PSEUDODIFFERENTIAL CALCULUS AND THERMODYNAMIC LIMITS.

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Abstract.

If $P(h)$ is a $h-$pseudidifferential operator in $\mathbf{R}^n$ associated to an holomorphic bounded symbol in some neighborhood of the real phase space, we describe the symbol of $e^{-tP(h)}$, by inequalities where the constants depend on the bounds for the symbol of $P(h)$, but not on the dimension n. Some applications to thermodynamic limits are given.

1. Introduction.

In [10], J. Sjöstrand describes the exponential of an $n-$dimensional semiclassical Schrödinger operator

$$P_n(h) \;=\; -h^2\Delta \;+\; V_n(x) \tag{1.1}$$

Since the dimension n is variable, the given object is rather a sequence $(V_n)_{n\geq 1}$, where $V_n \in C^\infty(\mathbf{R}^n)$, belonging to a suitable class of such sequences, called $0-$*standard*. This class is defined by inequalities where the constants are independent on the dimension n. Instead of recalling the definition of $0-$standard sequences, let us say that, if there are some constants $a > 0$ and $M > 0$, independent on n, such that V_n extends to an holomorphic function V_n on

$$\Omega_n(a) \;:=\; \{x \in \mathbf{C}^n,\; |Im\; x|_\infty < a\;\} \tag{1.2}$$

and such that

$$\left|\frac{\partial V_n}{\partial x_j}(x)\right| \;\leq\; M \qquad \forall x \in \Omega_n(a), \qquad \forall n \geq 1, \tag{1.3}$$

then $(V_n)_{(n\geq 1)}$ is $0-$standard, by Proposition 1.1 of [9]. Then, assuming also that V_n is real valued and lower bounded for real x, Sjöstrand gives (in [10], sect. 7), a description of the kernel of the operator $e^{-\frac{t}{h}P_n(h)}$, $(t > 0)$, near the diagonal, where the constants in the inequalities are independent on n, and he gives some applications to thermodynamic limits when $n \to +\infty$ (see below).

The aim of this paper is to replace the Schrödinger operator by pseudodifferential operators $P_n(h)$ associated, by the semiclassical Weyl calculus (see sect. 2), to symbols $p_n(x,\xi)$ which extend to

$$\Omega_{2n}(a) \;:=\; \{(x,\xi) \in \mathbf{C}^{2n},\; |Im\;(x,\xi)|_\infty < a\;\} \tag{1.4}$$

and satisfy

$$\left|\frac{\partial p_n}{\partial x_j}(x,\xi)\right| \ + \ \left|\frac{\partial p_n}{\partial \xi_j}(x,\xi)\right| \ \leq \ M \qquad \forall (x,\xi) \in \Omega_{2n}(a), \qquad \forall n \geq 1, \tag{1.5}$$

for some constants $a > 0$ and $M > 0$, independent on n. Moreover, we shall be interested in $e^{-tP_n(h)}$ instead of $e^{-\frac{t}{h}P_n(h)}$. Our aim is to give, assuming also that $Re\ p_n(x,\xi)$ is lower bounded, a description of the *symbol* of the operator $e^{-tP_n(h)}$, (and not of its *kernel*, like in [10]), where the constants will be independent on the dimension and, more precisely, to show that the symbol of this operator can be written $e^{-q_n(x,\xi,t,h)}$, where q_n satisfies inequalities similar to (1.5). This result will rely on the theorem of composition of pseudodifferential operators in large dimension, (where good estimations are not satisfied by the composed symbols, but by their logarithms), proved in a previous paper [1] with L. Amour and Ph. Kerdelhué. In [1], the results proved here were conjectured, and proved in the formal level.

Unfortunately, in this first paper, we have also to assume that $|p_n(x,\xi)| \leq Mn$ for all (x,ξ) in $\Omega_{2n}(a)$. With this hypothesis, the main result proved here is the theorem 2.2 below. In a work in progress, Ch. Royer is trying to prove a similar result without assuming that $p_n(x,\xi)$ is bounded, by an improvement of the technique. Even if this work is successful, the result will be limited essentially to first order operator (like Klein-Gordon).

Let us explain (in a particular case to simplify the notations), the physical model given by J. Sjöstrand ([10], sect.8) as an application of his results. At each point j of the one-dimensional lattice $\mathbf{Z}$, we consider a particle A_j described by a Schrödinger operator in $\mathbf{R}^k$ (where $k \geq 1$ is fixed)

$$P_1(h) \ = \ -h^2\Delta + V(x)$$

The assumptions of [10] are satisfied if V extends to an holomorphic function in $\Omega_k(a)$ $(a > 0)$, if its derivatives are bounded in $\Omega_k(a)$, if V is real for real x, and if $V(x)$ is greater than a positive power of $|x|$ for large $|x|$. The interaction between each particle and its neighbours in the lattice is given by a potential $W(x,y)$, and the hypotheses of [10] are satisfied if W extends to a bounded holomorphic function in $\Omega_{2k}(a)$, real for real (x,y), and such that $W(y,x) = W(x,y)$. For each $n \geq 1$, the Hamiltonian $P_n(h)$ describing the system of particles A_j $(-n \leq j \leq n)$ in interaction is defined by (1.1), (with n replaced by kn), where

$$V_n(x) \ = \ \sum_{j=-n}^{n} V(x^{(j)}) \ + \ \sum_{j=-n}^{n-1} W(x^{(j)}, x^{(j+1)}) \tag{1.6}$$

The variable of $\mathbf{R}^{(2n+1)k}$ is denoted by $x = (x^{(-n)}, \ldots, x^{(n)})$, with $x^{(j)} \in \mathbf{R}^k$. This sequence (V_n) satisfies (1.3). Such type of Hamiltonians, given at each point of a lattice, each of them interacting with its neighbors, is common in the literature (see Toda [11] in the classical mechanics). In E. Lieb [5] and B. Simon [8], a similar model (the *quantum Heisenberg model*), is studied, but with another Hamiltonian at each point of the lattice, and another type of interaction.

Then, Sjöstrand [10] proves that, for each $h > 0$ and $t > 0$, the sequence

$$\Lambda_n(t,h) \ := \ \frac{1}{(2n+1)} \ ln\ Tr \ \left(e^{-\frac{t}{h}P_n(h)}\right) \tag{1.7}$$

has a limit $\Lambda_\infty(t,h)$ when $n \to +\infty$, that

$$|\Lambda_n(t,h) \ - \ \Lambda_\infty(t,h)| \ \leq \ \frac{C}{n} \tag{1.8}$$

and, with some more hypotheses, that $\Lambda_\infty(t,h)$, (the *thermodynamic limit*), has an asymptotic expansion in powers of h when $h \to 0$. (See Ruelle [7] for the notion of thermodynamic limit, and also Helffer-Sjöstrand [3] for the Lemma used to prove its existence and (1.8)). In [10], the potentials are typically $\mathcal{O}(|x|)$ at infinity. For exactly quadratic potentials, see also Royer [6].

As an application of our theorem 2.2, we shall prove similar results when the particles A_j ($j \in \mathbf{Z}$) are no more described by a Schrödinger operator, and with $e^{-\frac{t}{h}P_n(h)}$ replaced by $e^{-tP_n(h)}$. We can hope that the work in progress of Ch. Royer will be applied, for example, to

$$P_n(h) \ := \ \sum_{j=-n}^{n} \sqrt{I - h^2\Delta_j} \ + \ V_n(x)$$

where Δ_j is the Laplacian for the variable $x^{(j)} \in \mathbf{R}^k$, and that similar results will be proved. In this paper, we can only consider a similar model (perhaps physically non realist), where the 1-particle Hamiltonian is periodic with respect to some lattice in $\mathbf{R}^{2k}$, and where the interaction is also periodic, like a system of Harper Hamiltonians in interaction. In this case, the trace in (1.7) does not exist in the usual sense, and has to be replaced by another notion of trace (see section 9). The similar result for the quantum Heisenberg model seems to be still an open problem. (In this model, h is the inverse of the spin, and Lieb [5] and Simon [8] proved only the continuity of $\Lambda_\infty(t,h)$ at $h = 0$). In [10], the asymptotic expansion of the thermodynamic limit in powers of h relies on a study of Laplace integrals in large dimension. In our sect. 9, it relies on the study of the greatest eigenvalue of some integral operator on the torus.

The results of Lascar [4] on pseudodifferential calculus in infinite dimension have another motivation, related to the quantum field theory.

Besides the main result (theorem 2.2) and its application (theorems 9.1 and 9.3), some propositions may have their own interest : composition (sect. 3), inversion (sect. 6) and exponential (sect. 7) of pseudodifferential operators in large dimension.

2. Statement of the result on the exponential.

For each $(x,\xi) \in \mathbf{R}^{2n}$, we set

$$\|(x,\xi)\|_\infty \ = \ \sup_{j \leq n} sup(|x_j|, |\xi_j|). \tag{2.1}$$

For each $a > 0$, the set $\Omega_{2n}(a)$ is defined in (1.4).

Definition 2.1. *For each $a > 0$, we denote by $\Sigma(a)$ the set whose elements are sequences $(f_n)_{(n \geq 1)}$, where $f_n = f_n(h) = f_n(.,h)$ is an holomorphic function in $\Omega_{2n}(a)$, depending on parameter h in an interval $]0,h_n]$, where $h_n > 0$, and satisfying the following condition. There exists $M > 0$, independent on n, such that, if $1 \leq j \leq n$ and if $0 < h \leq h_n$,*

$$|f_n(x,\xi,h)| \ \leq \ n \ M, \qquad \forall (x,\xi) \in \Omega_{2n}(a) \quad \forall h \in]0,h_n[, \tag{2.2}$$

(2.3) $$|\frac{\partial f_n}{\partial x_j}(x,\xi,h)| + |\frac{\partial f_n}{\partial \xi_j}(x,\xi,h)| \ \leq \ M, \qquad \forall (x,\xi) \in \Omega_{2n}(a) \quad \forall h \in]0,h_n[.$$

We set also

(2.4) $$S(a) \ = \ \bigcap_{0<b<a} \Sigma(b).$$

By the Weyl calculus, we associate to a suitable function $f \in C^\infty(\mathbf{R}^{2n})$ the $h-$pseudo-differential operator $Op_h(f)$ by

(2.5) $$(Op_h(f)u)(x) \ = \ (2\pi h)^{-n} \int_{\mathbf{R}^{2n}} e^{\frac{i}{h}(x-y).\xi} f(\frac{x+y}{2},\xi)u(y)\ dyd\xi \ .$$

We know, by the functional calculus of Helffer-Robert [3], that, for each fixed dimension n, the exponential $e^{tOp_h(p_n)}$ is also a $h-$pseudodifferential operator, and we want to describe its symbol uniformly with respect to the dimension n. The main result of this paper is the following.

