Jakobson's Theorem near saddle-node bifurcations

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Abstract

We discuss one parameter families of unimodal maps, with negative Schwarzian derivative, unfolding a saddle-node bifurcation. It was previously shown that for a parameter set of positive Lebesgue density at the bifurcation, the maps possess attracting periodic orbits of high period. We show that there is also a parameter set of positive density at the bifurcation, for which the maps exhibit absolutely continuous invariant measures which are supported on the largest possible interval. We prove that these measures converge weakly to an atomic measure supported on the orbit of the saddle-node point. Using these measures we analyze the intermittent time series that result from the destruction of the periodic attractor in the saddle-node bifurcation and prove asymptotic formulae for the frequency with which orbits visit the region previously occupied by the periodic attractor.

1 Introduction

1.1 Background

This article is a companion article for [HomYou00]. In that article we discussed intermittent dynamics associated with boundary crisis (homoclinic) bifurcations in families of unimodal maps. In the present work we treat saddle node bifurcations from the same perspective.

By Jakobson's celebrated work [Jak81], the logistic family $x \mapsto \mu x(1-x)$, $x \in [0,1]$, admits absolutely continuous invariant measures (a.c.i.m.'s) for μ from a set of positive measure. In fact, $\mu = 4$ is a (Lebesgue or full) density point of this set. A different argument for this result was given by Benedicks and Carleson [BenCar85], [BenCar91]. Their reasoning was generalized to unfoldings $\{f_\gamma\}$ of unimodal Misiurewicz maps f_0, with eventually periodic critical point c (and possessing negative Schwarzian derivative). It was shown that the bifurcation value $\gamma = 0$ is a density point of the set of parameter values for which f_γ admits an absolutely continuous invariant measure [MelStr93], [ThiTreYou94].

We generalize these results to unfoldings of saddle-node bifurcations in families $\{f_\gamma\}$ of unimodal maps with negative Schwarzian derivative. We establish that a saddle-node bifurcation value occurs as a point of positive density of the parameter set for which there are absolutely continuous invariant measures. Under the assumption that f_0 is not more than once renormalizable, we construct parameters for which f_γ possesses a.c.i.m.'s supported on the maximal interval $[f_\gamma^2(c), f_\gamma(c)]$, see Theorem A below. In contrast to the Misiurewicz bifurcation values, the saddle-node bifurcation value is not a full density point of this parameter set. This is because it is known that the parameter set for which the map has a periodic attractor has positive density at any saddle-node.

Following the construction of a.c.i.m.'s, we continue with a detailed discussion of the intermittency that occurs due to the saddle-node bifurcation. That saddle-node bifurcations can give rise to intermittency is known since [PomMan80], who called intermittency associated with a saddle-node bifurcation type I intermittency. Pomeau and Manneville studied type I intermittency in connection with the Lorenz model. In the model, simplifying (hyperbolicity) assumptions on the dynamics outside a neighborhood of the saddle-node periodic orbit are made. In perhaps the most basic example of type I intermittency, in families of unimodal maps, such simplifications are not justified. This is due to the presence of a critical point. Our discussion of absolutely continuous invariant measures allows us to give a rigorous treatment of intermittent time series, where we explain and prove quantitative aspects earlier discussed numerically in [HirHubSca82], see Theorems B and C below.

Diaz et. al. [DiaRocVia96] studied the unfoldings of saddle-node bifurcations in higher dimensional diffeomorphisms and their results imply in the present context that there exists a subset Ω_D which has positive density at 0, such that for each $\gamma \in \Omega_D$, f_γ has an absolutely continuous invariant measure. The measures produced there are supported on small periodic domains on which the map renormalizes to a Hénon-like family. In the present context this corresponds to parameter values inside periodic windows, for which f_γ is renormalizable. However, the invariant measures produced in Jakobson's work are supported on the maximal possible interval, $[f_\gamma^2(c), f_\gamma(c)]$ and the invariant measures that we construct are also supported on $[f_\gamma^2(c), f_\gamma(c)]$.

While editing the final draft of this paper, we learned of the results of Maria João Costa [Cos01], corresponding to part of her 1998 thesis, on related work in a similar context. She focused on the sink-horseshoe bifurcation in which a sink and a horseshoe collapse, see [Zee82, Cos98], and studied families $\{g_\gamma\}$ of unimodal interval maps to describe bifurcations. The class of families of interval maps studied in [Cos01] differs from ours; it consists of unfoldings of unimodal maps $g_0 : [0,1] \to [0,1]$, that possess a saddle node fixed point at $p \in (0,1)$, so that the critical point $c > p$ satisfies $g_0^2(c) < p$. For such families Costa derived the analogue of Theorem A below, stating that absolutely continuous invariant measures supported on the maximal interval occur with positive density at the bifurcation point. The set-up of Costa's proof corresponds to ours in that she also combines arguments originating from Benedicks and Carleson [BenCar85, BenCar91] with the introduction of induced maps, as discussed below.

1.2 Assumptions and statement of main results

Let $\{f_\gamma\}$ be a family of unimodal maps of the interval $[0,1]$, with critical point at c. Suppose that each f_γ is at least C^3 smooth and that $f_\gamma(x)$, $Df_\gamma(x)$, and $D^2 f_\gamma(x)$, are C^1 w.r.t. γ. Suppose that each f_γ has negative Schwarzian derivative (see [MelStr93]) and that $D^2 f_\gamma(c) < 0$. Further, suppose that $f_\gamma(1) = f_\gamma(0) = 0$ and that the fixed point at 0 is hyperbolic repelling. We say that $\{f_\gamma\}$ *unfolds a (quadratic) saddle-node* if,

- There is a q-periodic point a, with $Df_0^q(a) = 1$, and,
- $D^2 f_0^q(a) \frac{\partial}{\partial\gamma} f_\gamma^q(a) > 0$ at $\gamma = 0$.

