Spectral Analysis of Transport Operator: Functional Model Approach

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Abstract

This paper presents the spectral analysis of the dissipative one-speed transport operator for the cases of a finite slab and of a finite body, by means of the Szokefalvi-Nagy - Foias functional model. It is shown that the essential spectrum of the operator is absolutely continuous with one possible spectral singularity at point 0. The absolutely continuous component of the operator is studied in detail in the nontrivial case, when the spectral singularity does occur.

1 Introduction

The spectral theory of nonselfadjoint operators based on the functional model of Szokefalvi-Nagy and Foias [1] made significant progress towards understanding the structure of operators with discrete spectrum (see e.g. [2]). At the same time, the structure of operators with absolutely continuous spectrum is less understood and the theory is still short of applications. So far, these essentially amount to the Schrödinger operator with complex potential [3]. In the present paper we analyze an example of a dissipative operator with absolutely continuous spectrum coming from physics - the one-speed transport operator, and solve a relevant abstract problem in the theory applying then the result to the example.

It is well-known that the nonselfadjoint one-speed transport operator acting in the Hilbert space L^2 of distribution functions has rich essential spectrum [4] for various geometrical situations. As a rule this spectrum fills a set of positive measure in $\mathbb{C}$. The general theory of functional model is not developed far enough to be applied effectively in this case. Another situation arises if we consider the transport operator for a piece of multiplicative medium embedded in a pure absorbent filling the configurational space $\mathbb{R}^n$. Since the paper [5] it is known that in this case the essential spectrum coincides with the real line. The analysis in that paper was based on the direct study of the resolvent by elementary methods. The idea of the present paper to apply the functional model to the spectral analysis of the operator appears to be natural in this situation. This reduces the problems in spectral analysis of the operator to the questions of complex analysis. We fix our attention on two geometrical situations,

1. A slab of multiplicative medium of finite width.

2. A piece of muliplicative medium of finite diameter in the three-dimensional space.

Multiplicative properties of the medium are described by the local mean number of the secondaries per collision c, a nonnegative function on the configurational space Λ, $\Lambda = \mathbb{R}$ in case **1**, and $\Lambda = \mathbb{R}^3$ in case **2**. We suppose throughout that c is essentially bounded.

For case **1** it was shown in [5] that the spectrum of the transport operator for finite slab with constant c consists of finitely many eigenvalues lying on the imaginary axis and the essential spectrum which fills the real axis. Analysis of the proof shows that this result holds for arbitrary compactly supported $c \in L^\infty(\mathbb{R})$ and makes it possible to give an estimate of the Birman-Schwinger type for the dimension of the subspace corresponding to the discrete spectrum [6]. For c small in an appropriate sense, we also give an estimate of the angle between the discrete and essential spectrum subspaces of the operator in terms of c in a closed form.

In case **2**, for arbitrary compactly supported $c \in L^\infty(\mathbb{R}^3)$, the geometrical picture of the spectrum is shown to remain the same as for the slab geometry, and a Birman - Schwinger type estimate for the dimension of the discrete spectrum subspace is also provided.

All the other results are obtained by the study of the component corresponding to the essential spectrum by using the functional model. For both situations we show that the essential spectrum of the transport operator is purely absolutely continuous. It is proven that the corresponding component of the operator is similar to a selfadjoint operator if the function $c \notin \mathcal{E}$ for a certain singular set $\mathcal{E} \subset L^\infty(\Lambda)$. Moreover, if we take the operator corresponding to the transport in a perfect absorber for the generator of unperturbed dynamics, then the wave operators exist and are complete.

As is well-known [3, 7], an operator with absolutely continuous spectrum can fail to be similar to a selfadjoint one owing to the presence of so called spectral singularities. Roughly speaking, the spectral singularities are those points of the absolutely continuous spectrum of the operator at which the resolvent has growth of more than the first order. The differential operators with spectral singularities were studied in the papers by Naimark, Ljance and Pavlov (see [3, 8, 9] and references therein). In the situation under consideration for $c \in \mathcal{E}$ the operator is shown to have a unique point of spectral singularity at 0. We prove that the absolutely continuous component of the operator is similar to the orthogonal sum of a selfadjoint operator and an operator with spectrum of finite multiplicity $\mathcal{M}$, which is calculated in terms of c. For the spectral component of the transport operator corresponding to a neighborhood of the spectral singularity we also give an estimate of the angle between the corresponding invariant subspaces.

The proofs of these results are based on the analysis of the behaviour of the characteristic function of the operator. The characteristic function of a dissipative operator L is a contractive analytic operator function in the upper half plane which can be considered as an operator analog of the perturbation determinant for the pair (L, L^*). The information about boundary behavior of the characteristic function turns into conclusions about structure of operators by general theorems of functional model theory. One should mention that the questions arising here are more difficult than those of the selfadjoint theory where the singularities of the resolvent correspond to zeroes of R-functions [10]. The study of the characteristic function is reduced, to some extent, to that of a scalar analytic function by the presence of a scalar multiple. The proof of the absolute continuity of the spectrum essentially amounts to the verification of triviality of the singular inner factor for the scalar multiple. The existence and completeness of the wave operators in the situation under consideration are facts of the general theory [11]. The splitting of the absolutely continuous component is based on separation of some invariant subspaces defined in the model representation of the operator through the spectral decomposition of $\Delta = 1 - S^*S$, S being the characteristic function. These subspaces are defined in such a way that the estimate of the angle between them is reduced to the estimate of the function S^{-1} on the real axis. The suggested construction of the invariant subspaces corresponding to the absolutely continuous spectrum with the estimate of the angle between them is the main abstract result of the paper. It is presented in lemma 2.5 and corollary 2.6.

The proofs are completely parallel for the cases **1** and **2**, with some simplification in the second case.

One should emphasize that the functional model is not involved in the statement of results about the transport operator and is only used as a tool for their derivation. It should also be stressed that the appearance of the spectral singularity in our problem is not a kind of pathology. Namely, it will be seen that for any nonzero compactly supported $c \in L^\infty$ the function κc belongs to $\mathcal{E}$ for an infinite discrete set of values of the constant κ. This distinguishes the transport operator from the other examples of operators with spectral singularities known in mathematical physics, like the Schrödinger operator with complex potential [8, 9] where the spectral singularities appear *ad hoc*, for specially constructed potentials.

The structure of the paper is the following. In section 2 we give a brief description of the functional model of a dissipative operator. We use the symmetric form of the functional model (see [7, 12]). Then the abstract construction of invariant subspaces is described. Sections 3 and 4 are devoted to the spectral analysis of the transport operator for the slab and for the finite piece respectively. In the slab case for completeness we provide a derivation of the spectrum, further simplified as compared to [5, 6, 13], and of the estimate of the discrete spectrum from our paper [6] (Theorem 1, section 3.2). The spectrum of the transport operator for the finite body is derived in Theorem 3, section 4.2. The estimate of the angle between discrete and essential spectrum subspaces for the slab is given in proposition 3.1, section 3.2. The absolute continuity of the essential spectrum is established in propositions 3.6, 4.4. The remaining parts of sections 3.3 and 4.2 are concerned with the derivation of the results on the structure of the operators in the case $c \in \mathcal{E}$ which are resumed in Theorems 2, 4.

1.1 Notations and necessary standard facts

We use the same notation for analogous objects in sections 3, 4 wherever confusion is impossible. Throughout the paper $|\cdot|_\infty$ stands for the L^∞-norm of functions and $\|\cdot\|_p$ for the norm of operators in classes $\mathbf{S}^p$, $\widehat{f}$ for the Fourier transform of a function $f \in L^2(\mathbb{R})$, χ_M is the indicator of a set $M \subset \mathbb{R}$, $U_\delta(z) = \{z' \in \mathbb{C} : \ |z - z'| < \delta\}$,

$\omega_\delta(z) = U_\delta(z) \bigcap \mathbb{C}_+$, $\mathbb{C}_+$ being the open upper half plane; $\Pi_\epsilon = \{z : \ 0 < \Im z < \epsilon\}$, $[z_1, z_2) = \{z = tz_1 + (1-t)z_2, \ t \in (0,1]\}$ for $z_{1,2} \in \mathbb{C}$. The term measurable refers to the Lebesgue measure on $\mathbb{R}$. Parentheses $(\cdot,\cdot)$ stand for the angle between subspaces of a Hilbert space. For an operator A on a Hilbert space H we write $\sigma_+(A) = \sigma(A) \bigcap \mathbb{C}_+$. We adopt the definition of the essential spectrum of a closed operator used in [14], vol. 4.

In Section 2 below we introduce some spaces L^2 of vector functions with operator weights, defined in the following way. Let E be a Hilbert space and $\Xi : \mathbb{R} \to \mathbf{B}(E)$ be a bounded measurable function such that $\Xi \geq 0$. By $L^2(\mathbb{R}; \Xi)$, or $L^2(\Xi)$, we denote the Hilbert space obtained by the closure of the linear set of equivalence classes $L^2(\mathbb{R}, E)/\ker \Xi$ in the metric given by the weight Ξ. We do not distinguish in our notation an element $f \in L^2(\mathbb{R}, E)$ and the corresponding equivalence class in $L^2(\mathbb{R}; \Xi)$.

Let L be a dissipative operator, $L = L_0 + iV$, $V = \Im L$ is bounded, $R(z) = (L-z)^{-1}$, $R_0(z) = (L_0 - z)^{-1}$. Then the resolvent identity $R(z) - R_0(z) = -iR_0(z) VR(z)$ holds for $z \in \mathbb{C}_+ \setminus \sigma_+(L)$. Simple algebraic manipulations (see e. g. [15][1]) show that

$$R(z) = R_0(z) - iR_0(z)\sqrt{V}\left(I + i\sqrt{V}R_0(z)\sqrt{V}\right)^{-1}\sqrt{V}R_0(z), \tag{1.1}$$
$$\left(I + i\sqrt{V}R_0(z)\sqrt{V}\right)\left(I - i\sqrt{V}R(z)\sqrt{V}\right) = I.$$

It follows that $\sigma_+(L) = \{z \in \mathbb{C}_+ : \ -1 \in \sigma\left(Q(z)\right)\}$ where

$$Q(z) = i\sqrt{V}R_0(z)\sqrt{V} \tag{1.2}$$

is an analytic operator function on $\mathbb{C}_+$. The following is a version of the Weil theorem on relatively compact perturbations convenient for our proposes.

Sublemma. *If $Q(z)$ is a compact operator (at least at one point $z \in \mathbb{C}_+$ and then at all points) then $\sigma_{ess}(L) = \sigma_{ess}(L_0)$.*

Proof. First

$$\Re Q(z) = \frac{i}{2}\left(\sqrt{V}(R_0(z) - R_0(\overline{z}))\sqrt{V}\right) = -\Im z\sqrt{V}R_0(z)\cdot R_0(\overline{z})\sqrt{V} \tag{1.3}$$

is compact along with $Q(z)$. It follows that $R_0(z)\sqrt{V}$ is also compact for $\Im z \neq 0$. From the resolvent identity one now concludes that $R(z) - R_0(z)$ is compact for $z \in \rho(L) \cap \rho(L_0)$, and the result follows from the Weil theorem [14]. ■

In the following we preserve the notation $Q(z)$ for the restriction of the r.h.s. of (1.2) to $\overline{\operatorname{Ran} V}$.

[1] In [15] the perturbation is assumed to be selfadjoint, which is of no importance for the manipulations.

An analytic operator function $A(z)$ in the upper half plane $\mathbb{C}_+$ is called an R *-function* if $\Im A(z) \geq 0$ for all $z \in \mathbb{C}_+$. The following remark results from a general representation theorem for R - functions [10, Theorem 2.2, Remark 1].

Remark 1.1. If $A(z)$ is an R - function in $\mathbb{C}_+$ then $\sup_{|z|\leq 1} \Im z \, \|A(z)\|$ is finite.

Although the following simple remark also results immediately from the representation theorem, below we give an elementary derivation of it.

Remark 1.2. If an R - function $A(z)$ is analytic and selfadjoint on the negative real axis, then $A(k)$ is a monotone increasing function for $k < 0$.

Proof. It is enough to verify the assertion for scalar functions. Since for any $k_0 < 0$ the numbers $A(k_0)$ and $A'(k_0)$ are real, it follows that $A'(k_0) = \lim_{\varepsilon\to 0+} \varepsilon^{-1} \Im A(k_0 + i\varepsilon)$ is nonnegative. If $A'(k_0) = 0$, then for some $j \geq 2$ we would have $A(z) = A(k_0) + (z - k_0)^j A_j (1 + O(z - k_0))$ with an $A_j \neq 0$. Now considering $z = k_0 + \varepsilon e^{i\varphi}$, $\varepsilon \to 0$, for an appropriate $\varphi \in (0, \pi)$, one arrives at a contradiction with the property of being R -function. ■

Let $S : \mathbb{C}_+ \to \mathbf{B}(E)$, E being a Hilbert space, be a bounded analytic operator - function. A scalar function $m(z) \not\equiv 0$ in $\mathbb{C}_+$ is called a scalar multiple for S if there exists a bounded analytic operator - function $\Omega(z)$ in $\mathbb{C}_+$ such that $m(z)I = S(z)\Omega(z) = \Omega(z)S(z)$ for all $z \in \mathbb{C}_+$ [12].

The determinant $\det(I + A)$ is defined for any $A \in \mathbf{S}^1$ [16]. The function $A \mapsto \det(I + A)$ is continuous in the $\mathbf{S}^1$ - norm. Also

$$|\det(I + A)| \leq \exp\left(\|A\|_1\right). \tag{1.4}$$

If $\ker(I + A) \neq \{0\}$ then $F = \det(I + A)(I + A)^{-1}$ satisfies

$$\|F\| \leq \exp\left(\|A\|_1\right). \tag{1.5}$$

If $A(z)$ is an analytic[2] $\mathbf{S}^1$ - valued operator function on a domain $\mathcal{D} \subset \mathbb{C}$ then $a(z) = \det\left(I + A(z)\right)$ is a scalar analytic function on $\mathcal{D}$.

The following lemma is essentially proved in [17].

Lemma 1.3. *If $A(\cdot)$ is an $\mathbf{S}^p$ - valued ($p \in \mathbb{N}$) function on a domain $\mathcal{D} \subset \mathbb{C}$ analytic in the operator norm and satisfying $\sup_{z\in\mathcal{D}} \|A(z)\|_p < \infty$ then the function $s(z) = \det\left(I - A^p(z)\right)$ is a scalar multiple for $S(z) = I - A(z)$ if $s(z) \not\equiv 0$.*

[2]It is enough to verify that the function is analytic in the operator norm and locally bounded in the $\mathbf{S}^1$ - norm.