Theorem 2.2. *Let $b > 0$, and $p = (p_n)$ a family of symbols in $\Sigma(b)$, independent on h. Then, for each $a \in]0,b[$, and for each integer $m \geq 1$, there exist $\varepsilon_m > 0$ and a family of functions $q = (q_n)(x,\xi,t,h)$ in $\Sigma(a)$ $(0 \leq t \leq 1)$ such that*

(2.6) $$e^{tOp_h(p_n)} \ = \ Op_h(e^{q_n(.,t,h)}) \qquad if \quad 0 < nh^m \leq \varepsilon_m, \qquad n \geq 1$$

This function $q_n(.,t,h)$ satisfies (2.2) and (2.3) for some constant $M > 0$, for all $t \in [0,1]$, $(x,\xi) \in \Omega_{2n}(a)$ and $h > 0$ and $n \geq 1$ such that $nh^m \leq \varepsilon_m$. Moreover, the symbol q_n has an asymptotic expansion in powers of h

(2.7) $$q_n(x,\xi,t,h) \ = \ \sum_{j=0}^{m-1} E_n^{(j)}(x,\xi,t)h^j + h^m R_n(x,\xi,t,h)$$

where the families of functions $(E_n^{(j)})$ satisfy the inequalities (2.2) *and* (2.3)*, and and R_n satisfies (2.2) with the same conditions as above.*

The explicit construction of the functions $E_j^{(n)}$ will be precised in sections 5 (in general) and 9 (in some particular cases).

The paper is organized as follows. In sections 3, 6 and 7, we give some results on the Weyl calculus of holomorphic symbols, which have perhaps their own interest : composition (section 3), inversion (section 6), and exponential (section 7). In section 4, we recall the result of [1], stated here for the Weyl calculus. In section 5, we give the formal construction of the $E_n^{(j)}$ of (2.7), and in section 8, the end of the proof of Theorem 2.2. The precise statement for applications to thermodynamic limit is given and proved in section 9.

3. Composition of symbols in large dimension.

We denote by σ the symplectic form in $\mathbf{C}^{2n}$ defined by $\sigma\big((x,\xi),\ (y,\eta)\big) = y.\xi - x.\eta$. If f and g are C^∞ functions on $\mathbf{R}^{2n}$, bounded with all their derivatives, we denote by $f\sharp_h g$ the function defined by the oscillatory integral

$$(f\sharp_h g)(X) \ = \ (\pi h)^{-2n} \int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y,Z)} f(X+Y) g(X+Z) dY dZ \tag{3.1}$$

If f is a bounded function in $\Omega_{2n}(a)$ (defined in (1.4)), we set

$$\|f\|_a \ = \ \sup_{X\in\Omega_{2n}(a)} |f(X)| \tag{3.2}$$

If the derivatives are also bounded, we set

$$\|\nabla f\|_a \ = \ \sup_{X\in\Omega_{2n}(a)} \ \sup_{j\leq n} \ sup \left| \frac{\partial f}{\partial x_j}(X)\right|, \ \left|\frac{\partial f}{\partial \xi_j}(X)\right|$$

Theorem 3.1. *Let a and b be such that $0 < a < b$, and f and g be holomorphic bounded functions in $\Omega_{2n}(b)$. Then $f\sharp_h g$ is holomorphic and bounded in $\Omega_{2n}(a)$ and we have*

$$\|f\sharp_h g\|_a \ \leq \ \|f\|_b \ \|g\|_b \ \left(1 + \sqrt{\frac{2}{\pi}} \ \frac{\sqrt{h}}{b-a} e^{-\frac{(b-a)^2}{h}}\right)^{4n} \tag{3.3}$$

This theorem is a particular case of the following more general result, which will be used also in section 5.

Proposition 3.2. *Let $a > 0$ and $\rho > 0$, and $\Omega_{6n}(a,\rho)$ be the set of $(X,Y,Z) \in \mathbf{C}^{6n}$ such that X is in $\Omega_{2n}(a)$ and Y and Z in $\Omega_{2n}(\rho)$. Let F be a bounded holomorphic function in $\Omega_{6n}(a,\rho)$. Let $I_h(F)$ be the following oscillatory integral*

$$(I_h(F))\ (X) \ = \ (\pi h)^{-2n} \int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y,Z)} F(X,Y,Z)\ dY dZ \tag{3.4}$$

Then we have

$$\|I_h(F)\|_a \ \leq \ \sup_{(X,Y,Z)\in\Omega_{6n}(a,\rho)} |F(X,Y,Z)| \ \left(1 + \sqrt{\frac{2}{\pi}}\frac{\sqrt{h}}{\rho} e^{-\frac{\rho^2}{h}}\right)^{4n} \tag{3.5}$$

Proof. We set, for each $R > 0$ and $t \in \mathbf{R}$

$$\psi_R(t) \ = \ \begin{cases} = \ -R & if \quad t \leq -R \\ = t & if \quad -R \leq t \leq R \\ = R & if \quad R \leq t \end{cases}$$

We set, for each $t \in \mathbf{R}^n$

$$\varphi_R(t) \ = \ \big(\psi_R(t_1), \dots \psi_R(t_n)\big)$$

We define, for each $R > 0$, an integration contour γ_R in $\mathbf{C}^{4n}$ in the following way. For each $\theta = (s, t, \sigma, \tau) \in \mathbf{R}^{4n}$, $\gamma_R(\theta) = (Y_R(\theta), Z_R(\theta))$ is the point of $\mathbf{C}^{4n}$ defined by

$$\begin{array}{lll} \frac{1}{2}\left(y_R(\theta) + \zeta_R(\theta)\right) & = & s + i\varphi_R(s) \\ \frac{1}{2}\left(y_R(\theta) - \zeta_R(\theta)\right) & = & \sigma - i\varphi_R(\sigma) \end{array} \qquad \begin{array}{lll} \frac{1}{2}\left(z_R(\theta) - \eta_R(\theta)\right) & = & t + i\varphi_R(t) \\ \frac{1}{2}\left(z_R(\theta) + \eta_R(\theta)\right) & = & \tau - i\varphi_R(\tau) \end{array}$$

If $2R < \rho$ and $X \in \Omega_{2n}(a)$, then $(X, Y_R(\theta), Z_R(\theta))$ is in $\Omega_{6n}(a, \rho)$ for all θ, and we may replace $\mathbf{R}^{2n}$ by γ_R in (3.4) and write

$$\left(I_h(F)\right)(X) \;=\; (\pi h)^{-2n} \int_{\gamma_R} e^{-\frac{2i}{h}\sigma(Y,Z)} F(X, Y, Z)\; dY dZ$$

in other words

$$\left(I_h(F)\right)(X) \;=\; (\pi h)^{-2n} \int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y_R(\theta), Z_R(\theta))} F(X, Y_R(\theta), Z_R(\theta))\;\; det\; \gamma_R'(\theta)\; d\theta$$

Proposition 3.2 will follow from the next lemma (with $R = \rho/2$).

Lemma 3.3. *With the previous notations, we have, for all $R > 0$ and $h > 0$*

$$(\pi h)^{-2n} \int_{\mathbf{R}^{4n}} \left| e^{-\frac{2i}{h}\sigma(Y_R(\theta), Z_R(\theta))}\; \det\; \gamma_R'(\theta) \right| d\theta \;\leq\; \left(1 + \frac{\sqrt{h}}{R\sqrt{2\pi}} e^{-\frac{4R^2}{h}}\right)^{4n}$$

Proof of Lemma 3.3. It suffices to prove this lemma when $n = 1$. Let us set in this case

$$F_R(\theta) \;=\; \left| e^{-\frac{2i}{h}\sigma(Y_R(\theta), Z_R(\theta))}\; \det\; \gamma_R'(\theta) \right|$$

We see easily that, for each $\theta \in \mathbf{R}^4$, (denoting its components by $(\theta_1, \ldots, \theta_4)$ instead of (s, σ, t, τ))

$$F_R(\theta) \;\leq\; 4 \prod_{j=1}^{4} \left[1 + \varphi_R'(\theta_j)^2\right]^{1/2}\; e^{-\frac{4}{h}\theta\;\cdot\;\varphi_R(\theta)}$$

For each $\theta \in \mathbf{R}^4$, let us set $N(\theta) = \sharp\{j \leq 4,\; |\theta_j| \geq R\}$, and let A_j $(0 \leq j \leq 4)$ be the set of $\theta \in \mathbf{R}^4$ such that $N(\theta) = j$. We can see that, if $\theta \in A_0$

$$F_R(\theta) \;\leq\; 16\; e^{-\frac{4}{h}|\theta|^2}$$

If $\theta \in A_1$ and, for example, $\sup|\theta_j| = |\theta_1|$, we have

$$F_R(\theta) \;\leq\; 8\sqrt{2}\; e^{-\frac{4}{h}(R|\theta_1| + \theta_2^2 + \theta_3^2 + \theta_4^2)}$$

If $\theta \in A_2$ and, for example, $|\theta_1|$ and $|\theta_2| \geq R$, and $|\theta_3|$ and $|\theta_4| \leq R$, we have

$$F_R(\theta) \;\leq\; 8\; e^{-\frac{4}{h}(R|\theta_1| + R|\theta_2| + \theta_3^2 + \theta_4^2)}$$

If $\theta \in A_3$ and, for example, $|\theta_1|$, $|\theta_2|$ and $|\theta_3| \geq R$, and $|\theta_4| \leq R$, we have

$$F_R(\theta) \ \leq \ 4\sqrt{2}\ e^{-\frac{4}{h}(R|\theta_1|+R|\theta_2|+R|\theta_3|+\theta_4^2)}$$

If $\theta \in A_4$, we have

$$F_R(\theta) \ \leq \ 4e^{-\frac{4}{h}(R|\theta_1|+R|\theta_2|+R|\theta_3|+R|\theta_4|)}$$

Therefore, if $0 \leq j \leq 4$

$$(\pi h)^{-2}\int_{A_j} F_R(\theta)\ d\theta \ \leq \ (\pi h)^{-2}4(\sqrt{2})^{4-j}\ \ C_4^j\left[2\int_R^\infty e^{-\frac{4R\theta}{h}}\ d\theta\right]^j\ \ \left[\int_{-\infty}^{+\infty} e^{-\frac{4\theta^2}{h}}d\theta\right]^{4-j}$$

$$\ldots \ \leq \ (\pi h)^{-2}4(\sqrt{2})^{4-j}\ C_4^j\ \ \left[\frac{h}{2R}e^{-\frac{4R^2}{h}}\right]^j\ \ \left[\frac{\sqrt{\pi h}}{2}\right]^{4-j}$$

Therefore

$$(\pi h)^{-2}\int_{\mathbf{R}^4} F_R(\theta)\ d\theta \ \leq \ 4(\pi h)^{-2}\left[\sqrt{\frac{\pi h}{2}}\ +\ \frac{h}{2R}e^{-\frac{4R^2}{h}}\right]^4$$

The lemma is proved.

4. Composition of exponentials of symbols (result of [1]).

We shall use the theorems 2.4 and 2.6 of the paper [1] with L. Amour and Ph. Kerdelhué. Unfortunately, these results are written in [1] with the standard calculus, and here we use the Weyl calculus. It is useful to write here the statement for the Weyl calculus, and to recall the formal construction, but the proof, which is the same as in [1], will be omitted.