For the sake of clarity, we will assume that

$$\frac{\partial}{\partial\gamma} f_\gamma^q(a)|_{\gamma=0} > 0 \quad \text{and} \quad D^2 f_0^q(a) > 0.$$

With this convention, for $\gamma > 0$ the saddle-node point disappears and complicated dynamics may occur.

For a set A of parameter values, let $m(A)$ be its Lebesgue measure. A unimodal map f is called *renormalizable* if there is a proper subinterval $I \subset [0,1]$ containing the critical point c, so that $f^n(I) \subset I$ for some $n > 0$ and $f^i(I) \cap I = \emptyset$ for $0 < i < n$. A renormalizable map is called *once renormalizable* if the above property defines n uniquely.

Theorem A *Let $\{f_\gamma\}$ be as above, unfolding a saddle-node bifurcation at $\gamma = 0$. Assume that f_0 is once renormalizable. Consider the set Γ such that for each $\gamma \in \Gamma$, the map f_γ has an absolutely continuous invariant measure ν_γ whose support is the maximal interval $[f_\gamma^2(c), f_\gamma(c)]$. The set Γ has positive density at $\gamma = 0$:*

$$\liminf_{\gamma^* \searrow 0} \frac{m(\Gamma \cap (0, \gamma^*))}{\gamma^*} \quad > \quad 0.$$

It does not have full density:

$$\limsup_{\gamma^* \searrow 0} \frac{m(\Gamma \cap (0, \gamma^*))}{\gamma^*} \quad < \quad 1.$$

Careful numerical studies of the quadratic family, $x \mapsto \gamma x(1-x)$, predict that between the period doubling limit and $\gamma = 4$, excluding the period three window, less then about 15% of the parameter values correspond to the periodic windows [Hun01]. Numerical simulations also suggest that near a saddle-node bifurcation, there is a large set of parameter values for which the a.c.i.m.'s are supported on the maximal possible interval. This is illustrated in Figure 1.

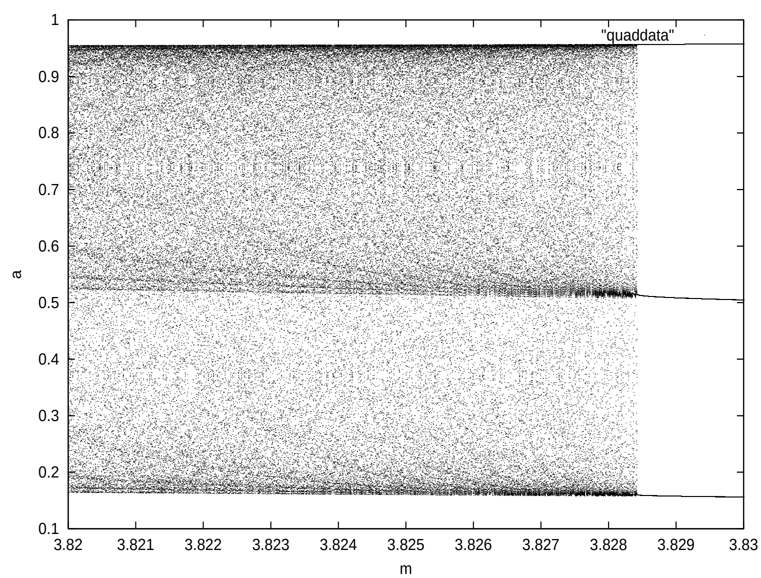

Figure 1: Bifurcation diagram for the quadratic family $x \mapsto \mu x(1-x)$. For $\mu = \mu_{sn} = 1 + 2\sqrt{2}$ the map undergoes a saddle-node bifurcation of a period 3 orbit. For most observed parameter values, $\mu < \mu_{sn}$, orbits appear to fill the interval $[f_\mu^2(1/2), f_\mu(1/2)]$.

Existence of the parameter set Γ follows from a similar approach as used by Benedicks and Carleson in their treatment of Jakobson's result. Below we comment in more detail on the construction of the parameter set Γ. We now first consider intermittency that results from the saddle-node bifurcation and introduce our second main result.

Intermittent dynamics manifests itself by alternating phases with different characteristics. In one phase, referred to as the laminar phase, the dynamics appear to be nearly periodic. While in the other phase, the relaminarization phase, the orbit makes large, seemingly chaotic excursions away from the periodic region. These excursions are called chaotic bursts. Let $\bar{E}$ be a neighborhood of the orbit, $\mathcal{O}(a)$ of a, not containing a critical point of f_0^q. Let $\chi_{\bar{E}}$ be defined as

$$\chi_{\bar{E}}(x,\gamma) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{\bar{E}}(f_\gamma^i(x)), \tag{1}$$

whenever the limit exists, where $1_{\bar{E}}$ is the usual indicator function of the set $\bar{E}$. That is, $\chi_{\bar{E}}(x,\gamma)$ is the relative frequency with which the orbit $\mathcal{O}(x)$ visits $\bar{E}$ (for those x for which the limit exists). The following theorem discusses $\chi_{\bar{E}}(x,\gamma)$ for γ near 0.

