Perron-Frobenius spectrum for random maps and its approximation

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Abstract. To study the convergence to equilibrium in random maps we developed the spectral theory of the corresponding transfer (Perron-Frobenius) operators acting in a certain Banach space of generalized functions. The random maps under study in a sense fill the gap between expanding and hyperbolic systems since among their (deterministic) components there are both expanding and contracting ones. We prove stochastic stability of the Perron-Frobenius spectrum and developed its finite rank operator approximations by means of a "stochastically smoothed" Ulam approximation scheme. A counterexample to the original Ulam conjecture about the approximation of the SBR measure and the discussion of the instability of spectral approximations by means of the original Ulam scheme are presented as well.

1 Introduction

Let $\{T_i\}$ be a collection of nonsingular maps from a d-dimensional smooth manifold X into itself with a metrics ρ on it, and let $\{p_i\}$ be a collection of nonnegative integers, such that $\sum_i p_i = 1$. A *random dynamical system* on the space X is a stationary stochastic process $T_1, T_2, \cdots : X \to X$ (i.e., with values in a space of maps), see for instance [17]. There are several approaches in further detalization of this object and in the sequel we shall use the following Markov one.

By a *random map* $\overline{T}$ we shall mean the Markov random process on X given by the following family of transition probabilities $\mathcal{P}(x,A) := \sum_i p_i \mathbf{1}_{T_i^{-1}(A)}(x)$ from a point $x \in X$ to a subset $A \subseteq X$. In other words, on every time step the map T_i is chosen from the collection $\{T_i\}$ independently from the previous choices with the probability given by the distribution $\{p_i\}$.

In the literature (especially physical) the above defined random map is often called by the *iterated function system* (with probabilities). Naturally, a pure deterministic setup was studied as well. This can be done as follows. According to the given collection of maps $\{T_i\}$ one can define a new multivalued map $\tilde{T} : X \to X$ as $\tilde{T}x := \cup_i T_i x$. Assuming that the maps T_i are continuous and strictly contracting one can show [13] that the multivalued map $\tilde{T}$ possesses a global attractor $X_{\tilde{T}} \subset X$, namely the Hausdorff distance between the sets $\tilde{T}^n Y$ and $X_{\tilde{T}}$ decreases exponentially fast for any closed nonempty set $Y \subset X$.

It is worth note that our aim in this paper is not to study the most general setup (for example, one can consider an infinite (continual) collection of maps $\{T_i\}$, while the distribution $\{p_i\}$ might be place dependent (on the space variable), but rather to give a deeper analysis of dynamical and statistical properties of these systems related to the convergence to equilibrium, i.e. on spectral problems of the corresponding transfer operators practically not studied in the literature (except [1]). Therefore we shall not consider well studied problems of the dimensional analysis of invariant sets and measures. The reader can find results and further references of this type, e.g. in [9].

One can always realize the random map under study as a deterministic one on the extended phase space. Denote by $\bar{\Omega}$ the space of one-side sequences $\bar{\omega} := \{\omega_1, \omega_2, \ldots\} \in \bar{\Omega}$, where each ω_i belongs to the set of indices of the collection T_i. On the space $\bar{\Omega}$ one defines the left-shift map $\sigma : \bar{\Omega} \to \bar{\Omega}$ according to the rule $(\sigma\bar{\omega})_i := \omega_{i+1}$. The topology in the space $\bar{\Omega}$ is defined as the direct product of discrete topologies, acting on the set of indices, and a Borel measure μ_p on $\bar{\Omega}$ is defined as the product of distributions $\{p_i\}$ on the set of indices. Naturally there exists an

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one-to-one correspondence between the sequences of maps T_i and the elements of the space $\bar{\Omega}$. On the extended space $\bar{\Omega} \times X$ one can define a new map – a skew product upon the left-shift map: $\tilde{T}(\bar{\omega}, x) := (\sigma\bar{\omega}, T_{\omega_1} x)$.

Let us give a few simple examples of random maps, demonstrating that even in the simplest cases the dynamics of random maps might be rather nontrivial. For the sake of simplicity in all our examples both here and in the sequel the collection of maps will consist of only two maps T_1, T_2 from the unit interval into itself and thus the corresponding distribution is described by the one parameter $p := p_1$.

Example 1.1 *Pure contractive maps:* $T_1(x) := x/2$, $T_2(x) := x/2 + 1/2$.

According to [13] (see also the next section) for each value of the parameter $0 < p < 1$ the corresponding random map $\overline{T}$ has the unique invariant measure, whose support typically (when $p \neq 1/2$) is a Cantor set.

Example 1.2 *A mixed case:* $T_1(x) := 1 - |2x - 1|$, $T_2(x) := x/2$.

In distinction to the previous example, depending on the choice of the parameter p the properties of the random map $\overline{T}$ differ qualitatively. Namely for $0 \leq p < \frac{1}{2}$ the random map possesses the unique invariant measure concentrated at 0. For $p > \frac{1}{2}$ the invariant measure becomes non unique but there exists a unique absolutely continuous invariant one, whose density $h_{\overline{T}}$ is such that $h_{\overline{T}}|I_k = \gamma_k$ for any $k = 1, 2, \ldots$ and intervals $I_k := (2^{-(k-1)}, 2^{-k}]$. However the constants γ_k are bounded on k while $2/3 < p < 1/2$ and go to infinity with k for $0 \leq p < 2/3$.

The above examples are typical for statistical problems related to random maps and in the next section we shall discuss separately properties of random maps expanding on average and contracting on average and thus we postpone the proof of the above claims till all needed technicalities will be introduced. It is worth note that our results related to the expanding on average random maps are very similar to those known for deterministic piecewise expanding maps [1] (see also results about the convergence to the Sinai-Bowen-Ruelle (**SBR**) measure for multidimensional expanding on average random maps in [16, 6]). Therefore in the corresponding sections we mainly demonstrate the similarities between these two types of systems. On the other hand, spectral analysis of the contracting on average random maps is completely new. In fact our interest in this type of systems is due to the analysis of spectral properties of Anosov maps in [4], where the presence of stable foliations relates the situation to our case. Some of the methods and ideas used in this paper were originated from the construction in [4] needed to study the behavior of the transfer operators 'along the stable foliation' and are based technically on the establishing of the so called Lasota-Yorke type inequalities. The nature of the system under study gives some advantages compare to the Anosov maps, in particular, the functional Banach spaces in our case do not depend on the finite structure of the map, the spectrum stability results are proven for a much broader class of random perturbations, and a more direct approximation scheme is elaborated.

The paper is organized as follows. In the following two sections (2 and 3) we introduce the notions of contractive and expanding on average random maps and discuss some of their basic properties slightly generalizing known results about them. The main results of the paper are described in section 4, where we construct and study the Perron-Frobenius spectrum for random maps. We prove also stochastic stability of isolated eigenvalues in this spectrum and construct by means of a special random smoothing two schemes of their finite rank operator approximation. To some extent these schemes generalize the well known Ulam approximation scheme [22] (see also discussion of its realization in [1, 7, 11, 18]). We discuss in detail why the spectrum approximation by means of the original Ulam scheme is not stable in the case of random maps (see also [1, 4]). Since the original Ulam conjecture about the approximation of the **SBR** measure (the leading element of the spectrum) still holds in our setting, in Lemma 4.13 we construct the first (to the best of our knowledge) counterexample to this conjecture. Note however that the map in this counterexample is not only discontinuous, but the discontinuity occurs in a periodic turning point (compare to instability results for general random perturbations in [1]).

2 Contracting on average random maps

An important and well known case of random maps corresponds to the situation when all the maps T_i are continuous and their *contracting constants*

$$\Lambda_{T_i} := \sup_{x,y\in X} \frac{|T(x)-T(y)|}{\rho(x,y)} \le \Lambda < 1.$$

Only the list of literature dedicated to various theoretical and applied aspects of the theory of this class of random maps would fill several pages, therefore we only give a reference to one of the first (and giving a very essential contribution) works in this field – [13], to the monograph [9], and to one of the recent publications [10], where the reader can find more references to recent results. Note that these papers are dedicated mainly to questions related to dimensional and multifractal properties of invariant sets and measures which we shall not touch in this work.

We start the analysis of ergodic properties of these systems using a weaker assumption, namely that the random map $\overline{T}$ is contracting on average.

Definition 2.1 We shall say that the random map $\overline{T}$ is *contracting on average* if its *contracting constant*

$$\Lambda_{\overline{T}} := \sum_i p_i \Lambda_{T_i} < 1.$$

Let $\mathcal{M}$ be the space of probability measures on X. Then the Markov operator associated with the random map $\overline{T}$ for each probabilistic measure $\mu \in \mathcal{M}$ can be written as

$$\overline{T}\mu := \sum_i p_i \mu \circ T_i^{-1}.$$

Following the standard scenario we introduce in the space $\mathcal{M}$ the special metrics (often called Hutchinson metrics) [13]:

$$\rho_H(\mu,\nu) := \sup\left\{\int h\,d\mu - \int h\,d\nu,\ h \in \mathbf{C}^0(X,\mathbb{R}),\ |h(x)-h(y)| \le \rho(x-y),\ \forall x,y \in X\right\},$$

where ρ is the metrics on our original phase space X. It is well known (see, for example, [13]) that the pair $(\mathcal{M},\rho_H)$ defines a compact metric space.

Lemma 2.1 *Let $h : X \to \mathbb{R}$ be a continuous function and let $\mu \in \mathcal{M}$. Then*

$$\int h\,d(\overline{T}\mu) = \sum_i p_i \int h\circ T_i\,d\mu.$$

Proof. For each continuous function $h : X \to \mathbb{R}$ there is a sequence of piecewise constant approximating functions h_n converging uniformly on X. Therefore

$$\begin{aligned}\int h_n\,d(\overline{T}\mu) &= \sum_i p_i \int h_n\,d(\mu\circ T_i^{-1}) \\ &= \sum_i p_i \int_{T_i X} h_n\,d(\mu\circ T_i^{-1}) = \sum_i p_i \int h_n\circ T_i\,d\mu.\end{aligned}$$

On the other hand, $\int h_n\,d(\overline{T}\mu) \to \int h\,d(\overline{T}\mu)$ as $n\to\infty$, while for each pair i,n the function $h_n\circ T_i$ is piecewise constant. Thus the sequence of functions $\{h_n\circ T_i\}_n$ converges uniformly on n to a function $h\circ T_i$ which yields the convergence

$$\sum_i p_i \int h_n\circ T_i\,d\mu \to \sum_i p_i \int h\circ T_i\,d\mu.$$

$\blacksquare$

The following result is a simple generalization of the well known Hutchinson Theorem [13].