Let us explain first the formal aspects. Let $b > 0$, and $f = (f_n)$ and $g = (g_n)$ be functions in $S(b)$, which are first supposed to be independent on h. We want to find first a formal series (in powers of h) $\varphi_n(.,h)$ such that $e^{\varphi_n(.,h)} \sim e^{f_n}\sharp_h e^{g_n}$. For that, we can find a unique formal serie

$$u(y,\eta,z,\zeta,h) \ \sim \ \sum_{k\geq 0} u_k(y,\eta,z,\zeta)h^k$$

which is a formal solution of the Cauchy problem (the substrict n being omitted)

$$2i\ \frac{\partial e^u}{\partial h} \ = \ \sigma\big((\partial_y,\partial_\eta),\ (\partial_z,\partial_\zeta)\big)\ e^u \tag{4.1}$$

$$u(y,\eta,z,\zeta,0) \ = \ f(y,\eta)+g(z,\zeta) \tag{4.2}$$

Then the formal serie $\varphi(.,h)$ will be given by $\varphi(x,\xi,h) = u(x,\xi,x,\xi,h)$. Let us prove now that the coefficients of this formal series are in $S(b)$. The equation (4.1) can be written

$$2i\frac{\partial u}{\partial h} \ \sim \ \sum_{j=1}^n \frac{\partial^2 u}{\partial z_j\partial\eta_j} \ - \ \frac{\partial^2 u}{\partial y_j\partial\zeta_j} \ + \ \frac{\partial u}{\partial z_j}\frac{\partial u}{\partial\eta_j} \ - \ \frac{\partial u}{\partial y_j}\frac{\partial u}{\partial\zeta_j}$$

Therefore the coefficients u_k are defined by $u_0(y,\eta,z,\zeta) = f(y,\eta) + g(z,\zeta)$ and, if $k \geq 1$, by

$$2iku_k \;=\; \sum_{j=1}^{n}\left[\frac{\partial^2 u_{k-1}}{\partial z_j \partial \eta_j} \;-\; \frac{\partial^2 u_{k-1}}{\partial y_j \partial \zeta_j} \;+\; \sum_{p+q=k-1}\Big(\frac{\partial u_p}{\partial z_j}\frac{\partial u_q}{\partial \eta_j} \;-\; \frac{\partial u_p}{\partial y_j}\frac{\partial u_q}{\partial \zeta_j}\Big)\right] \tag{4.3}$$

We set $c_k(f,g)(x,\xi) := u_k(x,\xi,x,\xi)$. Then the definition of $c_k(f,g)$ can be extended to functionf f and g depending on h (using the explicit expression (4.3), not the differential equation (4.1)). It is easy to see, using Cauchy inequalities and the Proposition 1.1 of [9], recalled below for reader's convenience, that the $c_k(f,g)$ are in $S(b)$. More precisely, if $0 < a < b$, $\|c_k(f,g)\|_a$ is bounded by some constant $C > 0$, depending only on k, $b-a$, $\|\nabla f\|_b$ and $\|\nabla g\|_b$. We remark also that, if f and g are polynomials in h of degree $\leq m$, then $c_k(f,g)$ is a polynomial of degree $\leq (k+1)m$.

Proposition 4.1 (Proposition 1.1 of [9]). *Let a and b such that $0 < a < b$. Then, there exists $C > 0$ (independent on n) with the following property. Let $f(y,\zeta)$ be an holomorphic function in $\Omega_{2n}(b)$ such that $|\nabla f|_\infty$ is bounded in $\Omega_{4n}(b)$. Then we have*

$$\|\sum_{j=1}^{n}\frac{\partial^2 f}{\partial y_j \partial \zeta_j}\|_a \leq Cn\|\nabla f\|_b \qquad \|\nabla\left(\sum_{j=1}^{n}\frac{\partial^2 f}{\partial y_j \partial \zeta_j}\right)\|_a \;\leq\; C\|\nabla f\|_b$$

Moreover, if $u = (u_1, \dots u_n)$ a vector, and $\rho > 0$ such that $a + \rho|u|_\infty < b$, we have

$$\|\sum_{j=1}^{n} u_j \frac{\partial f}{\partial y_j}\|_a \;\leq\; \frac{1}{\rho}\;\|f\|_{a+\rho|u|_\infty}$$

Now, instead of a formal series, we want to find a function φ_n satisfying $e^{f_n(.,h)}\sharp_h e^{g_n(.,h)}$ The answer is given by theorems 2.4 and 2.6 of [1] (or, more precisely, their analogues for the Weyl calculus), which can be written as follows.

Theorem 4.2. *Let f and g be functions in $\Sigma(b)$ $(b > 0)$, and let h'_n and h''_n be the sequences associated to them as in Definition 1.1. Let $a \in]0,b[$ and $m \geq 1$. Then, there exist an element φ of $\Sigma(a)$ and a constant ε_0 such that, if we set*

$$h_n \;=\; inf\;(h'_n, h''_n, \varepsilon_0 n^{-1/m}), \tag{4.4}$$

we have

$$e^{\varphi_n(h)} \;=\; e^{f_n(h)}\sharp_h e^{g_n(h)} \qquad if \quad 0 < h \leq h_n. \tag{4.5}$$

Moreover, if $c_k(f,g)$ is defined as above, we can write

$$\varphi(x,\xi,h) \;=\; \sum_{k=0}^{m-1} c_k(f,g)(x,\xi)h^k \;+\; h^m\; R(x,\xi,h) \tag{4.6}$$

and there exists $C_m > 0$, depending only on m, such that, if $0 < h \leq h_n$

$$\frac{1}{n}\|c_k(f,g)\|_a \;+\; \frac{1}{n}\|R(.,h)\|_a \;+\; \|\nabla c_k(f,g)\|_a \;+\; \|\nabla R(.,h)\|_a \;\leq\; C_m$$

It will be useful later to know how $c_k(f,g)$ is changed by a small perturbation of f or g (of course, the dependence is not linear).

Proposition 4.3. *Let $f=(f_n)$, $g=(g_n)$ and $\widetilde{g}=(\widetilde{g}_n)$ be functions in $\Sigma(b)$ $(b>0)$. Then, for each $a\in]0,b[$ and $k\geq 0$, there exists $C_k>0$, independent on n, such that*

$$\frac{1}{n}\|c_k(f,g+h^m\widetilde{g}) \;-\; c_k(f,g)\|_a \;+\; \|\nabla\left(c_k(f,g+h^m\widetilde{g}) \;-\; c_k(f,g)\right)\|_a \;\leq\; C_k h^m \tag{4.7}$$

Proof. Let u_k and w_k $(k\geq 0)$ be the sequences of functions defined by (4.3) and by

$$u_0(y,\eta,z,\zeta)=f(y,\eta)+g(z,\zeta) \qquad w_0(y,\eta,z,\zeta)=f(y,\eta)+g(z,\zeta)+h^m\widetilde{g}(z,\zeta)$$

Let $v_k=w_k-u_k$. We have $v_0(y,\eta,z,\zeta)=h^m\widetilde{g}(z,\zeta)$ and, if $k\geq 1$

$$2ikv_k \;=\; \sum_{j=1}^{n}\left[\frac{\partial^2 v_{k-1}}{\partial z_j\partial\eta_j} \;-\; \frac{\partial^2 v_{k-1}}{\partial y_j\partial\zeta_j} \;+\; \sum_{p+q=k-1}\Big(\frac{\partial u_p}{\partial z_j}\frac{\partial v_q}{\partial\eta_j} \;-\; \frac{\partial u_p}{\partial y_j}\frac{\partial v_q}{\partial\zeta_j} \;+\; \ldots\right.$$

$$\left.\ldots \;+\; \frac{\partial v_p}{\partial z_j}\frac{\partial u_q}{\partial\eta_j} \;-\; \frac{\partial v_p}{\partial y_j}\frac{\partial u_q}{\partial\zeta_j}+\frac{\partial v_p}{\partial z_j}\frac{\partial v_q}{\partial\eta_j} \;-\; \frac{\partial v_p}{\partial y_j}\frac{\partial v_q}{\partial\zeta_j}\Big)\right] \tag{4.8}$$

Now let p be an integer ≥ 1, and let $a_0,a_1,\ldots,a_p$ be real numbers such that $a=a_p<a_{p-1}<\ldots<a_1<a_0=b$. We shall prove that

$$\|\nabla v_k\|_{a_k} \;+\; \frac{1}{n}\|v_k\|_{a_k} \;\leq\; C_k h^m \tag{X_k}$$

for some constant C_k, independent on n. For $k=0$, (X_0) is only our hypothesis. If $k\geq 1$, and if the property is proved for integers $j\leq k-1$, the property (X_k) follows, using (4.8) and Proposition 4.1. Taking the restriction of (X_k) to the diagonal, we see that (4.7) is satisfied for all $k\leq p$. Since p is arbitrary, the Proposition is proved.

The next Proposition follows from Proposition 4.3, using also the estimations of $c_k(f,g)$ before Proposition 4.1.

Proposition 4.4. *With the notations of theorem 4.1, if f and g have asymptotic expansions of the following form*

$$f(x,\xi,h) \;=\; \sum_{k=0}^{m-1} f_k(x,\xi)h^k \;+\; h^m R_m(x,\xi,h)$$

where, for some constant C, independent on n

$$\frac{1}{n}\|f_k\|_b \;+\; \frac{1}{n}\|R_m(.,h)\|_b \;+\; \|\nabla f_k\|_b \;+\; \|\nabla R_m(.,h)\|_b \leq\; C,$$

then, if nh^m is small enough, the function φ of theorem 4.1 has a similar expansion (with b replaced by a).

5. Formal construction of the symbol in theorem 2.2.

The formal aspect of theorem 2.2 is very similar to the Théorème 2.2 of [1]. The only difference is that the standard calculus is now replaced by the Weyl one. For each functions f and g, C^∞ in $\mathbf{R}^{2n}$, let us set $(f\otimes g)(y,\eta,z,\zeta)=f(y,\eta)g(z,\zeta)$ and

$$a_k(f,g)(x,\xi) \;=\; \frac{i^k}{2^k\,k!}\left[\sigma\big((D_y,D_\eta),\;(D_z,D_\zeta)\big)^k(f\otimes g)\right](x,\xi,x,\xi) \tag{5.1}$$

We know that, if f and g are bounded with all their derivatives

$$(f\sharp_h g)(X) \;\sim\sum_{k\geq 0} a_k(f,g)(X)h^k \tag{5.2}$$

We can prove, exactly as in [1], the following proposition.

Proposition 5.1. *Let $p=(p_n)$ be a family of functions in $S(b)$ $(b>0)$ (independent on h) and m be an integer ≥ 0. Then there exist functions $\Phi^{(m)}(.,h,t)$ and $R^{(m)}(.,h,t)$ in $S(b)$ $(t\in\mathbf{R})$ such that $\Phi^{(m)}$ is a polynomial in h of degree $\leq m-1$ and, (omitting the subscript n)*

$$\frac{\partial}{\partial t}e^{\Phi^{(m)}(.,h,t)} \;=\sum_{k=0}^{m-1} a_k\left(p,e^{\Phi^{(m)}(.,h,t)}\right)h^k + h^m e^{\Phi^{(m)}(.,h,t)}R^{(m)}(.,h,t) \tag{5.3}$$

Proof. We take $\Phi^{(0)}(x,\xi,h,t)=tp(x,\xi)$. Once the polynomial $\Phi^{(m)}$ of degree $\leq m-1$ in h is constructed, we denote by $c_k^{(m)}(p,\Phi^{(m)})$ the coefficient of h^m in the polynomial $e^{-\Phi^{(m)}}a_k(p,e^{\Phi^{(m)}})$. Then we denote by $E^{(m)}(x,\xi,t)$ the function satisfying

$$\frac{\partial E^{(m)}}{\partial t} \;=\sum_{k=0}^{m} c_k^{(m)}(p,\Phi^{(m)}) \qquad E^{(m)}(x,\xi,0)=0$$

and we set $\Phi^{(m+1)}=\Phi^{(m)}+h^mE^{(m)}$. We see easily that the sequence $\Phi^{(m)}$ defined by this way satisfies (5.3) and, using the argument of the section 9 of [1], we see that the $E^{(j)}$ are in $S(b)$.