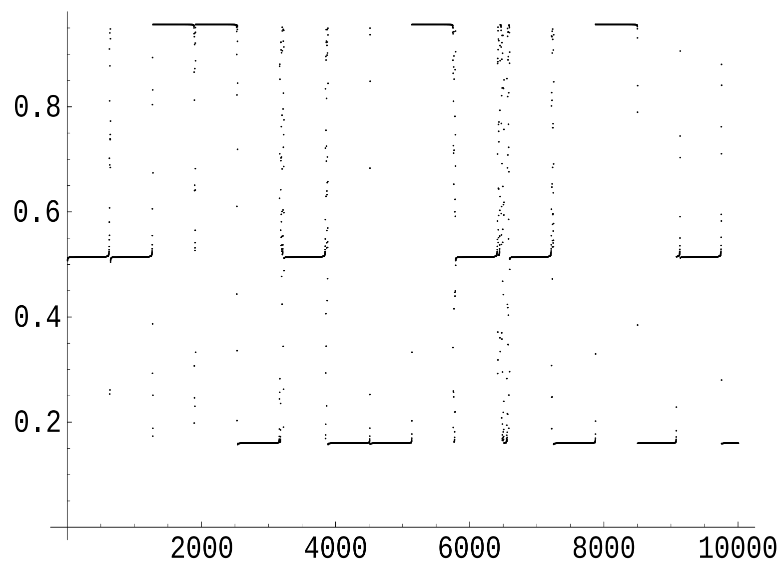

Figure 2: Time series for the quadratic map $x \mapsto \mu x(1-x)$ for $\mu = 3.828$ near the saddle-node bifurcation of the period three orbit. Both the laminar phase (nearly periodic) and chaotic bursts are clearly seen.

Denote by ν_0 the atomic measure supported on the orbit of a, given by

$$\nu_0 = \frac{1}{k} \sum_{i=0}^{k-1} \delta_{f_0^i(a)}. \tag{2}$$

We will denote the usual weak convergence of measures by the symbol $\rightharpoonup$.

Theorem B *Let $\{f_\gamma\}$ and Γ be as in Theorem A. There exist sets $\Omega \subset \Gamma$ of parameter values with positive density at $\gamma = 0$, so that*

$$\lim_{\gamma\in\Omega,\gamma\searrow 0} \nu_\gamma \quad \rightharpoonup \quad \nu_0.$$

Restricting to $\gamma \in \Omega$, $\chi_{\bar{E}}(x,\gamma)$ is a constant, $\chi_{\bar{E}}(\gamma)$, almost everywhere on $[0,1]$ and $\chi_{\bar{E}}(\gamma)$ depends continuously on γ at 0. There exist $K_1, K_2 > 0$ so that

$$K_1 \leq \lim_{\gamma\in\Omega,\gamma\searrow 0} \frac{1-\chi_{\bar{E}}(\gamma)}{\sqrt{\gamma}} \leq K_2. \tag{3}$$

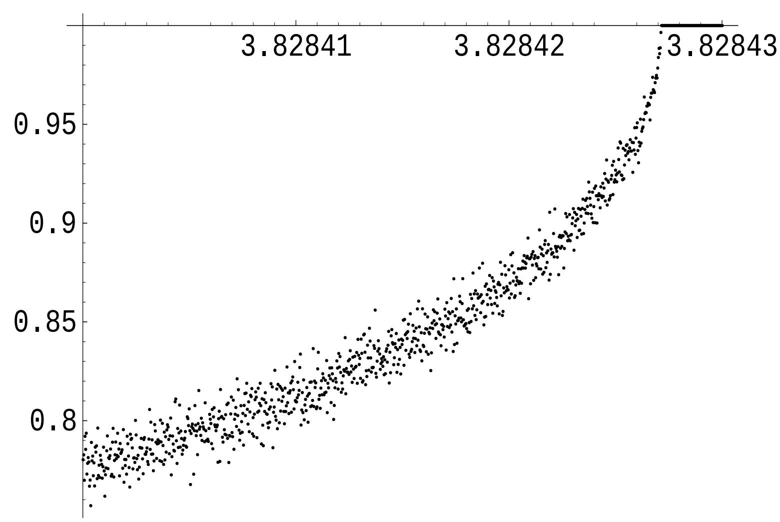

Figure 3: Numerical computation of $\chi_{\bar{E}}(\mu)$ as a function of μ for the logistic family $f_\mu(x) = \mu x(1-x)$ near the saddle-node bifurcation of a period three orbit at $\mu_{sn} = 1 + 2\sqrt{2}$.

Each set Ω from the above theorem is constructed as a union $\cup_{l\geq l_0}\Omega_l$ of sets Ω_l, where each Ω_l has, as a Lebesgue density point, a parameter value γ_l^* for which $f_{\gamma_l^*}^{ql+m}(c)$ hits a hyperbolic repelling periodic point $y^*_{\gamma_l^*}$, for some m (the map $f_{\gamma_l^*}$ is a Misiurewicz map). Different sets Ω result from considering points $y^*_{\gamma_l^*}$ in different periodic orbits. For the construction of Ω_l, we adapt Benedicks and Carleson's treatment of Jakobson's result (we follow in fact the approach taken by Luzzatto [Luz00]) and construct parameter values for which $f_\gamma(c)$ has a positive Lyapunov exponent. The main obstacle in the present case is the fact that when orbits fall into the saddle-node region they remain there for many iterations and the derivative of f_γ^q in that region is close to one. We get around this problem by defining an induced map which skips over the saddle-node region. We show that the induced map has uniform exponential growth of derivatives and that this implies exponential growth of derivatives (with weaker constants) for the original map. Careful estimates bound the measure of the sets Ω_l for increasing values of l, thus bounding the measure of their union Ω.

For comparison we include a result from [HomYou00], which builds on results in [AfrLiuYou96], [DiaRocVia96], and [AfrYou98], showing that also periodic attracting orbits are found for parameters from a set with positive density at the bifurcation point. The frequency with which the dynamics is in the laminar phase behaves in a similar way to the frequency in Theorem B. Note that the fact that the set Γ in Theorem A does not have full density at $\gamma = 0$, follows from the following result.