Proof. From (1.4) and the inequality $\|A^p(z)\|_1 \le \|A(z)\|_p^p$ one sees that $s(z)$ is a bounded analytic function on $\mathcal{D}$. Assume $s(z) \not\equiv 0$. Put $F(z) = s(z)\,(I - A^p(z))^{-1}$. We have

$$s(z)I = F(z)\,(I - A^p(z)) = F(z)\left(\sum_{j=0}^{p-1} A^j(z)\right)(I - A(z))\,.$$

Put $\Omega(z) = F(z)\sum_{j=0}^{p-1} A^j(z)$. In view of the estimate (1.5) the function $F(\cdot)$ is analytic and bounded in $\mathcal{D}$, and so is $\Omega(\cdot)$. Since $\Omega(z)$ and $A(z)$ commute, the proof is completed. ■

2 Functional Model and Angles

Let L be a closed dissipative operator in a Hilbert space H with bounded imaginary part $V = \Im L$ such that $\sigma_{ess}(L) \subset \mathbb{R}$, and let $E = \overline{\mathrm{Ran} V}$. The characteristic function $S(z):\ E \to E$, $z \in \mathbb{C}_+$, of the operator L is defined by the formula

$$S(z) = I + 2i\sqrt{V}\,(L^* - z)^{-1}\sqrt{V}.$$

This is a contractive analytic function in $\mathbb{C}_+$. This function has the nontangential boundary values $S(k) \equiv S(k + i0)$ on the real axis in the strong sense for a.e. $k \in \mathbb{R}$. Let L_{pure} be the completely nonselfadjoint part of L. Then the set $\sigma(L_{pure}) \bigcap \mathbb{R}$ conicides with the complement (in $\mathbb{R}$) of the union of open intervals on which $S(z)$ is analytic and unitary. For $z \in \mathbb{C}_+ \bigcap \rho(L)$ the operator $S(z)$ is boundedly invertible on E. It can easily be shown by manipulations with the resolvent identity that $S(z)$ is expressed in terms of the operator $Q(z)$ given by (1.2) as follows [18]:

$$S(z) = \frac{I + Q(z)}{I - Q(z)}, \quad z \in \mathbb{C}_+. \tag{2.1}$$

Remark that the operator $I - Q(z)$ has the bounded inverse since $\Re Q(z) \le 0$ for all $z \in \mathbb{C}_+$. This formula shows, in particular, that $S(z) - I \in \mathbf{S}^p$ whenever $Q(z) \in \mathbf{S}^p$ and that $S(z)$ is analytic at a point on the real axis if so does $Q(z)$.

The matrix $\begin{pmatrix} I & S^*(k) \\ S(k) & I \end{pmatrix}$ defines a nonnegative operator in $E \oplus E$ for a. e. $k \in \mathbb{R}$ which allows us to define the Hilbert space $\mathcal{X} = L^2\begin{pmatrix} I & S^* \\ S & I \end{pmatrix}$. Let $H^2_\pm(E)$ be the Hardy classes of E-valued functions f analytic in $\mathbb{C}_\pm$, respectively, and satisfying

$\sup_{\varepsilon>0}\int_{\mathbb{R}}\|f(k\pm i\varepsilon)\|_E^2\,dk<\infty$. Define for a representative $(\tilde{g},g)\in L^2(\mathbb{R},E\oplus E)$ from the equivalence class of an element of $\mathcal{X}$

$$P_{\mathcal{K}}\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}=\begin{pmatrix}\tilde{g}-P_+(\tilde{g}+S^*g)\\ g-P_-(S\tilde{g}+g)\end{pmatrix} \tag{2.2}$$

where $P_\pm$ are the Riesz projections on $H^2_\pm(E)$ in $L^2(E)$. Then [17], $P_{\mathcal{K}}$ is a correctly defined operator in $\mathcal{X}$ and its closure in $\mathcal{X}$ which will be denoted with the same letter, is an orthogonal projection. Let $\mathcal{K}$ be the range of $P_{\mathcal{K}}$ in $\mathcal{X}$. Define a unitary group $\mathcal{U}_t$ of operators in $\mathcal{X}$ by the formula $(\mathcal{U}_tf)(k)=e^{ikt}f(k)$. Then the completely nonselfadjoint part of L is unitarily equivalent to the generator of the contraction semigroup $Z_t=P_{\mathcal{K}}\mathcal{U}_t\mid_{\mathcal{K}}$. This generator is called the functional model of the operator L_{pure}.

We use theorems on dissipative operators corresponding to theorems stated for contractions in the book [1] without special explanations. Also the definitions of the functional - theoretic objects related to the upper half plane like inner and outer functions etc. are adopted which correspond naturally to those introduced in [1] for the unit circle. For example, a bounded analytic function $\Theta:\mathbb{C}_+\to\mathbf{B}(E)$ is called *outer* if $\overline{\Theta H^2_+(E)}=H^2_+(E)$. Then, the characteristic function admits the *canonical factorization* in a product of two contractive analytic $\mathbf{B}(E)$ - valued functions of the form $S=S_iS_e$ where S_e is an outer function, and $S_i(k)$ is isometric for a.e. $k\in\mathbb{R}$.

Throughout the paper, by an *invariant subspace* of an operator we mean a regular invariant subspace; that is, a subspace $\mathcal{H}$ is called an invariant subspace of L if $\overline{(L-\lambda)^{-1}\mathcal{H}}=\mathcal{H}$ for all $\lambda\in\rho(L)$.

Remark 2.1. Every regular invariant subspace $\mathcal{H}$ is invariant in the following natural sense,

(i) $\mathcal{D}(L)\bigcap\mathcal{H}=(L-\lambda)^{-1}\mathcal{H}$, $\lambda\in\rho(L)$,

(ii) $L(\mathcal{D}(L)\bigcap\mathcal{H})\subset\mathcal{H}$.

Proof. First, (ii) implies (i) in view of the identity $(L-z)^{-1}Lf-z(L-z)^{-1}f=f$. Then $in(L+in)^{-1}\to I$ as $n\to+\infty$ in the strong sense by the functional calculus [1, Theorem III.2.1]. Hence $in(L+in)^{-1}Lf\to Lf$ for any $f\in\mathcal{D}(L)\bigcap\mathcal{H}$. Now, the identity $(L+in)^{-1}Lf=f-in(L+in)^{-1}f$ shows that the left hand side belongs to $\mathcal{H}$ for all $n>0$ by the regular invariance of $\mathcal{H}$, and (ii) follows. ■

It follows that the restriction $L_{\mathcal{H}}=L|_{\mathcal{H}}$ with the domain $\mathcal{D}(L)\bigcap\mathcal{H}$ is closed and densely defined.

In the following, we define some invariant subspaces of the completely nonselfadjoint part of L in terms of its functional model. In doing so, to keep notation at minimum, we omit in formulas the operator that accomplishes the unitary equivalence between L_{pure} and its functional model.

Let $\Delta(k) = I - S^*(k)S(k)$. Since $\Delta(k) \geq 0$ for a.e. $k \in \mathbb{R}$, one can define the space $L^2(\mathbb{R};\Delta)$. Note that for any $\tilde{g} \in L^2(\mathbb{R};\Delta)$ the vector $\begin{pmatrix} \tilde{g} \\ -S\tilde{g} \end{pmatrix}$ defines an element of $\mathcal{X}$ of the norm $\|\tilde{g}\|_{L^2(\mathbb{R};\Delta)}$. *The absolutely continuous subspace* $\mathcal{N}_e \subset \mathcal{K}$ of the operator L [7, 19] is defined as the closure of the linear set $\widetilde{\mathcal{N}_e}$ of smooth vectors:

$$\mathcal{N}_e = \overline{\widetilde{\mathcal{N}_e}}, \qquad \widetilde{\mathcal{N}_e} \equiv \left\{ P_{\mathcal{K}} \begin{pmatrix} \tilde{g} \\ -S\tilde{g} \end{pmatrix}, \ \tilde{g} \in L^2(\mathbb{R};\Delta) \right\}.$$

Then $\mathcal{N}_e$ coincides with the invariant subspace of L corresponding to the canonical factorization of the characteristic function in the sense that the characteristic function of $L|_{\mathcal{N}_e}$ coincides with the pure part of S_e (see [7]). We call the spectrum of L *purely absolutely continuous* if $H = H_0 \oplus \mathcal{N}_e$ where H_0 is an invariant subspace of L such that $L|_{H_0}$ is a selfadjoint operator with absolutely continuous spectrum in the sense of spectral theory of selfadjoint operators [14]. The set of smooth vectors of a completely nonselfadjoint dissipative operator admits the following description in model-free terms, [12, Theorem 4, Corollary 1] $\widetilde{\mathcal{N}_e} = \left\{ u \in H : \ \sqrt{V}\,(L - \cdot)^{-1} u \in H_2^+(E) \right\}$. This description allows to motivate the suggested definition of the absolutely continuous subspace from the viewpoint of the theory smooth perturbations [11]. Another interpretation of the definition stems from the fact [20] that a suitable normalized system of generalized eigenfunctions corresponding to the spectrum of $L|_{\mathcal{N}_e}$ is complete in $\mathcal{N}_e$. These eigenfunctions are constructed in terms of generalized limits of the resolvent $R(k \pm i0)$ on the real axis. We refer the reader to [21, 22] and references therein for the further discussion of this definition.

Let us define a bounded operator $W : L^2(\mathbb{R};\Delta) \to \mathcal{K}$ by setting

$$W : \tilde{g} \mapsto P_{\mathcal{K}} \begin{pmatrix} \tilde{g} \\ -S\tilde{g} \end{pmatrix}. \tag{2.3}$$

The norm of a smooth vector $W\tilde{g}$ is given by[3] [20]

$$\|W\tilde{g}\|^2 = \left\| S\sqrt{\Delta}\tilde{g} \right\|^2 + \|P_- \Delta \tilde{g}\|^2. \tag{2.4}$$

Then, [12, Theorem 4, Corollary 1],

$$(L - z)^{-1} W \ = \ W\,(A_0 - z)^{-1}, \quad z \in \rho(L) \tag{2.5}$$

where A_0 is the operator of multiplication by the independent variable in $L^2(\mathbb{R};\Delta)$. Note that $\mathcal{N}_e \subset H_{ess}$ where H_{ess} is an invariant subspace of L naturally associated

[3]In this section for E-valued functions on the real axis $\|\cdot\|$ assumes the norm in $L^2(\mathbb{R}, E)$, unless otherwise specified.

with the real spectrum as follows: $H_{ess} = \bigcap \ker \mathcal{P}_d$ where $\mathcal{P}_d$ ranges over the Riesz projections for isolated finite portions of the spectrum $\sigma_+(L)$. Indeed, it follows from (2.5) by the integration over a closed contour in $\mathbb{C}_+$ encircling a portion that $\mathcal{P}_d W = 0$ for each $\mathcal{P}_d$, and the closure gives $\mathcal{P}_d \mathcal{N}_e = 0$.

The following lemma describes a class of invariant subspaces of the operator L in $\mathcal{N}_e$. Let $\{X(k)\}$ be a measurable family of subspaces of E defined for a. e. $k \in \mathbb{R}$, and let $X = \{f \in L^2(\mathbb{R}, E) : \ f(k) \in X(k) \text{ for a.e. } k\}$. Define H to be the closure of X in $L^2(\mathbb{R}; \Delta)$. For an operator A in a Hilbert space $\mathcal{A}$ a subspace $\mathcal{J} \subset \mathcal{A}$ is called generating if $\mathcal{A} = \bigvee_{\lambda \in \rho(A)} (A - \lambda)^{-1} \mathcal{J}$. We define the *multiplicity of the spectrum* of an operator A as the number $m(A) = \inf \dim \mathcal{N}$, where $\mathcal{N}$ ranges over the generating subspaces of A.

Lemma 2.2. *$\mathcal{H} = \overline{W\mathsf{H}}$ is an invariant subspace of the operator L, and*

$$m\left(L_{\mathcal{H}}\right) \leq ess\ sup_{k \in \mathbb{R}} \dim X(k). \tag{2.6}$$

If $S(k)$ is invertible a. e. on the real axis in the wide sense, that is, $\ker S(k) = \{0\}$ for a. e. $k \in \mathbb{R}$, then

$$m\left(L_{\mathcal{H}}\right) = ess\ sup_{k \in \mathbb{R}} \dim \Delta(k) X(k). \tag{2.7}$$

If $ess\ sup_{k \in \mathbb{R}} \left\| \left(S(k)|_{\Delta(k)X(k)}\right)^{-1} \right\| < \infty$ then $W|_{\mathsf{H}}$ has a bounded inverse, and thus $L_{\mathcal{H}}$ is similar to a selfadjoint operator.

P r o o f. Since the space H reduces A_0, the invariance of $\mathcal{H}$ follows from the intertwining relation (2.5).

Let $\mathcal{J}$ be a generating subspace for $A_0|_{\mathsf{H}}$, thus $\mathsf{H} = \bigvee_{\Im\lambda \neq 0} (A_0 - \lambda)^{-1} \mathcal{J}$. Since obviously $\bigvee_{\lambda \in \mathbb{C}_+} (A_0 - \lambda)^{-1} \mathcal{J} = \bigvee_{\lambda \in \rho(L) \cap \mathbb{C}_+} (A_0 - \lambda)^{-1} \mathcal{J}$, one obtains that $\overline{W\mathcal{J}}$ is a generating subspace for $L_{\mathcal{H}}$. This implies (2.6).

By (2.4) the bounded operator $W^{-1} : \mathcal{N}_e \to L^2(\mathbb{R}; S^* S \Delta)$ is defined. Let $\widetilde{A}_0$ be the operator of multiplication by the independent variable in $L^2(\mathbb{R}; S^* S \Delta)$. One sees from the intertwining relation

$$W^{-1} (L - z)^{-1} = \left(\widetilde{A}_0 - z\right)^{-1} W^{-1}, \quad z \in \rho(L), \tag{2.8}$$

that for any generating subspace $\mathcal{N}$ of $L_{\mathcal{H}}$ $\overline{W^{-1}\mathcal{N}}$ is a generating subspace for the restriction of $\widetilde{A}_0$ to H_*, H_* being the closure of X in $L^2(\mathbb{R}; S^* S \Delta)$. Thus

$$m\left(L_{\mathcal{H}}\right) \geq m\left(\widetilde{A}_0\Big|_{\mathsf{H}_*}\right) = m\left(A|_{\overline{S^* S \Delta X}}\right) = \text{ess sup}_{k \in \mathbb{R}} \dim \Delta(k) X(k)$$

since $\dim (S^*S\Delta)(k)X(k) = \dim \Delta(k)X(k)$ for a. e. $k \in \mathbb{R}$ by the assumption about S. Here A is the operator of multiplication by the independent variable in $L^2(\mathbb{R}, E)$.

The last assertion of the lemma results from the following estimate valid for $\tilde{v} \in \mathsf{H}$ by (2.4)

$$\|W\tilde{v}\| \geq \left\|S\sqrt{\Delta}\tilde{v}\right\| \geq \left\|\left(S|_{\sqrt{\Delta}X}\right)^{-1}\right\|^{-1} \|\tilde{v}\|_{L^2(\mathbb{R};\Delta)}.$$

■

Note that a special case of the formula (2.7) was obtained in paper [19] when studying the triangle model of a dissipative operator.