We can write the first coefficients of the formal series : $E^{(0)}(x,\xi,t)=tp(x,\xi)$, $E^{(1)}=0$ and

$$E^{(2)} \;=\; \frac{t^2}{8}\sum_{jk}\left[\frac{\partial^2 p}{\partial x_j\partial x_k}\,\frac{\partial^2 p}{\partial\xi_j\partial\xi_k} \;-\; \frac{\partial^2 p}{\partial x_j\partial\xi_k}\,\frac{\partial^2 p}{\partial\xi_j\partial x_k}\right]+\dots \tag{5.4}$$

$$\dots \;+\; \frac{t^3}{24}\;\sum_{jk}\left[\frac{\partial^2 p}{\partial x_j\partial x_k}\,\frac{\partial p}{\partial\xi_j}\,\frac{\partial p}{\partial\xi_k} \;+\; \frac{\partial^2 p}{\partial\xi_j\partial\xi_k}\,\frac{\partial p}{\partial x_j}\,\frac{\partial p}{\partial x_k} \;-\; 2\,\frac{\partial^2 p}{\partial x_j\partial\xi_k}\,\frac{\partial p}{\partial\xi_j}\,\frac{\partial p}{\partial x_k}\right]$$

We know that there exists another function $\widetilde{R}^{(m)}(.,h,t)$ in $S(b)$ such that

$$\frac{\partial}{\partial t}e^{\Phi^{(m)}(.,h,t)} \;=\sum_{k=0}^{m-1} a_k\left(e^{\Phi^{(m)}(.,h,t)},p\right)h^k + h^m e^{\Phi^{(m)}(.,h,t)}\widetilde{R}^{(m)}(.,h,t) \tag{5.5}$$

Now we shall obtain the following consequence of this proposition.

Proposition 5.2. *With the notations of proposition 5.1, there exist a family $S^{(m)}(.,h,t)$ in $S(b)$ such that*

$$\frac{\partial}{\partial t}e^{\Phi^{(m)}(.,h,t)} \;=p\;\sharp_h\; e^{\Phi^{(m)}(.,h,t)} \;+\; h^m e^{\Phi^{(m)}(.,h,t)}S^{(m)}(.,h,t) \tag{5.6}$$

More precisely, for each $a \in]0,b[$, there exists $\varepsilon_0 > 0$ and $M > 0$, independent on n, such that $S^{(m)}$ satisfies (H_1) and (H_2) in $\Omega_{2n}(a)$ if $0 < h \leq \varepsilon_0(1 + Log\ n)^{-1}$ and $t \in [-1,1]$.

Proof. First step. For each function $\Phi \in C^\infty(\mathbf{R}^{2n})$ let us set, for each $(x,\xi) \in \mathbf{R}^{2n}$

$$H_\Phi(x,\xi) \;=\; \left(-\frac{\partial\Phi}{\partial\xi},\; \frac{\partial\Phi}{\partial x}\right) \tag{5.7}$$

and, for each X and Y in $\mathbf{R}^{2n}$

$$Q_\Phi(X,\psi) \;=\; \int_0^1 H_\Phi(X+tY)\;dt \tag{5.8}$$

Thus we have

$$\Phi(X+Y) \;-\; \Phi(X) \;=\; \sigma(Q_\Phi(X,Y),\; Y) \tag{5.9}$$

With these notations, we can write, for each $\Phi \in S(b)$

$$e^{-\Phi}\Big[p\;\sharp_h e^{\Phi}\Big](X) \;=\; (\pi h)^{-2n}\int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y+\frac{ih}{2}Q_\Phi(X,Z),Z)}p(X+Y)\;dYdZ \tag{5.9}$$

$$\ldots \;=\; (\pi h)^{-2n}\int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y,Z)}F_h(X,Y,Z)\;dYdZ$$

where

$$F_h(X,Y,Z) \;=\; p(X+Y-\frac{ih}{2}Q_\Phi(X,Z)) \tag{5.11}$$

Of course, this is not a change of variables, but a change of contour. Let a , a_1 and a_2 such that $a < a_1 < a_2 < b$. We remark that there exists a constant $C > 0$, independent on n, such that, if X and $X+Z$ are in $\Omega_{2n}(a_2)$, then $|Q_\Phi(X,Z)|_\infty \leq C$. Therefore, the change of contour is allowed, and there is some other constant $C > 0$, independent on h, such that, for h small enough,

$$|F_h(X,Y,Z)| \;\leq\; Cn \qquad |\nabla F_h(X,Y,Z)|_\infty \;\leq\; C \tag{5.11}$$

if X, $X+Y$ and $X+Z$ are in $\Omega_{2n}(a_2)$.

Second step. Let L be the following operator in $\mathbf{R}^{4n}$

$$L \;=\; \frac{i}{2}\;\sigma(D_Y,\; D_Z)$$

For the integral defined in (5.10), we have, by Taylor formula

$$e^{-\Phi}\Big[p\ \sharp_h e^{\Phi}\Big](X)\ =\ \sum_{k=0}^{m-1}\frac{1}{k!}h^k(L^kF_h)(X,0,0)\ +h^mT^{(m)}(X)$$

where

$$T^{(m)}(X)\ =\ \int_0^1\frac{(1-\theta)^{m-1}}{(m-1)!}(\pi h)^{-2n}\int_{\mathbf{R}^{4n}}e^{-\frac{2i}{h}\sigma(Y,Z)}(L^mF_h)(X,\theta Y,\theta Z)dYdZd\theta$$

We can write, for each $k\leq m-1$

$$(L^kF_h)(X,0,0)\ =\ \sum_{j=0}^{m-k-1}h^jg_j(X)\ +\ h^{m-k}R_{mk}(X,h)$$

where g_j and R_{mk} are in $S(a)$. Therefore, we have

$$e^{-\Phi}\ \sum_{k=0}^{m-1}a_k\left(p,e^{\Phi(.)}\right)h^k\ -\ \sum_{k=0}^{m-1}\frac{1}{k!}h^k(L^kF_h)(X,0,0)\ =\ h^mR'^{(m)}(.,h)$$

where $R'^{(m)}$ is in $S(a)$. If we apply that to the function $\Phi^{(m)}$ of Proposition 5.1, we obtain the equality (5.6) with $S^{(m)}=R^{(m)}+R'^{(m)}-T^{(m)}$. It remains to prove that $T^{(m)}$ is in $S(a)$. By (5.12) and Proposition 4.1, there exists $C>0$, independent on n, such that the function F_h defined in (5.11) with $\Phi=\Phi^{(m)}$ satisfies

$$|\sigma(D_Y,\ D_Z)^mF(X,Y,Z)|\leq Cn\qquad\qquad|\nabla\Big(\sigma(D_Y,\ D_Z)^mF(X,Y,Z)\Big)|_\infty\ \leq\ C$$

if X, $X+Y$ and $X+Z$ are in $\Omega_{2n}(a_1)$. Therefore, it follows from Proposition 3.2 that

$$\|T^{(m)}\|_a\ \leq\ Cn\left(1+\frac{\sqrt{h}}{a_1-a}e^{-\frac{(a_1-a)^2}{h}}\right)^{4n}$$

If $0<h\leq(a_1-a)^2/(1+Log\ n)$, it follows that $\|T^{(m)}\|_a\leq Cne^4$, and we check similarly the hypothesis (2.3) of Definition 2.1. The Proposition is proved.

From (5.5) we obtain, by the arguments of Proposition 5.2, the following equality

$$\frac{\partial}{\partial t}e^{\Phi^{(m)}(.,h,t)}\ =\ e^{\Phi^{(m)}(.,h,t)}\ \sharp_h\ p\ +\ h^me^{\Phi^{(m)}(.,h,t)}\widetilde{S}^{(m)}(.,h,t)\tag{5.13}$$

where $\widetilde{S}^{(m)}$ has the same properties as S_m in Proposition 5.2.

6. Inversion of a symbol.

Proposition 6.1. *Let a and b such that $0<a<b$. Then there exists $\varepsilon_0>0$ (independent on n), with the following properties. If f is a bounded holomorphic function in $\Omega_{2n}(b)$ $(n\geq 1)$ such*

that $\|f\|_b \leq \varepsilon_0$, *and if* $h > 0$ *satisfies* $h \leq \varepsilon_0(1 + Log\ n)^{-3}(b-a)^2$, *then there exists a bounded holomorphic function* g_h *in* $\Omega_{2n}(a)$ *such that*

$$(1+f)\ \sharp_h\ (1+g_h)\ =\ 1 \tag{6.1}$$

$$\|g_h\|_a\ \leq 5\|f\|_b \tag{6.2}$$

The proof will rely on the following lemma, which is stated in a more general form since it will be used also in Section 7. The role of the integer $4n+1$ will appear there.

Lemma 6.2. *There exists* $\varepsilon_0 > 0$ *(independent on n), with the following properties. If* $0 < a < b$, *if* $f_1, \ldots f_k$ $(1 \leq k \leq 4n+1)$ *are bounded holomorphic functions in* $\Omega_{2n}(b)$ *such that*

$$\varepsilon := \sup_{j \leq k} \|f_j\|_b \leq \varepsilon_0, \tag{6.3}$$

if $h > 0$ *satisfies* $h \leq \varepsilon_0(1 + Log\ n)^{-3}(b-a)^2$, *then*

$$\|f_1\ \sharp_h\ \ldots\ \sharp_h\ f_k\|_a\ \leq\ \varepsilon^{(k+1)/2}$$

Proof of Lemma 6.2. Let us set

$$\varepsilon_0\ =\ inf\left(\left(Log\ \frac{10}{9}\right)^2,\ e^{-8}\right) \qquad a_j\ =\ b\ -\ (b-a)\ \frac{Log\ j}{Log\ (4n+1)}$$

Therefore $a = a_{4n+1} < \ldots < a_1 = b$. We shall prove, by induction on j $(1 \leq j \leq k)$, the following property.