Theorem C *Let $\{f_\gamma\}$ be as above, unfolding a saddle node bifurcation at $\gamma = 0$. There exist sets $\mathcal{A}$ of parameter values of positive measure and positive density at 0, i.e.*

$$\lim_{\gamma \searrow 0} \frac{m\left(\mathcal{A} \cap [0,\gamma)\right)}{\gamma} \quad > \quad 0, \tag{4}$$

so that for each $\gamma \in \mathcal{A}$, f_γ has an attracting periodic orbit. Further,

$$\lim_{\gamma \in \mathcal{A}, \gamma \searrow 0} \nu_\gamma \quad \rightharpoonup \quad \nu_0,$$

where ν_γ, $\gamma \in \mathcal{A}$, is the invariant measure supported on the periodic orbit. Restricting to $\gamma \in \mathcal{A}$, $\chi_{\bar{E}}(x,\gamma)$ is a constant, $\chi_{\bar{E}}(\gamma)$, almost everywhere on $[0,1]$ and $\chi_{\bar{E}}(\gamma)$ depends continuously on γ at 0. There exists $K > 0$ so that

$$\lim_{\gamma \in \mathcal{A}, \gamma \searrow 0} \frac{1 - \chi_{\bar{E}}(\gamma)}{\sqrt{\gamma}} \quad = \quad K.$$

Different sets $\mathcal{A}$ lead to different limit values K in Theorem C. In fact, the proof of Theorem C makes clear that arbitrary large numbers occur as the limit values K. This fact, together with Theorems A and C lead us to conjecture that there is a parameter set Λ which has $\gamma = 0$ as a Lebesgue density point, so that

$$\lim_{\gamma \searrow 0, \gamma \in \Lambda} \frac{\ln(1 - \chi_{\bar{E}})}{\ln \gamma} = \frac{1}{2}.$$

It was shown in [HomYou00] that such a limit cannot hold without restricting the parameter set.

2 The saddle node: local embedding flows

Denote by E a small neighborhood of a on which f_0^q is invertible. Let $W_{loc}^s(a)$ and $W_{loc}^u(a)$ denote the usual local stable and local unstable sets for a.

Proposition 2.1 *Let $\{f_\gamma\}$ be a C^1 family of C^r, $r \geq 2$, maps unfolding a saddle-node. Then there exists a family of C^r flows, $\{\phi_\gamma^t\}_{0\leq\gamma<\bar{\gamma}}$, on E such that $f_\gamma^q \equiv \phi_\gamma^1$ for each $\gamma \geq 0$. Further, $\phi_\gamma^t(\cdot) \to \phi_0^t(\cdot)$ in the C^1 topology on E and in the C^r topology on compact intervals away from the fixed point. The flow ϕ_0^t is uniquely determined by f_0.*

Proof. The C^∞ version of this theorem is due to Takens [Tak73]. The C^r result follows from Part 2 of [Yoc95]. The case $\gamma = 0$ follows from Appendix 3 of [Yoc95]. The case $\gamma > 0$ and the convergences as $\gamma \searrow 0$ follow from Theorem IV.2.5 and Lemma IV.2.7 of the same. □

Remark 2.2 *This result is known as the Takens Embedding Theorem. A version of it appears in [IlyLi99]. They proved that one may obtain $\phi_\gamma^t(x)$ which depends C^r smoothly on both x and γ, even at the fixed point, if one requires that $(x, \gamma) \mapsto f_\gamma(x)$ be $C^{R(r)}$ smooth, where $R(r)$ may be larger than r. Proposition 2.1 allows for our weaker hypotheses and its implications are sufficient for our purposes.*

Choose $d \in W_{loc}^s(a)$ and let

$$I_\gamma^s = [d, f_\gamma^q(d)].$$

Also, choose a point $e \in W_{loc}^u(a)$ so that

$$I_\gamma^u = [e, f_\gamma^q(e)] \subset E.$$

For the sake of convenience we restrict E to be the interval

$$E = [d, e],$$

so that $W_{loc}^s(a) = [d, a)$ and $W_{loc}^u(a) = (a, e]$. We will use the embedding flow on the interval $E \cup I_\gamma^u = [d, f_\gamma^q(e)]$.

Given $\gamma \geq 0$ and $x \in I_\gamma^u$, define $\tau_\gamma^u(x)$ to be the unique number for which

$$\phi_\gamma^{\tau_\gamma^u(x)}(e) = x.$$

For $\gamma \geq 0$ and $x \in I_\gamma^s$, let $\tau_\gamma^s(x)$ be defined by

$$\phi_\gamma^{\tau_\gamma^s(x)}(d) = x.$$

It follows from the smoothness of $\phi^t_\gamma(x)$ that for each $\gamma \geq 0$, the functions $\tau^{s,u}_\gamma$ are C^r diffeomorphisms from $I^{s,u}_\gamma$ to [0,1]. We will use $\tau^{s,u}_\gamma$ as coordinates on $I^{s,u}_\gamma$. In the following, we will associate $[0,1]$ with the unit circle $\mathbf{S}^1$ by making the identification $0 \sim 1$. We also treat $I^{s,u}_\gamma$ as circles through the coordinates $\tau^{s,u}_\gamma$.

Given d and e as above, let $\{\gamma_l\}_{l=l_0}^\infty$ be the sequence, $\bar{\gamma} > \gamma_{l_0} > \gamma_{l_0+1} > \ldots$, defined by:

$$f^{ql}_{\gamma_l}(d) = e.$$

For each $l \geq l_0$ let $g_l : [0,1] \to [\gamma_{l+1}, \gamma_l]$, be the reparameterization map defined by

$$\phi^{l+\theta}_{g_l(\theta)}(d) = e.$$

We have that $g_l(0) = \gamma_l$ and $g_l(1) = \gamma_{l+1}$. We may invert $g_l(\cdot)$, for each l, to obtain maps $\theta_l : [\gamma_{l+1}, \gamma_l] \to [0,1]$.

Proposition 2.3 *The reparameterization maps g_l are smooth monotone decreasing functions with uniformly small distortion: given $\varepsilon > 0$ there is $N \in \mathbb{N}$ so that for every $l \geq N$ and every $\theta \subset [0,1]$,*

$$(1-\varepsilon) \leq \frac{Dg_l(\theta)}{|\gamma_l - \gamma_{l+1}|} \leq (1+\varepsilon).$$

PROOF. Diaz et. al. proved this result under the hypothesis that $(x,\gamma) \mapsto f_\gamma(x)$ is $C^{R(r)}$, using Il'yashenko and Li's embedding result (Remark 2.2). A proof of this result under the current hypotheses appears in [AfrYou98] based on [Jon90]. $\square$

We remark that $l^2\gamma_l$ converges as $l \to \infty$ (see [MisKaw90]), so that $\gamma_{l+1}/\gamma_l \to 1$ as $l \to \infty$. This fact, together with Proposition 2.3 imply the next proposition [AfrYou98].