Lemma 2.2 *$\rho_H(\overline{T}\mu, \overline{T}\nu) \le \Lambda_{\overline{T}} \cdot \rho_H(\mu,\nu)$ for any measures $\mu,\nu \in \mathcal{M}$. In particular contraction on average yields the strict ergodicity of the random map $\overline{T}$.*

Proof. Introduce the notation

$$\mathcal{H} := \left\{h \in \mathbf{C}^0(X,\mathbb{R}),\ |h(x)-h(y)| \le \rho(x,y),\ \forall x,y \in X\right\}.$$

Then

$$\begin{aligned}\rho(\overline{T}\mu, \overline{T}\nu) &= \sup\left\{\int h\, d(\overline{T}\mu) - \int h\, d(\overline{T}\nu),\ h \in \mathcal{H}\right\} \\ &= \sup\left\{\sum_i p_i \int h \circ T_i\, d\mu - \sum_i p_i \int h \circ T_i\, d\nu,\ h \in \mathcal{H}\right\}.\end{aligned}$$

Consider a function $\tilde{h} := \frac{1}{\Lambda_{\overline{T}}} \sum_i p_i h \circ T_i$. For each pair of points $x, y \in X$ we have

$$\begin{aligned}|\tilde{h}(x) - \tilde{h}(y)| &\le \frac{1}{\Lambda_{\overline{T}}} \sum_i p_i |h \circ T_i(x) - h \circ T_i(y)| \\ &\le \frac{1}{\Lambda_{\overline{T}}} \sum_i p_i \rho(T_i(x), T_i(y)) \le \frac{1}{\Lambda_{\overline{T}}} \sum_i p_i \Lambda_{T_i} \cdot \rho(x,y) = \rho(x,y).\end{aligned}$$

Hence $\tilde{h} \in \mathcal{H}$. Introducing another set of functions

$$\tilde{\mathcal{H}} := \left\{\tilde{h} \in \mathcal{H}:\ \exists h \in \mathcal{H}:\ \tilde{h} = \frac{1}{\Lambda_{\overline{T}}} \sum_i p_i h \circ T_i\right\},$$

we can rewrite the distance between the images of the measures as follows

$$\rho_H(\overline{T}\mu, \overline{T}\nu) = \sup\left\{\frac{1}{\Lambda_{\overline{T}}} \int \tilde{h}\, d\mu - \frac{1}{\Lambda_{\overline{T}}} \int \tilde{h}\, d\nu:\ \tilde{h} \in \tilde{\mathcal{H}}\right\}.$$

Now, since $\tilde{\mathcal{H}} \subset \mathcal{H}$, we come to the desired estimate

$$\rho_H(\overline{T}\mu, \overline{T}\nu) \le \Lambda_{\overline{T}} \cdot \rho_H(\mu,\nu),$$

and thus the contraction on average yields the uniform contraction in the space of measures. ∎

Note that the example 1.1 satisfies the conditions of Lemma 2.2, while for the example 1.2 the conditions of Lemma 2.2 hold only when $0 < p < 1/3$. Indeed,

$$\sum_i p_i \Lambda_i = 2p + \frac{1}{2}(1-p) = \frac{3}{2}p + \frac{1}{2} < 1.$$

On the first sight it seems that the continuity of the maps T_i was not used in the proof, however it plays a very important role in it. One can easily construct an example when the absence of this property leads to the nonuniqueness of the invariant measure.

Example 2.3 $T_1(x) := \frac{x}{2}\mathbf{1}_{[0,1/2]}(x) + \frac{x+1}{2}\mathbf{1}_{(1/2,1]}(x)$, $T_2(x) := x$.

For each $p \in (0,1)$ the random map corresponding to example 2.3 possesses exactly two ergodic invariant measures concentrated at points 0 and 1 respectively.

The contraction on average, that we assume in this section, does not prevent some of the maps T_i to be expanding. Therefore, despite the fact that some of the maps T_i may possess several (not necessary a finite number) ergodic invariant measures, the random map $\overline{T}$ under the assumptions of Lemma 2.2 is strictly ergodic.

3 Expanding on average random maps

Results obtained in the previous section are based technically on the contraction on average property. Now we are going to show that the opposite assumption about the expansion on average leads to the ideologically close result – the existence of the absolutely continuous invariant measure.

For the sake of simplicity we shall restrict ourselves here to the analysis of piecewise C^2-smooth maps of the unit interval $[0,1]$ into itself with nondegenerate *expanding constants*

$$\lambda_{T_i} := \inf_x |T_i'(x)| \geq \lambda > 0.$$

It is straightforward to show that the transfer operator corresponding to the random map $\overline{T}$ in the space $\mathbf{L}_1$ can be written as

$$\mathbf{P}_{\overline{T}} := \sum_i p_i \mathbf{P}_i,$$

where $\mathbf{P}_i$ is the Perron-Frobenius operator, corresponding to the map T_i and describing the dynamics of densities of measures under its action (see a detailed discussion of properties of these operators for example in [1]).

Denoting by Ω the set of values of the index i, we consider for a given number Λ the sequence of sets

$$\Omega^{(n)}(\Lambda) := \{\omega \in \Omega: \ |(\overline{T}^n_\omega x)'| > \Lambda^n \text{ for a.a. } x \in X\},$$

where

$$\overline{T}^n_\omega x := T_{\omega_n} \circ T_{\omega_{n-1}} \circ \ldots \circ T_{\omega_1} x$$

is the n-th point of a realization of a trajectory of the random map $\overline{T}$ starting from the point x, and introduce the following *regularity assumption*: there exist two constants $\Lambda > 1$ and $C < \infty$ such that

$$\mathcal{P}\{\Omega^{(n)}(\Lambda)\} \geq 1 - Ce^{-\sqrt{n}} \tag{3.1}$$

for each positive integer n.

Denote by $\operatorname{var}(\cdot)$ the standard one-dimensional variation o a function and by $\mathbf{BV}$ the space of functions of bounded variation equipped with the norm $||\cdot||_{\mathbf{BV}} := \operatorname{var}(\cdot) + ||\cdot||_{\mathbf{L}_1}$.

The following result gives the decomposition for the transfer operator under the considered assumptions.

Theorem 3.1 *[19, 1] Let the regularity assumption (3.1) holds. Then for each pair of positive integers n,k the following decomposition takes place for the random map $\overline{T}$:*

$$\mathbf{P}^n_{\overline{T}} = P_{n,k} + Q_{n,k},$$

and

$$\operatorname{var}(P_{n,k}h) \leq \text{Const}\left(\alpha^n \operatorname{var}(h) + \beta^k ||h||\right), \tag{3.2}$$

$$||Q_{n,k}h|| < \text{Const}\sqrt{k}e^{-\sqrt{k}}||h||, \tag{3.3}$$

for each function $h \in \mathbf{BV}$ and $0 < \alpha < 1 < \beta < \infty$. All constants above depend on the choice of the map T_i, but do not depend on n and k.

One can find in the literature dedicated to the question of the existence of absolutely continuous invariant measures in our setting two types of sufficient conditions for this existence. The first of these conditions obtained in [21] corresponds to the strong expansion on average

$$\sum_i \frac{p_i}{\lambda_{T_i}} < 1, \tag{3.4}$$

while the second, described in [19], is a weaker condition:

$$\prod_i \lambda_{T_i}^{p_i} > 1. \tag{3.5}$$

One can easily show that the first of these assumption yields the second one. On the other hand, as we shall show there is an important difference between properties of invariant measures and respectively random maps under these assumptions. To explain the difference let us return to our regularity assumption and show that it is even more general with respect the inequality (3.5). For this purpose we shall need the following simple technical estimate.

Lemma 3.1 *Let $\{\xi_i\}_i$ be a sequence of independent identically distributed (iid) random variables having exponential moments up to some positive order s_0, i.e. $\mathbf{E}\left[e^{s\xi}\right] < \infty$ for all $0 < s < s_0$. Then for each number $R > \mathbf{E}[\xi_i]$ there are constants $a < 1, A < \infty$ such that*

$$\mathcal{P}\{\frac{1}{n}\sum_{i=1}^{n}\xi_i > R\} < Aa^n$$

for each positive n.

Proof. By the exponential Chebyshev inequality

$$\mathcal{P}\{\frac{1}{n}\sum_{i=1}^{n}\xi_i > R\} \le e^{-sR}\mathbf{E}\left[e^{s\xi}\right]$$

for each positive number s. For our purpose it is enough to show that the right hand side of this inequality decreases exponentially fast. Note that for each number x the following inequality holds

$$|e^x - 1 - x| \le e^{|x|} - 1 - |x|.$$

Indeed, this is trivial for $x \ge 0$ (since $e^x \ge +x$ and $e^{-x} \le e^x$), while for $x < 0$ we have $e^x \le +x$. Thus the inequality can be reduced to

$$-e^x + 1 + x \le e^{-x} - 1 + x \qquad \text{or} \qquad e^{-x} + e^x \ge 2,$$

which is evidently correct. Therefore

$$|e^{s\xi} - 1 - s\xi| \le e^{|s\xi|} - 1 - |s\xi|.$$

Assume first, that the values ξ_i are bounded from below. Then

$$\mathbf{E}\left[e^{|s\xi|}\right] - 1 - \mathbf{E}\left[|s\xi|\right] < \infty$$

for $|s| < s_0$, which yields the negativity of the left hand side of the previous inequality. Therefore there are such positive constants $s \in (0, \tilde{s}_0)$ and C that

$$\mathbf{E}\left[e^{s\xi}\right] \le 1 + \mathbf{E}[\xi] + Cs^2 \le e^{s\mathbf{E}[\xi]+Cs^2}$$

Thus, setting

$$s = (R - \mathbf{E}[\xi])/(2C), \quad a = e^{-(R-\mathbf{E}[\xi]^2)/(4C))},$$

we get the desired estimate. Observe that $s < \tilde{s}_0$ by the consruction. Therefore our inequalities make sense only if

$$R \le R_0 := \mathbf{E}[\xi] + 2C\tilde{s}_0.$$

This means that larger values of R should be changed to R_0. To finish the proof note that if the random values ξ_i are not bounded from below it is enough to 'cut' them from below by means of some constant and to apply the above argument to the result. ∎

Lemma 3.2 *The inequality (3.5) implies the regularity assumption (3.1).*

Proof. It is enough to apply Lemma 3.1 to the sequence of iid random values $\xi_i := \ln\lambda_{T_i}$. ∎

On the other hand, the following result shows that the opposite statement does not hold.

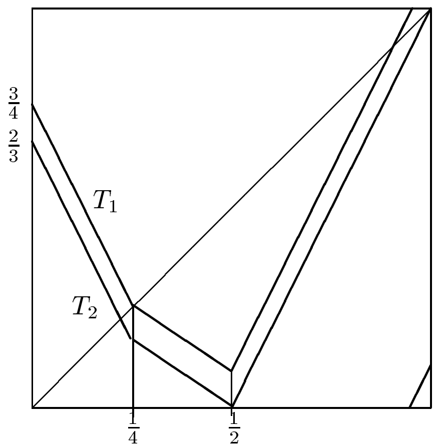

Figure 1:
Example when the regularity condition holds while the condition (3.5) breaks down

Example 3.3

$$T_1(x) := \begin{cases} \frac{3}{4} - 2x, & \textit{if } 0 \le x < \frac{1}{4} \\ \frac{1}{4} - \frac{2}{3}x, & \textit{if } \frac{1}{4} \le x < \frac{1}{2} \\ 2 - 2x \,(\bmod 1)\,, & \textit{otherwise,} \end{cases}$$

while $T_2(x) := T_1(x) + 1/12 \,(\bmod 1)$. Graphs of these maps are shown on Fig. 1.

Lemma 3.4 *The random map $\overline{T}$ in the example 3.3 for any $0 < p < 1$ satisfies the regularity assumption, while the condition (3.5) breaks down. It is interesting that in this example the stronger assumption (3.4) holds for the second iterate $\overline{T}^2$.*

Proof. Both maps T_i are piecewise linear and the moduli of their derivatives take only two values 2 (on the intervals $(0, \frac{1}{4})$ and $(\frac{1}{2}, 1)$) and $\frac{2}{3}$ (on the interval $(\frac{1}{2}, 1)$). Since both these maps transform the interval $(\frac{1}{4}, \frac{1}{2})$ into $(0, \frac{1}{4})$ and on the remaining interval $(\frac{1}{2}, 1)$ the derivatives of both maps is equal to 2, it follows that for any $i, j \in \{1, 2\}$ the inequality

$$|(T_i T_j x)'| \ge 2 \cdot \frac{2}{3} = \frac{4}{3} > 1$$

holds. Thus we have checked the regularity assumption. Now observe that the derivatives of both maps is strictly less than 1 on the interval $x \in (\frac{1}{4}, \frac{1}{2})$, which contradicts to the condition (3.5).

It remains to check the condition (3.4) for the second iterate $\overline{T}^2$, which turns out to be a consequence of the fact that expanding constants for the maps $(T_i T_j x)$ are not less than $\frac{4}{3} > 1$ (according to the inequality above). ∎

Note that since the right hand side of the inequality (3.2) contains the term β^k with $\beta > 1$ the Theorem 3.1 guarantees only estimates of the type

$$\mathrm{var}(P_{n,k}\overline{h}) \le \text{ Const } \frac{\beta^k}{1 - \alpha},$$

which means that despite the fact that the density of the invariant measure is integrable, it might be not a function of bounded variation. In fact, the example 2 from Section 1 demonstrate this phenomenon when the parameter p belongs to $(\frac{1}{2}, \frac{2}{3})$. Moreover, in this example the density not only not a function of bounded variation, but it goes to infinity in the vicinity of the origin.