(X_j) If F_j is a product (for the composition law $\sharp_h$) of k functions chosen among $f_1,\ \ldots\ , f_k$ (which satisfy (6.3)), and if $0 < h \leq \varepsilon_0(1 + Log\ (4n+1))^{-3}(b-a)^2$, then we have

$$\|F_j\|_{a_j}\ \leq\ \varepsilon^{(j+1)/2}$$

For $j = 1$, (X_1) is our hypothesis. Suppose that $j \geq 2$, and that (X_i) is proved for all $i < j$. Then we can find an integer i such that $\frac{j}{2} \leq i \leq \frac{9j}{10}$. Let F_j a function as in the statement of X_j. We can find a function F_i (resp. F_{j-i}), which is a composed of i functions (resp. of $j-i$ functions) chosen among $f_1,\ \ldots\ , f_k$, such that $F_j = F_i\ \sharp_h F_{j-i}$. By theorem 3.1, we have

$$\|F_j\|_{a_j}\ \leq\ \|F_i\|_{a_i}\ \|F_{j-i}\|_{a_i}\ \left(1 + \frac{\sqrt{h}}{|a_i - a_j|} e^{-\frac{(a_i-a_j)^2}{h}}\right)^{4n}$$

With our choice of i, we have

$$|a_i - a_j|\ =\ \frac{b-a}{Log\ (4n+1)}\ Log\ \left(\frac{j}{i}\right)\ \geq \frac{b-a}{Log\ (4n+1)}\ Log\ \frac{10}{9}$$

If $0 < h \leq \varepsilon_0(1 + Log\ (4n+1))^{-3}(b-a)^2$, it follows that $\sqrt{h} \leq |a_i - a_j|$ and $e^{-\frac{(a_i-a_j)^2}{h}} \leq \frac{1}{n}$, and therefore

$$\left(1 + \frac{\sqrt{h}}{|a_i - a_j|} e^{-\frac{(a_i-a_j)^2}{h}}\right)^{4n} \leq \left(1 + \frac{1}{n}\right)^{4n} \leq e^4$$

By our induction hypothesis, we have

$$\|F_i\|_{a_i} \leq \varepsilon^{(i+1)/2}$$

Since $i \geq j - i$, and therefore $a_i \leq a_{j-i}$, we have also

$$\|F_{j-i}\|_{a_i} \leq \|F_{j-i}\|_{a_{j-i}} \leq \varepsilon^{(j-i+1)/2}$$

Therefore

$$\|F_j\|_{a_j} \leq \varepsilon^{(i+1)/2}\ \varepsilon^{(j-i+1)/2}\ e^4 \leq \varepsilon^{(j+1)/2}$$

since $\varepsilon \leq \varepsilon_0 \leq e^{-8}$. Therefore (X_j) is proved for all $j \leq k$. The lemma follows directly from (X_k), with another ε_0.

Proof of Proposition 6.1. Let us denote by $f^{(j)}$ the composed symbol $f^{(j)} = f \sharp_h f \ldots \sharp_h f$ (j times). Let us set $a_1 = a + \frac{1}{3}(b-a)$ and $a_2 = a + \frac{2}{3}(b-a)$. Let ε_0 be the constant of Lemma 6.2 (applied to a_2 and b). If $\varepsilon := \|f\|_b \leq \varepsilon_0$ and $h \leq \varepsilon_0(b-a_2)^2(1 + Log\ n)^{-3}$, we have $\|f^{(j)}\|_{a_2} \leq \varepsilon^{(j+1)/2}$ for all $j \leq n$. Therefore, if we set $G_1 = \sum_{k=1}^{n-1}(-1)^k f^{(k)}$, we have

$$(1+f) \sharp_h (1+G) = 1 + (-1)^{n+1} f^{(n)} \qquad \|G\|_{a_2} \leq \sum_1^\infty \varepsilon^{(k+1)/2} \leq 2\varepsilon$$

the last inequality being true if $\varepsilon \leq 1/4$. Now, let $R = (-1)^n f^{(n)}$ and $b_k = a_1 + \frac{a_2 - a_1}{k}$. If $\varepsilon \leq 200^{-8}$ and $0 < h \leq (a_2 - a_1)^2$ (these conditions being added to the previous ones), we shall prove, by induction on k, the following property

$$(Y_k) \qquad \|R^{(k)}\|_{b_k} \leq \frac{\varepsilon^{(n+1)/2}}{k^{4n}}$$

For $k = 1$, (Y_1) follows from Lemma 6.2 since $\|R\|_{b_1} = \|f^{(n)}\|_{a_2} \leq \varepsilon^{(n+1)/2}$. Suppose that $k \geq 2$, and that (Y_j) is proved for all $j < k$. We can find an integer j such that $\frac{k}{2} \leq j \leq \frac{9k}{10}$. By Theorem 3.1, we have, if $0 < h \leq (a_2 - a_1)^2$

$$\|R^{(k)}\|_{b_k} \leq \|R^{(j)}\|_{b_j}\ \|R^{(k-j)}\|_{b_j}\ \left(1 + \frac{a_2 - a_1}{|b_j - b_k|}\right)^{4n}$$

With our choice of j, we have $|b_j - b_k| \geq (a_2 - a_1)\left(\frac{1}{j} - \frac{1}{k}\right) \geq \frac{a_2 - a_1}{9k}$. Therefore

$$\left(1 + \frac{a_2 - a_1}{|b_j - b_k|}\right)^{4n} \leq (1 + 9k)^{4n} \leq (10k)^{4n}$$

By our induction hypothesis

$$\|R^{(j)}\|_{b_j} \leq \frac{\varepsilon^{(n+1)/2}}{j^{4n}} \leq \frac{2^{4n}\varepsilon^{n/2}}{k^{4n}}$$

Since $j \geq k-j$, and therefore $b_j \leq b_{k-j}$, we have also

$$\|R^{(k-j)}\|_{b_j} \;\leq\; \|R^{(k-j)}\|_{b_{k-j}} \;\leq\; \frac{\varepsilon^{(n+1)/2}}{(k-j)^{4n}} \;\leq\; \frac{10^{4n}\varepsilon^{n/2}}{k^{4n}}$$

Therefore

$$\|R^{(k)}\|_{b_k} \;\leq\; \frac{2^{4n}\varepsilon^{(n+1)/2}}{k^{4n}}\;\frac{10^{4n}\varepsilon^{(n+1)/2}}{k^{4n}}(10k)^{4n} \;\leq\; \frac{\varepsilon^{(n+1)/2}}{k^{4n}}$$

if $\varepsilon \leq (200)^{-8}$. Therefore (Y_k) is proved for all k. Since $b_k > a_1$ we have $\|R^{(k)}\|_{a_1} \leq \varepsilon^{(n+1)/2}k^{-4n}$. Therefore $G_1 = \sum_{k=1}^{\infty} R^{(k)}$ satisfies

$$(1-R)\;\sharp_h\;(1+G_1) \;=\; 1 \qquad \|G_1\|_{a_1} \;\leq\; \varepsilon^{(n+1)/2}\sum_1^{\infty}\frac{1}{k^{4n}} \;\leq\; 2\varepsilon^{(n+1)/2}$$

Therefore the function $g = G + G_1 + G\sharp_h G_1$ satisfies $(1+f)\sharp_h(1+g) = 1$ and

$$\|g\|_a \;\leq\; \|G\|_{a_1} \;+\; \|G_1\|_{a_1} \;+\; \|G\|_{a_1}\|G_1\|_{a_1}\left(1+\frac{\sqrt{h}}{a_1-a}e^{-\frac{(a_1-a)^2}{h}}\right)^{4n}$$

If $h \leq (a_1-a)^2/(1+Log\ n)$ and $\varepsilon < (200)^{-8}$, it follows that $\|g\|_a \leq 5\varepsilon = 5\|f\|_b$. The proposition is proved.

7. Exponential of a small operator.

Proposition 7.1. *There exists $\varepsilon_0 > 0$ and $C > 0$ (independent on the dimension n) with the following properties. If $0 < a < b$, if $f(x,\xi,t)$ and $u_0(x,\xi)$ are bounded holomorphic functions in $\Omega_{2n}(b)$ $(n \geq 1)$ such that $\|f\|_b \leq \varepsilon_0$, and f depends continuously on $t \in [0,1]$, if $0 < h \leq \varepsilon_0(1+Log\ n)^{-3}(b-a)^2$ and $0 \leq t \leq 1$, then, there exists a bounded holomorphic function $u(x,\xi,t,h)$ in $\Omega_{2n}(a)$, depending in a C^1 way of $t \in [0,1]$, such that*

$$\frac{\partial u}{\partial t}(.,t,h) \;=\; f(.,t)\;\sharp_h\;u(.,t,h) \qquad u(.,0,h) = u_0(.) \tag{7.1}$$

$$\|u(.,t,h) - u_0(.)\|_a \;\leq\; C\sup_{0\leq t\leq 1}\|f(.,t)\|_b\;\|u_0\|_b \tag{7.2}$$

Proof. For each α and β such that $0 < \alpha < \beta$, let $E(\alpha,\beta)$ be the space of fonctions v_t on $[0,1]$ such that, for all $t \in [0,1]$, v_t is a bounded holomorphic function in $\Omega_{2n}(a(t))$, with $a(t- = \beta - t(\beta-\alpha)$, such that v_t depends continuously on t. The space $E(\alpha,\beta)$ is endowed with the norm

$$\|u\|_{E(\alpha,\beta)} \;=\; \sup_{0\leq t\leq 1}\|u_t\|_{a(t)}$$

Let $a_j = a + j\frac{b-a}{4}$ $(1 \leq j \leq 3)$. We shall write now E instead of $E(a_1,a_2)$. Let us define, formally, for each $u \in E$

$$(T_h u)(.,t) \;=\; \int_0^t f(.,s)\;\sharp_h u(.,s)\;ds$$

The Cauchy problem (7.1)) can be written

$$u \ - \ T_h u = u_0 \tag{7.3}$$

Unfortunately, the operator T_h is not bounded in E, but we shall prove that T_h^{4n+1} is a contractive map from E to E. We have, for all $u \in E$, and for all $k \geq 0$,

$$(T_h^k u)(.,t) \ = \ \int_{\Delta_k(t)} \Big[f(.,t_k) \ \sharp_h \ \ldots \ \sharp_h f(.,t_1) \Big] \ \sharp_h \ u(.,t_1) \ dt_1 \ldots dt_k$$

where $\Delta_k(t) = \{(t_1, \ldots t_k), \quad 0 \leq t_1 \leq \ldots \leq t_k \leq t\}$. By Lemma 6.2, applied to a_3 and b, we have, if $k \leq 4n+1$, $\sup_{0\leq t\leq 1} \|f(t)\|_b \leq \varepsilon_1$ and $0 < h \leq \varepsilon_1 (1 + Log\ n)^{-3} (b - a_3)^2$ (where ε_1 is the constant of lemma 6.2 in this case)

$$\|f(.,t_k) \ \sharp_h \ \ldots \ \sharp_h f(.,t_1)\|_{a_3} \leq \varepsilon^{(k+1)/2} \qquad \varepsilon := \sup_{0\leq t\leq 1} \|f(t)\|_b$$

Therefore, by theorem 3.1, for each $u \in E$, $t \in [0,1]$, $(t_1, t_2, \ldots, t_k) \in \Delta_k(t)$ and $h \leq (a_2 - a_1)^2$, setting $a(t) = a_2 - t(a_2 - a_1)$

$$\|f(.,t_k) \ \sharp_h \ \ldots \ \sharp_h f(.,t_1) \ \sharp_h u(.,t_1)\|_{a(t)} \ \leq \ \varepsilon^{(k+1)/2} \ \|u\|_E \ \left(1 + \frac{a_2 - a_1}{a(t_1) - a(t)}\right)^{4n}$$

$$\ldots \ \leq \ \varepsilon^{(k+1)/2} \ \|u\|_E \left(\frac{2}{(t - t_1)}\right)^{4n}.$$