Proposition 2.4 *Let Ω be a measurable subset of $[0,\bar{\gamma})$ and denote $\Omega_l = \Omega \cap [\gamma_l, \gamma_{l-1}]$. If the limit*

$$\lim_{l \to +\infty} m(\theta_l(\Omega_l))$$

exists and equals Δ, then

$$\lim_{\gamma \searrow 0} \frac{m\left(\Omega \cap [0,\gamma)\right)}{\gamma} = \Delta.$$

We remark that, with Ω_l satisfying the assumption in the above proposition, $m(\Omega_l)/\gamma_l\sqrt{\gamma_l}$ converges as $l \to \infty$.

Let $\mathcal{L}_{l,\theta}$ denote the local (first hit) map from $I^s_{g_l(\theta)}$ to $I^u_{g_l(\theta)}$ induced by $f_{g_l(\theta)}$. The convenience of using τ^s_γ and τ^u_γ as coordinates on I^s_γ and I^u_γ is seen in the following proposition.

Proposition 2.5 *For each $l \geq l_0$ and each $\theta \in [0,1]$*

$$\mathcal{L}_{l,\theta} = (\tau^u_{g_l(\theta)})^{-1} \circ R_{-\theta} \circ \tau^s_{g_l(\theta)}, \tag{5}$$

where R_θ denotes a rigid rotation by angle θ.

PROOF. This follows from Proposition 2.1 and the definitions of $\tau^s_{g_l(\theta)}$ and $\tau^u_{g_l(\theta)}$ as the time variables for the embedding flow for $f^q_{g_l(\theta)}$. $\square$

In addition to $\tau^s_\gamma : I^s_\gamma \to [0,1]$ defined above, it is also convenient to define an extension, $\bar{\tau}^s_\gamma$, by

$$\phi_\gamma^{\bar{\tau}^s_\gamma(x)}(d) = x.$$

Thus $\bar{\tau}^s_0 : [d,a) \to [0,\infty)$ and $\bar{\tau}^s_\gamma : [d,e] \to [0,\infty)$ for $\gamma > 0$. Furthermore, we will denote

$$\tau^s_\gamma(x) = \bar{\tau}^s_\gamma(x) \mod 1$$

wherever $\bar{\tau}^s_\gamma$ is defined. Similarly, $\bar{\tau}^u_\gamma$ is given by

$$\phi_\gamma^{\bar{\tau}^u_\gamma(x)}(e) = x.$$

Let $c_i(\gamma) = f^i_\gamma(c)$, $i \geq 1$. It follows from the assumptions that there is an integer j such that $f^j_0(c) \in I^s_0$. By choosing d so that $f^j_0(c)$ is in the interior of I^s_0 we have that $f^j_\gamma(c) \in I^s_\gamma$ for all γ small. Denote this point by $c^s(\gamma)$. It then follows that for any l, either

$$c_{j+lq}(\gamma) \in I^u_\gamma \qquad \text{or} \qquad c_{j+(l+1)q}(\gamma) \in I^u_\gamma.$$

Denote this intersection of $\{c_i(\gamma)\}$ with I^u_γ by $c^u(\gamma)$. Note that for a fixed l the function $\theta \mapsto c^u(g_l(\theta))$ will have a jump discontinuity at which the value will jump from one endpoint of I^u_γ to the other. Denote by $\theta^\sharp_l$ the point at which the discontinuity takes place.

Lemma 2.6 *There exists a limit*

$$\lim_{l \to \infty} \theta^\sharp_l \to \theta^\sharp_\infty.$$

As $l \to \infty$ the sequence of maps $\theta \mapsto c^u(g_l(\theta))$ converges in the C^r topology on compact sets not containing $\theta^\sharp_\infty$.

PROOF. This easily follows from Proposition 2.1 and the definition of $g_l(\theta)$. $\square$

Specifically, we will make use of the implication that the derivatives of $c^u(g_l(\theta))$ with respect to θ converge uniformly as $l \to \infty$ for θ in compact intervals away from the discontinuity. Later, this will allow us to make estimates of derivatives along $\{c_i(\gamma)\}$ which are uniform in l.

3 The saddle-node: global analysis

In this section, we apply the local analysis near the saddle-node from the previous section to obtain expressions for global return maps. We study the occurrence of Misiurewicz maps for small parameter values.

3.1 The Mather invariant and return maps

We begin with a simple but, useful lemma.

Lemma 3.1 $W^u(a) = [f_0^2(c), f_0(c)]$.

PROOF. Let N be the interval bounded by a such that $f_0^q(N) = N$. Consider

$$M = \bigcup_{i \geq 0} f_0^{iq}(W_{loc}^u(a)).$$

Note that M is an invariant set. We have $f_0^q(N \cup M) = N \cup M$. If M does not cover $[f_0^2(c), f_0(c)]$, then for some $0 \leq i < q$, $f_0^i(N \cup M)$ is disjoint from $N \cup M$. This contradicts the assumption that f_0 is once renormalizable. $\square$

We will use the freedom in the choice of e and take e so that c is not in $\mathcal{O}(e)$. By this choice we have that c will be in the interior of $f_0^i(I_0^u)$ for some i.

