scenario. This leaves only $n - k$ actions to control the internal frequencies $\omega_1, \ldots, \omega_n$. Since the Diophanticity of a frequency vector $(\omega_1, \ldots, \omega_n)$ is to a large extent determined by the frequency ratio $[\omega_1 : \ldots : \omega_n]$ we could formulate simple non–degeneracy criteria in Section 4.1 above to obtain persistence. In fact, this does not yield the persistence of a given torus with Diophantine frequency vector, but only ensures the existence of an invariant torus with the same frequency ratio in the perturbed system.

For $1 < k < n$ we now relax the control on the perturbed tori even further and only aim to show that a perturbed system does have Cantor families of invariant n-tori, without trying to explicitly connect the perturbed tori to the unperturbed ones. The proper requirement is that the collection of $\binom{n+2}{n}$ vectors

$$\frac{\partial^{|\ell|}\nu}{\partial y^{\ell}} \quad , \quad |\ell| \leq 2$$

spans $\mathbb{R}^k \times \mathbb{R}^n$. This implies that the image of the mapping

$$\nu := (\mu, \omega) : \quad \mathsf{Y} \quad \longrightarrow \quad \mathbb{R}^k \times \mathbb{R}^n \tag{10}$$

is "sufficiently curved" and does not lie in any linear hyperplane in $\mathbb{R}^{k+n}$ passing through the origin. Correspondingly, the constant τ in the Diophantine condition (7) has to be chosen somewhat larger to satisfy $\tau > 2n - 1$. See [Broer,Huitema,Sevryuk;96] for more details on how this Diophantine approximation of dependent quantities helps to overcome the lack of parameter problem.

In this way we use $\mu : \mathsf{Y} \longrightarrow \mathbb{R}^k$ to pull back the bifurcation diagram to the space of actions. The remaining first derivatives together with the higher derivatives of $\nu = (\mu, \omega)$ then ensure that most frequencies perturbed from the $\omega(y)$ are Diophantine and, hence, yield invariant tori in the perturbed system. This is used in [Broer,Hanßmann,You;02] to show that the bifurcation scenarios derived from (1) persist on pertinent Cantor sets, with a similar result announced for the bifurcation scenarios derived from (2) and (3).

The Hamiltonian flip and Hopf bifurcations are different in that the internal–normal resonance only affects the normal behaviour of the bifurcating torus. Accordingly, it has been shown in [Bourgain;94] that those bifurcating tori satisfying the Diophantine conditions (7) on the internal frequencies do persist, with no assumptions on the higher order terms necessary. Thus, future work on the quasi–periodic Hamiltonian Hopf bifurcation can be focussed on the normal behaviour of the bifurcating n–torus and in particular on the structure of the families of subordinate tori of dimensions $n+1$ and higher, the latter being subject to monodromy.

The quasi–periodic Hamiltonian flip bifurcations can be considered as $\mathbb{Z}_2$–symmetric bifurcations of normally parabolic invariant tori on a $2:1$ covering space. The structure–preserving nature of the proof in [Broer,Hanßmann,You;02] allows to carry the π–rotation (5) through all steps of the iteration procedure. This yields persistence of the bifurcation scenario on pertinent Cantor sets in the (μ, ω)–space $\mathbb{R}^{k+n}$ and thus in the action space

Y provided that the image of the mapping (10) is sufficiently curved. In particular a non–zero determinant of the linearization of (9) yields persistence in the co–dimension 1 case, as stated in Section 4.1.

5 Reducibility

Floquet's theorem allows to transform the normal linear behaviour of a periodic orbit to constant coefficients, with a transformation that may have twice the period of the periodic orbit. Similarly, a transformation of an invariant torus to Floquet form may only be defined on some finite covering of that torus. For a 2-dimensional *symplectic normal* z–space S the possible matrices Ω all commute and this does not pose additional problems, cf. [Broer,Hanßmann,Jorba,Villanueva,Wagener;02]. However, this might further complicate the quasi–periodic Hamiltonian Hopf bifurcation, as well as normally elliptic bifurcations or more degenerate bifurcations related to *e.g.* zero being an eigenvalue with multiplicity 4 .

Linear (non–Hamiltonian) systems

$$\begin{aligned} \dot{x} &= \omega \\ \dot{z} &= \Omega^{\lambda}(x) z \end{aligned}$$

with z taking values in some compact semi–simple Lie group G are considered in [Krikorian;99]. Denoting by $\mathfrak{g}$ the Lie algebra of G , and by $\Lambda \subseteq \mathbb{R}$ a bounded interval, the family $\Omega^{\lambda} : \mathbb{T}^n \longrightarrow \mathfrak{g}$ is given by $\Omega^{\lambda}(x) = \lambda\Omega_0 + F(x)$, $\lambda \in \Lambda$ with F analytic and small. The system is then shown to be reducible to Floquet form for almost every λ .

One may speculate what can be proven for families $\Omega^{\lambda}_{\mu}(x) = \lambda\Omega_{\mu} + F_{\mu}(x)$, where the movement of the eigenvalues of Ω_{μ} initiates a bifurcation (which in turn requires well–determined higher order terms in z , to have finite co–dimension). In a Hamiltonian setting the actions y conjugate to the toral angles x would then have to assume control over λ as well.

We end with an example where the obstruction to reducibility is of topological nature, see [Wagener;98] for the dissipative analogue. It is simpler to formulate this as a (discrete) mapping, which may be suspended afterwards to a continuous flow.

Thus, let $x \in S^1$ denote a single angle and $y \in \mathbb{R}$ its conjugate action. On the 4-dimensional symplectic normal space S we write $z = (p_1, q_1, p_2, q_2)$. Superposing the rigid rotation

$$x \mapsto x + \omega$$

with the x–dependent rotation

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \mapsto \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}$$

it is necessary to translate y according to

$$y \quad \mapsto \quad y + \frac{p_1^2 + q_1^2}{2}$$

to ensure that the symplectic form $\mathrm{d}x \wedge \mathrm{d}y \, + \, \mathrm{d}q_1 \wedge \mathrm{d}p_1$ remains invariant. In this way there is no recurrence except for $(p_1, q_1) = (0, 0)$. Therefore we further superpose

$$\begin{pmatrix} q_2 \\ p_2 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} q_2 \\ p_2 \end{pmatrix}$$

and correspondingly translate with

$$y \quad \mapsto \quad y \; + \; \frac{p_1^2 + q_1^2}{2} \; - \; \frac{p_2^2 + q_2^2}{2}$$

to end up with what might be termed a "skew Hamiltonian Hopf" bifurcation. Both the decisive higher order terms and the necessary unfolding terms (in particular their number) have yet to be determined.

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