Corollary 2.3. *[20] For a Borel set $\omega \subset \mathbb{R}$ let χ_ω be the operator of multiplication by the indicator of the set ω in $L^2(\mathbb{R};\Delta)$. Define $\mathcal{H}_\omega = \overline{Ran\, W\chi_\omega}$. Then $\mathcal{H}_\omega$ is an invariant subspace of L. If ess $\sup_{k\in\omega} \|S^{-1}(k)\| < \infty$ then $L|_{\mathcal{H}_\omega}$ is similar to a selfadjoint operator, and $\sigma\left(L|_{\mathcal{H}_\omega}\right) \subset \overline{\omega}$. Moreover, in this case the projection $\mathcal{P}_\omega$ on the subspace $\mathcal{H}_\omega$ parallel to $\mathcal{H}_{\mathbb{R}\setminus\omega}$ is bounded and coincides with the closure of $W\chi_\omega W^{-1}$.*

The operators $\mathcal{P}_\omega$ defined in this corollary, can be considered as the spectral projections of the operator L for the sets ω if one can guarantee that the spectrum of the restriction of L to $\mathrm{Ran}(I - \mathcal{P}_\omega) = \mathcal{H}_{\mathbb{R}\setminus\omega}$ is contained in $\overline{\mathbb{R}\setminus\omega}$. The operator $I - \mathcal{P}_\omega$ in this case gives the spectral projection for the set $\mathbb{R}\setminus\omega$.

Remark 2.4. [1, 20] If the function S has a scalar multiple then for any Borel set $\omega \subset \mathbb{R}$

(i) $\sigma\left(L|_{\mathcal{H}_\omega}\right) \subset \overline{\omega}$.

(ii) any invariant subspace $\mathcal{H} \subset \mathcal{N}_e$ of L such that $\sigma\left(L|_{\mathcal{H}}\right) \subset \overline{\omega}$ is contained in $\mathcal{H}_\omega$.

A point $k \in \mathbb{R}$ is called a *proper point of the operator* L if $\sup_{z\in\omega_\delta(k)} \|S_e^{-1}(z)\| < \infty$ for some $\delta > 0$. Following [7] we call a point $k \in \mathbb{R}$ a *spectral singularity* if is not proper. Applying lemma 2.2 to $\mathsf{H} = L^2(\mathbb{R}, \Delta)$ one gets the if part of the following

Nagy - Foias criterion. *$L|_{\mathcal{N}_e}$ is similar to a selfadjoint operator if and only if* $\sup_{k\in\mathbb{R}} \|S^{-1}(k)\| < \infty$.

In the usual form of the criterion [1] the supremum is taken over the upper half plane and the assertion refers, correspondingly, to the operator L itself, which is equivalent but less convenient for our proposes.

Given X and defining the corresponding invariant susbspace $\mathcal{H}$ as in lemma 2.2, one can always construct another invariant subspace $\mathcal{H}^\sim$ of L such that $\mathcal{N}_e = \overline{\mathcal{H}\dot{+}\mathcal{H}^\sim}$ by the formula $\mathcal{H}^\sim = \overline{W\mathsf{H}^\sim}$, $\mathsf{H}^\sim$ being the closure of $X^\perp = \{f \in L^2(\mathbb{R}, E) : f(k) \in X(k)^\perp$ for a.e. $k\}$ in $L^2(\mathbb{R};\Delta)$. In general, the angle $(\mathcal{H}, \mathcal{H}^\sim)$ can hardly be estimated. We now choose $\{X(k)\}$ in a special way so that the angle $(\mathcal{H}, \mathcal{H}^{\sim})$ admits an estimate in terms of S.

Define the function $D = S^*S$. Given a measurable function $k \mapsto \gamma_k$, $\gamma_k \in [0,1]$, $k \in \mathbb{R}$, let $P_1(k)$, $P_2(k)$ be the spectral projections of $D(k)$ for the intervals $[0,\gamma_k)$, $[\gamma_k, 1]$, respectively, and let $X_{1,2}(k) = \mathrm{Ran} P_{1,2}(k)$, hence $E = X_1(k) \oplus X_2(k)$ since $0 \le D \le I$. Set[4] $\widetilde{S}_1 = \sqrt{D} \mid_{X_1} \oplus I \mid_{X_2}$, $\widetilde{S}_2 = I \mid_{X_1} \oplus \sqrt{D} \mid_{X_2}$. In what follows the contractions defined by the functions $D(\cdot)$, $\widetilde{S}_{1,2}(\cdot)$ in $L^2(\mathbb{R}, E)$ will be denoted with the same letters without arguments. Analogously we write $X_{1,2} = \{f \in L^2(\mathbb{R}, E) : f \in X_{1,2}(k) \text{ for a.e. } k \in \mathbb{R}\}$. By construction the pair of orthogonal subspaces $X_{1,2} \subset L^2(\mathbb{R}, E)$ reduces Δ.

Let $\mathsf{H}_{1,2}$ be the closures of $X_{1,2}$ in $L^2(\mathbb{R}; \Delta)$, hence $L^2(\mathbb{R}; \Delta) = \mathsf{H}_1 \oplus \mathsf{H}_2$. By lemma 2.2, $\mathcal{H}_{1,2} = \overline{W\mathsf{H}_{1,2}}$ are invariant subspaces of the operator L.

Lemma 2.5.

$$\sin(\mathcal{H}_1, \mathcal{H}_2) \ge \operatorname*{ess\,inf}_{k \in \mathbb{R}} \left\| \left(S(k) \mid_{X_2(k)} \right)^{-1} \right\|^{-1}. \tag{2.9}$$

Proof. The assertion is nontrivial if $\left\| \widetilde{S}_2^{-1} \right\| = \operatorname{ess\,sup}_{k\in\mathbb{R}} \left\| \left(S(k) \mid_{X_2(k)} \right)^{-1} \right\|$ is finite. For $\tilde{u} \in X_1$, $\tilde{v} \in X_2$ we have

$$\|W\tilde{v}\|^2 \le \|\tilde{v}\|^2_{L^2(\mathbb{R};\Delta)} = \left\| \sqrt{\Delta}\tilde{v} \right\|^2 \le \left\| \widetilde{S}_2^{-1} \right\|^2 \left\| \widetilde{S}_2 \sqrt{\Delta} \tilde{v} \right\|^2$$

and

$$\left\| \widetilde{S}_2 \sqrt{\Delta}\tilde{v} \right\|^2 = \left\| \sqrt{D\Delta}\tilde{v} \right\|^2 \le \left\| \sqrt{D\Delta}\tilde{u} \right\|^2 + \left\| \sqrt{D\Delta}\tilde{v} \right\|^2 = \left\| \sqrt{D\Delta}(\tilde{u} - \tilde{v}) \right\|^2 =$$
$$= \left\| S\sqrt{\Delta}(\tilde{u} - \tilde{v}) \right\|^2 \le \| W(\tilde{u} - \tilde{v}) \|^2 .$$

Here we used the first equality in (2.4) and the fact that $X_{1,2}$ reduce Δ. Thus $\left\| \widetilde{S}_2^{-1} \right\|^{-1} \|W\tilde{v}\| \le \|W(\tilde{u} - \tilde{v})\|$. Now, using the definition of the angle between subspaces X, Y of a Hilbert space [16], $\sin(X, Y) = \inf_{u \in X, v \in Y : \|v\| = 1} \|u - v\|$, we obtain

$$\sin(\mathcal{H}_1, \mathcal{H}_2) = \inf_{\tilde{u} \in X_1, \tilde{v} \in X_2 : \|W\tilde{v}\| = 1} \|W\tilde{u} - W\tilde{v}\| \ge \left\| \widetilde{S}_2^{-1} \right\|^{-1}$$

since $X_{1,2}$ are dense in $\mathsf{H}_{1,2}$. ■

We shall use $P_{[0,\delta)}(k)$ for the spectral projection of $D(k)$ corresponding to the interval $[0, \delta)$.

[4] We omit the argument k after all the spaces and operators in the following equalities which are assumed to hold for a. e. $k \in \mathbb{R}$.

Note that it is obvious from the proof that this lemma actually bears local character, that is, for an interval $\omega \subset \mathbb{R}$ the angle $\left(\overline{\mathcal{H}_\omega \cap \mathcal{H}_1}, \overline{\mathcal{H}_\omega \cap \mathcal{H}_2}\right)$ admits the estimate (2.9) with the infimum taken over ω.

In the standard setup of the Nagy-Foias theory (see [1, Theorem VII.1.1]) invariant subspaces are related to regular factorizations of the characteristic function in the products of contractive analytic functions. We refer to our recent paper [23] where the above proof of lemma 2.5 is taken from, for the explanation of the correspondence between the constructed invariant subspaces $\mathcal{H}_{1,2}$ and the regular factorizations.

For any $\beta \in (0,1)$ one can always construct a decomposition of $L_e = L|_{\mathcal{N}_e}$ into a linear sum $L_e = L_1^\beta \dot{+} L_2^\beta$ of operators $L_{1,2}^\beta = L|_{\mathcal{H}_{1,2}^\beta}$ acting in the invariant subspaces $\mathcal{H}_{1,2}^\beta$ such that (i) L_2^β is similar to a selfadjoint operator, and (ii) $\sin(\mathcal{H}_1^\beta, \mathcal{H}_2^\beta) \geq \beta \neq 0$, by setting $\gamma_k \equiv \beta$ in the construction above. If the function S is continuous up to the real axis and at ∞ in the operator norm, and the perturbation V is relatively compact which is equivalent to saying that $S(z) \in I + \mathbf{S}^\infty$, then $m\left(L_1^\beta\right) < \infty$ for any $\beta \in (0,1)$. According to the following corollary, in this case one can choose β appropriately to give $m\left(L_1^\beta\right)$ its smallest value and thus obtain a decomposition of L_e with minimal possible multiplicity of the component L_1^β such that L_2^β is still similar to a selfadjoint operator, at the penalty of loss of quantitative estimate of the angle $(\mathcal{H}_1, \mathcal{H}_2)$. Let $\mathcal{P}_\delta$ be the spectral projection of L for the δ - neighborhood σ_0^δ of the set of spectral singularities, which is defined for $\delta > 0$ as a bounded operator, $\mathcal{P}_\delta = I - \mathcal{P}_\omega$, $\omega = \mathbb{R} \setminus \sigma_0^\delta$.

Corollary 2.6. *Suppose that S is norm continuous in the closed upper halfplane, $S(k) - I \in \mathbf{S}^\infty$ for all $k \in \mathbb{R}$, and $S(k) \to I$ in the operator norm as $|k| \to \infty$. Then $\mathcal{N}_e$ can be represented as a linear sum $\mathcal{N}_e = \mathcal{H}_1 \dot{+} \mathcal{H}_2$ of invariant subspaces $\mathcal{H}_{1,2}$ of the operator L such that*

1. L_1 has spectrum of multiplicity $\max_{k\in\mathbb{R}} \dim\ker S(k) < \infty$,

2. L_2 is similar to a selfadjoint operator.

3. $(\mathcal{H}_1, \mathcal{H}_2) > 0$.

4. For any $\delta > 0$ small enough the angle between the subspaces $\mathcal{P}_\delta\mathcal{H}_1$ and $\mathcal{P}_\delta\mathcal{H}_2$ satisfies

$$(\mathcal{P}_\delta\mathcal{H}_1, \mathcal{P}_\delta\mathcal{H}_2) \geq \frac{p}{\sqrt{2}}, \quad p = \min_k \mathit{dist}\left(0, \sigma\left(\sqrt{D(k)}\right) \setminus \{0\}\right)$$

where k ranges over the set $\sigma_0 = \{k : \ker S(k) \neq 0\}$ of spectral singularities.

Proof. Under the formulated assumptions the eigenvalues $\xi_j(k)$, $0 \leq \xi_j(k) \leq \xi_{j+1}(k)$, $j \geq 1$ of $D(k)$ are continuous in k. Put $n(k) = \dim\ker S(k)$. Then the set $\{k : \; n(k) \leq m\}$ is open for any $m \geq 0$. From this fact and the asymptotic of

S at infinity one gets that $n = \max_{k\in\mathbb{R}} n(k) < \infty$ and $\xi_{n+1}(k) \geq 2\beta > 0$. Define the subspaces $\mathcal{H}_{1,2}$ according to the construction above with $\gamma_k \equiv \beta$. Then, since $\operatorname{rank}\left(P_{[0,\beta)}(k)\right) = n$ for $k \in \omega$ for an interval $\omega \subset \mathbb{R}$ and $\ker S(k) = \{0\}$ for a. e. $k \in \mathbb{R}$ (see e. g. [11]), one gets assertions 1 through 3 from (2.7) and lemma 2.5. Now choose δ from the condition $\rho(\delta) < p^2/2$ where ρ is the modulus of continuity of the function D on σ_0, and put $\beta = p^2/2$. Assertion 4 then follows from the remark after lemma 2.5. ■

Remark that the subspace $\mathcal{H}_1$ in this corollary can be chosen to belong to $\mathrm{R}an\mathcal{P}_\delta$. It suffices to put $\gamma_k = \frac{p^2}{2}\chi^\delta(k)$ in the construction, where χ^δ is the indicator of σ_0^δ.

3 Transport Operator for Slab

3.1 Preliminaries

The phase space of the one-speed transport problem for the slab is $\Gamma = \mathbb{R} \times \Omega$, $\Omega \equiv [-1, 1]$. For a particle at a point $(x, \mu) \in \Gamma$, the number $x \in \mathbb{R}$ is the position and $\mu \in \Omega$ is the cosine of the angle between the particle momentum and the coordinate axis. A density of particles is a complex-valued function on Γ belonging to the Hilbert space $H = L^2(\mathbb{R} \times \Omega)$. The physical parameters in the problem are the total cross-section $\sigma \geq 0$, the mean number of secondaries per collision $c : \mathbb{R} \to \mathbb{R}_+$ and the collision operator $K \in \mathbf{B}L^2(\Omega)$. We take the operator K in the form $K = \frac{1}{2}\int_\Omega \cdot \, d\mu'$ which corresponds to the isotropic scattering of the secondaries. Here $d\mu$ stands for the Lebesgue measure on Ω. It is assumed that $c \in L^\infty(\mathbb{R})$.

We do not distinguish the operator K and the operator $I \otimes K$ in $L^2(\mathbb{R} \times \Omega) = L^2(\mathbb{R}) \otimes L^2(\Omega)$ in our notation. We say that a positive function $d \in L^\infty(\mathbb{R})$ belongs to the class L_0^+ if there exists an $a > 0$ such that $d(x) = 0$ for a. e. $x \notin [-a, a]$. Then $\mathbf{1}$ stands for the indicator of the set Ω.

The evolution of a density in the space H is given by the solution of the Cauchy problem for the Boltzmann equation [5],

$$\begin{cases} -i\partial_t u = Lu \\ u\big|_{t=0} = u_0. \end{cases} \tag{3.1}$$

The generator L, which is called the one-speed transport operator, is defined in H by the following expression,

$$L = i\mu\partial_x + i\sigma(1 - c(x)K), \tag{3.2}$$

on the domain $\mathcal{D} = \{f \in H : (1) f(\cdot, \mu)$ is absolutely continuous for a.e. $\mu \in \Omega$; (2) $\mu \partial_x f \in H\}$. The imaginary part of L is bounded while its real part $L_0 = i\mu\partial_x$ is a selfadjoint operator on the domain $\mathcal{D}$ which corresponds to the evolution $U_t = \exp iL_0 t$, $(U_t f)(x, \mu) = f(x - \mu t, \mu)$, of densities in vacuum.