For $\gamma = 0$, let $\bar{\mathcal{G}}$ be the first hit map from I_0^u to $W_{loc}^s(a) = [d, a)$. Define $\bar{\mathbf{M}} : [0,1] \to \mathbb{R}$ by

$$\bar{\mathbf{M}} = \bar{\tau}_0^s \circ \bar{\mathcal{G}} \circ (\tau_0^u)^{-1},$$

and define $\mathbf{M} : [0,1] \to [0,1]$ by $\mathbf{M} = \bar{\mathbf{M}} \mod 1$. By identifying the endpoints of $[0,1]$ we may consider $\bar{\mathbf{M}}$ as a map from a subset $D(\bar{\mathbf{M}})$ of $\mathbf{S}^1$ into $\mathbb{R}$ and $\mathbf{M}$ as a map from $D(\mathbf{M}) = D(\bar{\mathbf{M}})$ into $\mathbf{S}^1$. Following [Yoc95], we call $\bar{\mathbf{M}}$ the *Mather invariant* for f_0. One may show that $\mathbf{M}$ is a modulus of smooth conjugation, in other words, it is invariant under differentiable changes of variables.

For each τ in the domain, $\mathcal{D}(\mathbf{M})$, of $\mathbf{M}$ and $x \in I_0^u$ such that $\tau = \tau_0^u(x)$ associate the integer

$$\mathbf{R}(\tau) = \min_{x_i \in \mathcal{O}(x) \cap (a, f_0^q(e))} \lfloor \bar{\tau}_0^u(x_i) \rfloor.$$

Some $x \in I_0^u$ will return (perhaps many times) to $(a, f_0^q(e)]$ before eventually being trapped in $W^s(a) = [d, a)$ and the function $\mathbf{R} : [0,1] \to \mathbb{Z}^-$ measures the depth of the deepest return.

Finally, to each τ in the domain of $\mathbf{M}$ and the corresponding point $x \in I_0^u$ associate a positive integer

$$\mathbf{N}(\tau) = \min\{i | f_0^i(x) \in W^s(a)\}.$$

Given j, denote by $V(j)$ the subset of $\mathcal{D}(\mathbf{M})$ defined by

$$V(j) = \{\tau \in \mathcal{D}(\mathbf{M}) : \bar{\mathbf{M}}(\tau) < j, |\mathbf{R}(\tau)| < j \text{ and } \mathbf{N}(\tau) < j\}.$$

The points in $(\tau_0^u)^{-1}(V(j)) \in I_0^u$ are those whose forward orbits enter $W_{loc}^s(a)$ in a bounded number iterations and which do not come too close to a in the process, either by re-entering $W_{loc}^u(a)$ too close to a or by landing in $W_{loc}^s(a)$ too close to a. Since the forward orbit of almost every $x \in W_{loc}^u(a)$ has $\mathcal{O}(a)$ as its omega limit set, the measure $m(V(j))$ may be made close to 1 by taking j to be large.

We will let $\bar{E} = \cup_{i=0}^{q-1} f_0^i(E)$, so that $\bar{E}$ is a neighborhood of the orbit of a for $\gamma = 0$. Denote by $e_{-i} = \left(f_\gamma^{-iq}|_E\right)^{-1}(e)$ and

$$E_\gamma^{-i} = [e_{-i-1}, e_{-i}). \tag{6}$$

Note that orbits associated with $V(i-1)$ do not reenter E_0^{-i} before they hit $[d, a)$. Also note that for i fixed, the intervals E_γ^{-i} approach E_0^{-i} as $\gamma \to 0$.

Consider the first return map, $\kappa_{i,\gamma}$, of the interval E_γ^{-i} and let $\tilde{\kappa}_{i,\gamma}$ be the normalized map given by

$$\tilde{\kappa}_{i,\gamma} = \tau_\gamma^u|_{E^{-i}} \circ \kappa_{i,\gamma} \circ (\tau_\gamma^u|_{E^{-i}})^{-1}.$$

Identifying the endpoints of $[0,1]$, we may consider $\tilde{\kappa}_{i,\gamma}$ as a map on the circle.

Proposition 3.2 *Given any i,*

$$\lim_{l\to\infty} \left| \tilde{\kappa}_{i,g_l(\theta)}|_{V(i-1)} - R_{-\theta} \circ \mathbf{M}|_{V(i-1)} \right|_{C^r} \to 0$$

for each $0 \le \theta < 1$.

PROOF. Denote $d_i = f_\gamma^{iq}(d)$ and $D^i = [d_i, d_{i+1})$. Let $\mathcal{G}_{i,\gamma}$ denote the global (first hit) map from E^{-i} to D^iinduced by f_γ. Let $\mathcal{L}_{i,\gamma}$ denote the local map. Note that $\kappa_{i,\gamma}$ is not equal to $\mathcal{L}_{i,\gamma} \circ \mathcal{G}_{i,\gamma}$ since some points in E^{-i} will return to E^{-i} before hitting D^i (by landing in $[d_{i+1}, e_{-i})$). However, the two maps do agree when restricted to $(\tau^u|_{E^{-i}})^{-1}(V(i-1))$ since those points will in fact hit D^i before returning to E^{-i}. Since we are only considering a finite number of iterations from E^{-i} to

D^i, it follows from the construction of $\mathbf{M}$ that $\tau^s_{i,\gamma} \circ \mathcal{G}_{i,\gamma} \circ (\tau^u_{i,\gamma})^{-1}$ converges to $\mathbf{M}$ on the restricted set $V(i-1)$. Proposition 2.5 then implies that,

$$\tau^u_\gamma \circ \mathcal{L}_{j,\gamma} \circ \mathcal{G}_{i,\gamma} \circ (\tau^u_\gamma|_{E^{-i}})^{-1} \tag{7}$$

converges to $R_{-\theta} \circ \mathbf{M}$ on $V(i-1)$. □

3.2 Misiurewicz maps

In this section we will identify parameter values θ for which $f_{g_l(\theta)}$ maps the critical point c onto some repelling hyperbolic periodic point and thus c does not return to a neighborhood of itself, i.e. $f_{g_l(\theta)}$ satisfies the *Misiurewicz condition*. There are two ways that hyperbolic periodic points can occur: as periodic points whose orbits pass through $\bar{E}$, or as continuations of periodic points for $\gamma = 0$ (outside of $\bar{E}$). In the former case, the periodic orbits exist for parameter values within subintervals of $[\gamma_{l+1}, \gamma_l]$ for each l. In the latter case the periodic orbits exist for all $\gamma > 0$ sufficiently small. We confine ourselves to a discussion of this case. We show that parameters for which c is mapped onto the periodic orbits under consideration occur in decreasing sequences $\{\gamma_l^*\}$ of values approaching 0 as $l \to \infty$. Later we will show that such a sequence typically gives rise to the parameter sets predicted in Theorem A.