Therefore, with $k = 4n+1$,

$$\|(T_h^{4n+1} u)(.,t)\|_{a(t)} \ \leq \ \left(2\varepsilon^{1/2}\right)^{4n} \ \|u\|_E \ \int_{\Delta_{4n+1}} \frac{dt_1 \ldots dt_{4n+1}}{|t - t_1|^{4n}}$$

Since the last integral is smaller that 1, T_h^{4n+1} is a contractive map in E, with ratio $\leq 1/2$, if $\varepsilon \leq 1/16$. If u_0 is a bounded holomorphic function in $\Omega_{2n}(b)$ (independent on t), we prove similarly that, if $1 \leq k \leq 4n$ and $0 < h \leq (1 + Log\ n)^{-1} (a_3 - a_2)^2$

$$\|T_h^k u_0\|_E \ \leq \ e^4 \varepsilon^{(k+1)/2} \|u_0\|_b.$$

Therefore the function $\sum_{k=0}^{4n} T_h^k \ u_0$ is in E, with a norm bounded by $2e^4 \|u_0\|_b$ if $\varepsilon < 1/4$. Hence, there exists a function u_h in E, such that

$$(1 - T_h^{4n+1}) \ u_h \ = \ \sum_{k=0}^{4n} T_h^k \ u_0$$

Moreover, we have $\|u_h\|_E \leq 4e^4 \|u_0\|_b$, if n and h satisfy the previous relations. Let us prove that u_h satisfies (7.3). We have $E = E(a_1, a_2) \subset E(a_0, a_1)$ and, by theorem 3.1, T_h^k is a continuous map from E to $E(a_0, a_1)$. Let $v_h := (I - T_h) u_h - u_0$. This function is in $E(a_0, a_1)$ and $(I - T_h^{4n+1}) v_h = 0$. With our choice of parameters, the norm of T_h^{4n+1} in $\mathcal{L}(E(a_0, a_1))$ is $\leq 1/2$, and therefore $v_h = 0$, and u_h is a solution in $E = E(a_1, a_2)$ of (7.3), or of the Cauchy problem (7.1). We have, if $t \in [0,1]$, by (7.3) and Theorem 2.1, if $h \leq (a_1 - a)^2 / (1 + Log\ n)$

$$\|u_h(.,t) - u_0\|_a \ \leq \ \sup_{0\leq s\leq 1} \|f(.,s) \ \sharp_h \ u(.,s)\|_a \ \leq \|u\|_E \ \sup_{0\leq s\leq 1} \|f(.,s)\|_b \ \left(1 + \frac{\sqrt{h}}{a_1 - a} e^{-\frac{(a_1 - a)^2}{h}}\right)^{4n}$$

$$\ldots \;\leq\; e^4\|u\|_E \sup_{0\leq s\leq 1}\|f(.,s)\|_b \;\leq\; 4e^8 \sup_{0\leq s\leq 1}\|f(.,s)\|_b\;\|u_0\|_b$$

The proposition is proved.

8. Proof of theorem 2.2.

Lemma 8.1. *Let $\Phi^{(m)}$ be the function of Propositions 5.1 and 5.2. If $0<a<b$, there exist $C>0$ and $\varepsilon_0>0$ (independent on n), and a function U_m in $\Sigma(a)$ such that*

$$e^{\Phi^{(m)}(.,h,t)} \sharp_h e^{\Phi^{(m)}(.,h,-t)} \;=\; 1+h^m U_m(.,h,t) \tag{8.1}$$

satisfies

$$\frac{1}{n}\,\|U_m(.,h,t)\|_a \;+\; \|\nabla U_m(.,h,t)\|_a \;\leq\; C \tag{8.2}$$

is $nh^m\leq\varepsilon_0$ and $|t|\leq 1$.

Proof. By theorem 4.2, there exists $\Psi^{(m)}(.,h,t)$ in $\Sigma(a)$ such that, if nh^m is small enough,

$$e^{\Phi^{(m)}(.,h,t)} \sharp_h e^{\Phi^{(m)}(.,h,-t)} \;=\; e^{\Psi^{(m)}(.,h,t)} \tag{8.3}$$

Moreover, $\Psi^{(m)}$ has an asymptotic expansion

$$\Psi^{(m)}(.,h,t) \;=\; \sum_{k=0}^{m-1}\psi_k(x,\xi,t)h^k \;+\; h^m\widetilde{\Psi}_m(x,\xi,h,t)$$

where $\|\widetilde{\Psi}_m\|_a\leq Cn$. In order to find the ψ_k, we differentiate (8.3) with respect to t, using (5.5) and (5.13)(5.13) (with t replaced by $-t$). We obtain, for each (x,ξ,t)

$$e^{\Psi^{(m)}(x,\xi,h,t)}\frac{\partial\Psi^{(m)}}{\partial t}(x,\xi,h,t) \;=\; \mathcal{O}(h^m)$$

In other words, $\psi_k=0$ for $k\leq m-1$, and $\Psi^{(m)}=h^m\widetilde{\Psi}^{(m)}$. Therefore, if nh^m is small enough, there exists a function $U_m(.,h,t)$ in $\Sigma(a)$ such that $e^{h^m\widetilde{\Psi}^{(m)}} = 1+h^mU_m(.,h,t)$, and (8.1) is satisfied. The lemma is proved.

End of the proof of Theorem 2.2. Let $a\in]0,b[$. Let a_1, a_2 and a_3 such that $a<a_1<a_2<a_3<b$. Let U_m be the function of Lemma 8.1. By Proposition 6.1, there exist $\varepsilon_0>0$, $C>0$, and a function $V_m(.,h,t)$ in $\Sigma(a_3)$ such that, if $nh^m\leq\varepsilon_0$

$$\big(1+h^mU_m(.,h,t)\big)\,\sharp_h\,\big(1+h^mV_m(.,h,t)\big) \;=\; 1 \qquad \|V_m(.,h,t)\|_{a_3}\;\leq Cn \tag{8.4}$$

Let us set

$$f_m(.,h,t) \;=\; e^{\Phi^{(m)}(.,h,-t)}\,\sharp_h\,(1+h^mV_m(.,h,t))\,\sharp_h\,\Big(h^m e^{\Phi^{(m)}(.,h,t)}S^{(m)}(.,h,t)\Big)$$

where $S^{(m)}$ is the function of Proposition 5.2. If nh^m is small enough, we can write

$$1+h^mV_m(.,h,t) \;=\; e^{h^m\widetilde{V}_m(.,h,t)} \qquad 1+h^mS_m(.,h,t) \;=\; e^{h^m\widetilde{S}_m(.,h,t)}$$

where $\widetilde{V}_m$ and $\widetilde{S}_m$ are in $\Sigma(a_3)$. Therefore

$$1 + f_m(.,h,t) \;=\; e^{\Phi^{(m)}(.,h,-t)} \,\sharp_h\, e^{h^m \widetilde{V}_m(.,h,t)} \,\sharp_h\, e^{\Phi^{(m)}(.,h,t)+h^m \widetilde{S}_m(.,h,t)} \tag{8.5}$$

By Theorem 4.2 and Proposition 4.4, if nh^m is small enough, there exists $F^{(m)}(.,h,t)$ in $\Sigma(a_2)$ such that $1 + f_m(.,h,t) = e^{F^{(m)}(.,h,t)}$. Moreover, $F^{(m)}$ has an asymptotic expansion

$$F^{(m)}(x,\xi,h,t) \;=\; \sum_{k=0}^{m-1} a_k(x,\xi,t)h^k \;+\; h^m \widetilde{F}^{(m)}(.,h,t)$$

where $\|\widetilde{F}^{(m)}(.,t,h)\|_{a_2} \leq Cn$. By Proposition 4.3, the coefficients a_k are the same as if $\widetilde{V}_m$ and $\widetilde{S}_m$ were replaced by 0 in (8.5). Hence, by Lemma 8.1, we have $a_k = 0$ for $k \leq m-1$. Therefore

$$\|f_m(.,h,t)\|_{a_2} \;=\; \|e^{h^m \widetilde{F}^{(m)}(.,h,t)} - 1\|_{a_2} \;\leq\; Cnh^m$$

Therefore, by Proposition 7.1, if nh^m is small enough, there exists an holomorphic function $v_m(.,h,t)$ in $\Omega_{2n}(a_1)$ such that

$$\frac{\partial v_m}{\partial t}(.,h,t) \;=\; -f_m(.,h,t) \,\sharp_h\, v_m(.,h,t) \qquad v_m(.,h,0) \;=\; 1 \tag{8.6}$$

$$\|v_m(.,h,t) \;-\; 1\|_{a_1} \;\leq\; Cnh^m \tag{8.7}$$

where $C > 0$ is independent on n. By (5.6), (8.6), (8.5), (8.1) and (8.4), the function $u_m(.,h,t) = e^{\Phi^{(m)}(.,h,t)} \sharp_h v_m(.,h,t)$ satisfies

$$\frac{\partial u_m}{\partial t}(.,h,t) \;=\; p \,\sharp_h\, u_m(.,h,t) \qquad u_m(.,h,0) \;=\; 1$$

Hence, $u_m(.,h,t)$ is the symbol of the operator $e^{tP_n(h)}$. Now, let us study the symbol

$$w_m(.,h,t) \;=\; e^{-\Phi^{(m)}(.,h,t)} \,u_m(.,h,t) \;=\; e^{-\Phi^{(m)}(.,h,t)} \left[e^{\Phi^{(m)}(.,h,t)} \sharp_h v_m(.,h,t)\right]$$

For that, we use the argument of the first step in the proof of Proposition 5.2, and the function $Q_{\Phi^{(m)}}$ satisfying (5.9), (with $\Phi = \Phi^{(m)}$). With these notations, we can write

$$w_m(X,h,t) \;=\; (\pi h)^{-2n} \int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y,Z)} \; e^{\sigma(Q_{\Phi^{(m)}}(X,Y),Y)} \; v_m(X+Z,h,t) \; dY\,dZ$$

and, by a change of contour which is allowed here

$$w_m(X,h,t) \;-\; 1 \;=\; (\pi h)^{-2n} \int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y,Z)} \left[v_m(X+Z+\frac{ih}{2}Q_{\Phi^{(m)}}(X,Y),h,t) \;-\; 1\right] dY\,dZ$$

By (8.7), there exist $C > 0$ and $\rho > 0$ such that

$$|v_m(X+Z+\frac{ih}{2}Q_{\Phi^{(m)}}(X,Y),h,t) \;-\; 1| \;\leq\; Cnh^m$$

if X is in $\Omega_{2n}(a)$ and Y and Z in $\Omega_{2n}(\rho)$. By Proposition 3.2, it follows that

$$\|w_m(.,h,t) \ - \ 1\|_a \ \leq \ Cnh^m \ \left(1 \ + \ \frac{\sqrt{h}}{\rho} e^{-\frac{\rho^2}{h}}\right)^{4n}$$

If nh^m is small enough, it follows that

$$\|w_m(.,h,t) \ - \ 1\|_a \ \leq \ Ce^4 nh^m \ \leq \ \frac{1}{2}$$

Therefore we can write $w_m = e^{h^m \widetilde{w}_m}$, where $\widetilde{w}_m$ is in $\Sigma(a)$. The symbol of $e^{tP_n(h)}$ can be written $e^{\Phi^{(m)}(x,\xi,h,t)+h^m \widetilde{w}_m(x,\xi,h,t)}$. The Theorem 2.2 is proved.