Instead of L, it is convenient to deal with the dissipative operator $T = L^* + i\sigma = i\mu\partial_x + i\sigma c(x)K$. Without loss of generality, one can set $\sigma = 1$; then $V = \Im T = c(x)K$. The spectral analysis of L is reduced to that of T since L and $-T + i$ are unitarily equivalent, $L = J(-T + i)J^*$ with $(Jf)(x, \mu) = f(x, -\mu)$.

3.2 Spectral Analysis: Discrete Spectrum

We henceforth assume that the slab has a finite width, i.e., $c \in L_0^+$. Depending on the context, we use $\sqrt{c}$ either for the operator of multiplication by the function $\sqrt{c(x)}$ in $L^2(\Gamma)$, or for the corresponding element of $L^2(\Gamma)$. One can naturally identify the range of K in H with the space $L^2(\mathbb{R})$ of functions of the variable x, $KH = L^2(\mathbb{R}) \otimes \mathbf{1} \simeq L^2(\mathbb{R})$. Thus $\sqrt{c}$ acts in $KH = L^2(\mathbb{R})$ and leaves invariant $E \equiv \overline{\operatorname{Ran} V} \subset L^2(\mathbb{R})$. In this section we use $\langle \cdot, \cdot \rangle$ for the inner product in $L^2(\mathbb{R})$.

Set $R_0(z) = (L_0 - z)^{-1}$. The kernel of the operator $Q(z) = i\sqrt{V} R_0(z) \sqrt{V}\Big|_E$, $z \in \mathbb{C}_+$, acting in the space E, has the form [5][5] $-\frac{1}{2}\sqrt{c(x)} E(-iz|x - y|) \sqrt{c(y)}$, where $E(s) = \int_1^\infty e^{-st} \frac{dt}{t}$ for $\Re s > 0$. The function $E(s)$ admits the representation

$$E(s) = -\ln s - \gamma + \theta(s) \tag{3.3}$$

where $\theta(s) = -\sum_{m=1}^\infty \frac{(-s)^m}{m! m}$ is an entire function and γ is the Euler constant. Since c is compactly supported, it follows that $Q(z)$ is of the Hilbert - Schmidt class $\mathbf{S}^2$ for all $z \in \mathbb{C}_+$. Let $\{\eta_n(z)\}_{n=1}^\infty$, $|\eta_n| \geq |\eta_{n+1}| > 0$, be the eigenvalues of the operator $Q(z)$. Using (3.3), one can represent $Q(z)$ as

$$Q(z) = \widetilde{Q}(z) + \frac{1}{2}\Theta(z) \tag{3.4}$$

where $\widetilde{Q}(z)$ is the operator with kernel $\frac{1}{2}\sqrt{c(x)}(\ln(-iz|x - y|) + \gamma)\sqrt{c(y)}$; thus, $\Theta(z)$ is an entire function and $\widetilde{Q}(z)$ is an analytic function in $\mathcal{O} = \mathbb{C} \setminus \{-it,\ t \geq 0\}$. According to this formula, the function $Q(z)$ has an analytical continuation from $\mathbb{C}_+$

[5]This integral kernel is actually calculated in [5, §2] for the case $c = q\chi_{[-a,a]}$ with constant q. Below we use an estimate on Q obtained in [13, 25] for the case $c = q\chi_{[-a,a]}$, since the estimate for general $c \in L_0^+$ follows from it.

to $\mathcal{O}$. The Hilbert - Schmidt norm of $\Theta(z)$ admits the following estimate,

$$\|\Theta(z)\|_2 \leq \left(\int c(x)c(y)\left|\sum_{m=1}^{\infty}\frac{|x-y|^m|z|^m}{m!m}\right|^2 dx\,dy\right)^{\frac{1}{2}} \leq |c|_1\sum_1^{\infty}\frac{(2a|z|)^m}{m!} = |c|_1\left(e^{2a|z|}-1\right), \quad (3.5)$$

where $|c|_1 = \int_{\mathbb{R}} c(x)dx$.

Theorem 1. *Suppose that $c \in L_0^+$. Then the nonreal spectrum $\sigma_+(T)$ of the operator T consists of finitely many eigenvalues lying on the imaginary axis. Moreover, T has no associate vectors. The dimension $N(c)$ of the invariant subspace H_d corresponding to $\sigma_+(T)$ satisfies the estimate*

$$N(c) \leq 1 + \frac{1}{4}\iint \ln^2|x-y|c(x)c(y)dxdy. \quad (3.6)$$

The essential spectrum of T coincides with the real axis: $\sigma_{ess}(T) = \mathbb{R}$.

Proof. First $\sigma_{ess}(T) = \sigma_{ess}(L_0) = \mathbb{R}$ by Sublemma.

We derive the inclusion $\sigma_+(T) \subset i\mathbb{R}$ from the fact that $Q(z)$ has no real eigenvalues (and, particularly the eigenvalue -1) provided that $k \equiv \Re z \neq 0$. It suffices to show that the operator $\Im Q(z)$ is strictly positive or negative for $k \neq 0$. Consider the operator $\Xi(z) = iKR_0(z)|_{KH}$ acting in the space KH. In the Fourier representation with respect to the variable x, this operator acts as the multiplication by the function

$$\xi(p,z) = \frac{i}{2}\int_{-1}^{1}\frac{d\mu}{p\mu - z}. \quad (3.7)$$

Then

$$r_z(p) \equiv \Im\xi(p,z) = \frac{1}{2}\int_{-1}^{1}\frac{p\mu - k}{(p\mu-k)^2 + (\Im z)^2}d\mu = \frac{1}{2|p|}\int_{-|p|-k}^{|p|-k}\frac{\tau d\tau}{\tau^2 + (\Im z)^2} = \frac{1}{4p}\ln\frac{(p-k)^2+(\Im z)^2}{(p+k)^2+(\Im z)^2}. \quad (3.8)$$

Clearly, $r_z(p) < 0$ or $-r_z(p) < 0$ and hence $\Im\Xi(z) < 0$ or $-\Im\Xi(z) < 0$ depending on sign k. Since $\Im Q(z) = \sqrt{c}\Im\Xi(z)\sqrt{c}|_E$, it follows that for $k \neq 0$ either $\Im Q(z) < 0$ or $-\Im Q(z) < 0$. Note that the last equality in (3.8) is not necessary for the conclusion that r_z has definite sign but will be used in due time.

This property also implies immediately the absence of associate vectors for the operator. Indeed, consider the function $G(w) = \left(I + Q(\sqrt{w})\right)^{-1}$. This an R - function in the upper half plane, and applying remark 1.1 one concludes that $|\Re z| \left\| (I + Q(z))^{-1} \right\|$ is bounded as z approaches a $z_0 \in \sigma_+(T)$. If follows that $(I + Q(z))^{-1}$ has simple poles at the points of $\sigma_+(T)$, and the same is true of $(T - z)^{-1}$ by (1.1).

We shall now estimate $N(c)$. Let $N_\varepsilon(c)$, $\varepsilon > 0$, be the dimension of the spectral subspace corresponding to the interval $[i\varepsilon, i\infty)$. Set $z = i\varepsilon$. Then $r_z(p) \equiv 0$, and thus $Q(i\varepsilon)$ is selfadjoint. Applying now remark 1.2 to the function $-Q(\sqrt{w})$, one concludes that $Q(i\varepsilon)$ is a monotone *increasing* function of ε and a negative operator for each $\varepsilon > 0$. It follows that $N_\varepsilon(c) \equiv \#\{s \geq \varepsilon : \ker(I + Q(is)) \neq 0\} = \#\{\tau \in (0, 1] : \ker(I + \tau Q(i\varepsilon)) \neq 0\}$, counting multiplicity. According to (3.4),

$$2Q(i\varepsilon) = Q^0 + \Theta(i\varepsilon) + (\gamma + \ln \varepsilon)\ \mathbf{P}_1, \tag{3.9}$$

where $\mathbf{P}_1 = \langle \cdot, \sqrt{c} \rangle \sqrt{c}$ and Q^0 is the operator with kernel $\sqrt{c(x)c(y)} \ln |x - y|$. Since $\mathbf{P}_1$ has unit rank, it follows that

$$\begin{aligned} \#\{\tau \in (0, 1] : \ \ker(I + \tau Q(i\varepsilon)) \neq 0\} \leq 1 + \#\{\tau \in (0, 1] : \\ \ker(I + \tau Q_1(i\varepsilon)) \neq 0\} \leq 1 + \|Q_1(i\varepsilon)\|_2^2 \end{aligned}$$

where $Q_1(i\varepsilon) = \frac{1}{2}(Q^0 + \Theta(i\varepsilon))$. From this using (3.5) one obtains

$$\begin{aligned} N_\varepsilon(c) \leq 1 + \|Q_1(i\varepsilon)\|_2^2 = 1 + \frac{1}{4} \left\| Q^0 \right\|_2^2 + O(\varepsilon) = 1 + \\ \frac{1}{4} \iint \ln^2 |x - y| c(x)c(y)dxdy + O(\varepsilon). \end{aligned}$$

Then, $N(c) = \lim_{\varepsilon \downarrow 0} N_\varepsilon(c)$, which gives (3.6). ■

Note that $\sigma_+(T)$ is not empty for any nonzero function $c \in L_0^+$, since $\|Q(it)\| \to \infty$ as $t \to 0$ and $\|Q(it)\| \to 0$ as $t \to \infty$. It is also clear that $N(\kappa c) \to \infty$ as the coupling constant $\kappa \to \infty$.

All the assertions of this theorem, except for the estimate (3.6), has been proven in [5, 13] for the case $c = q\chi_{[-a,a]}$. The basic observation about sign-definiteness properties of $\Im Q(z)$, with an implication that $\sigma_+(T) \subset i\mathbb{R}$, was made in these papers. It has not been realized, however, that the monotonicity of $Q(i\varepsilon)$ and the absence of associate vectors also follow immediately. Instead, in [5, 25] these facts were established by calculating the Fourier representation of $Q(i\varepsilon)$ explicitly and observing that $Q'(i\varepsilon)$ times a scalar constant is strictly positive. The latter can be shown to imply the required first order estimate for $(I + Q(z))^{-1}$ (see Appendix). Although this arguing is also avaiable in the case of general $c \in L_0^+$, it is remarkable that all these tedious calculations specific for the problem are redundant[6], and the

[6]Below (Proposition 3.1), however, we employ them for another purpose.

result follows at once from the abstract theory. The finiteness of $N(c)$ was deduced in [5] from the monotonicity of $Q(i\varepsilon)$ and the fact that the divergent term in the asymptotics of $Q(i\varepsilon)$ at 0 has finite rank but it has not been realized that these circumstances allow also to estimate this number. A proof of the theorem more closed to the original Lehner-Wing arguments can be found in our paper [6].

Define $\mathcal{P}_d$ to be the Riesz projection corresponding to $\sigma_+(T)$, so $H_d = \mathcal{P}_d H$. Then $H = H_d \dot{+} H_{ess}$ where $H_{ess} = (I - \mathcal{P}_d)H$ is the invariant subspace of T corresponding to the essential spectrum. Since H_d is finite dimensional, the angle $(H_d, H_{ess}) > 0$. In the following Proposition we give an estimate of this angle for small c announced in [6] without proof.

Proposition 3.1. *Let c be such that the integral in (3.6) < 4 and hence* $\dim H_d = 1$. *Then*

$$\sin\left(H_d, H_{ess}\right) \geq \frac{e^{-\gamma}}{4+\pi^2} \frac{1}{a|c|_\infty} \exp\left(-\frac{2}{|c|_1}\right).$$

Proof. We rely on the following abstract formula valid in the case $\dim H_d = 1$,

$$\sin(H_d, H_{ess}) = \Im z_0 \left|\langle Q'(z_0)e, e^*\rangle\right| \tag{3.10}$$

where $\{z_0\} = \sigma_+(T)$, $e \in \ker(I + Q(z_0))$, $e^* \in \ker(I + Q^*(z_0))$, $\|e\| = \|e^*\| = 1$. Indeed, for $\dim H_d = 1$ $\mathcal{P}_d = \frac{1}{\langle u, u^*\rangle} \langle \cdot, u^*\rangle\, u$ where $u \in H_d$, $u^* \in \ker(T^* - \overline{z_0})$. We have $u = -iR_0(z_0)Vu$, $u^* = iR_0\left(\overline{z_0}\right)Vu^*$. Multiplying this by $\sqrt{V}$, one obtains that $\sqrt{V}u$ and $\sqrt{V}u^*$ differ from e and e^*, respectively, by scalar factors. Then

$$\begin{aligned} \left|\langle u, u^*\rangle\right| = \left|\langle R_0(z_0)Vu, R_0(\overline{z_0})Vu^*\rangle\right| = \left|\left\langle \sqrt{V}R_0^2(z_0)Vu, \sqrt{V}u^*\right\rangle\right| = \\ = \left|\left\langle Q'(z_0)\sqrt{V}u, \sqrt{V}u^*\right\rangle\right|. \end{aligned}$$

By the definitions of u and u^* one has $\Im z_0 \left\|u\right\|^2 = \left\|\sqrt{V}u\right\|^2$, $\Im z_0 \left\|u^*\right\|^2 = \left\|\sqrt{V}u^*\right\|^2$. We now get (3.10) from the identity

$$\sin(H_d, H_{ess}) \equiv \left\|\mathcal{P}_d\right\|^{-1} = \frac{\left|\langle u, u^*\rangle\right|}{\|u\|\,\|u^*\|}.$$

In our case $Q(z_0)$ is selfadjoint and thus $e = e^*$. We shall estimate separately the factors in (3.10) from below. One has $Q(i\varepsilon) = \sqrt{c}\,\Xi(i\varepsilon)\sqrt{c}|_E$, where the operator $\Xi(i\varepsilon) = \Xi^*(i\varepsilon)$ in the Fourier representation acts as the multiplication by the function

$$\xi_\varepsilon(p) = -\frac{\varepsilon}{2}\int_{-1}^{1} \frac{d\mu}{\left|p\mu + i\varepsilon\right|^2} = -\frac{1}{|p|}\arctan\frac{|p|}{\varepsilon}. \tag{3.11}$$

One can now calculate the operator $Q'(i\varepsilon) = -i\,\sqrt{c}\left(\frac{d}{d\varepsilon}\Xi(i\varepsilon)\right)\sqrt{c}\big|_E$ in the Fourier representation by differentiation. This gives

$$|\langle Q'(i\varepsilon)e,e\rangle| = \int \frac{\left|\widehat{\sqrt{c}e}(p)\right|^2}{p^2+\varepsilon^2}\,dp. \tag{3.12}$$

Define the operator $G(\varepsilon) = \sqrt{c}\Xi^2(i\varepsilon)\sqrt{c}|_E$ acting in E. By (3.11) we have $\langle G(\varepsilon)e,e\rangle = \left\langle \xi_\varepsilon^2\widehat{\sqrt{c}e},\widehat{\sqrt{c}e}\right\rangle$. On the other hand ($z_0 = i\varepsilon_0$)

$$\frac{1}{|c|_\infty} = \frac{|\langle Q^2(i\varepsilon_0)e,e\rangle|}{|c|_\infty} = \frac{\|\sqrt{c}\Xi(i\varepsilon_0)\sqrt{c}e\|^2}{|c|_\infty} \le \left\|\Xi(i\varepsilon_0)\sqrt{c}e\right\|^2 = \langle G(\varepsilon_0)e,e\rangle\,.$$