It turns out that under a stronger assumption (3.4) the standard Lasota-Yorke inequality is valid for the random map and thus the invariant density is a function of bounded variation.

Theorem 3.2 *[21] Let for a.a. $x \in [0, 1]$ the inequality*

$$\sum_i \frac{p_i}{|T_i'(x)|} \le \gamma < 1$$

holds. Then there are constants $C, \beta < \infty$ such that for each $n \in \mathbb{Z}_+$ and a function $h \in \mathbf{BV}$ the Lasota-Yorke inequality holds:

$$\mathrm{var}(\mathbf{P}_{\overline{T}}^{n} h) \leq C\gamma^n \,\mathrm{var}(h) + \beta ||h||, \tag{3.6}$$

from where (as usual) it follows that for each nonnegative function $h \in \mathbf{L}^1$ with $||h|| = 1$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{P}_{\overline{T}}^{k} h =: h_{\overline{T}},$$

exists and

$$\mathbf{P}_{\overline{T}} h_{\overline{T}} = h_{\overline{T}}, \qquad \mathrm{var}(h_{\overline{T}}) \leq \mathrm{Const}\,,$$

while which respect to the absolutely continuous invariant measure $\mu_{\overline{T}}$ with the density $h_{\overline{T}}$ the correlations decay exponentially.

Now we are able to finish the analysis of the example 2 from Section 1. Observe that for $p > \frac{1}{2}$ we have

$$\prod_k \lambda_{T_k}^{p_k} = 2^p \cdot \left(\frac{1}{2}\right)^{1-p} = 2^{2p-1} > 1,$$

which yields the condition (3.5) and, hence, our regularity assumption. Therefore for $p > 1/2$ there exists an absolutely continuous $\overline{T}$-invariant measure. On the other hand, for $\frac{2}{3} < p < 1$ the condition (3.4):

$$\sum_k \frac{p_k}{|T_k'(x)|} = \frac{p}{2} + 2(1-p) = 2 - \frac{3}{2}p < 1$$

holds for a.a. $x \in [0,1]$. Thus we can apply Theorem 3.2, whereis the boundedness of the density of the invariant measures follows.

4 Perron-Frobenius spectrum (PF-spectrum)

In the previous sections we have restricted the analysis of statistical features of random maps to the properties of their invariant measures and more specifically Sinai-Bowen-Ruelle (**SBR** measures). From a more general point of view the SBR measure is the eigenfunction of the Perron-Frobenius (transfer) operator of our random map in a suitable Banach space corresponding to the leading eigenvalue (1). Therefore it is very natural to extend the analysis of the dynamics to the complete spectrum of this operator.

It is worth note that the interest to the PF-spectrum is based to a large extent on the fact that the subleading elements of the spectrum define the rate of mixing (convergence to the **SBR** measure, correlation decay, etc.).

4.1 Definition of the PF-spectrum

Let us start with the short description of objects related to the notion of the spectrum which we shall need further. Let $\mathbf{P} : \mathcal{B} \to \mathcal{B}$ be a bounded linear operator in a complex Banach space $(\mathcal{B}, ||\cdot||_{\mathcal{B}})$. As usual we denote by $\mathrm{sp}_{\mathcal{B}}(\mathbf{P})$ its *spectrum*, which is defined as a complement to the set of regular elements, i.e. to the points $z \in \mathbf{C}$ such that *the resolvent* $(zI - \mathbf{P})^{-1}$ of the operator is defined in the entire space and hence is bounded. The maximal (on modulus) element of the spectrum is called the *spectral radius*:

$$\rho_{\mathcal{B}}(\mathbf{P}) := \sup\{|z| :\ z \in \mathrm{sp}_{\mathcal{B}}(\mathbf{P})\}.$$

As it is well known [8] the spectral radius can be calculated by the following formula:

$$\rho_{\mathcal{B}}(\mathbf{P}) = \lim_{n\to\infty} ||\mathbf{P}^n||_{\mathcal{B}}^{1/n}.$$

Browder [5] introduced the notion of *essential spectrum* ${}_{\mathrm{ess}}\mathrm{sp}_{\mathcal{B}}(\mathbf{P})$ of a bounded linear operator $\mathbf{P}$ as a union of the elements of the spectrum $z \in \mathrm{sp}_{\mathcal{B}}(\mathbf{P})$ such that at least one of the following properties holds:

1. The region of values of the operator $zI - \mathbf{P}$ is not bounded in $\mathcal{B}$.
2. $\cup_{n \ge 0} \ker((zI - \mathbf{P})^n)$ is infinite dimensional.
3. z is a limit point of the spectrum $\mathrm{sp}_{\mathcal{B}}(\mathbf{P})$.

Outside of the essential spectrum only a countable number of *isolated* eigenvalues of the operator $\mathbf{P}$ may occur. Naturally the *essential* spectral radius $\rho_{\mathcal{B},\mathrm{ess}}(\mathbf{P})$ of the operator $\mathbf{P}$ is defined as the minimal nonnegative number such that all elements of the spectrum $\mathrm{sp}_{\mathcal{B}}(\mathbf{P})$ outside of the disc $\{z \in \mathbf{C} : \ |z| \le \rho_{\mathcal{B},\mathrm{ess}}(\mathbf{P})\}$ are isolated eigenvalues of finite multiplicity. It turns out [20] that the essential spectral radius can be calculated by a formula similar to the one for the usual spectral radius:

$$\rho_{\mathcal{B},\mathrm{ess}}(\mathbf{P}) = \lim_{n \to \infty} ||\mathbf{P}^n||_{\mathcal{B},\mathrm{ess}}^{1/n}, \tag{4.1}$$

but for the specialized seminorm:

$$||\mathbf{P}||_{\mathcal{B},\mathrm{ess}} := \inf\{||\mathbf{P} - K||_{\mathcal{B}} : \ K : \mathcal{B} \to \mathcal{B} \ \text{ – a compact operator}\}.$$

Clearly for each $\varepsilon > 0$ the set

$$\mathrm{sp}(\mathbf{P}) \cap \{z \in \mathbf{C} : \ |z| \ge \rho_{\mathcal{B},\mathrm{ess}}(\mathbf{P}) + \varepsilon\}$$

consists of a finite number eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{M(\varepsilon)}$ of the operator $\mathbf{P}$. Schematically on the complex plane the spectrum can be represented as a disk of radius $\rho_{\mathcal{B},\mathrm{ess}}(\mathbf{P})$ centered at the origin (describing the essential spectrum) and (not more than countable) collection of points between this disk and the circle of radius $\rho_{\mathcal{B}}(\mathbf{P})$ also centered at the origin.

4.2 Spectrum for the case of contracting on average random maps

In Section 2 it was shown that a contracting on average random map $\overline{T}$ possesses the only one invariant measure $\mu_{\overline{T}}$ to which the sequence of iterations $\{\overline{T}^n \mu\}$ converges (exponentially fast in the Hutchinson metrics) for each probabilistic initial measure μ. ¿From the point of view of dynamical system theory the following question after this is the analysis of the spectrum of this convergence. The problem here is that typically the limit measure $\mu_{\overline{T}}$ is not absolutely continuous, which rules out the description of the transfer operator $\mathbf{P}_{\overline{T}}$ in a space of a reasonably 'good' functions, for example, in the space of functions of bounded variation. To overcome this difficulty we shall study the action of the operator $\mathbf{P}_{\overline{T}}$ in a much larger space of generalized functions equipped with a norm induced by the Hutchinson metrics.

Let (X, ρ) be a d-dimensional smooth manifold with a finite collection of continuous maps $\{T_i\}$ having bounded Lipschitz constants Λ_{T_i} from X into itself, and a collection of probabilities $\{p_i\}$, defining the random map $\overline{T}$. Remind that the contraction on average means that

$$\Lambda_{\overline{T}} := \sum_i p_i \Lambda_{T_i} < 1.$$

Before to define our space of generalized functions we need first to define the class of test-functions $\varphi : X \to \mathbb{R}^1$. For this purpose we introduce the following functionals:

$$\begin{aligned} H_\alpha(\varphi) &:= \sup_{x,y \in X, \ \rho(x,y) \le \nu} \frac{|\varphi(x) - \varphi(y)|}{\rho^\alpha(x,y)}, \\ V_\alpha(\varphi) &:= H_\alpha(\varphi) + |\varphi|_\infty, \end{aligned}$$

where $|\varphi|_\infty := \mathrm{esssup}|\varphi|$, and the constant $\nu \in (0, 1]$. The first of these functionals is the Hölder constant with the exponent $\alpha < 1$, while the second one for each finite nonnegative value of the parameter a is the norm in the Banach space of α-Hölder functions on X, which we shall denote by $\mathbf{C}^\alpha$. Without the loss of generality we shall assume that the diameter of the phase space $\sup_{x,y \in X} \rho(x, y) \le 1$. Note that the another restriction $\rho(x, y) \le \nu$ is introduced only to be able to work with the exponential map on general smooth manifolds: in the case of a flat torus this restriction can be omitted (or simply one can set $\nu = 1$).

Consider now the space of generalized functions $\mathcal{F}$ on X with the norm defined in terms of the test-functions from the space $\mathbf{C}^\alpha$:

$$||h||_{(\alpha)} := \sup_{V_\alpha(\varphi)\le 1} \int h\varphi.$$

The proof that the functional $||\cdot||_{(\alpha)}$ is indeed a norm in this space is standard and we leave it for the reader.

Denote by $\mathcal{F}_\alpha$ the closure of the set of bounded in the norm $||\cdot||_{(\alpha)}$ generalized functions from $\mathcal{F}$.

Lemma 4.1 *$\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$ for any numbers $0 < \alpha \le \beta \le 1$ and each function $\varphi \in \mathbf{C}^\beta$ the inequality $V_\beta(\varphi) \le V_\alpha(\varphi)$ holds.*

Proof. Indeed,

$$\frac{|\varphi(x)-\varphi(y)|}{\rho^\beta(x,y)} = \frac{|\varphi(x)-\varphi(y)|}{\rho^\alpha(x,y)}\rho^{\beta-\alpha}(x,y) \le H_\alpha(\varphi).$$

∎

Extending the standard definition of the Perron-Frobeniusa operator to the action in the space of generalized functions we get the representation

$$\int \mathbf{P}_{\overline{T}} h \cdot \varphi = \int h \cdot \sum_i p_i\, \varphi \circ T_i =: \int h \cdot (\varphi \circ \overline{T})$$

for each test-function from $\varphi \in \mathbf{C}^\alpha$.

Let us fix some constants $0 < \alpha < \beta \le 1$, $0 < a < \infty$, whose exact values we shall define later. Our first aim is to derive a version of the Lasota-Yorke inequality for the action in the space $\mathcal{F}_\alpha$. For $q \in [0,1]$ define a function

$$\Lambda_{\overline{T}}(q) := \sum_i p_i \Lambda_{T_i}^q,$$

which we shall need in this derivation.

Introduce additionally the notation $\overline{G} = \{G_i, p_i\}$ for the random map defined by the collection of maps G_i and the distribution $\{p_i\}$.