9. Periodic symbols and thermodynamic limits.

At each point j of the lattice $\mathbf{Z}$, we consider a particle A_j described, when there is no interaction, by an hamiltonian $A(x,\xi) \in C^\infty(\mathbf{R}^{2k})$, where $k \geq 1$ is a fixed integer. The interaction between A_j and A_{j+1} will be described by a function $B(x,\xi,y,\eta) \in C^\infty(\mathbf{R}^{4k})$, ($A$, B and k are independent on j).

We assume that A (resp. B) extends to a bounded holomorphic function in $\Omega_{2k}(a)$, (resp. $\Omega_{4k}(a)$), defined as in (1.4). We assume also that there exist a lattices Γ in $\mathbf{R}^{2k}$, such that $\mathbf{R}^{2k}/\Gamma$ is compact, ans such that A (resp. B) is periodic with respect to Γ (resp. $\Gamma \times \Gamma$). We assume also that A and B are real for real (x,ξ), and that $B(y,\eta,x,\xi) = B(x,\xi,y,\eta)$. We shall write now the hamiltonian describing the system of particles A_j ($|j| \leq n$) with interaction. If there is no interaction between A_n and A_{-n}, this hamiltonian can be written, setting $X = (x,\xi)$ and denoting by $(X^{(-n)}, \ldots, X^{(n)})$ the variable of $\mathbf{R}^{2k(2n+1)}$ (with $X^{(j)} = (x^{(j)}, \xi^{(j)}) \in \mathbf{R}^{2k}$)

$$\widetilde{p}_n(X) \ = \ \sum_{j=-n}^{n} A(X^{(j)}) \ + \ \sum_{j=-n}^{n-1} B(X^{(j)}, X^{(j+1)}) \tag{9.1}$$

If there is an interaction between A_n and A_{-n} (case some formulas will be simpler), the hamiltonian becomes, setting $X^{(n+1)} = X^{(-n)}$,

$$p_n(X) \ = \ \sum_{j=-n}^{n} \left(A(X^{(j)}) \ + B(X^{(j)}, X^{(j+1)})\right) \tag{9.1}$$

We denote by $P_n(h)$ and $\widetilde{P}_n(h)$ the $h-$pseudodifferential operator in $L^2(\mathbf{R}^{k(2n+1)})$, associated to the symbols p_n and $\widetilde{p}_n$ by the Weyl calculus (2.5). We see easily that the sequences (p_n) and $(\widetilde{p}_n)$ are in $\Sigma(a)$.

We want to associate, to the sequence of operators $P_n(h)$, a notion of thermodynamic limit. A classical possible definition of a thermodynamic limit is the limit of (1.7), if it exists. Unfortunately, $e^{-tP_n(h)}$ is not here of trace class. By theorem 2.2, we know that $e^{-tP_n(h)}$ is an $h-$pseudodifferential operator associated, by the Weyl calculus, to a symbol of the form $e^{-q_n(x,\xi,t,h)}$.

$$e^{-tP_n(h)} \ = \ Op_h\left(e^{-q_n(.,t,h)}\right) \tag{9.3}$$

and we have also a symbol $\widetilde{q}_n(x,\xi,t,h)$ with similar properties for $\widetilde{P}_n(h)$. We see easily that q_n and $\widetilde{q}_n$ are periodic with the lattice $\Gamma_n = (\Gamma)^{2n+1}$ of $\mathbf{R}^{2k(2n+1)}$.

It is natural to associate to a $h-$pseudodifferential operator $A_n(h)$ associated, by the Weyl calculus, to a symbol $a_n(x,\xi)$ which is periodic with respect to Γ_n, not the usual trace, but the following one

$$T\big(A(h)\big) \;=\; \frac{1}{|E_n|}\int_{E_n} a_n(x,\xi)\; dxd\xi \tag{9.4}$$

where E_n is a fundamental domain of Γ_n. If p and q are C^∞, periodic with respect to Γ_n, and if $P(h)$ and $Q(h)$ are the corresponding $h-$pseudodifferential operators, we have

$$T(P(h)\circ Q(h)) \;=\; \frac{1}{|E_n|}\int_{E_n} p(x,\xi)q(x,\xi)\; dxd\xi \;=\; T(Q(h)\circ P(h)) \tag{9.5}$$

If, moreover, $P(h)$ is a positive self-adjoint operator, we have

$$|T(P(h)\circ Q(h))| \;\leq\; T(P(h))\;\|Q(h)\| \tag{9.6}$$

where $\|Q(h)\|$ is the norm of $Q(h)$ as an operator in $L^2(\mathbf{R}^n)$.

This notion of trace is often used in the literature of C^*-algebras. Therefore, it is natural to define, for our sequence $P_n(h)$, the thermodynamic limit as the limit, if it exists, of

$$\Lambda_n(t,h) \;=\; \frac{1}{2n+1} Log\Big[T\big(e^{-tP_n(h)}\big)\Big] \tag{9.7}$$

where T is the 'trace' defined in (9.4). We define also a similar sequence $\widetilde{\Lambda}_n(t,h)$ for the operator $\widetilde{P}_n(h)$.

The main results of this section are the theorems 9.1 (existence of the thermodynamic limit) and 9.3 (asymptotic expansion in powers of h).

Theorem 9.1. *With the previous notations, for each $h>0$ and $t>0$, the sequence $\Lambda_n(t,h)$ defined in (9.7) and its analogue $\widetilde{\Lambda}_n(t,h)$ for the operator $\widetilde{P}_n(h)$ have a common limit $\Lambda(t,h)$ when $n\to\infty$. There is a constant $C>0$ such that*

$$|\Lambda_n(t,h) \;-\; \Lambda(t,h)| \;+\; |\widetilde{\Lambda}_n(t,h) \;-\; \Lambda(t,h)| \;\leq\; \frac{C}{n} \qquad \forall h\in]0,1], \qquad \forall t\in[0,1] \tag{9.8}$$

The proof relies on the following Lemma.

Lemma 9.2. *There exists $C>0$ such that, for each m and $n\geq 2$, for each h and t in $]0,1]$*

$$\Delta_{mn}(t,h) \;:=\; |(2m+2n+1)\widetilde{\Lambda}_{m+n}(t,h) \;-\; (2m+1)\widetilde{\Lambda}_m(t,h) \;-\; (2n+1)\widetilde{\Lambda}_n(t,h)| \;\leq\; C \tag{9.9}$$

Proof of the Lemma. For $m \geq 1$ and $n \geq 1$, let us denote by $(X,Y) = (X^{(-m-n)}, \ldots, X^{(m+n)}, Y)$ the variable of $\mathbf{R}^{2k(2m+2n+2)}$. We define two functions in $\mathbf{R}^{2k(2m+2n+2)}$ by

$$a_{mn}(X,Y) \;=\; \widetilde{p}_{m+n}(X) \;+\; A(Y)$$

$$b_{mn}(X,Y) \;=\; B(X^{(-m-1)}, X^{(-m)}) \;+\; B(X^{(m)}, X^{(m+1)}) - B(X^{(-m-1)}, Y) \;-\; B(Y, X^{(m+1)})$$

We denote by $A_{mn}(h)$ and $B_{mn}(h)$ the $h-$operators in $L^2(\mathbf{R}^{k(2m+2n+2)})$ associated to the symbols $a_{mn}(x,\xi)$ and $b_{mn}(x,\xi)$, and by $P_0(h)$ the $h-$operator associated to $A(y,\eta)$. We see easily that

$$(2m+2n+1)\widetilde{\Lambda}_{m+n}(t,h) \;=\; Log\left[T\left(e^{-tA_{mn}(h)}\right)\right] \;-\; Log\left[T\left(e^{-tP_0(h)}\right)\right]$$

$$(2m+1)\widetilde{\Lambda}_m(t,h) + (2n+1)\widetilde{\Lambda}_n(t,h) = \; Log\left[T\left(e^{-tA_{mn}(h)+tB_{mn}(h)}\right)\right]$$

If we set $F(\theta) = Log\Big(T\left(e^{-tA_{mn}(h)+\theta tB_{mn}(h)}\right)\Big)$, we can write, for the left hand side $\Delta_{mn}(t,h)$ of (9.9)

$$\Delta_{mn}(t,h) \;\leq\; |F(1)-F(0)| \;+\; \left|Log\; T\left(e^{-tP_0(h)}\right)\right|$$

Using (9.5), we obtain

$$|F(1)-F(0)| \leq \sup_{\theta\in[0,1]} |F'(\theta)| \;=\; \sup_{\theta\in[0,1]} \frac{|T\left(tB_{mn}(h)e^{-tA_{mn}(h)+t\theta B_{mn}(h)}\right)|}{|T\left(e^{-tA_{mn}(h)+t\theta B_{mn}(h)}\right)|}$$

By (9.6), since $e^{-tA_{mn}(h)+t\theta B_{mn}(h)}$ is a positive self-adjoint operator, we have

$$|T\left(C_{mn}(h)e^{-tA_{mn}(h)+t\theta B_{mn}(h)}\right)| \;\leq\; \|B_{mn}(h)\|\; |T\left(e^{-tA_{mn}(h)+t\theta B_{mn}(h)}\right)|$$

Since the norm of $B_{mn}(h)$ is bounded independently on m, n and $h \in]0,1]$, the Lemma is proved.

Proof of Theorem 9.1. Since the sequence $\widetilde{\Lambda}_n(t,h)$ satisfies (9.10), it follows classically (cf. Helffer-Sjöstrand [3], lemma 2.5) that $\widetilde{\Lambda}_n(t,h)$ has a limit $\Lambda(t,h)$ when $n \to \infty$, and that

$$|\widetilde{\Lambda}_n(t,h) \;-\; \Lambda(t,h)| \;\leq\; \frac{C}{n} \qquad \forall n \geq 1 \quad \forall h \in]0,1]$$

The proof of Lemma 9.2 shows also that, for some constant $C > 0$

$$|\Lambda_n(t,h) \;-\; \widetilde{\Lambda}_n(t,h)| \;\leq\; \frac{C}{n} \qquad \forall n \geq 1$$

Therefore (9.8) is proved.

Let us prove now that $\Lambda(t,h)$ has an asymptotic expansion in powers of h. In order to define the first term of this expansion, we need the following integral operator $S_0(t)$ in $L^2(E)$ (where E is a fundamental domain of the lattice Γ), defined by

$$(9.10) \qquad (S_0(t)u)(X) \;=\; \frac{1}{|E|} \int_E e^{-\frac{t}{2}(A(X)+2B(X,Y)+A(Y))}\; u(Y)\; dY \qquad \forall u \in L^2(E)$$

This operator is self-adjoint, of trace class. The norm $\|S_0(t)\|$ is a simple eigenvalue of $S_0(t)$ (by Krein-Rutman theorem), and all other eigenvalues have a modulus strictly smaller than $\|S_0(t)\|$.