From this and (3.12) using the elementary inequality

$$\frac{1}{p^2+t^2} \ge \frac{1}{1+\frac{\pi^2}{4}}\xi_t^2(p)$$

we find

$$|\langle Q'\left(i\varepsilon_0\right)e,e\rangle| \ge \frac{1}{\left(1+\frac{\pi^2}{4}\right)|c|_\infty}. \tag{3.13}$$

We now estimate ε_0 from below through the variational principle according to which $-\eta_1(i\varepsilon_0) = \max_{u\in E;\|u\|=1}\left(-\langle Q(i\varepsilon_0)u,u\rangle\right) = 1$. By monotonicity of the function E one has

$$1 \ge -\frac{\langle Q(i\varepsilon_0)\tilde{u},\tilde{u}\rangle}{\|\tilde{u}\|^2} = \frac{1}{2|c|_1}\int\!\!\int c(x)c(y)E(\varepsilon_0|x-y|)dx\,dy \ge \frac{|c|_1}{2}E(2a\varepsilon_0)$$

for $\tilde{u} = \sqrt{c}$. The following estimate results from the identity

$$E(s) = -\gamma - \ln s + \int_0^s \frac{1-e^{-t}}{t}dt$$

valid by the definition of E,

$$E(s) \ge -\gamma - \ln s, \qquad s>0.$$

From this one finds

$$\varepsilon_0 \ge \frac{1}{2a}\exp\left(-\gamma - \frac{2}{|c|_1}\right). \tag{3.14}$$

Substituting this and (3.13) in (3.10), one comes to the required estimate. ■

Let us explain the meaning of this proposition in terms of the dynamics generated by L. Since L and $-T+i$ are intertwined by J, JH_d, JH_{ess} are invariant subspaces of L. The further study of T by tools of the functional model shows that under the assumption made in the proposition the restriction of T to H_{ess} is similar to a selfadjoint operator (see remark 3.9). From this it is easy to see that $e^{iLt}f = e^{(\varepsilon_0-1)t}\widetilde{\mathcal{P}_d}f + O\left(e^{-t}\right)$ where $\widetilde{\mathcal{P}_d} = J\mathcal{P}_d J^*$. Since $\left\|\widetilde{\mathcal{P}_d}\right\|^{-1} = \sin\left(H_d, H_{ess}\right)$, the estimate obtained provides us with an upper bound for the factor before exponent in the leading term of the time asymptotics.

Of course, the estimate (3.14) for ε_0 is rather crude and can be improved in many ways. Our only aim here is to demonstrate the possibility of estimating the angle (H_d, H_{ess}) explicitly in terms of c, and (3.14) has been chosen for its simplicity. Let us remark, however, that estimating ε_0 is a question of some interest of its own, since a lower bound for ε_0 provides us with a lower bound on the rate of the slowest decay in the corresponding physical problem related to the operator L.

3.3 Spectral Analysis: Essential Spectrum

We now proceed to study the component $T_{ess} = T|_{H_{ess}}$ by means of the functional model. Let S be the characteristic function of the operator T. Then S is analytic on the real axis except at the point 0.

Lemma 3.2. *S has a scalar multiple.*

Proof. We first see that $T(z) \equiv S(z) - I \in \mathbf{S}^2$ for all $z \in \mathbb{C}_+$. According to lemma 1.3, to conclude that the function $m(z) \equiv \det\left(I - T^2(z)\right)$ is a scalar multiple for S, it suffices to show that $\|T(z)\|_2$ is uniformly bounded in $\mathbb{C}_+$. We first estimate it in the vicinity of 0. By (3.3), the following representation holds:

$$T(z) = (A + \Theta(z))\left(I - Q(z)\right)^{-1} + M(z),$$

where rank $M(z) = 1$ and $A \in \mathbf{S}^2$. Since $\|T(z)\| \leq 2$, we have $\|M(z)\|_2 = \|M(z)\| \leq 2 + \|A\| + C_\delta|z| \leq C_{1,\delta}$ for $|z| \leq \delta$. It follows that $\sup_{z\in\omega_\delta(0)}\|T(z)\|_2 \leq C_\delta$, since $\|\Theta(z)\|_2$ is bounded for $|z| \leq \delta$ by (3.5). Then, Lemma 5 in [25] states, in our notation, that[7]

$$\|Q(z)\|_2^2 \leq \frac{C_\varepsilon}{1 + |\Re z|} \tag{3.15}$$

for $z \in \Pi_\varepsilon \bigcap \{z : |\Re z| \geq 1\}$ for any $\varepsilon > 0$. Choosing an arbitrary $\varepsilon \in (0, 1]$ and taking $\delta > \sqrt{2}$, we obtain $\|T(z)\|_2 \leq C$ for $z \in \Pi_\varepsilon$. Then, by the obvious inequality

[7] This estimate was essentially established in the course of the proof. The statement of Lemma 5 in [25] is weakened to the estimate of the operator norm of $Q(z)$.

$|E(s)| \leq E(\Re s) \leq E(\Re s_0)$ valid for $0 < \Re s_0 \leq \Re s$, which results from the definition of E, one has $\|Q(z)\|_2 \leq \|Q(i\varepsilon)\|_2$ for $\Im z \geq \varepsilon$, and thus $\sup_{z\in\mathbb{C}_+\setminus\Pi_\varepsilon} \|T(z)\|_2 < \infty$. Combining the estimates obtained we get $\sup_{z\in\mathbb{C}_+} \|T(z)\|_2 < \infty$. ■

The estimate (3.15) was used in [25] for the derivation of some pointwise asymptotics of solutions of the Boltzmann equation. However, the derivation itself contains an error (see below).

Let $\mathcal{B}_c = \{k_n = \lim_{\varepsilon\downarrow 0} \eta_n(i\varepsilon) : \ |k_n| < \infty \}$. We define the singular set $\mathcal{E} \subset L_0^+$ as follows: $\mathcal{E} = \{c \ : \ -1 \in \mathcal{B}_c\}$. Note that for any $c \in L_0^+$ the function κc belongs to $\mathcal{E}$ for the infinite discrete set $\{-k_n^{-1},\ k_n \in \mathcal{B}_c\}$ of values of the constant $\kappa > 0$. The following lemma motivates the definition of the set $\mathcal{E}$.

Lemma 3.3. *If $c \notin \mathcal{E}$, then $\sup_{z\in\omega_\delta(0)} \|S^{-1}(z)\| < \infty$ for sufficiently small δ.*

Proof. By (2.1) we have $S^{-1}(z) = -I + 2\,(I + Q(z))^{-1}$ for $z \in \mathbb{C}_+ \cap \rho(T)$, and thus the desired estimate is equivalent to the uniform boundedness of $\left\|(I + Q(z))^{-1}\right\|$ in $z \in \omega_\delta(0)$. Suppose, on the contrary, that $(I + Q(z_n))\,\varphi_n \to 0$ as $z_n \to 0$ for some sequence $\{\varphi_n\}, \|\varphi_n\| = 1$. Then $\ln|z_n|\ \mathbf{P}_1\varphi_n$ remains bounded as $n \to \infty$ (see (3.4)), and thus $\mathbf{P}_1\varphi_n \to 0$. It follows that $(Q(z_n) - Q(i|z_n|))\,\varphi_n \to 0$. This gives $(I + Q(i|z_n|))\varphi_n \to 0$ which contradicts the assumption that $c \notin \mathcal{E}$. ■

Corollary 3.4. *0 is a spectral singularity if and only if $c \in \mathcal{E}$.*

Proof. For $c \in \mathcal{E}$ we have $S(k)f(k) \to 0$ as $k \to 0$ where $f(k)$ is the normalized eigenfunction of $Q(i|k|)$, $Q(i|k|)f(k) = \eta(|k|)f(k)$, such that $\eta(|k|) \to -1$ as $k \to 0$. Indeed,

$$\|(I + Q(k))f(k)\| = \|(1 + \eta(|k|))f(k) + (Q(k) - Q(i|k|))f(k)\| \leq \\ |1 + \eta(|k|)| + \frac{1}{2}\frac{\pi}{2}\|\mathbf{P}_1 f(k)\| + O(|k|).$$

The first term tends to 0 by the choice of $f(k)$ and $\mathbf{P}_1 f(k) \to 0$ because $\ln|k|\ \mathbf{P}_1 f(k)$ remains bounded as $k \to 0$ by (3.9). Since $\|S(k)u\| = \|S_e(k)u\|$ for a.e. $k \in \mathbb{R}$, this implies the if part of the assertion. The only if part follows from lemma 3.3 since $S_e^{-1} = S^{-1}S_i$ is bounded whenever S^{-1} is bounded. ■

Note that it follows from the proof that for $c \in \mathcal{E}$ we have $S(z)f(\pm|z|) \to 0$ as $z \to 0$ uniformly in $\arg z \in [0, \pi]$. Thus for $c \in \mathcal{E}$ the function $S^{-1}(\cdot)$ is unbounded at any neighborhood of 0. A priori the inner factor S_i of the characteristic function can also have a zero at the point 0 [7]. In Proposition 3.6 below, we show that this does not take place in fact. Let us first obtain another description of the set $\mathcal{E}$.

Lemma 3.5. *$S(\cdot)$ is continuous at 0 in the operator norm and*

$$c \in \mathcal{E} \Longleftrightarrow \ker\left(I - \widetilde{Q}_0^2 - \frac{2}{\vartheta_c}\left\langle \cdot, \sqrt{c}\right\rangle \sqrt{c}\right) \neq 0, \tag{3.16}$$

where $\widetilde{Q}_0$ is the integral operator with kernel $\frac{1}{2}\sqrt{c(x)}\ln\left(\frac{|x-y|}{2a}\right)\sqrt{c(y)}$ and $\vartheta_c = \left\langle \left(I - \widetilde{Q}_0\right)^{-1}\sqrt{c}, \sqrt{c}\right\rangle$.

Proof. Since $\Re Q(z) \leq 0$, it follows from (3.4) that $\left(I - \widetilde{Q}(z)\right)^{-1}$ exists and is bounded for $z \in \omega_\delta(0)$ provided that δ is sufficiently small. Rewriting (2.1) as $S(z) = -I + 2(I - Q(z))^{-1}$, we see from the identity

$$(I - Q(z))^{-1} - \left(I - \widetilde{Q}(z)\right)^{-1} = -(I - Q(z))^{-1}\frac{\Theta(z)}{2}\left(I - \widetilde{Q}(z)\right)^{-1} \tag{3.17}$$

that $S(z)$ is continuous at 0 whenever so is $\left(I - \widetilde{Q}(z)\right)^{-1}$. Fix an arbitrary positive δ such that $\widetilde{Q}(i\delta) < 1/2$. We then have $\widetilde{Q}(z) = \widetilde{Q}(i\delta) + \alpha(z)\ \mathbf{P}_1$ where $\alpha(z) = \frac{1}{2}\ln\left(-\delta^{-1}iz\right)$. The operator $I - \widetilde{Q}(i\delta)$ has a bounded inverse, and $\vartheta_c(\delta) = \left\langle \left(I - \widetilde{Q}(i\delta)\right)^{-1}\sqrt{c}, \sqrt{c}\right\rangle \neq 0$. A straightforward computation now gives

$$\left(I - \widetilde{Q}(z)\right)^{-1} = \left(I - \widetilde{Q}(i\delta)\right)^{-1} + \frac{\alpha(z)}{1 - \alpha(z)\vartheta_c(\delta)}\left(I - \widetilde{Q}(i\delta)\right)^{-1}\mathbf{P}_1\left(I - \widetilde{Q}(i\delta)\right)^{-1}. \tag{3.18}$$

The right hand side is obviously continuous at 0. Thus, S is continuous at 0.

Set $\tau = \frac{1}{2ae^\gamma}$. It is easy to see that the operator[8] $\widetilde{Q}_0 \equiv \widetilde{Q}(i\tau) < 0$. Indeed, one can insert the indicator of the set $|x-y| \leq 2a$ into the definition of the kernel $\widetilde{Q}_0$, so $\widetilde{Q}_0 = \sqrt{c}\,\widetilde{\Xi}_0\sqrt{c}\Big|_E$ where the operator $\widetilde{\Xi}_0$ in the Fourier representation acts as the multiplication by the function

$$\frac{1}{2\sqrt{2\pi}}\int_{-2a}^{2a} e^{ips}\ln\frac{|s|}{2a}\,ds = \frac{1}{\sqrt{2\pi}}\,2a\int_0^1 \cos(2aps)\ln s\,ds$$

which is obviously negative for real p. Thus, (3.18) holds with $\delta = \tau$, and passing to the limit $z \to 0$ one gets

$$S(0) = \frac{I + \widetilde{Q}_0}{I - \widetilde{Q}_0} - \frac{2}{\vartheta_c}\left(I - \widetilde{Q}_0\right)^{-1}\mathbf{P}_1\left(I - \widetilde{Q}_0\right)^{-1}.$$

[8] This fact, with a somewhat lengthy derivation, has first been observed in [13, Lemma 7] for the case $c = q\chi_{[-a,a]}$ when proving the finiteness of $\sigma_+(T)$, and is actually redundant for that purpose.

Multiplying this equality on the left and right by $I - \widetilde{Q}_0$, one obtains (3.16), since $c \in \mathcal{E}$ iff $\ker S(0) \neq \{0\}$. ■

Let $\mathcal{N}_e$ be the absolutely continuous subspace of T and H_0 be the maximal reducing subspace of T on which it coincides with L_0.

Proposition 3.6. *The component T_{ess} of the operator T has purely absolutely continuous spectrum, $H_{ess} = \mathcal{N}_e \oplus H_0$.*

Proof. First, the space $\mathcal{N}_e \subset H_{ess} \ominus H_0$ (see the remark made after (2.5)). According to the general theory [7], $H \ominus H_0 = \mathcal{N}_e \dot{+} \mathcal{N}_i$ where $\mathcal{N}_i$ is the inner subspace of T (see [7] for the definition), provided that the angle $(\mathcal{N}_e, \mathcal{N}_i) > 0$. Our aim will be to show that $\mathcal{N}_i = H_d$.

It is obvious from the representation (3.4) that the analytical continuation of the function $Q(z)$ across the real axis to the set $\mathcal{O}$ also holds in the $\mathbf{S}^2$ - norm. This implies that the scalar multiple $m(z)$ is analytic on the real axis outside 0 since so is $T(z)$ in the $\mathbf{S}^2$ - norm. This means that the singular component m_s in the canonical factorization of m has the form

$$m_s(z) = e^{i\mu_1 z - \mu_2 \frac{i}{z}}$$

with some $\mu_1, \mu_2 \geq 0$.