Lemma 4.2 *The superposition of any pair of random maps $\overline{G} = \{G_i, p_i\}$ and $\overline{G}' = \{G'_i, p'_i\}$ acting on the same manifold (X,ρ) and satisfying the Lipschitz condition is again the random map $\overline{G}' \circ \overline{G} := \{G'_i \circ G_j, p'_i p_j\}$ and $\Lambda_{\overline{G}'\circ\overline{G}}(q) \le \Lambda_{\overline{G}'}(q) \cdot \Lambda_{\overline{G}}(q)$ for each $q \in [0,1]$.*

Proof. Indeed,

$$\begin{aligned}\Lambda_{\overline{G}'\circ\overline{G}}(q) &= \sum_i\sum_j p'_i p_j \Lambda^q_{G'_i\circ G_j} \le \sum_i\sum_j p'_i p_j \Lambda^q_{G'_i}\cdot\Lambda^q_{G_j} \\ &= \left(\sum_i p'_i \Lambda^q_{G'_i}\right)\cdot\left(\sum_j p_j \Lambda^q_{G_j}\right) = \Lambda_{\overline{G}'}(q)\cdot\Lambda_{\overline{G}}(q).\end{aligned}$$

∎

Theorem 4.1 *For each number $\kappa > 2$ and for any $h \in \mathcal{F}_\alpha$ and $n \in \mathbb{Z}_+$ the Lasota-Yorke inequality holds:*

$$\left|\left|\mathbf{P}^n_{\overline{T}} h\right|\right|_{(\alpha)} \le \kappa \Lambda^n_{\overline{T}}(\alpha)\, ||h||_{(\alpha)} + \mathrm{Const}\cdot(\kappa-2)^{-1/\alpha}\, ||h||_{(\beta)}. \tag{4.2}$$

Proof. We start from the proof of the following two inequalities:

$$\left|\left|\mathbf{P}_{\overline{T}}h\right|\right|_{(\beta)} \leq ||h||_{(\beta)}, \tag{4.3}$$

$$\left|\left|\mathbf{P}_{\overline{T}}h\right|\right|_{(\alpha)} \leq \kappa\Lambda_{\overline{T}}(\alpha)\,||h||_{(\alpha)} + \text{Const}\cdot(\kappa-2)^{-1/\alpha}\,||h||_{(\beta)}. \tag{4.4}$$

By the definition of the Hölder constant we have:

$$\frac{|\varphi(T_i x)-\varphi(T_i y)|}{\rho^\alpha(x,y)} = \frac{\rho^\alpha(T_i x, T_i y)}{\rho^\alpha(x,y)}\frac{|\varphi(T_i x)-\varphi(T_i y)|}{\rho^\alpha(T_i x, T_i y)} \leq \Lambda^\alpha_{T_i} H_\alpha(\varphi).$$

Hence,

$$\frac{\sum_i p_i|\varphi(T_i x)-\varphi(T_i y)|}{\rho^\alpha(x,y)} \leq \sum_i p_i \Lambda^\alpha_{T_i} H_\alpha(\varphi).$$

Thus,

$$H_\alpha\left(\sum_i p_i(\varphi\circ T_i)\right) \leq \Lambda_{\overline{T}}(\alpha)H_\alpha(\varphi) < H_\alpha(\varphi).$$

On the other hand, since

$$|\varphi\circ T_i|_\infty \leq |\varphi|_\infty,$$

then

$$\sum_i p_i|\varphi\circ T_i|_\infty \leq |\varphi|_\infty.$$

Therefore for each $\beta\in(0,1]$ we have

$$V_\beta\left(\sum_i p_i|\varphi\circ T_i|\right) \leq V_\beta(\varphi),$$

which implies the inequality (4.3) for each $\beta\in(0,1]$ and a function $h\in\mathcal{F}_\beta$.

To prove the second inequality (4.4) we need more delicate estimates.

Introduce the following notation:

$$B_\delta(x) := \{y\in X:\ \rho(x,y)\leq\delta\} \qquad B_\delta := \{\xi\in\mathcal{T}_x X:\ |\xi|\leq\delta\},$$

i.e. $B_\delta(x)$ is the ball of radius δ centered at the point x in the space X, while B_δ is the ball of radius δ centered at the origin in the tangent space $\mathcal{T}_x X$. For each point $x\in X$ consider the exponential map

$$\Psi_x := \exp_x : B_\delta \subseteq \mathcal{T}_x X \to X.$$

Choosing $\delta>0$ small enough we always can assume that $\Psi_x B_\delta \subset B_\nu(x)$.

Denoting by m the Lebesgue measure on X, we introduce the following smoothing operator

$$Q_\delta\varphi(x) := \frac{\int_{\Psi_x B_\delta}\varphi(y)m(dy)}{m(\Psi_x B_\delta)} = \frac{\int_{B_\delta}\varphi(\Psi_x z)\cdot J\Psi_x(z)\,dz}{\int_{B_\delta}J\Psi_x(z)\,dz},$$

where $J\Psi_x$ is the Jacobian of the map Ψ_x.

Let us estimate the Hölder constant of the function $Q_\delta\varphi$:

$$\begin{aligned}
|Q_\delta\varphi(x)-Q_\delta\varphi(y)| &\leq \left|\int_{B_\delta}J\Psi_x(z)\,dz - \int_{B_\delta}J\Psi_y(z)\,dz\right|\cdot\left|\frac{\int_{B_\delta}\varphi(\Psi_y z)\cdot J\Psi_y(z)\,dz}{\int_{B_\delta}J\Psi_x(z)\,dz\cdot\int_{B_\delta}J\Psi_y(z)\,dz}\right| \\
&\quad + \frac{1}{\int_{B_\delta}J\Psi_x(z)\,dz}\cdot\left|\int_{B_\delta}\varphi(\Psi_x z)\cdot J\Psi_x(z)\,dz - \int_{B_\delta}\varphi(\Psi_y z)\cdot J\Psi_y(z)\,dz\right| \\
&\leq 2\left(\sup_x J\Psi_x\right)^2\cdot|\varphi|_\infty\cdot\frac{|B_\delta\setminus\Psi_x^{-1}\Psi_y B_\delta| + |\Psi_x^{-1}\Psi_y B_\delta\setminus B_\delta|}{|B_\delta|} \\
&\leq \frac{1}{\delta}C_1|\varphi|_\infty\rho(x,y),
\end{aligned}$$

where the constant C_1 depends only on the properties of the manifold X.

Now we estimate how much the operator Q_δ differs from the identical operator:

$$|Q_\delta\varphi(x) - \varphi(x)| \le \frac{\int_{B_\delta} |\varphi(\Psi_x(z)) - \varphi(x)| \cdot J\Psi_x(z)\, dz}{\int_{B_\delta} J\Psi_x(z)\, dz} \le C_2\delta^\alpha H_\alpha(\varphi),$$

where the constant C_2 also depends only on the properties of the manifold X.

On the other hand,

$$\begin{aligned} &H_\alpha\left(\sum_i p_i(\varphi\circ T_i) - Q_\delta\left(\sum_i p_i(\varphi\circ T_i)\right)\right) \\ &\le H_\alpha\left(\sum_i p_i(\varphi\circ T_i)\right) + H_\alpha\left(Q_\delta\left(\sum_i p_i(\varphi\circ T_i)\right)\right) \\ &\le 2\Lambda_{\overline{T}}(\alpha)H_\alpha(\varphi). \end{aligned}$$

Thus,

$$\begin{aligned} &V_\alpha\left(\sum_i p_i(\varphi\circ T_i) - Q_\delta\left(\sum_i p_i(\varphi\circ T_i)\right)\right) \\ &\qquad \le 2\Lambda_{\overline{T}}(\alpha)H_\alpha(\varphi) + C_2\delta^\alpha H_\alpha(\varphi). \end{aligned}$$

Gathering the obtained estimates and using Lemma 4.1 we come to

$$\begin{aligned} \left|\left|\mathbf{P}_{\overline{T}}h\right|\right|_{(\alpha)} &\le \sup_{V_\alpha(\varphi)\le 1}\int h\cdot Q_\delta\left(\sum_i p_i(\varphi\circ T_i)\right) + \sup_{V_\alpha(\varphi)\le 1}\int h\cdot\sum_i p_i\left(\varphi\circ T_i - Q_\delta(\varphi\circ T_i)\right) \\ &\le \sup_{V_\beta(\varphi)\le 1} V_\alpha\left(Q_\delta\left(\sum_i p_i(\varphi\circ T_i)\right)\right)||h||_{(\beta)} \\ &\quad + \sup_{V_\alpha(\varphi)\le 1} V_\alpha\left(\sum_i p_i\left(\varphi\circ T_i - Q_\delta(\varphi\circ T_i)\right)\right)||h||_{(\alpha)} \\ &\le \frac{C_1}{\delta}||h||_{(\beta)} + \left(2\Lambda_{\overline{T}}(\alpha) + C_2\delta^\alpha\right)||h||_{(\alpha)}. \end{aligned}$$

Therefore choosing the value of the parameter δ such small that $C_2\delta^\alpha = \kappa - 2$, we get the inequality (4.4).

Now according to Lemma 4.2 and above inequalities we get

$$\begin{aligned} \left|\left|\mathbf{P}^n_{\overline{T}}h\right|\right|_{(\alpha)} &= \left|\left|\mathbf{P}_{\overline{T}^n}h\right|\right|_{(\alpha)} \le \kappa\Lambda_{\overline{T}^n}(\alpha)\,||h||_{(\alpha)} + \mathrm{Const}\,\delta^{-1}\,||h||_{(\beta)} \\ &\le \kappa\Lambda^n_{\overline{T}}(\alpha)\,||h||_{(\alpha)} + \mathrm{Const}\cdot(\kappa-2)^{-1/\alpha}\,||h||_{(\beta)}, \end{aligned}$$

which finishes the proof of the inequality (4.2). ∎

Lemma 4.3 *The function $\Lambda_{\overline{T}}(\cdot)$ is convex, takes values strictly less than 1 in the interval $(0,1]$, and under the condition*

$$\prod_i \Lambda_{T_i}^{p_i\Lambda_{T_i}} < 1 \qquad (4.5)$$

its unique point of minima either lies inside of this interval, or is larger than 1 otherwise.

Proof. By the definition of the contraction on average we have $\Lambda_{\overline{T}}(1) = \Lambda_{\overline{T}} < 1$. On the other hand, $\Lambda_{\overline{T}}(0) = 1$, and

$$\frac{d^2}{dq^2}\Lambda_{\overline{T}}(q) = \sum_i p_i\Lambda^q_{T_i}\cdot(\ln\Lambda_{T_i})^2 > 0.$$

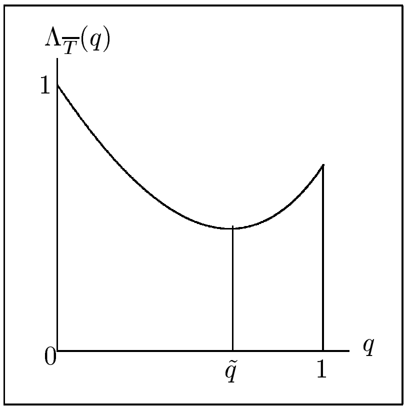

Figure 2:
A typical behavior of the function $\Lambda_{\overline{T}}(q)$.

Thus the function $\Lambda_{\overline{T}}(q)$ is strictly convex and for any $q \in (0,1]$ it is less than 1. Let us show that this function strictly decreases at 0. Indeed,

$$\frac{d}{dq}\Lambda_{\overline{T}}(0) = \sum_i p_i \ln \Lambda_{T_i}.$$

On the other hand, from the contraction on average

$$\sum_i p_i \Lambda_{T_i} < 1$$

using the convexity of the logarithmic function, we get:

$$\sum_i p_i \ln \Lambda_{T_i} < \ln 1 = 0,$$

which proves that $\frac{d}{dq}\Lambda_{\overline{T}}(0) < 0$. It remains to check the last statement.

$$\frac{d}{dq}\Lambda_{\overline{T}}(1) = \sum_i p_i \Lambda_{T_i} \cdot \ln \Lambda_{T_i} = \sum_i \ln \Lambda_{T_i}^{p_i \Lambda_{T_i}} = \ln\left(\prod_i \Lambda_{T_i}^{p_i \Lambda_{T_i}}\right).$$

Thus due to the inequality (4.5) the unique (due to the strict convexity of the function $\Lambda_{\overline{T}}(\cdot)$) solution of the equation $\frac{d}{dq}\Lambda_{\overline{T}}(q) = 0$ (the point of minima of the function $\Lambda_{\overline{T}}(\cdot)$) belongs to the interval $(0,1)$. On the other hand, if the inequality (4.5) does not hold this solution is greater than 1. ∎

A typical behavior of the function $\Lambda_{\overline{T}}(q)$ is shown Fig. 2. Denote by $\tilde q$ the value of the parameter q corresponding to the unique (due to the strict convexity) minima of the function $\Lambda_{\overline{T}}(q)$. The value $\tilde q$ is positive since $\frac{d}{dq}\Lambda_{\overline{T}}(0) < 0$. The position of $\tilde q$ with respect to 1 is defined by the sign of the derivative of the function $\Lambda_{\overline{T}}(q)$ at 1 which might be both positive and negative. For example, if all the maps are contractive then this sign is negative and the function $\Lambda_{\overline{T}}(q)$ strictly decreases on the interval $[0,1]$. On the other hand, the map in the example 1.2 satisfies the condition (4.5) if $0 < p < 1/5$ and in this case $\tilde q = \frac{1}{2}\log_2(1/p - 1)$. We describe the properties of the function $\Lambda_{\overline{T}}(q)$ in such detail because the fact that it can grow in the vicinity of the point 1 plays an important role in further calculations.