Theorem 9.3. *The thermodynamic limit $\Lambda(t,h)$ of Theorem 9.1 has an asymptotic expansion when $h \to 0$*

$$\Lambda(t,h) \ \sim \ \sum_{j\geq 0} \gamma_j(t) h^j \tag{9.11}$$

where $\gamma_j(t)$ are real numbers, and $\gamma_0(t) = Log(\|S_0(t)\|)$, where $S_0(t)$ defined in (9.10).

If $B = 0$, (when there is no interaction), the 'classical' thermodynamic limit $\gamma_0(t)$ is given by

$$\gamma_0(t) \ = \ Log \ \left(\frac{1}{|E_0|} \int_{E_0} e^{-tA(X)} dX \right)$$

Formal construction of the $\gamma_j(t)$. By theorem 2.2, we know that $q_n(x,\xi,t,h)$, (the symbol satisfying (9.3)), has the asymptotic expansion (2.7) with coefficients $E_n^{(j)}(x,\xi,t)$. By the construction of section 5, we know that $E_n^{(0)} = t\widetilde{p}_n$, $E_n^{(1)} = 0$, and that $E_n^{(2)}$ is given like in (5.4), where p is replaced by p_n defined in (9.2). By (5.4) and (9.2), we see that there is a function $f^{(2)} \in C^\infty(\mathbf{R}^{8k} \times \mathbf{R})$ such that

$$E_n^{(2)}(x,\xi,t) \ = \ \sum_{j=1}^{n} f^{(2)}(X^{(j)}, X^{(j+1)}, X^{(j+2)}, X^{(j+3)}, t)$$

We have set $X^{(n+j)} = X^{(-n+j-1)}$ for $j \geq 1$. As a function of t, $f^{(2)}$ is a polynomial. As a function of X, it is in $S(a)$ (holomorphic with bounded derivatives in $\{X \in \mathbf{C}^{8k}, |Im\ X|_\infty < b\}$ for all $b \in]0,a[$). More generally, if we follow the induction in the proof of Proposition 5.1, we see that, for each integer m, there is some integer $d_m \geq 1$, and a function $F^{(m)}(.,t,h) \in C^\infty(\mathbf{R}^{2k(1+d_m)})$ such that

$$q_n^{(m)}(x,\xi,t,h) \ := \ \sum_{j=0}^{m} E_j(x,\xi,t) h^j \ = \ \sum_{j=-n}^{n} F^{(m)}(X^{(j)}, \ldots, X^{(j+d_m)}, t, h) \tag{9.12}$$

For example, we have $d_0 = d_1 = 1$, $d_2 = 3$. The approximation $q_n^{(m)}$ can be written in the form (9.12), but not in the unique way. For $m = 0$, $F^{(0)}$ (which is independent on h) can be written

$$F^{(0)}(X,t) \ = \ t \left(\frac{A(X^{(1)})}{2} \ + \ B(X^{(1)}, X^{(2)}) \ + \ \frac{A(X^{(2)})}{2} \right)$$

For m arbitrary and $h = 0$, we can take, for all X in $\mathbf{R}^{2k(1+d_m)}$

$$F^{(m)}(X^{(j)}, \ldots, X^{(j+d_m)}, t, 0) \ = \ \frac{t}{d_m + 1} \ \sum_{j=1}^{1+d_m} A(X^{(j)}) \ + \ \frac{t}{d_m} \ \sum_{j=1}^{d_m} B(X^{(j)}, X^{(j+1)}) \tag{9.13}$$

Using this form of the coefficients $E_j(X,t)$, we shall define, for each m, an integral operator whose greatest eigenvalue will have, up to the order m, the same asymptotic expansion in h that the

exponential of the thermodynamic limit. For that, we introduce the operator $S_m(t,h)$ in $L^2(E^{d_m})$ (where E is a fondamental domain of Γ) defined, for all $u \in L^2(E^{d_m})$ by

$$(S_m(t,h)u)(Y) \;=\; \frac{1}{|E|^{d_m}} \int_{E^{d_m}} e^{-\Phi_m(X,Y,t,h)} u(X)\, dX \tag{9.14}$$

where we set $X = (X^{(1)}, \ldots, X^{(d_m)})$, $Y = (Y^{(1)}, \ldots, Y^{(d_m)})$, $X^{(j+d_m)} = Y^{(j)}$ and:

$$\Phi_m(X,Y,t,h) \;=\; \sum_{j=1}^{d_m} F^{(m)}(X^{(j)}, \ldots, X^{(j+d_m)}, t, h) \tag{9.15}$$

We shall prove that $S_m(t,0)$ is almost isospectral to the power $S_0(t)^{d_m}$ of $S_0(t)$ defined in (9.10). We remark that, for each integer n, $S_0(t)$ satisfies

$$Tr(S_0(t)^n) \;=\; \frac{1}{|E|^n} \int_{E^n} e^{-t\Psi_n(X)} dX$$

where Tr is the usual trace of operators, (here in $L^2(E^n)$), and

$$\Psi_n(X) \;=\; B(X^{(n)}, X^{(1)}) \;+\; \sum_{j=1}^{n} A(X^{(j)}) \;+\; \sum_{j=1}^{(n-1)} B(X^{(j)}, X^{(j+1)}) \qquad \forall X \in \mathbf{R}^{2nk}$$

We remark also that $S_m(t,h)$ defined in (9.14) is of trace class and that, for each integer q, we have

$$Tr(S_m(h)^q) \;=\; \frac{1}{|E|^{qd_m}} \int_{E^{qd_m}} e^{-\theta_q(X,t,h)} dX \tag{9.16}$$

where, setting now $X^{(qd_m+j)} = X^{(j)}$ for $j \geq 1$

$$\theta_q(X,t,h) \;=\; \sum_{j=1}^{qd_m} F^{(m)}(X^{(j)}, \ldots, X^{(j+d_m)}, t, h) \tag{9.17}$$

It follows from (9.13) that $\theta_q(X,t,0) = t\Psi_{qd_m}(X)$ for all X in $\mathbf{R}^{2kqd_m}$, and therefore $S_m(t,0)$ is related to the operator $S_0(t)$ of (9.10), for each integer q, by

$$Tr(S_m(t,0)^q) \;=\; Tr(S_0(t)^{qd_m})$$

Therefore, $S_m(t,0)$ has $\|S_0(t)\|^{d_m}$ as a simple eigenvalue, and all others eigenvalues have a modulus strictly smaller than $\|S_0(t)\|^{d_m}$. Since $S_m(t,h)$ depends analytically on h, by the classical theory of perturbation of a single eigenvalue, $S_m(t,h)$ has an eigenvalue $\lambda_m(t,h)$ which depends analytically on h, such that $\lambda_m(t,0) = \|S_0(t)\|^{d_m}$. For h small enough, $Log\ \lambda_m(t,h)$ is well defined. Since m is arbitrary, the theorem 9.3 will follow from

Lemma 9.4. *With the previous notations, we have, for each integer m, and for h small enough*

$$\Lambda(t,h) \;=\; \frac{1}{d_m} Log\ (\lambda_m(t,h)) \;+\; \mathcal{O}(h^m) \tag{9.18}$$

Proof. First step. It is natural to approximate $\Lambda_n(t,h)$ (defined in (9.7)), by

$$\Lambda_n^{(m)}(t,h) \;=\; \frac{1}{2n+1}\; Log\; \left[\frac{1}{|E_n|}\quad \int_{E_n} e^{-q_n^{(m)}(x,\xi,t,h)}\; dxd\xi\right]. \tag{9.19}$$

where $q_n^{(m)}$ is defined in (9.12). We shall prove that

$$\left|\Lambda_n^{(m)}(t,h)\;-\frac{1}{d_m}Log\;(\lambda_m(t,h))\right|\;\leq\;\frac{C}{n} \tag{9.20}$$

for some constant C. Following the proof of Lemma 9.2, we see that $\Lambda_n^{(m)}(t,h)$ satisfies an inequality analogous to (9.9). By the Lemma 2.5 of [3], $\Lambda_n^{(m)}(t,h)$ has a limit $\mu(t,h)$ when $n\to+\infty$ and, for some constant $C>0$, we have $|\Lambda_n^{(m)}(t,h)-\mu(t,h)|\leq C/n$. Let us show that $\mu(t,h)=(1/d_m)Log\;\lambda_m(t,h))$. By (9.18), (9.12), (9.17) and (9.16), we can write, when $2n+1=kd_m$

$$\Lambda_n^{(m)}(t,h)\;=\;\frac{1}{kd_m}\;Log\;\left[\;Tr\;(S_m(h)^k)\right]\qquad\qquad 2n+1=kd_m.$$

Since $S_m(h)$ is of trace class, since $\lambda_m(t,h)$ is a single eigenvalue and since all the other eigenvalues have, for h small enough, a modulus strictly smaller than $|\lambda_m(t,h)|$, it follows that

$$\lim_{k\to+\infty}\frac{1}{k}Log\left(Tr\;(S_m(h)^k)\right)\;=\;Log\;(\lambda_m(t,h))$$

and, therefore, $\mu(t,h)=(1/d_m)Log\;\lambda_m(t,h))$, and (9.20) is proved.

Second step. Theorem 2.2 shows that there is $K_m>0$ and $\varepsilon_m>0$, independent on n, such that, for each h, n and t satisfying $0\leq t\leq 1$, $0<h\leq 1$ and $nh^m\leq\varepsilon_m$, and for real (x,ξ), we have

$$|q_n(x,\xi,t,h)\;-\;q_n^{(m)}(x,\xi,t,h)|\;\leq\;K_m nh^m \tag{9.21}$$

where q_n is the symbol satisfying (9.3) and $q_n^{(m)}$ is its approximation, defined in (9.12). Since we used the Weyl calculus and since $e^{-tP_n(h)}$ is self-adjoint, the function $q_n(x,\xi,t,h)$ is real-valued. Therefore, it follows from (9.21) that, for some constants $\varepsilon_m>0$ and $K_m>0$

$$nh^m\;\leq\;\varepsilon_m\;\Longrightarrow|\Lambda_n(h)\;-\;\Lambda_n^{(m)}(h)|\;\leq\;K_m h^m \tag{9.22}$$

For each $h\in]0,1]$, let $n=n(h)$ be the integer part of ε_m/h^m, where ε_m is a constant such that (9.22) is valid. Then we have

$$|\Lambda(t,h)\;-\;\frac{1}{d_m}Log\;\lambda_m(t,h))|\;\leq\;|\Lambda(t,h)\;-\;\Lambda_n(h)|\;+\;|\Lambda_n(h)\;-\;\Lambda_n^{(m)}(h)|\;+\;\ldots$$

$$\ldots\;+\;|\Lambda_n^{(m)}(h)\;-\;\frac{1}{d_m}Log\;\lambda_m(t,h))|$$

Using (9.8), (9.22) and (9.20), it follows that

$$|\Lambda(t,h)\;-\;\frac{1}{d_m}Log\;\lambda_m(t,h))|\;\leq\;K_m h^m\;+\;\frac{2C}{n}\;\leq\;\left(K_m+\frac{4C}{\varepsilon_m}\right)h^m$$

The lemma is proved, and Theorem 9.3 follows directly.

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