It follows from (3.18) that $\left(I - \widetilde{Q}(z)\right)^{-1}$ is a bounded analytic function in $U_\tau(0) \setminus [0, -i\tau)$. Then, by taking into account the fact that $\|\Theta(z)\| = o(1)$ as $z \to 0$, we conclude that $(I - Q(z))^{-1}$, and therefore $S(z)$, extends to a bounded analytic function on $\widetilde{U}_\delta \equiv U_\delta(0) \setminus [0, -i\delta)$ for sufficiently small δ. Arguing as in the proof of lemma 3.2, we now see that the analytic continuation of $T(z)$ into $\widetilde{U}_\delta$ is also bounded in the $\mathbf{S}^2$ - norm, and therefore, m admits analytic continuation into $\widetilde{U}_\delta$ which is bounded. It follows that $\mu_2 = 0$. Indeed, otherwise an application of the Carlson theorem [26] to the function $g(z^{-1})$, where $g(z) = e^{\frac{\mu_2}{2}\frac{i}{z}} m(z)$, implies $g \equiv 0$. Thus, the inner factor, m_i, of m has the form $m_i = b e^{i\mu_1 z}$ where b is a Blaschke product.

The characteristic function S_{i*} of the restriction $T|_{\mathcal{N}_i}$ has the scalar multiple m_i, the inner factor of m. By [1, Theorem VI.5.1] it follows that $T|_{\mathcal{N}_i}$ has the minimal function m_T, which is the greatest common inner divisor of scalar multiples for S_{i*}. Assume that $\theta(z) = e^{idz}$ is an inner divisor of m_T for some $d > 0$. Then ([1, Theorem III.6.3]) there exists an invariant subspace of T such that θ is the minimal function for the restriction T' of T to this subspace. This means that $\sigma(T')$ is empty which is impossible because T, and therefore T', has bounded imaginary part. Since m_T is a divisor of m_i, in our case this implies that b is a scalar multiple for S_{i*}. Hence $\mathcal{N}_i$ is generated by the root vectors of T, and so $\mathcal{N}_i = H_d$. Comparing the decompositions $H \ominus H_0 = (H_{ess} \ominus H_0) \dot{+} H_d$ and $H \ominus H_0 = \mathcal{N}_e \dot{+} \mathcal{N}_i$, we now obtain the result. ■

Let us mention that the inner factor m_i is in fact the finite Blaschke product corresponding to the set $\sigma_+(T)$, that is, $\mu_1 = 0$ as well, since $\|Q(i\varepsilon)\|_2 \to 0$ as $\varepsilon \to +\infty$ and therefore $m(i\varepsilon) \to 1$. This arguing which avoids references to abstract facts about the minimal function, was used in the proof of this proposition in our earlier paper [6].

Proposition 3.7. *T_{ess} is similar to a selfadjoint operator if and only if $c \notin \mathcal{E}$.*

Proof. According to the Nagy-Foias criterion, one needs to show that $\sup_{k\in\mathbb{R}} \|S^{-1}(k)\| < \infty$ if $c \notin \mathcal{E}$ (the only if part is already proven). From (3.8) we have $r_z(p) \to \frac{1}{2p} \ln \left|\frac{p-k}{p+k}\right|$ as $\Im z \to 0$, $\Re z = k \neq 0$. Since the Fourier transform of $\sqrt{c}u$, $u \in L^2(\mathbb{R})$, is continuous owing to the compactness of the support of c, and the singularity of r_k at $p = \pm k$ is weak, one can pass to the limit $\Im z \to 0$ in the quadratic form of $\Im Q(z)$, which gives $\langle \Im Q(k)u, u\rangle = \left\langle r_k \widehat{\sqrt{c}u}, \widehat{\sqrt{c}u}\right\rangle$. This implies that $(\text{sign } k)\Im Q(k) < 0$ and so $\ker S(k) = \{0\}$ for real $k \neq 0$. Then the estimate (3.15) shows that $S(k) \to I$ as $|k| \to \infty$ in the operator norm, and it follows that

$$\sup_{k\in\mathbb{R}\setminus[-\delta,\delta]} \left\|S^{-1}(k)\right\| < \infty. \tag{3.19}$$

Then, $\sup_{k\in[-\delta,\delta]} \|S^{-1}(k)\| < \infty$ for $c \notin \mathcal{E}$ by lemma 3.3, and we are done. ■

Corollary 3.8. *If $c \notin \mathcal{E}$ then the wave operators*

$$W_\pm(L, L_0 + i\sigma) \equiv W_\pm(T^*, L_0) = s\text{ - }lim_{t\to\pm\infty} e^{iT^*t} e^{-iL_0 t}$$

exist on H and are complete, $Ran\ W_\pm = JH_{ess} \equiv H_d^\perp$.

Proof. According to theorem 6 in [11], the fact that $S(z) - I$ is compact for $\Im z \geq 0$ yields that the wave operators $W_\pm(T, L_0)$ exist on a dense set, $\mathcal{R}$, in $H \ominus H_0$ and are complete, $\overline{W_\pm(T, L_0)\mathcal{R}} = \mathcal{N}_e$, for all $c \in L_0^+$. For $c \notin \mathcal{E}$ they are bounded and, therefore, exist on the whole of H, and admit inverse. It remains to note that obviously $JW_\pm(T^*, L_0)J = W_\mp(T, L_0)$. ■

We thus have the following picture of the behaviour of the operator in coupling constant implied by monotonicity of the function $Q(i\varepsilon)$. In the remark below $c = \kappa\widetilde{c}$ with a fixed nonzero $\widetilde{c} \in L_0^+$.

Remark 3.9. For $\kappa > 0$ small enough the operator T has one eigenvalue, and the component T_{ess} is similar to a selfadjoint operator. In particular, $c \notin \mathcal{E}$ if the integral in (3.6) is < 4. Then, at $\kappa = \kappa_1 = -1/\eta_2(+i0)$, a spectral singularity occurs at the point 0, and a bunch of eigenvalues monotone increasing in κ along the imaginary axis emerges[9] from 0 for $\kappa > \kappa_1$: if $c \in \mathcal{E}$ then $N((1+\varepsilon)c) > N(c)$ for any $\varepsilon > 0$. The spectral singularity disappears for $\kappa > \kappa_1$, and T_{ess} is similar to a selfadjoint operator as long as $\kappa \in (\kappa_1, \kappa_2)$, $\kappa_2 = -1/\eta_3(+i0)$. At $\kappa = \kappa_2$ the singularity occurs again, and a new bunch of eigenvalues emerges, and so on.

We now restrict our consideration to the case $c \in \mathcal{E}$. Then, in view of (3.19), the point 0 is the only spectral singularity of T. As we have seen, the function S satisfies the assumptions of corollary 2.6. Thus, one can decompose T_{ess} into a linear sum, one component of which is similar to a selfadjoint operator while the other has spectrum of multiplicity $\mathcal{M} = \dim\ker S(0)$. Moreover, we have a local estimate of the angle between the corresponding invariant subspaces in the sense given by item 4 of corollary 2.6 with $p = \operatorname{dist}\,(0, \sigma(S(0)) \setminus \{0\})$. According to the construction suggested, it suffices to choose $\delta \neq 0$ small enough so that the norm of $D(k) - D(0)$ be lesser than $p^2/2$ for all $k \in (-\delta, \delta)$. Specific for our problem is the fact that one can estimate the modulus of continuity of D at 0 and, thus, find a sufficient δ explicitly in terms of c and p.

Lemma 3.10. *Let $|c|_2 = \|c\|_{L^2(\mathbb{R})}$ and*

$$\delta_* = \frac{1}{2a} \exp\left(-\frac{64 a \Upsilon^2 |c|_2^2}{p^2 |c|_1} - \gamma \right) \tag{3.20}$$

where $\Upsilon = \left(\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \ln^4 |x-y| dx dy \right)^{1/4}$. Then $\|D(k) - D(0)\| < p^2/2$ for all $k \in (-\delta_*, \delta_*)$.

Proof. Since $\widetilde{Q}_0 < 0$, we have $\Re \widetilde{Q}(z) < 0$ for $z \in U^\circ \equiv \overline{\omega_\tau(0)}$, $z \neq 0$. This allows to define for $z \in U^\circ$ the contractive function

$$\Sigma(z) = \frac{I + \widetilde{Q}(z)}{I - \widetilde{Q}(z)} = -I + 2\left(I - \widetilde{Q}(z) \right)^{-1}$$

which is continuous in z and satisfies

$$\|\Sigma(z) - \Sigma(0)\| = 2 \left\| \left[\frac{\alpha(z)}{1 - \alpha(z)\vartheta_c} + \frac{1}{\vartheta_c} \right] \left(I - \widetilde{Q}_0 \right)^{-1} \mathbf{P}_1 \left(I - \widetilde{Q}_0 \right)^{-1} \right\| \leq$$
$$2 \left\| \left(I - \widetilde{Q}_0 \right)^{-1} \right\|^2 \|\mathbf{P}_1\| \left| \frac{1}{\vartheta_c (1 - \alpha(z)\vartheta_c)} \right| \leq \frac{2|c|_1}{\vartheta_c |1 - \alpha(z)\vartheta_c|} \leq \frac{2|c|_1}{\vartheta_c^2 |\alpha(i|z|)|}.$$

[9] point 0 itself is never an eigenvalue

Thus

$$\left\|(\Sigma^*\Sigma)(z) - \Sigma^2(0)\right\| \le \frac{4|c|_1}{\vartheta_c^2|\alpha(i|z|)|} = -\frac{8|c|_1}{\vartheta_c^2 \ln(2ae^{\gamma}|z|)}. \tag{3.21}$$

Then,

$$S(z) - \Sigma(z) = 2\left((I - Q(z))^{-1} - \left(I - \widetilde{Q}(z)\right)^{-1}\right).$$

By (3.17) this implies $\|S(z) - \Sigma(z)\| \le \|\Theta(z)\|$ as $\left\|\left(I - \widetilde{Q}(z)\right)^{-1}\right\| \le 1$, and thus

$$\|(S^*S)(z) - (\Sigma^*\Sigma)(z)\| \le 2\|\Theta(z)\|$$

for $z \in U^\circ$. Combining this with (3.21), one gets

$$(S^*S)(z) - S^2(0) = (S^*S)(z) - (\Sigma^*\Sigma)(z) + (\Sigma^*\Sigma)(z) - S^2(0) \equiv \Omega_1(z) + \Omega_2(z)$$

where $\|\Omega_1(z)\| \le 2|c|_1\left(e^{2a|z|} - 1\right)$ by (3.5) and $\|\Omega_2(z)\|$ is estimated from above by the expression in the right hand side of (3.21). It follows that

$$\left\|(S^*S)(z) - S^2(0)\right\| < \frac{p^2}{2}$$

if $|z| < \delta$ with a δ satisfying the following system of elementary inequalities,

$$2|c|_1\left(e^{2a\delta} - 1\right) \le \frac{p^2}{4}; \qquad -\frac{8|c|_1}{\vartheta_c^2 \ln\left(2ae^{\gamma}\delta\right)} \le \frac{p^2}{4}.$$

To make the estimate of δ more effective we take into account that

$$\vartheta_c \ge \frac{|c|_1}{1 + \left\|\widetilde{Q}_0\right\|} \ge \frac{|c|_1}{2\left\|\widetilde{Q}_0\right\|}$$

because $c \in \mathcal{E}$ and thus $\left\|\widetilde{Q}_0\right\| \ge 1$. Then, $\left\|\widetilde{Q}_0\right\| \le \left\|\widetilde{Q}_0\right\|_2 \le \frac{\Upsilon}{2}\sqrt{2a}|c|_2$ by the Schwartz inequality. From this we get that the required estimate holds if

$$|z| < \delta \le \frac{1}{2a}\min\left\{\ln\left(1 + \frac{p^2}{8|c|_1}\right), e^{-\frac{32\Upsilon^2|c|_1}{p^2}\left(\frac{\sqrt{2a}|c|_2}{|c|_1}\right)^2 - \gamma}\right\}.$$

Using the inequality $e^x - 1 > x$ for $x > 0$ with $x = \frac{8|c|_1}{p^2}$ one can see that the second entry is not greater than the first one, since $\frac{\sqrt{2a}|c|_2}{|c|_1} \ge 1$ by the Schwartz inequality and $\Upsilon > 1$. ∎

Choose a positive $\delta \le \delta_*$ and define the invariant subspaces $\mathcal{H}_{1,2}$ according to the construction of Section 2 with $\gamma_k = \frac{p^2}{2}\chi_{[-\delta,\delta]}(k)$. Let $\mathcal{P}_\delta$ be the spectral projection of

T for the interval $[-\delta, \delta]$ which is defined to be $\mathcal{P}_\delta = I - \mathcal{P}_\omega$ with $\omega = \mathbb{R} \setminus [-\delta, \delta]$. Applying corollary 2.6, we thus get the following

Theorem 2. *If $c \in \mathcal{E}$, then for any positive $\delta \leq \delta_*$, where δ_* is given by (3.20), the operator T_{ess} can be represented as the linear sum $T_{ess} = T_1 \dot{+} T_2$ of operators $T_{1,2}$ acting in invariant subspaces $\mathcal{H}_{1,2}$ such that*

1. T_1 has the spectrum $\sigma(T_1) = [-\delta, \delta]$ of multiplicity $\mathcal{M} = \dim\ker S(0)$,

2. T_2 is similar to a selfadjoint operator and $\sigma(T_2) = \mathbb{R}$,

3. The angle $(\mathcal{H}_1, \mathcal{H}_2) > 0$. The angle $(\mathcal{H}_1, \mathcal{P}_\delta \mathcal{H}_2)$ admits the estimate

$$\sin(\mathcal{H}_1, \mathcal{P}_\delta \mathcal{H}_2) \geq \frac{p}{\sqrt{2}} \tag{3.22}$$

where p is defined before lemma 3.10.

Proof. It remains to verify the assertions about the spectrum. Both follow from the intertwining relation (2.5). Indeed, it is obvious from (2.1) that $\|S(k)f\| < \|f\|$ for $f = (I - Q(k))\varphi$ with $\Re Q(k)\varphi \neq 0$. Since the operator $\Re Q(k)$ has a logarithmic singularity at zero, it cannot have finite rank. Therefore, $\operatorname{rank}\Delta(k) = \infty$ for all $k \in \mathbb{R}$, and the assertion $\sigma(T_2) = \mathbb{R}$ is immediate. That $\sigma(T_1) \subset [-\delta, \delta]$ is implied by the existence of a scalar multiple, according to remark 2.4; the equality now results from the intertwining relation in the form (2.8). ■

There is a partial case when the spectrum of T_1 is simple (that is, $\mathcal{M} = 1$), and the number p admits an explicit lower bound. For $\xi > 0$ let $p(\xi) = \operatorname{dist}\left(-1, \sigma(\widetilde{Q}(i\xi))\right)$.

Remark 3.11. Assume that $p(\xi) \neq 0$ for some $\xi > 0$. Then $\mathcal{M} = 1$ and

$$p \geq \frac{p(\xi)}{3} \tag{3.23}$$

for any $\xi \in (0, \tau)$.

Proof. If $\mathcal{M} \neq 1$ then there exists a nonzero $\varphi \in \ker(I + \widetilde{Q}(i\varepsilon))$ such that $\langle \varphi, \sqrt{c} \rangle = 0$ which contradicts the assumption $p(\xi) \not\equiv 0$.