Note that the assumptions on f_0 imply that the nonwandering set, L_0, of f_0 restricted to $[0,1]\backslash\bar{E}$ is a hyperbolic set (see Theorem III.5.1 in [MelStr93]). If a point $x \in I^u_0$ is mapped onto L_0 then $\mathbf{M}$ will have a discontinuity at $\tau^u_0(x)$. By Lemma 3.1, for any $y^* \in L_0$ there is a point $x^* \in I^u_0$ which is mapped onto y^* by f_0. Recall that $c_j(\gamma) \in I^s_\gamma$ is the first point in the orbit of c that hits I^s_γ. Let $\tau^c_0 = \tau^s_0(c_j(0))$.

Lemma 3.3 *Suppose that y^*_0 is a periodic point contained in the nonwandering set L_0 of $[0,1] \setminus \bar{E}$. Denote by y^*_γ the continuation of y^*_0 for $\gamma \geq 0$. Let τ^* be a point of discontinuity of $\mathbf{M}$ corresponding to y^*_0 and suppose that θ^* is such that τ^c_0 is mapped onto τ^* by $(R_{-\theta^*} \circ \mathbf{M})^k \circ R_{-\theta^*}$ for some $k \geq 0$. Suppose further that*

$$\frac{\partial}{\partial \theta}\bigg|_{\theta=\theta^*} \Big((R_{-\theta} \circ \mathbf{M})^k \circ R_{-\theta}(\tau^c_0) - \tau^* \Big) \neq 0. \tag{8}$$

Then there is a sequence of parameter values $\{\gamma^*_l\}$, $\gamma^*_l \in [\gamma_{l+1}, \gamma_l]$, such that c is mapped onto $y^*_{\gamma^*_l}$ by iterates of $f_{\gamma^*_l}$.

Proof. The point $c_j(\gamma) = f^j_\gamma(c)$ is in I^s_γ for all small γ, and, $\tau^c_\gamma = \tau^s_\gamma(c_j(\gamma))$ approaches τ^c_0 as $\gamma \searrow 0$. Thus, $c^u(\gamma) = f^{j+ql}_\gamma(c) \in I^u_\gamma$. Let x^* be the point in I^u_0 which is mapped onto y^*_0 by an iterate of f_0. If $k = 0$, then $c^u(g_l(\theta^*))$ approaches x^* as $l \to \infty$ and (8) implies the result. If $k > 0$ then it follows there is an m such that $(R_{-\theta^*} \circ \mathbf{M})^i(\tau^*) \in V(m-1)$ for all $0 \leq i \leq k-1$. Applying Proposition 3.2 to D^m and E^{-m} then implies that $c^u(l, \theta)$ is mapped by the k-return map into $E^{-m}_{g_l(\theta^*)}$ arbitrarily close the mq-th preimage of x^* for l large. The result then follows from (8). $\square$

The following corollary provides the parameter values which will be used in the proof of Theorem A.

Proposition 3.4 *Given any $y^* \in L_0$, there exists integers $m > 0$ and $l_0 > 0$ and a sequence of parameter values $\gamma_l^* \in [\gamma_{l+1}, \gamma_l]$, $l \geq l_0$ such that*

$$f_{\gamma_l^*}^{lq+m}(c) = y^*_{\gamma_l^*}.$$

Further, there is a number $\theta^* \in (0,1)$ such that $\theta_l(\gamma_l^*) \to \theta^*$ as $l \to \infty$.

Proof. For each $y^* \in L_0$ there is a point $x^* \in I_0^u$ such that x^* is mapped to y^* by an iterate of f_0. Let $\tau^* = \tau_0^u(x^*)$. If we let $\theta^* = 1 - \tau^* + \tau^c$ then $R_{-\theta^*}(\tau^c) = \tau^*$ and

$$\frac{\partial}{\partial \theta}\bigg|_{\theta=\theta^*} (R_{-\theta}(\tau_0^c) - \tau^*) = 1.$$

Then Lemma 3.3 implies the result. $\square$

4 Parameter values with bounded recurrence

We start the proof of Theorem A. In this section we construct a set Ω with positive density at $\gamma = 0$, so that f_γ for $\gamma \in \Omega$ has bounded recurrence (see Definition 4.2; the definition is in terms of an induced map which is defined below). We deduce that f_γ for $\gamma \in \Omega$ has an absolutely continuous invariant measure.

Considering f_γ for $\gamma \in [\gamma_{l+1}, \gamma_l]$, we reparameterize and use the parameter $\theta \in [0,1]$ defined by $g_l(\theta) = \gamma$, see Section 2. Denote the resulting family of unimodal maps by $f_{l,\theta}$. By Proposition 3.4, there exists a converging sequence of parameter values $\theta_l^* = \theta_l(\gamma_l^*)$ so that $f_{l,\theta_l^*}^{ql+m}(c)$ is a hyperbolic periodic point. The map f_{l,θ_l^*} is a Misiurewicz map and by an extension of Jakobson's Theorem, see Theorem V.6.1 in [MelStr93], θ_l^* is a Lebesgue density point of a set of parameters Θ_l for which $f_{l,\theta}$ supports an absolutely continuous invariant measure. Theorem A can be proved by establishing that the measure of Θ_l is bounded away from 0 uniformly in l. Proposition 2.4 guarantees that the union $\cup_l g_l(\Theta_l)$ has positive Lebesgue measure and positive density at $\gamma = 0$. Combined with the statement that $f_{l,\theta}$, $\theta \in \Theta_l$, admits an absolutely continuous invariant measure, this proves Theorem A.