Lemma 4.4 *The unit disk in the strong norm $||\cdot||_{(\alpha)}$ is a compact set in the weak norm $||\cdot||_{(\beta)}$*

The **proof** of this statement follows immediately from standard results on the enclosure of the spaces of Hölder functions.

Above statements imply by the Ionescu-Tulchea and Marinescu Theorem [14] the quasicompactness of the operator $\mathbf{P}_{\overline{T}}$, and the validity of the following based on Nussbaum Theorem [20] estimate of its essential spectral radius.

Lemma 4.5 *[12] The essential spectal radius of the operator $\mathbf{P}_{\overline{T}} : \mathcal{F}_\alpha \to \mathcal{F}_\alpha$ belongs to the disk or radius $\Lambda_{\overline{T}}(\alpha)$ centered at zero.*

Note that the estimates leading to the Lasota-Yorke type inequalities depend sensitively on the choice of the value of the parameter α. Thus it is reasonable to choose the value of α which yields the smallest (and hence the best) available estimate of the essential spectral radius. Normally (compare to [4]) this value is equal to 1 which is unavailable since we consider only Hölder continuous test functions. However in our case Lemma 4.3 shows that under the condition (4.5) the optimal value of α may be strictly less than 1.

An immediate corollary to Lemma 4.5 is the existence of a constant $\gamma \in [\Lambda_{\overline{T}}(\alpha), 1)$ such that the set $\mathrm{sp}(\mathbf{P}_{\overline{T}}) \setminus \{|z| \le \gamma\}$ consists of a finite number of periferal eigenvalues $r_1, \dots, r_N$ of finite multiplicity. Denote by $P_1, \dots, P_N$ the corresponding spectral projectors and set $P := 1_{\mathcal{F}_\alpha} - \sum_{j=1}^N P_j$. Then the $\mathrm{rank}(P_j) < \infty$, $\mathbf{P}_{\overline{T}} P_j = r_j P_j$ $(j = 1, \dots, N)$, and the spectral radius of the operator $\mathbf{P}_{\overline{T}} P$ does not exceed γ.

Besides,

- If $|r| = 1$ the operator

$$P_r := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r^{-k} \mathbf{P}^k_{\overline{T}} = \sum_{j=1}^{N} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{r_j}{r}\right)^k P_j = \begin{cases} P_j, & \text{if } r = r_j \\ 0, & \text{otherwise} \end{cases} \tag{4.6}$$

  is well defined in the $||\cdot||_{(\alpha)}$-norm. In particular, $P_j = P_{r_j}$, and since $\int |P_{r_j} f| \le \int |f|$ for all $f \in \mathbf{C}^1(X, \mathbb{R}^1)$, then operators P_j can be extended continuously to the entire space $\mathbf{L}^1$.

- For any function $f \in P_j \mathcal{F}_\alpha$ there is a finite Borel signed measure μ_f on X such that $\langle f, \varphi \rangle = \int \varphi \, d\mu_f$ for all $\varphi \in \mathbf{C}^1(X, \mathbb{R}^1)$. $r_1 := 1 \in \mathrm{sp}(\mathbf{P}_{\overline{T}})$, $\mu := \mu_{P_1 1}$ – a positive measure, $\mu(X) = m(X)$, and all signed measures μ_f are absolutely continuous with respect to μ.

  One can interpret these statements as follows: for $f, \varphi \in \mathbf{C}^1(X, \mathbb{R}^1)$ and $|r| = 1$,

$$\langle P_r f, \varphi \rangle = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r^{-k} \langle \mathbf{P}^k_{\overline{T}} f, \varphi \rangle = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r^{-k} \int \varphi \circ \overline{T}^k \cdot f. \tag{4.7}$$

  Hence, $\langle P_r f, \varphi \rangle \le |\varphi|_\infty \cdot \int |f|$, so $P_r f$ can be extended by continuity to a continuous linear functional on $\mathbf{C}^0(X, \mathbb{R}^1)$, and by Riss Theorem there is a measure $\mu_{P_r f}$, such that $\langle P_r f, \varphi \rangle = \int \varphi \, d\mu_{P_r f}$. If $r = 1$ and $f, \varphi \ge 0$ then $\int \varphi \, d\mu_{P_1 f} = \langle P_1 f, \varphi \rangle \ge 0$ and $\langle P_1 f, 1 \rangle = \int f$ according to (4.7). $\mu_{P_1 1}$ is a positive measure and $r = 1$ is an eigenvalue of the operator $\mathbf{P}_{\overline{T}}$. Finally, it follows from (4.7) that for any $\varphi \ge 0$ we have

$$|\langle P_j f, \varphi \rangle| \le \lim_{\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int \varphi \circ \overline{T}^k \cdot |f| \le |f|_\infty \langle P_1 1, \varphi \rangle = |f|_\infty \int \varphi.$$

  Besides, $\mu_{P_j f}$ is absolutely continuous with respect to μ.

  It remains to show that $P_j \mathcal{F}_\alpha = V_j := P_j(\mathbf{C}^1(X, \mathbb{R}^1))$. Since $V_j \subseteq P_j \mathcal{F}_\alpha$ and they are finite dimensional linear subspaces in $\mathcal{F}_\alpha$, then this statement immediately follows from the denseness of the space $\mathbf{C}^1(X, \mathbb{R}^1)$ in $\mathcal{F}_\alpha$.

- $\overline{T}^* \mu = \mu$, since $\int \varphi \, d(\overline{T}^* \mu) = \int \varphi \circ \overline{T} \, d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int \varphi \circ \overline{T}^{k+1} = \int \varphi \, d\mu$ for all $\varphi \in \mathbf{C}^1(X, \mathbb{R}^1)$.

- If $r = 1$ is a simple eigenvalue and there are no other eigenvalues equal to 1 on modulus, then $P_1 f = \langle f, 1 \rangle \cdot P_1 1$ for all $f \in \mathcal{F}_\alpha$. This follows immediately from the fact that $\langle P_1 f, 1 \rangle = \langle f, 1 \rangle$ (see (4.7).

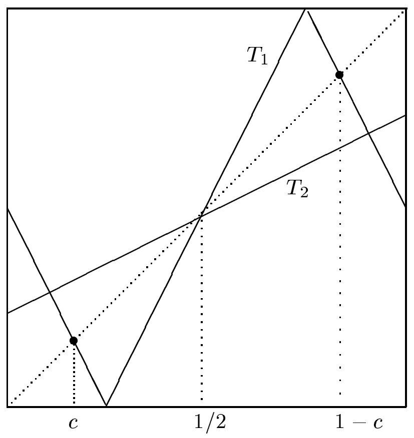

Figure 3:
Example of a random map with a nontrivial isolated eigenvalue

- μ is a **SBR** measure, since for each $\varphi \in \mathbf{C}^1(X, \mathbb{R}^1)$

$$\lim_{n\to\infty} \int \varphi \, d\left(\frac{1}{n}\sum_{k=0}^{n-1} \overline{T}^{*k} m\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \int \varphi \circ \overline{T}^k = \langle P_1 1, \varphi\rangle = \int \varphi \, d\mu \ .$$

So far the only isolated eigenvalue that we were able to identify explicitly was the unit eigenvalue (the leading one). Let us discuss the following argument. According to Lemma 4.5 the essential spectral radius of the operator $\mathbf{P}_{\overline{T}}$ is not larger than $\Lambda_{\overline{T}}(\alpha)$. On the other hand, the Lemma 2.2 garantees the convergence to the limit measure with the rate at least $1 > \Lambda_{\overline{T}} > \Lambda_{\overline{T}}(\alpha)$ (by Lemma 4.3). Thus if there is a function $f \in \mathcal{F}_\alpha$ such that $\mathbf{P}_{\overline{T}} f = \Lambda_{\overline{T}} f$, then $\Lambda_{\overline{T}}$ is an isolated eigenvalue.

Example 4.6 *$T_1(x) := 1/2 + \mathrm{sign}(x-1/2)\cdot[1/2 - |2\mathrm{sign}(x-1/2)\cdot(x-1/2) - 1/2|]$, where $\mathrm{sign}(x)$ is the sign of the number x, and $T_2(x) := x/2$.*

The maps T_i are shown on Fig. 3. Note that this random map is contractive on average when $0 < p < 1/3$. By the construction the map T_1 has three unstable fixed points $c = 1/6$, $1/2$ and $1-c = 5/6$ and the point $1/2$ is the only fixed point of the contractive map T_2. Besides, the trajectories of the random map starting at points c and $1-c$ are symmetric and consist of a countable number of points $\{2^{-1} - 2^{-n}c\}_n$ and $\{2^{-1} + 2^{-n}(1-c)\}_n$ respectively. Consider a family of generalized functions

$$f_{\bar a}(x) := \sum_{k=0}^{n} a_k \cdot \mathbf{1}_{2^{-1}-2^{-k}c}(x) - \sum_{k=0}^{n} a_k \cdot \mathbf{1}_{2^{-1}+2^{-k}(1-c)}(x),$$

parametrized by the sequence of coefficients $\bar a = \{a_k\}_k$. Due to the previous remark this family is invariant with respect to the action of $\mathbf{P}_{\overline{T}}$ and our aim is to find such a sequence $\bar a$ that $\mathbf{P}_{\overline{T}} f_{\bar a} = \Lambda_{\overline{T}} f_{\bar a}$.

Lemma 4.7 *In the example 4.6 for each $0 < p < 1/5$ there is a sequence $\bar a = \bar a(p)$ such that $\mathbf{P}_{\overline{T}} f_{\bar a} = \Lambda_{\overline{T}} f_{\bar a}$. However $||f_{\bar a}||_{(\alpha)} = \infty$ for any $\alpha \in (\frac{1}{2}\log_2(1/p-1), 1)$.*

Proof. Observe that $\Lambda_{\overline{T}} = (1+3p)/2 < 1$. Therefore rewriting the eigenvalue relation in terms of the weights a_k we obtain the following recurrent relations:

$$\begin{aligned} ((1+3p)/2)\cdot a_0 &= pa_0 + pa_1 \\ ((1+3p)/2)\cdot a_k &= (1-p)a_{k-1} + pa_{k+1} \end{aligned}$$

for each $k \ge 1$. Solving these equations with respect to the variables with higher indices we get:

$$a_1 = \frac{1+p}{2p} a_0, \qquad a_{k+1} = \frac{1+3p}{2p} a_k - \frac{1-p}{p} a_{k-1} \qquad \text{for} \quad k \ge 1.$$

The eigenvalues of the matrix $A = \begin{pmatrix} \frac{1+3p}{2p} & -\frac{1-p}{p} \\ 1 & 0 \end{pmatrix}$, controlling the growth of the coefficients, are equal to 2 and $(1-p)/(2p)$ respectively. Now since for $0 < p < 1/5$ the second eigenvalue is larger and since $a_1/a_0 = (1+p)/(2p) > 2$ we deduce that the constants a_k grow as $((1-p)/(2p))^k$ as $k \to \infty$.