Put $\widetilde{\mathcal{B}_c} = \{\widetilde{\eta}_n(i0+)\}$, $\widetilde{\eta}_n(z)$ being the set of eigenvalues of $\widetilde{Q}(z)$, hence $\widetilde{\mathcal{B}_c} = \mathcal{B}_c$. By the functional calculus we have for some $\widetilde{\eta}_* \in \mathcal{B}_c$,

$$p = \frac{|1 + \widetilde{\eta}_*|}{1 - \widetilde{\eta}_*} \geq \min\left\{\frac{1}{3}, \frac{p(\xi)}{3}\right\},$$

where the cases having been considered separately are $\widetilde{\eta}_* \leq -2$ and $\widetilde{\eta}_* \geq -2$. It remains to note that $p(\xi) < 1$ since $\widetilde{Q}(i\xi)$ is a compact operator of infinite rank. ■

Under the assumption of this remark, substituting (3.23) in (3.22) and (3.20), one gets weaker estimates which are, however, more explicit with respect to c since they do not require calculating a matrix element of the inverse operator $\left(I-\widetilde{Q}_0\right)^{-1}$, which is incorporated in p through ϑ_c.

Let us mention that given c it may happens that $p(\xi)=0$ for all $\xi>0$ which means that $\langle\varphi,\sqrt{c}\rangle=0$ for a nonzero $\varphi\in\ker\left(I+\widetilde{Q}_0\right)$. This situation arises, for instance, if $c\in\mathcal{E}$ is of the form $c=\kappa\widetilde{c}$ where $\widetilde{c}$ is an even function from L_0^+, under the appropriate choice of the constant $\kappa>0$, owing to the commutation of $\widetilde{Q}$ with the reflection operator R, $(Rf)(x)=R(-x)$. In this case the logarithmic estimate for the norm of $(I+Q(z))^{-1}$ at the vicinity of 0 wrongly asserted by Lemma 6 in [25] obviously fails to hold.

The main difficulty in estimating $(\mathcal{H}_1,\mathcal{H}_2)$, that is, in getting a "global" estimate of the angle, is that we do not know the way to estimate $\|S^{-1}(k)\|$ in the intermediate range i. e. for those k outside the vicinity of 0 given by lemma 3.10 that the estimate (3.15) is not yet essential.

Finally, in view of the untary equivalence between T and $-T^*$, all the assertions of Theorem 2 hold for the restriction of $L-i\sigma$ to its invariant subspace JH_{ess}. Strictly speaking, we have defined the absolutely continuous subspace for dissipative operators only while $L-i\sigma$ is "minus dissipative". However if we adopt the natural generalization of the notion of the absolutely continuous subspace to the nondissipative case suggested in [12] then the spectrum of $(L-i\sigma)|_{JH_{ess}}$ automatically becomes absolutely continuous, that is, $JH_{ess}=\mathcal{N}_e(T^*)\oplus H_0$.

4 Three Dimensional Transport Operator

4.1 Preliminaries

The phase space of the one-speed transport problem in the three dimensional case is $\Gamma=\mathbb{R}^3\times\mathbb{S}^2$. For a particle at a point $(q,\widehat{p})\in\Gamma$, the vectors $q\in\mathbb{R}^3$ and $\widehat{p}\in\mathbb{S}^2$ are the position and the momentum direction, respectively. The physical parameters in the problem are the number $\sigma\geq 0$ and the function $c:\mathbb{R}^3\to\mathbb{R}_+$ having the same physical meaning as before. We suppose that the distribution of the secondaries is isotropic i.e. the collision operator $K\in\mathbf{B}L^2\left(\mathbb{S}^2\right)$ is given by $K=\frac{1}{4\pi}\int_{\mathbb{S}^2}\cdot\, d\Omega$, $d\Omega$ being the Lebesgue measure on the unit sphere. It is assumed throughout that the function c is bounded ($c\in L^\infty(\mathbb{R}^3)$) and compactly supported.

Define the operator T in the Hilbert space $H = L^2(\Gamma)$ by the following expression,

$$T = i\widehat{p} \cdot \nabla_q + i\sigma c(q)K,$$

on the domain $\mathcal{D} = \{f \in H : (1)\ F(t) = f(q + t\widehat{p}, \widehat{p})$ is absolutely continu- ous for a. e. $(q, \widehat{p}) \in \Gamma$; (2) $\left.\frac{dF}{dt}\right|_{t=0} \in H\}$ of its real part $L_0 = i\widehat{p} \cdot \nabla_q$. Then T is a dissipative operator with a bounded imaginary part. The evolution of a density $u_0 \in H$ is given by solution of the Cauchy problem of the form (3.1) where the one-speed transport operator L is linked with the operator T by the relation $T = L^* + i\sigma$. The operator L_0 corresponds to the evolution $U_t = \exp iL_0 t$, $(U_t f)(q, \widehat{p}) = f(q - \widehat{p}t, \widehat{p})$, of densities in vacuum. The spectral analysis of L is reduced to that of T since L and $-T + i$ are unitarily equivalent, $L = J(-T + i)J^*$ with $(Jf)(x, \widehat{p}) = f(x, -\widehat{p})$. Without loss of generality, we set $\sigma = 1$; then $V = \Im T = c(q)K$. One can naturally identify the range of K in H with the space $L^2(\mathbb{R}^3)$ of functions of the variable q, $KH = L^2(\mathbb{R}^3) \otimes \mathbf{1} \simeq L^2(\mathbb{R}^3)$ where $\mathbf{1} \in L^2(\mathbb{S}^2)$ is the indicator of the unit sphere.

Remark that the transport operator studied in Section 3 is obtained from T when separating variables in the case of function c depending on one space coordinate only.

4.2 Spectral Analysis

We use the representation of the resolvent $R_0(z) = (L_0 - z)^{-1}$ for $z \in \mathbb{C}_+$ given by the Laplace transform of U_t,

$$(R_0(z)f)(q, \widehat{p}) = i\int_0^\infty e^{izt} f(q + \widehat{p}t, \widehat{p})dt.$$

Let us calculate the kernel of the operator $Q(z) = i\sqrt{V}R_0(z)\sqrt{V}\Big|_E$, $z \in \mathbb{C}_+$, acting in the space $E \equiv \overline{\operatorname{Ran} V} \subset L^2(\mathbb{R}^3)$. We have

$$\begin{aligned}(Q(z)f)(q) = -\frac{1}{4\pi}\sqrt{c(q)}\int_{\mathbb{S}^2} d\Omega_p \int_0^\infty e^{izt} f(q + \widehat{p}t)\sqrt{c(q + \widehat{p}t)}dt = \\ -\frac{1}{4\pi}\sqrt{c(q)}\int_{\mathbb{R}^3} e^{iz|q'|} f(q + q')\sqrt{c(q + q')}\frac{d^3q'}{|q'|^2} = \\ -\frac{1}{4\pi}\sqrt{c(q)}\int_{\mathbb{R}^3} \frac{e^{iz|q'-q|}}{|q' - q|^2}\sqrt{c(q')}f(q')d^3q'.\end{aligned}$$

Thus,

$$(Q(z)f)(q) = -\frac{1}{4\pi}\sqrt{c(q)}\int_{\mathbb{R}^3} \frac{e^{iz|q'-q|}}{|q' - q|^2}\sqrt{c(q')}f(q')d^3q'. \tag{4.1}$$

The actual integration in this formula is over a ball of finite radius. The kernel of $Q(z)$ has a weak $(2<3)$ singularity for all $z \in \mathbb{C}$. Initially defined in $\mathbb{C}_+$, the function $Q(z)$ thus extends to an entire function. By the lemma on iteration of kernels with weak singularity [27] $Q^2(z)$ is of the Hilbert-Schmidt class for any $z \in \mathbb{C}$, and hence $Q(z) \in \mathbf{S}^4$ for all $z \in \mathbb{C}$.

Theorem 3. *The nonreal spectrum $\sigma_+(T)$ of the operator T consists of finitely many eigenvalues lying on the imaginary axis. Moreover, T has no associate vectors. The dimension $N(c)$ of the invariant subspace H_d corresponding to $\sigma_+(T)$ satisfies the estimate*

$$N(c) \le \frac{1}{256\pi^4} \iint c(q)c(q') \left(\int \frac{c(\tau) d^3\tau}{|q-\tau|^2 |q'-\tau|^2} \right)^2 d^3q d^3q'. \tag{4.2}$$

The essential spectrum of T coincides with the real axis: $\sigma_{ess}(T) = \mathbb{R}$.

Proof. First, $\sigma_{ess}(T) = \sigma_{ess}(L_0) = \mathbb{R}$ by Sublemma. Remark that the finiteness of $N(c)$ follows immediately from the analyticity of Q on the real axis by the analytical Fredholm theorem.

Consider the operator $\widetilde{\Xi}(z) = iKR_0(z)|_{KH}$ acting in the space $KH = L^2(\mathbb{R}^3)$. In the Fourier representation with respect to the variable q this operator acts as the multiplication by the function

$$\widetilde{\xi}(s,z) = \frac{i}{4\pi} \int_{\mathbb{S}^2} \frac{d\Omega_p}{\widehat{p}s - z} =$$

By the rotation symmetry the integral does not depend on direction of $\hat{s} = \frac{s}{|s|}$. We continue the equality taking $\hat{s}$ for the axis of a spherical coordinate system,

$$= \frac{i}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{\sin\theta d\theta d\varphi}{|s|\cos\theta - z} = \frac{i}{2} \int_{-1}^{1} \frac{d\mu}{|s|\mu - z}.$$

Comparing this with (3.7) one sees that $\widetilde{\xi}(s,z) = \xi(|s|,z)$. In particular, $\Im\widetilde{\xi}(s,z) = r_z(|s|)$. Therefore $\Im Q(z) = \sqrt{c}\,\Im\widetilde{\Xi}(z)\sqrt{c}\Big|_E$ is sign-definite for $\Re z \neq 0$, and $\Im Q(z) = 0$ for $\Re z = 0$. Now reproducing verbatim the arguing following (3.8), one concludes that $\sigma_+(T) \subset i\mathbb{R}$, T has no associate vectors, and that $Q(i\varepsilon)$ is a negative operator and a monotone increasing function of ε. Let $N_\varepsilon(c)$ be the dimension of the spectral subspace corresponding to the interval $[i\varepsilon, i\infty)$. We then have

$$\begin{aligned} N_\varepsilon(c) \equiv \#\{s \ge \varepsilon :\ & \ker(I + Q(is)) \neq 0\} = \#\{\tau \in (0,1] : \\ & \ker(I + \tau Q(i\varepsilon)) \neq 0\} \le \|Q(i\varepsilon)\|_4^4 = \left\|Q^2(i\varepsilon)\right\|_2^2 . \end{aligned} \tag{4.3}$$

This gives $N(c) = \lim_{\varepsilon\downarrow 0} N_\varepsilon(c) \leq \|Q^2(0)\|_2^2$, which is (4.2). ■

Contrary to the slab geometry case, one sees that $\sigma_+(T) = \emptyset$ for small coupling. Comparing (4.2) and (3.6), one should mention the following drawback of (3.6). Consider $c(x) = c_1(x-b) + c_1(x+b)$ where c_1 is a compactly supported function and b is a large parameter. Then, obviously, the right hand side of (3.6) diverges as $b \to \infty$. This is obscure from the viewpoint of physics, and, indeed, one can show that $N(c)$ is actually uniformly bounded in b. At the same time, the right hand side of the estimate (4.2) is clearly uniformly bounded in b as $b \to \infty$ in the analogous problem for the three-dimensional operator.

A different statement of the transport problem was used in the paper [24]. Under the assumption of convexity of the set $\{x : c(x) \neq 0\}$, it led again to the study of the equation $(I+Q(z))\varphi = 0$ for determination of the discrete modes. Note that this equation was studied in [24] for all $z \in \mathbb{C}$. In fact, the properties of monotonicity and continuity of $Q(i\varepsilon)$ in ε which are sufficient for the conclusion that $N(c)$ is finite, have already been mentioned in [24]. The estimate (4.2) for $N(c)$ is apparently new.

Let $S(z)$ be the characteristic function of the operator T. This function is analytic on the real axis and satisfies $S(z) - I \in \mathbf{S}^\infty$. Define $\mathcal{E} = \{c : \ker(I+Q(0)) \neq 0\}$. One immediately concludes from the continuity of S on the real axis that 0 is a spectral singularity point for $c \in \mathcal{E}$. The following lemma provides us with an estimate of the function S^{-1} in a neighbourhood of zero.

Lemma 4.1. *1. If $c \notin \mathcal{E}$, then $\sup_{z\in\Pi_\epsilon} \|S^{-1}(z)\| < \infty$ for ϵ sufficiently small.*

2. If $c \in \mathcal{E}$, then $C_1|z|^{-1} \leq \|S^{-1}(z)\| \leq C|z|^{-1}$ as $z \in \omega_\delta(0)$ for sufficiently small δ.

Proof. As well as in the proof of lemma 3.3, it suffices to obtain the desired upper estimates for $(I+Q(z))^{-1}$.

1. Arguing as in the proof of Proposition 3.7, one concludes that $\ker S(k) = \{0\}$ for $k \neq 0$. Hence we are done if we show that $\|\Re Q(z)\| \to 0$ as $|\Re z| \to \infty$ uniformly in $\Im z$ for $0 \leq \Im z \leq \epsilon$. We have $\Re Q(z) = \sqrt{c}\Re\widetilde{\Xi}(z)\sqrt{c}\Big|_E$ where $\Re\widetilde{\Xi}(z)$ in the Fourier representation acts as the multiplication by the function $\Re\xi(|s|, z) = -\Im\left[\frac{1}{|s|}\int_{-|s|}^{|s|}\frac{d\tau}{\tau - z}\right]$. Then for $x \in [0, C]$, $C < \Re z$, we have $|\Re\xi(x,z)| \leq \frac{2}{|C - \Re z|}$. Since $\left|\Im\left(\ln\frac{z-x}{z+x}\right)\right| \leq 4\pi$ one estimates $|\Re\xi(x,z)| \leq \frac{4\pi}{x} \leq \frac{4\pi}{C}$ for $x \in [C, +\infty)$. Taking $C \to \infty$, $C = o(\Re z)$ for $|\Re z| \to \infty$, one gets $\left\|\Re\widetilde{\Xi}(z)\right\| \to 0$.

2. For $z \in U_\delta(0)$ we have $Q(z) = Q(0) - izA + O\left(|z|^2\right)$ in the operator norm. Then $Q'(i\varepsilon) = -i\sqrt{c}\left(\frac{d}{d\varepsilon}\widetilde{\Xi}(i\varepsilon)\right)\sqrt{c}\Big|_E$, $\varepsilon > 0$. In the Fourier representation, the operator in parentheses acts as the multiplication by the function $\left(|p|^2 + \varepsilon^2\right)^{-1}$. Thus, $iQ'(i\varepsilon)$

is a positive operator and a monotone decreasing function of ε. Hence $A > 0$. The upper bound now follows from lemma A.1. The lower bound is obvious since $\|(I + Q(z))\varphi\| = O(|z|)$ for $\varphi \in \ker(I + Q(0))$. ■

Note that had we tried to obtain the upper bound in part 2 by applying abstract remark 1.1, say, to the function $(I + Q(\sqrt{w}))^{-1}$, we would have got an estimate of a worse order due to the singularity of the conformal transformation.