It remains to estimate the $||\cdot||_{(\alpha)}$-norm of the generalized function $f_{\bar{a}}$:

$$||f_{\bar{a}}||_{(\alpha)} = 2c^\alpha \sum_{k=0}^{\infty} a_k 2^{-k\alpha} < \infty$$

if and only if $((1-p)/(2p))2^{-k\alpha} < 1$. On the other hand, $(1-p)/(2p) > 2$ for $0 < p < 1/5$ which contradicts to the convergence. ■

4.3 Spectrum for the case of expanding on average random maps

Let T_i for each i be a map from the unit interval into itself and let the following condition holds:

$$\gamma_{\overline{T}} := \sup_x \sum_i \frac{p_i}{|T_i'(x)|} < 1,$$

where the supremum is taken over all points $x \in X$ where the derivatives of maps T_i are well defined. Then, according to Theorem 3.2, the Lasota-Yorke inequality is valid:

$$\mathrm{var}(\mathbf{P}_{\overline{T}}^n h) \le C\gamma_{\overline{T}}^n \mathrm{var}(h) + \beta||h||.$$

Therefore all known results on the spectral properties of the Perron-Frobenius operator obtained for piecewise expanding maps based on a similar inequalities remain valid as well (see for example the detailed discussion in [1]).

As we already mentioned the condition (3.5) does not imply the inequality of Lasota-Yorke type and moreover there are examples when under the condition (3.5) there is no exponential correlation decay. Therefore the question how to extend the description of the spectrum to this case remains open.

4.4 Stochastic stability

In this section we shall study random perturbations of random maps under consideration. Since under the condition (3.4) perturbations of expanding on average random maps can be considered exactly as in the case of deterministic piecewise expanding maps (see [2, 3] and general discussion in [1]) we shall restrict ourselves only to the case of contracting on average systems.

As usual under the randomly perturbed system we shall mean the superposition of the original system and a Markov process acting on the same phase space and defined by the family of transition operators Q_ε (here ε stands for the 'size' of perturbation).

To simplify the calculations we shall start from the case $X = \mathbf{Tor}^d$ and then shall explain how the corresponding arguments should be changed in the case of a general smooth manifold.

Consider two families of operators: integral operators $Q_\varepsilon : \mathcal{F}_\alpha \to \mathbf{C}^1$ and the dual ones $Q_\varepsilon^* : \mathbf{C}^1 \to \mathbf{C}^1$:

$$Q_\varepsilon f(x) := \int q_\varepsilon(z,x) f(z)\, dz,$$

$$Q_\varepsilon^* \varphi(x) := \int q_\varepsilon(x,z)\varphi(z)\, dz$$

with the family of nonnegative kernels $q_\varepsilon(\cdot,\cdot)$, with respect to which we shall assume that for some $1 < M < \infty$ the following conditions hold:

$$\int q_\varepsilon(x,y)\, dy = 1, \quad q_\varepsilon(x,y) = 0 \quad \forall \rho(x,y) > \varepsilon, \tag{4.8}$$

$$\int |q_\varepsilon(x,z) - q_\varepsilon(y, z+y-x)|\, dz \le M\rho(x,y), \tag{4.9}$$

We start the analysis of the operator Q_ε^* from the following simple estimates.

Lemma 4.8 *For each $\varphi \in \mathbf{C}^\alpha$ we have*

$$|Q_\varepsilon^*\varphi|_\infty \le |\varphi|_\infty, \tag{4.10}$$

$$|Q_\varepsilon^*\varphi - \varphi|_\infty \le \varepsilon^\alpha H_\alpha(\varphi). \tag{4.11}$$

Proof. The proof is straightforward:

$$\begin{aligned}
|Q_\varepsilon^*\varphi(x)| &= |\int q_\varepsilon(x,z)\varphi(z)\,dz| \le \int q_\varepsilon(x,z)|\varphi(z)|\,dz \le |\varphi|_\infty. \\
|Q_\varepsilon^*\varphi(x) - \varphi(x)| &= |\int q_\varepsilon(x,z)\varphi(z)\,dz - \varphi(x)| \\
&\le \int q_\varepsilon(x,z)|\varphi(z) - \varphi(x)|\,dz \\
&\le \varepsilon^\alpha H_\alpha(\varphi),
\end{aligned}$$

since only the points $z \in B_\varepsilon(x)$ should be taken into account. ∎

Now let us estimate the norm of the operator Q_ε.

Lemma 4.9 *Let $M_1(\varepsilon) := \max\{2\varepsilon^{\alpha/2},\ M\varepsilon^{(1-\alpha)/2}\}$. Then $||Q_\varepsilon||_{(\alpha)} \le 1 + M_1(\varepsilon)$.*

Proof. Our aim is to show that $Q_\varepsilon^*\varphi$ is a valid test function and to estimate the values of $V_\alpha(Q_\varepsilon^*\varphi)$. We already estimated the supremum norm of the function $Q_\varepsilon^*\varphi$. To get the estimate of the Hölder constant we consider two different situations: when the points x, y are close, i.e. $\rho(x,y) \le \sqrt{\varepsilon}$, and the opposite case when they are far apart. In the first case we proceed as follows:

$$\begin{aligned}
|Q_\varepsilon^*\varphi(x) - Q_\varepsilon^*\varphi(y)| &= |\int q_\varepsilon(x,z)\varphi(z)\,dz - \int q_\varepsilon(y,z)\varphi(z)\,dz| \\
&= |\int q_\varepsilon(x,z)\varphi(z)\,dz - \int q_\varepsilon(y,z+y-x)\varphi(z+y-x)\,dz| \\
&\le |\int q_\varepsilon(x,z)\varphi(z)\,dz - \int q_\varepsilon(x,z)\varphi(z+y-x)\,dz| \\
&\quad + |\int q_\varepsilon(x,z)\varphi(z+y-x)\,dz - \int q_\varepsilon(y,z+y-x)\varphi(z+y-x)\,dz| \\
&\le \int q_\varepsilon(x,z)|\varphi(z) - \varphi(z+y-x)|\,dz \\
&\quad + \int |q_\varepsilon(x,z) - q_\varepsilon(y,z+y-x)| \cdot |\varphi(z+y-x)|\,dz \\
&\le \rho^\alpha(x,y)H_\alpha(\varphi) + M\rho(x,y)|\varphi|_\infty \\
&\le \left[H_\alpha(\varphi) + M\varepsilon^{(1-\alpha)/2}|\varphi|_\infty\right]\rho^\alpha(x,y).
\end{aligned}$$

In the opposite case, when $\rho(x,y) > \sqrt{\varepsilon}$ we shall proceed in a different way:

$$\begin{aligned}
|Q_\varepsilon^*\varphi(x) - Q_\varepsilon^*\varphi(y)| &\le |Q_\varepsilon^*\varphi(x) - \varphi(x)| + |Q_\varepsilon^*\varphi(y) - \varphi(y)| + |\varphi(x) - \varphi(y)| \\
&\le 2|Q_\varepsilon^*\varphi - \varphi|_\infty + \rho^\alpha(x,y)H_\alpha(\varphi) \\
&\le 2\varepsilon^\alpha H_\alpha(\varphi) + \rho^\alpha(x,y)H_\alpha(\varphi) \\
&\le (1 + 2\varepsilon^{\alpha/2})\rho^\alpha(x,y)H_\alpha(\varphi).
\end{aligned}$$

Hence we have

$$\begin{aligned}
V_\alpha(Q_\varepsilon^*\varphi) &= H_\alpha(Q_\varepsilon^*\varphi) + |Q_\varepsilon^*\varphi|_\infty \\
&\le (1 + 2\varepsilon^{\alpha/2})H_\alpha(\varphi) + M\varepsilon^{(1-\alpha)/2}|\varphi|_\infty \\
&\le (1 + \max\{2\varepsilon^{\alpha/2},\ M\varepsilon^{(1-\alpha)/2}\})V_\alpha(\varphi).
\end{aligned}$$

Thus, setting $M_1(\varepsilon) := \max\{2\varepsilon^{\alpha/2},\, M\varepsilon^{(1-\alpha)/2}\}$, we get

$$\begin{aligned} ||Q_\varepsilon f||_{(\alpha)} &= \sup_{V_\alpha(\varphi)\le} \int Q_\varepsilon f \cdot \varphi = \sup_{V_\alpha(\varphi)\le} \int f \cdot Q_\varepsilon^* \varphi \\ &\le \sup_{V_\alpha(\varphi)\le 1} V_\alpha(Q_\varepsilon^*\varphi) \cdot \sup_{V_\alpha(\psi)\le 1} \int f \cdot \psi \\ &\le (1 + M_1(\varepsilon)) \cdot ||f||_{(\alpha)}. \end{aligned}$$

Observe that in several places we used the estimates of the supremum norms obtained in Lemma 4.8.
■

Lemma 4.10 *Let $G : \mathcal{F}_\alpha \to \mathcal{F}_\alpha$ be a linear operator, and let $G^* : \mathbf{C}^1 \to \mathbf{C}^1$ be dual to it, i.e. $\int Gf \cdot \varphi = \int f \cdot G^*\varphi$. Then for all $f \in \mathcal{F}_\beta$*

$$||Gf||_{(\beta)} \le \left(\sup_{V_\beta(\varphi)\le 1} V_\alpha(G^*\varphi) \right) \cdot ||f||_{(\alpha)}.$$

Proof. Indeed,

$$\begin{aligned} ||Gf||_{(\beta)} &= \sup_{V_\beta(\varphi)\le 1} \int Gf \cdot \varphi = \sup_{V_\beta(\varphi)\le 1} \int f \cdot G^*\varphi \\ &\le \left(\sup_{V_\beta(\varphi)\le 1} V_\alpha(G^*\varphi) \right) \cdot \sup_{V_\alpha(\psi)\le 1} \int f \cdot \psi. \end{aligned}$$

■

Lemma 4.11

$$|||Q_\varepsilon - 1||| \equiv ||Q_\varepsilon - 1||_{(\beta\to\alpha)} := \sup_{||f||_{(\alpha)}\le 1} ||Q_\varepsilon f - f||_{(\beta)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Proof. Applying Lemma 4.10 to the operator $G = Q_\varepsilon - 1$ we get that the sufficient condition of the validity of the desired statement is the convergence of

$$\sup_{V_\beta(\varphi)\le 1} V_\alpha(Q_\varepsilon^*\varphi - \varphi) \to 0$$

as $\varepsilon \to 0$. Let us prove this convergence. Observe that since $\beta > \alpha$ and $\varphi \in \mathbf{C}^\beta$ we can get a stronger estimate compare to Lemma 4.8:

$$\begin{aligned} |Q_\varepsilon^*\varphi(x) - \varphi(x)| &= |\int q_\varepsilon(x,z)\varphi(z)\,dz - \varphi(x)| \\ &\le \int q_\varepsilon(x,z)|\varphi(z) - \varphi(x)|\,dz \\ &\le \varepsilon^\beta H_\beta(\varphi). \end{aligned}$$

Applying now estimates similar to ones used in the proof of Lemma 4.9 and taking into account that we consider more smooth test-functions $\varphi \in \mathbf{C}^\beta$ in the case $\rho(x,y) \le \varepsilon$ we get

$$\begin{aligned} &|(Q_\varepsilon^*\varphi(x) - \varphi(x)) - (Q_\varepsilon^*\varphi(y) - \varphi(y))| \\ &\le |Q_\varepsilon^*\varphi(x) - Q_\varepsilon^*\varphi(y)| + |\varphi(x) - \varphi(y)| \\ &\le \rho^\beta(x,y)H_\beta(\varphi) + M\rho(x,y)|\varphi|_\infty + \rho^\beta(x,y)H_\beta(\varphi) \\ &\le \left[2\varepsilon^{\beta-\alpha}H_\beta(\varphi) + M\varepsilon^{1-\alpha}|\varphi|_\infty\right]\rho^\alpha(x,y). \end{aligned}$$

While in the opposite case, when $\rho(x,y) > \varepsilon$, using the same argument as in the proof of Lemma 4.9 we get

$$\begin{aligned} |\,(Q_\varepsilon^*\varphi(x) - \varphi(x)) - (Q_\varepsilon^*\varphi(y) - \varphi(y))\,| &\le 2|Q_\varepsilon^*\varphi - \varphi|_\infty \le 2\varepsilon^\beta H_\beta(\varphi) \\ &\le 2\varepsilon^{\beta-\alpha} H_\beta(\varphi)\rho^\alpha(x,y). \end{aligned}$$

Hence for $\varphi \in \mathbf{C}^\beta$

$$H_\alpha(Q_\varepsilon^*\varphi - \varphi) \le 2\varepsilon^{\beta-\alpha} H_\beta(\varphi) + M\varepsilon^{1-\alpha}|\varphi|_\infty,$$

which yields the following estimate

$$\begin{aligned} V_\alpha(Q_\varepsilon^*\varphi - \varphi) &\le 2\varepsilon^{\beta-\alpha} H_\beta(\varphi) + M\varepsilon^{1-\alpha}|\varphi|_\infty + \varepsilon^\beta H_\beta(\varphi) \\ &\le 3\varepsilon^{\beta-\alpha} H_\beta(\varphi) + M\varepsilon^{1-\alpha}|\varphi|_\infty \le (3 + M\varepsilon^{1-\beta})\varepsilon^{\beta-\alpha} V_\alpha(\varphi) \to 0 \end{aligned}$$

as $\varepsilon \to 0$. ∎

The properties of the transition operator obtained above together with Theorem 4.1 under the additional assumption that $\Lambda_{\overline{T}}(\alpha) < 1/2$ make it possible to use results about the spectral stability of transfer operators satisfying Lasota-Yorke type inequalities [15] and to obtain the following stability result.

Theorem 4.2 *Let the conditions (4.8, 4.9) be satisfied and let $\Lambda_{\overline{T}}(\alpha) < 1/2$ for some $\alpha \in (0,1)$. Then all elements of the spectrum $\mathrm{sp}_{\mathcal{F}_\alpha}(\mathbf{P}_{\overline{T}})$ outside of the disk of radius $\Lambda_{\overline{T}}(\alpha)$ are stochastically stable and the corresponding eigenprojectors of the perturbed system converge to the genuine ones.*

Proof. First let us show that the transfer operator for the stochastically perturbed system satisfies a Lasota-Yorke type inequality. A straightforward calculation shows that this operator is equal to $Q_\varepsilon \mathbf{P}_{\overline{T}}$. Combining the results of Lemma 4.9 and Theorem 4.1 we get for any $h \in \mathcal{F}_\alpha$ that

$$\begin{aligned} \left|\left|Q_\varepsilon \mathbf{P}_{\overline{T}} h\right|\right|_{(\alpha)} &\le (1 + M_1(\varepsilon)) \left|\left|\mathbf{P}_{\overline{T}} h\right|\right|_{(\alpha)} \\ &\le (1 + M_1(\varepsilon))\kappa\Lambda_{\overline{T}}(\alpha) \left|\left|h\right|\right|_{(\alpha)} + \mathrm{Const} \cdot (\kappa - 2)^{-1/\alpha} \left|\left|h\right|\right|_{(\beta)} . \end{aligned}$$

Therefore, if $\Lambda_{\overline{T}}(\alpha) < 2$ for some $0 < \alpha < 1$, then the number $\gamma := (1 + M_1(\varepsilon))\kappa\Lambda_{\overline{T}}(\alpha) < 1$ for the value of $\kappa > 2$ guaranteed by Theorem 4.1. Since all other assumptions of the abstract spectral stability result in [15] were alredy checked during the analysis of our Banach spaces of generalized functions, we come to the desired statement. ∎

To proceed further we need to generalize the notion of the periodic turning point, well known in the one-dimensional dynamics. Namely, a point $x \in X$ is called the *periodic turning point* for the map $T : X \to X$ if $T^n x = x$ for some $n \in \mathbb{Z}_+$ and the derivative of the map T is not well defined at the point x.

Definition 4.1 A point $x \in X$ is called the *periodic turning point* for the *random* map $\overline{T} : X \to X$ if there is a finite collection of indices $i_1, i_2, \ldots, i_k$ such that the point x is the periodic turning point for the deterministic map $T_{i_1} \circ T_{i_2} \circ \ldots T_{i_k}$.

For example, the point $x = 1/2$ is the periodic turning point for the random map in the example 2 for any nontrivial distribution $(0 < p < 1)$.

Theorem 4.3 *The assumption $\Lambda_{\overline{T}}(\alpha) < 1/2$ can be replaced by the following: either all the maps T_i are bijective and C^1-differentiable, or they are piecewise C^1-differentiable and have no periodic turning points. Then all isolated eigenvalues are stochastically stable.*

Proof. The key idea here is to consider another representation of the perturbed operator $(Q_\varepsilon \mathbf{P}_{\overline{T}})^n = \tilde{Q}_{\varepsilon(n)} \mathbf{P}_{\overline{T}}^n$ and to show that the new operator $\tilde{Q}_{\varepsilon(n)}$ satisfies the same assumptions (4.8, 4.9) as the operator Q_ε (except that the value of $\varepsilon(n)$ might be in $1/\min\{\Lambda_{T_i}\}$ times larger). Therefore choosing n large enough we can always get $\Lambda_{\overline{T}}^{n}(\alpha) < 1/2$. This idea was first applied in the case of piecewise expanding maps in [2, 3] and then in the case of hyperbolic maps in [4]. If all maps T_i are bijective this can be done by a simple change of variables, while in the second case the construction is more involved but is completely similar to the one in [2, 3]. ∎

Let us show now what should be changed in the case of a general smooth manifold. Since locally in a neighborhood of a point $x \in X$ one can introduce local coordinates by means of the exponential map Ψ_x, the tangent linear space $\mathcal{T}_x X$ can be isometrically mapped into $\mathbb{R}^d$. Let $\nu > 0$ be a number such that for each point $x \in X$ the ball (v the metrics ρ) of radius ν centered at this point belongs to the domain of values of the exponential map Ψ_x. Note that we already have introduced the restriction on the distance between the points in the definition of the Hölder constant needing to be content with the domain of definition of the exponential map. In fact, the first difference appears only in the analysis of random perturbations, in particular, the condition (4.9) should be rewritten as

$$\int |q_\varepsilon(x,y) - q_\varepsilon(\Psi_x(\Psi_x^{-1}(x)+t), \Psi_y(\Psi_y^{-1}(y)+t))| \, dy \le \rho(x, \Psi_x(\Psi_x^{-1}(x)+t))M,$$

where $t \in \mathbb{R}^d$ and $|t| \le \nu$.

Assuming now that $\varepsilon < \nu$ and replacing the expressions of type $z + y - x$ to

$$\Psi_z\left(\Psi_z^{-1}(z) + \Psi_z^{-1}(y) - \Psi_z^{-1}(x)\right),$$

we obtain the same estimates as in the flat case (when X is the unit torus). Therefore all results of this section remain valid for the case of a general smooth manifold.

4.5 Finite rank approximations

Let us discuss now finite dimensional approximations of transfer operators. Again due to the same reason as in the previous section we shall restrict the analysis to the case of contracting on average random maps.

Let $\{\Delta_i\}_i$ be a finite partitions of the phase space X into domains (cells) Δ_i of diameter not larger than $\delta > 0$. For a point $x \in X$ by Δ_x we denote the element of the partition containing it. Under these notation the so called Ulam approximation can be described as an operator

$$\tilde{Q}_\delta f(x) := \frac{1}{|\Delta|}\int_{\Delta_x} f.$$

Note that this operator is selfdual, i.e. $\tilde{Q}_\delta = \tilde{Q}_\delta^*$. One can easily check also that the dimension of the space $\tilde{Q}_\delta \mathcal{F}_\alpha$ coincides with the number of elements in the Ulam partition.

Lemma 4.12 $||Q_\delta||_{(\alpha)} = \infty$.

Proof. Let a point $y_0 \in X$ belongs to the boundary between two elements of the Ulam partition, and let the points $y(\varepsilon)$ and $y'(\varepsilon)$ belong to neighboring elements of the partition both on the distance ε from y_0. For the function

$$f_\varepsilon(x) := \mathbf{1}_{y(\varepsilon)}(x) + \mathbf{1}_{y'(\varepsilon)}(x),$$

where $\mathbf{1}_y$ means the δ-function at the point y, the following inequalities hold:

$$||f_\varepsilon||_{(\alpha)} \le \text{Const}\,\varepsilon^\alpha,$$

$$||Q_\delta f_\varepsilon||_{(\alpha)} \ge \text{Const} > 0.$$

The first of these inequalities follows from the definition of the norm $||\cdot||_{(\alpha)}$, while the second one is a consequence of the fact that the function $Q_\delta f_\varepsilon$ is the characteristic function of the union of two neighboring elements of the partition containing the points $y(\varepsilon)$ and $y'(\varepsilon)$. Thus, $||Q_\delta f_\varepsilon||_{(\alpha)} / ||f_\varepsilon||_{(\alpha)} \to \infty$ as $\varepsilon \to 0$. ∎

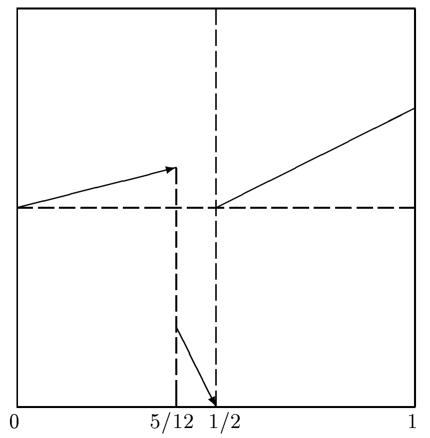

Figure 4:
Counterexample for the original Ulam construction.

This result shows that the original Ulam approximation scheme cannot be immediately applied to the spectral analysis of random maps.

What is still possible is that the leading eigenfunction – **SBR** measure may be stable for the class of maps we consider (the above example does not contradict to this – the **SBR** measure is preserved). I believe there should be very deep reasons explaining the stability of the leading eigenfunction, while all others are not stable, however presently we do not have the adequate explanation. In the literature (see, for example, [1] and further references therein) the stability of the **SBR** measure is proven for the class of piecewise expanding maps. Moreover numerous numerical studies confirm this stability for a much broader class of dynamical systems. To the best of our knowledge the following simple example of a one-dimensional discontinuous map represent the first counterexample to the original Ulam hypothesis.

Lemma 4.13 *The map*

$$Tx := \begin{cases} \frac{x}{4} + \frac{1}{2} & \text{if } 0 \le x < \frac{5}{12} \\ -2x + 1 & \text{if } \frac{5}{12} \le x < \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4} & \text{otherwise.} \end{cases}$$

from the unit interval into itself is uniquely ergodic, but the leading eigenvector of the Ulam approximation $\Pi_{1/n}\mathbf{P}_T$ does not converge weakly to the only T-invariant measure.

Observe that the situation when the **SBR** measure becomes unstable with respect to Ulam scheme is indeed very exotic and the map in the corresponding example is not only discontinuous (see Fig. 4), but this discontinuity occurs in a periodic turning point (compare to instability results about general random perturbations in [3]).