Define $\mathcal{P}_d$ to be the Riesz projection corresponding to $\sigma_+(T)$, then $H_{ess} = (I - \mathcal{P}_d)H$ is the invariant subspace of T corresponding to the essential spectrum. Part 2 of the above lemma implies the following conclusion about the group $Z_t^e = \exp(iT_{ess}t)$, $T_{ess} = T|_{H_{ess}}$.

Corollary 4.2. *If $c \in \mathcal{E}$ then*

$$\left\|Z_{-t}^e\right\| \le C_1 + C_2 t, \quad t \ge 0,$$

and this estimate is exact in the power scale, in the sense that for any $\varepsilon \in (0,1)$ there exists $f \in H_{ess}$ such that $\left\|Z_{-t}^e f\right\| = C_\varepsilon t^{1-\varepsilon}(1 + o(1))$, $C \neq 0$, for $t \to +\infty$.

This corollary is a partial case of an abstract result related to the operators with spectral singularities of finite power order. Its proof, which also uses the functional model, will be published elsewhere.

Lemma 4.3. *S has a scalar multiple.*

Proof. Examining the proof of lemma 4.1, part 1, one concludes that the norm of $S^{-1}(z)$ is bounded at any strip Π_ϵ, $\epsilon > 0$, if $z \notin U^\circ(\delta)$ where $U^\circ(\delta)$ is the union of an arbitrary open neighborhood of $\sigma_+(T)$ and $\omega_\delta(0)$, regardless to the condition $c \notin \mathcal{E}$. Then $S(z) \to I$ as $\Im z \to +\infty$ uniformly in $\Re z$ since $\|Q(z)\| \le \frac{\|V\|}{\Im z}$. Taking this into account, one obtains for all $\delta > 0$

$$\sup_{z \in \mathbb{C}_+ \setminus U^\circ(\delta)} \left\|S^{-1}(z)\right\| < \infty. \tag{4.4}$$

It now follows from lemma 4.1, part 2, that any function of the form $m(z) = b(z)z\widetilde{m}(z)$ where $\widetilde{m} \in H^\infty$ and satisfies $|\widetilde{m}(z)| \le C\left(1 + |z|\right)^{-1}$, and b is the Blaschke product with simple zeroes at the points of $\sigma_+(T)$, is a scalar multiple for S, since the poles of $(L - z)^{-1}$ at the points of $\sigma_+(T)$ are simple. ■

Taking $\widetilde{m}$ to be outer one gets the canonical factorization $m = bm_e$, $m_e(z) = z\widetilde{m}(z)$, of the scalar multiple. We thus arrive at the following

Proposition 4.4. *The component T_{ess} of the operator T has purely absolutely continuous spectrum, $H_{ess} = \mathcal{N}_e \oplus H_0$.*

The proof of this assertion for the slab case required more analysis due to the fact that we were not able to estimate S^{-1} at 0. In fact, to know that S is analytic at 0 would be enough for the above proof of Proposition 4.4, and the exact estimate of lemma 4.1 is redundant for that purpose.

Proposition 4.5. *T_{ess} is similar to a selfadjoint operator if and only if $c \notin \mathcal{E}$. If $c \notin \mathcal{E}$ then the wave operators*

$$W_{\pm}(L, L_0 + i\sigma) \equiv W_{\pm}(T^*, L_0) = s\text{ - }lim_{t\to\pm\infty} e^{iT^*t} e^{-iL_0 t}$$

exist on H and are complete, Ran $W_{\pm} = JH_{ess} \equiv H_d^{\perp}$.

Proof. The first assertion follows immediately from item 1 of lemma 4.1 by the Nagy-Foias criterion. The statement about wave operators follows from theorem 6 in [11], as it does in the slab case. ■

The picture of the behaviour of the operator T in coupling constant remains the same as in the slab case: remark 3.9 can be taken verbatim except, of course, for the fact that in the situation under consideration T has no eigenvalues for small κ.

The following lemma plays a role quite analogous to that of lemma 3.10. Let $|c|_R^2 = \int \frac{c(q)c(q')}{|q-q'|^2} d^3q d^3q'$ be the Rolnick norm of c [14].

Lemma 4.6. *Set $\tilde{p} =$ dist $(-1, \sigma(Q(0)) \setminus \{-1\})$, $\mathcal{M} = \dim\ker(I + Q(0))$ and*

$$\delta_* = \frac{\pi}{18} \frac{\tilde{p}^2}{|c|_R}. \tag{4.5}$$

Then $\|D(k) - D(0)\| < p^2/2$ for all $k \in (-\delta_*, \delta_*)$.

Proof. Let $\Theta(z) = Q(z) - Q(0)$. We then have

$$\|S(z) - S(0)\| = 2 \left\|(I - Q(z))^{-1}\, \Theta(z)\, (I - Q(0))^{-1}\right\| \le 2\|\Theta(z)\|$$

and thus

$$\|(S^*S)(z) - S^2(0)\| \le 4\|\Theta(z)\|. \tag{4.6}$$

The kernel of $\Theta(z)$ has the form $-\frac{1}{4\pi}\sqrt{c(q)}\frac{e^{iz|q'-q|}-1}{|q'-q|^2}\;\sqrt{c(q')}$. Using the inequality $|e^{ix} - 1| \le |x|$, one estimates for $k \in \mathbb{R}$,

$$\|\Theta(k)\|^2 \le \|\Theta(k)\|_2^2 = \frac{1}{16\pi^2} \iint c(q) \frac{\left|e^{ik|q'-q|} - 1\right|^2}{|q'-q|^4} c(q') d^3q d^3q' \le$$
$$\le \frac{|k|^2}{16\pi^2} \iint \frac{c(q)c(q')}{|q'-q|^2} d^3q d^3q' \equiv \frac{|k|^2}{16\pi^2} |c|_R^2 .$$

Substituting this in (4.6), we obtain for $k \in (-\delta_*, \delta_*)$

$$\|D(k) - D(0)\| \leq \frac{|c|_R}{\pi}|k| < \frac{\tilde{p}^2}{18}.$$

It remains to note that by the functional calculus

$$p^2 \equiv \operatorname{dist}\left(0, \sigma\left(S^2(0)\right) \setminus \{0\}\right) \geq \left(\min\left\{\frac{1}{3}, \frac{\tilde{p}}{3}\right\}\right)^2 \geq \frac{\tilde{p}^2}{9}$$

since $Q(0) \leq 0$, and thus $\tilde{p} \leq 1$. ■

Reproducing the arguing followed by lemma 3.10 and theorem 2, we now get the following

Theorem 4. *If $c \in \mathcal{E}$ then for any positive $\delta \leq \delta_*$ (see (4.5)), the operator T_{ess} can be represented as the linear sum $T_{ess} = T_1 \dot{+} T_2$ of operators $T_{1,2}$ acting in invariant subspaces $\mathcal{H}_{1,2}$ such that*

1. T_1 has the spectrum $\sigma(T_1) = [-\delta, \delta]$ of multiplicity $\dim \ker(I + Q(0))$,

2. T_2 is similar to a selfadjoint operator and $\sigma(T_2) = \mathbb{R}$,

3. The angle $(\mathcal{H}_1, \mathcal{H}_2) > 0$. The angle $(\mathcal{H}_1, \mathcal{P}_\delta \mathcal{H}_2)$ admits the estimate

$$\sin(\mathcal{H}_1, \mathcal{P}_\delta \mathcal{H}_2) \geq \frac{\tilde{p}}{3\sqrt{2}}$$

where $\tilde{p}$ is defined in lemma 4.6 and $\mathcal{P}_\delta \equiv I - \mathcal{P}_\omega$, $\omega = \mathbb{R} \setminus [-\delta, \delta]$, is the spectral projection of T for the interval $[-\delta, \delta]$ defined after corollary 2.3.

The assertions of this theorem are extended to the operator $L - i\sigma$ in the same way as it takes place in the slab case, see the remark made at the end of section 3.

5 Concluding remarks

Let us discuss an exactly solvable limit case, when $c(x) \equiv \kappa$ with a $\kappa > 0$ for all $x \in \mathbb{R}$, of the transport operator for slab geometry. After Fourier transformation with respect to the variable x the operator T is fibred. Its fibres $T_p = p\mu + i\frac{\kappa}{2}\langle\cdot, \mathbf{1}\rangle\mathbf{1}$, $p \in \mathbb{R}$, act in $L^2(\Omega)$ and are rank 1 perturbations of the selfadjoint multiplication operators $L_{o,p} = p\mu\cdot$. Thus, the essential spectrum of T_p is $[-p, p]$. The spectrum of T_p in $\mathbb{C}_+$ is determined from the equation $1 + \xi(p, z) = 0$ to give that T_p has one eigenvalue $\lambda_p = ip\cot\frac{p}{\kappa}$ for $|p| < \frac{\pi}{2}\kappa$ and has none for $|p| \geq \frac{\pi}{2}\kappa$. From this it follows that the spectrum $\sigma(T) = \bigcup_p \sigma(T_p) = \mathbb{R} \bigcup [i\kappa, 0)$ (that $\sigma(T)$ cannot get larger is obvious since $\xi(p,z) \to 0$ for fixed $z \in \mathbb{C}_+$ as $|p| \to \infty$ (see (3.7))). In this way, let us remark that each $\lambda \in [i\kappa, 0)$ can be shown to be an accumulation point of eigenvalues of the operators T corresponding to $c = \kappa\chi_{[-a,a]}$ when $a \to \infty$.

It should be noted that in this limit case T is not a spectral operator (see [28]) *for all* $\kappa > 0$. Indeed the spectral projection $\mathcal{P}_p$ with respect to the eigenvalue λ_p of the operator T_p looks like

$$\mathcal{P}_p = -\frac{p^2}{2\sin^2\frac{p}{\kappa}} \left\langle \cdot, \overline{\Psi_p} \right\rangle_{L^2(\Omega)} \Psi_p$$

where $\Psi_p(\mu) = \frac{1}{p\mu - i\rho(p)}$, $\rho(p) = p\cot\frac{p}{\kappa}$. Obviously, the operator $\mathcal{P}_\alpha = \int_{-\frac{\pi}{2}\alpha}^{\frac{\pi}{2}\alpha} \oplus \mathcal{P}_p$ is bounded whenever $\alpha < \kappa$ and gives the spectral projection of T for the interval $\left[i\rho\left(\frac{\pi}{2}\alpha\right), i\kappa\right]$. However, $\|\mathcal{P}_\alpha\| \to \infty$ as $\alpha \to \kappa$ which contradicts (B) property of spectral operators.

A special interest from the part of physics is paid to the solutions of the problem (3.1) exponentially increasing in time. Obviously, such solutions may only exist when $|c|_\infty > 1$ (the supercritical transport problem), otherwise the operator L is dissipative. Since L and $-T + i$ are unitarily equivalent, one observes that $\sup_{\varepsilon>0} N_{1+\varepsilon}(c)$ is precisely the number of exponentially increasing modes for the supercritical transport problem. Thus, for $|c|_\infty > 1$ one can use (3.6) and the estimate obtained by taking $\varepsilon = 1$ in (4.3) as estimates on this number. In this respect, the estimate (3.6) appears to be rather crude due to the fact that in the slab case the estimate of $N_\varepsilon(c)$, that is, before going to the limit $\varepsilon \to 0$, is worse than that of $N(c)$.

Notice that the perturbations treated in this paper are not of relative trace class: the condition $(H - z)^{-1} - (H_0 - z)^{-1} \in \mathbf{S}^1$ obviously fails for the pair $(L, L_0 + i\sigma)$ because $\Re Q(z) \notin \mathbf{S}^1$, and therefore $R_0(z)VR_0(z) \notin \mathbf{S}^1$ (see (1.3)).

It should also be noted that, once we have established the similarity of T_{ess} to a selfadjoint operator for $c \notin \mathcal{E}$, the existence and completeness of the wave operators follow directly from the theory of smooth perturbations [14, 29]. For the three - dimensional problem this is straightforward, since obviously $\|Q(z)\| \leq \|Q(0)\|$ for $\Im z \geq 0$, and thus the operator $\sqrt{V}$ is L_0 - smooth [14, Theorem XIII.25]. That the operator $\sqrt{V}$ is T_{ess} - smooth whenever $c \notin \mathcal{E}$ is an abstract fact: the operator $\sqrt{\Im D}$ is D - smooth for any dissipative operator D similar to a selfadjoint one. For the slab problem the operator $\sqrt{V}$ is not L_0 - smooth since $Q(i\varepsilon) = \Re Q(i\varepsilon)$ is unbounded as $\varepsilon \to 0$ (see [14, Theorem XIII.25, it. 4]); however, it is L_0 - smooth locally with respect to any closed interval not containing zero, and this suffices.

Finally, it should be mentioned that the main difficulty in physical interpretation of the results obtained is that the natural statement of the transport problem assumes $L^1(\Gamma)$ for the space of distribution functions rather than L^2. It can be shown, however, that for the slab case the sets of L^2 and L^1 - eigenfunctions of the operator

L coincide, hence the estimate (3.6) remains valid for the operator L in L^1. It is also believed that if we will manage to prove the existence and completeness of the wave operators in L^1 for $c \notin \mathcal{E}$ then the stationary approach for calculating the scattering operator $\mathcal{S} = W_+^{-1}W_-$ could be applied. In particular, it is easy to see that if the wave operators exist for the problem in L^1 with $c \notin \mathcal{E}$ then they are consistent with the ones defined in Proposition 4.5, that is, coincide on $L^2 \cap L^1$.

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Appendix

Lemma A.1. *Let $A(z)$ be an analytic operator function on a domain $\mathcal{D} \subset \mathbb{C}$ whose values are compact operators in a Hilbert space K and* $\ker(I + A(z_0)) \neq \{0\}$ *for some $z_0 \in \mathcal{D}$. Assume $cA'(z_0) > 0$ with some $c \in \mathbb{C}$ and $A(z_0) = A^*(z_0)$. Then*

$$\left\| (I + A(z))^{-1} \right\| \leq C \left| z - z_0 \right|^{-1}$$

for $z \in U_\delta(z_0)$, δ small enough.

Proof. Put $\mathcal{N} = \ker(I + A(z_0))$. Representing operators by 2×2 matrices with respect to the decomposition $K = \mathcal{N} \oplus \mathcal{N}^\perp$ one has

$$I + A(z) = \begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix} + (z - z_0) \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} + O\left((z - z_0)^2\right) =$$
$$\begin{pmatrix} (z - z_0)B_{00} & 0 \\ 0 & A_0 \end{pmatrix} \left[\begin{pmatrix} I & B_{00}^{-1}B_{01} \\ 0 & I \end{pmatrix} + O(z - z_0) \right].$$

Here we took into account that the assumption about $A'(z_0)$ guarantees the invertibility of B_{00} since $\dim \mathcal{N} < \infty$ by compactness of $A(z)$. Then

$$(I + A(z))^{-1} = \left[\begin{pmatrix} I & -B_{00}^{-1}B_{01} \\ 0 & I \end{pmatrix} + O(z - z_0) \right] \begin{pmatrix} (z - z_0)^{-1} B_{00}^{-1} & 0 \\ 0 & A_0^{-1} \end{pmatrix}.$$

■

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