Proof. Denote by $v^{(n)}$ the normalized leading eigenvector of the matrix $\Pi_{1/n}\mathbf{P}_T$. A straightforward calculation shows that for each $n \in \mathbb{Z}_+^1$ all entries of the vector $v^{(2n+1)}$ are zeros except the first entry, which is equal to $1/3$ and the $(n+1)$-th one, which is equal to $2/3$. Compare this to the only invariant measure of the map T – the unit mass at the point $1/2$. ∎

To overcome this difficulty we consider another 'smoothed' approximation scheme. In each element of the partition $\{\Delta_i\}$ (of diametr $\le \delta$) we fix an arbitrary point (its 'center') $x_i \in \Delta_i$. Now for a given smooth enough kernel $q_\varepsilon(\cdot,\cdot)$ satisfying the assumptions from the previous section we define the following finite dimensional operator:

$$Q_{\varepsilon,\delta} f(x) := \sum_i 1_{\Delta_i}(x) \int q_\varepsilon(z, x_i) f(z)\, dz.$$

Observe that the dual operator is equal to:

$$Q^*_{\varepsilon,\delta}\varphi(x) := \sum_i q_\varepsilon(x, x_i) \int_{\Delta_i} \varphi(z)\, dz.$$

Indeed,

$$\begin{aligned}\int Q_{\varepsilon,\delta} f(x)\cdot\varphi(x)\;dx &= \int\sum_i 1_{\Delta_i}(x)\int q_\varepsilon(z,x_i)f(z)\;dz\cdot\varphi(x)\;dx\\ &= \int f(z)\sum_i q_\varepsilon(z,x_i)\left(\int 1_{\Delta_i}(x)\varphi(x)\;dx\right)\;dz\\ &= \int f(z)\sum_i q_\varepsilon(z,x_i)\int_{\Delta_i}\varphi(x)\;dx\;dz = \int f(z)\cdot Q^*_{\varepsilon,\delta}\varphi(z)\;dz.\end{aligned}$$

Lemma 4.14 *Let additionally to the assumptions (4.8, 4.9) for any points $x, y, z \in X$ the inequality*

$$|q_\varepsilon(x,y) - q_\varepsilon(x,z)| + |q_\varepsilon(y,x) - q_\varepsilon(z,x)| \le M\varepsilon^{-d-1}\rho(y,z). \tag{4.12}$$

holds. Then

$$||Q_\varepsilon f - Q_{\varepsilon,\delta} f||_{(\alpha)} \le 5M\varepsilon^{-d-1}\delta^{1-\alpha}\,||f||_{(\alpha)}\,.$$

Proof.

$$\int (Q_\varepsilon f - Q_{\varepsilon,\delta} f)\cdot\varphi = \int f\cdot(Q^*_\varepsilon\varphi - Q^*_{\varepsilon,\delta}\varphi).$$

Denote

$$\Phi(x) := Q^*_\varepsilon\varphi(x) - Q^*_{\varepsilon,\delta}\varphi(x) = \sum_i\int_{\Delta_i}(q_\varepsilon(x,z) - q_\varepsilon(x,x_i))\varphi(z)\;dz.$$

Then

$$\begin{aligned}|\Phi|_\infty &\le |\varphi|_\infty\cdot\sup_x\sum_i\int_{\Delta_i}|q_\varepsilon(x,z) - q_\varepsilon(x_i,x)|\;dz\\ &\le |\varphi|_\infty\cdot\sup_x\sum_i\int_{\Delta_i}\left(|q_\varepsilon(x,z) - q_\varepsilon(x,x_i)| + |q_\varepsilon(x,x_i) - q_\varepsilon(x_i,x)|\right)\;dz\\ &\le 2M\varepsilon^{-d-1}\delta|\varphi|_\infty,\end{aligned}$$

since $x, z \in \Delta_i$ and, hence, $\max\{\rho(z,x_i),\rho(x,x_i)\} \le \delta$.

Let us estimate $H_\alpha(\Phi)$. If $\rho(x,y) \le \delta$ then

$$\begin{aligned}|\Phi(x) - \Phi(y)| &\le |Q^*_\varepsilon\varphi(x) - Q^*_\varepsilon\varphi(y)| + |Q^*_{\varepsilon,\delta}\varphi(x) - Q^*_{\varepsilon,\delta}\varphi(y)|\\ &\le \int |q_\varepsilon(x,z) - q_\varepsilon(y,z)|\cdot|\varphi(z)|\;dz + \sum_i|q_\varepsilon(x,x_i) - q_\varepsilon(y,x_i)|\int_{\Delta_i}|\varphi|\\ &\le M\varepsilon^{-d-1}\rho(x,y)|\varphi|_\infty + M\varepsilon^{-d-1}\rho(x,y)\cdot|\varphi|_\infty = 2M\varepsilon^{-d-1}\delta^{1-\alpha}\rho^\alpha(x,y)\cdot|\varphi|_\infty.\end{aligned}$$

Otherwise if $\rho(x,y) > \delta$ we apply another estimate

$$|\Phi(x) - \Phi(y)| \le 2|\Phi|_\infty \le 4M\varepsilon^{-d-1}\delta|\varphi|_\infty \le 4M\varepsilon^{-d-1}\rho^\alpha(x,y)\cdot\delta^{1-\alpha}\cdot|\varphi|_\infty.$$

Thus,

$$H_\alpha(\Phi) \le 4M\varepsilon^{-d-1}\delta^{1-\alpha}\cdot|\varphi|_\infty,$$

and hence

$$V_\alpha(\Phi) \le 5M\varepsilon^{-d-1}\delta^{1-\alpha}\cdot V_\alpha(\varphi),$$

which yields the desired statement. ∎

Theorem 4.4 *Let the family of kernels $\{q_\varepsilon(\cdot,\cdot)\}$ satisfies the conditions (4.8, 4.9,4.12). Then*

$$\begin{aligned}||Q_{\varepsilon,\delta}||_{(\alpha)} &\le 1 + M_1(\varepsilon) + 5M\varepsilon^{-d-1}\delta^{1-\alpha},\\ |||Q_{\varepsilon,\delta} - 1||| &\le (5M + 3 + M\varepsilon^{1-\beta})(\varepsilon^{-d-1}\delta^{1-\alpha} + \varepsilon^{\beta-\alpha}) \to 0 \quad \text{as } \varepsilon^{-d-1}\delta^{1-\alpha} + \varepsilon^{\beta-\alpha} \to 0.\end{aligned}$$

Hence for the case $\Lambda_{\overline{T}}(\alpha) < 1/2$ the isolated eigenvalues and the corresponding eigenprojectors of the operator $\mathbf{P}_{\overline{T}}$ are stable with respect to the considered approximation.

Proof. According to Lemmas 4.9 and 4.14

$$||Q_{\varepsilon,\delta}||_{(\alpha)} \le ||Q_\varepsilon||_{(\alpha)} + ||Q_{\varepsilon,\delta} - Q_\varepsilon||_{(\alpha)} \le 1 + M_1(\varepsilon) + 5M\varepsilon^{-d-1}\delta^{1-\alpha},$$

which proves the first statement.

Similarly but using Lemma 4.11 instead of Lemma 4.9, we get

$$\begin{aligned} V_\alpha(Q_{\varepsilon,\delta}\varphi - \varphi) &\le V_\alpha(Q_{\varepsilon,\delta}\varphi - Q_\varepsilon\varphi) + V_\alpha(Q_\varepsilon\varphi - \varphi) \\ &\le \left(5M\varepsilon^{-d-1}\delta^{1-\alpha} + (3 + M\varepsilon^{1-\beta})\varepsilon^{\beta-\alpha}\right) \cdot V_\alpha(\varphi), \end{aligned}$$

which finishes the proof. ∎

In fact the finite dimensional approximation defined by the two-parameter family of operators $\{Q_{\varepsilon,\delta}\}_{\varepsilon,\delta}$ one can consider as a smoothed version of the original Ulam construction, which corresponds to the case $\varepsilon = 0$. Observe that in our approximations the relation between the parameters is completely different – it is necessary that $\varepsilon \gg \delta$.

We consider also another (seeming more natural) finite rank approximation scheme. Denote by Π_δ the pure Ulam approximation operator corresponding to the partition into domains $\{\Delta_i\}$ whose diameters do not exceed δ:

$$\Pi_\delta f(x) := \frac{1}{|\Delta_x|} \int_{\Delta_x} f(s)\ ds,$$

where Δ_x stands for the element of the partition containing the point x. Note that this operator is self adjoint. We shall approximate our transfer operator $\mathbf{P}_{\overline{T}}$ by $\Pi_\delta Q_\varepsilon \mathbf{P}_{\overline{T}}$. To study the properties of this approximation we need as usual to analyze properties of the adjoint operator, i.e. of the operator

$$Q_\varepsilon^* \Pi_\delta^* \varphi(x) = \int q_\varepsilon(x,z) \frac{1}{|\Delta_z|} \int_{\Delta_x} \varphi(s)\ ds\ dz.$$

Lemma 4.15 $||Q_\varepsilon - \Pi_\delta Q_\varepsilon||_{(\alpha)} \le (3 + 2M)\varepsilon^{-d-1}\delta^{\alpha(1-\alpha)} \to 0$ *as* $\varepsilon^{-d-1}\delta^{\alpha(1-\alpha)} \to 0$.

Proof. Denote

$$\begin{aligned} \Phi(x) &:= Q_\varepsilon^*\varphi(x) - Q_\varepsilon^*\Pi_\delta^*\varphi(x) = \int q_\varepsilon(x,z) \left(\varphi(z) - \frac{1}{|\Delta_z|}\int_{\Delta_x} \varphi(s)\ ds\right)\ dz \\ &= \int q_\varepsilon(x,z) \frac{1}{|\Delta_z|} \int_{\Delta_x} (\varphi(z) - \varphi(s))\ ds\ dz. \end{aligned}$$

Since $\varphi \in \mathbf{C}^\alpha$ and the diameter of the elements of the partition does not exceed δ, we have

$$|\Phi|_\infty \le \delta^\alpha H_\alpha(\varphi) \sup_x \int q_\varepsilon(x,z)\ dz = \delta^\alpha H_\alpha(\varphi).$$

Now we are going to estimate the Hölder constant of the function Φ. which we shall do in two steps. First, we consider the case when $\rho(x,y) \le \delta^\alpha$:

$$\begin{aligned} |\Phi(x) - \Phi(y)| &\le |Q_\varepsilon^*\varphi(x) - Q_\varepsilon^*\varphi(y)| + |Q_\varepsilon^*\Pi_\delta^*\varphi(x) - Q_\varepsilon^*\Pi_\delta^*\varphi(y)| \\ &\le \int |q_\varepsilon(x,z) - q_\varepsilon(y,z)| \cdot |\varphi|_\infty\ dz + \int |q_\varepsilon(x,z) - q_\varepsilon(y,z)| \cdot \frac{1}{|\Delta_z|} \int_{\Delta_z} |\varphi(s)|\ ds\ dz \\ &\le 2M\varepsilon^{-d-1}\rho(x,y)|\varphi|_\infty \le 2M\varepsilon^{-d-1}\delta^{\alpha(1-\alpha)}|\varphi|_\infty \rho^\alpha(x,y). \end{aligned}$$

In the opposite case, when $r(x,y) > \delta^\alpha$ we use a different estimate:

$$|\Phi(x) - \Phi(y)| \le 2|\Phi|_\infty \le 2\delta^\alpha H_\alpha(\varphi) \le 2\delta^{\alpha(1-\alpha)} H_\alpha(\varphi)\rho^\alpha(x,y).$$

Thus

$$\begin{aligned} V_\alpha(\Phi) &\le 2\delta^{\alpha(1-\alpha)} H_\alpha(\varphi) + 2M\varepsilon^{-d-1}\delta^{\alpha(1-\alpha)}|\varphi|_\infty + \delta^\alpha H_\alpha(\varphi) \\ &\le 3\delta^{\alpha(1-\alpha)} H_\alpha(\varphi) + 2M\varepsilon^{-d-1}\delta^{\alpha(1-\alpha)}|\varphi|_\infty \le (3 + 2M)\varepsilon^{-d-1}\delta^{\alpha(1-\alpha)} V_\alpha(\varphi) \to 0 \end{aligned}$$

as $\varepsilon^{-d-1}\delta^{\alpha(1-\alpha)} \to 0$. ■

Observe that the rate of convergence in this approximation is lower compare to the previous one, however the numerical application of the 2nd scheme is more straitforward.

Corollary 4.16 *Again as in Theorem 4.3 and due to the same reason the assumption $\Lambda_{\overline{T}}(\alpha) < 1/2$ can be replaced by either the bijectivity of the maps T_i or the absence of its periodic turning points.*

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