In the following section we define induced maps. The actual construction of the sets Θ_l is described in Section 4.3.

4.1 Induced maps

As before, a denotes a saddle-node periodic point of f_0, of period q. We may assume that a is nearest to the critical point c of all points in $\mathcal{O}(a)$, so that f_0^q is a homeomorphism on (c, a).

We may suppose that there is a periodic point $z_0 \in L_0$ and an integer j such that $f_0^j(e) = z_0$. Let z_γ be the periodic continuation of z_0 and e_γ the continuation of e_0 such that $f_0^j(e_\gamma) = z_\gamma$. We will suppress the γ dependence of e in the notation.

Let $E_{l,\theta}^{-i}$ be the fundamental intervals given by (e_{-i}, e_{-i+1}). Denote

$$\tilde{E}_{l,\theta} \quad = \quad \cup_{i=0}^{l} E_{l,\theta}^{-i} \subset E.$$

Note that $[d_1, e) \subset \tilde{E}_{l,\theta}$. Before proceeding with the definition and construction of Θ_l, we introduce an induced map $\tilde{f}_{l,\theta}$.

Namely, $\tilde{f}_{l,\theta}$ will be defined from $f_{l,\theta}$ in the following way, $\tilde{f}_{l,\theta}$ will equal $f_{l,\theta}$, except for points in $\tilde{E}_{l,\theta}$. Points which fall in this set will be mapped ahead to the image of $I_{l,\theta}^u = E_{l,\theta}^0$. That is,

$$\tilde{f}_{l,\theta}(x) = \begin{cases} f_{l,\theta}^{iq+1}(x), & \text{if } x \in E_{l,\theta}^{-i} \\ f_{l,\theta}(x), & \text{otherwise.} \end{cases} \tag{9}$$

Thus points in $\tilde{E}$ are mapped by the first hit map to the interval $f_{l,\theta}(I_{l,\theta}^u)$. For $\gamma \in (\gamma_{l+1}, \gamma_l)$, $\tilde{f}_{l,\theta}$ will have $l+1$ discontinuities. Recall that for $\gamma > 0$ and $x \in [d, e)$ the function $\tau_{l,\theta}^s$ is defined by $\tau^s(x) = \bar{\tau}^s(x) \mod 1$ where

$$\phi_{l,\theta}^{\bar{\tau}_{l,\theta}^s(x)}(d) = x.$$

Observe that $\tilde{f}_{l,\theta}$ equals

$$\tilde{f}_{l,\theta}(x) = \begin{cases} f_{l,\theta} \circ (\tau_{g_l(\theta)}^u)^{-1} \circ \tau_{g_l(\theta)}^s(x), & \text{if } x \in \tilde{E}_{l,\theta} \\ f_{l,\theta}(x), & \text{otherwise.} \end{cases} \tag{10}$$

Write

$$\tilde{E}_{\infty,\theta} = \left((\bar{\tau}_0^s)^{-1} \circ R_\theta \circ \bar{\tau}_0^u (f_{l,\theta}|_E)^{-q}(e), e \right).$$

Let $\tilde{f}_{\infty,\theta}$ denote the map defined from f_0 in the following way. If $x \notin \tilde{E}_{\infty,\theta}$ then $\tilde{f}_{\infty,\theta}(x) = f_0(x)$. If $x \in (a,e)$ then $\tilde{f}_{\infty,\theta}(x)$ is given be the first hit map from (a,e) to $f_0(I_0^u)$ induced by f_0. If $x \in (d,a) \cap \tilde{E}_{\infty,\theta}$ then let $\tau_0^s(x)$ be defined as above and let

$$\tilde{f}_{\infty,\theta}(x) = f_0 \circ (\tau_0^u)^{-1} \circ R_{-\theta} \circ \tau_0^s(x). \tag{11}$$

In other words, if $x \in (d,a) \cap \tilde{E}_{\infty,\theta}$ then x is mapped to $f_0(I_0^u)$ using the time coordinate shifted by $-\theta$. Note that $\tilde{f}_{\infty,\theta}(x)$ will have countably many discontinuities accumulating at a from both sides. In (a,e) these occur at $\{e_{-i}\}_{i=0}^{\infty}$. In (d,a) these occur at those x for which

$$\tau_0^s(x) = \theta.$$

Proposition 4.1 *For each θ the sequence of maps $\{\tilde{f}_{l,\theta}\}_{l=l_0}^{\infty}$ converges to the map $\tilde{f}_{\infty,\theta}$, uniformly on compact sets which are disjoint from the discontinuities of $\tilde{f}_{\infty,\theta}$.*

Proof. Given that $\tilde{f}_{l,\theta}$ satisfies (10) and $\tilde{f}_{\infty,\theta}$ is defined by (11), this follows from Proposition 2.5. □

Observe that, with θ^* and m as in Proposition 3.4, $\tilde{f}^m_{\infty,\theta^*}(c) = y^*$.

4.2 Iterating intervals

In the following we will consider iterates of intervals. The map $\tilde{f}_{l,\theta}$, $l_0 \leq l < \infty$, is discontinuous along backward iterates of e in E, so that the intervals in the image of an interval under $\tilde{f}_{l,\theta}$ might be arbitrarily small, regardless of the size of the original interval. We therefore slightly adjust the definition of $\tilde{f}_{l,\theta}$ to avoid this problem. Consider an interval $I \subset [0,1]$. For $0 \leq i \leq l$, denote $I^{-i} = I \cap E_{l,\theta}^{-i}$ and let I^{-l-1} be the component of $I \backslash E$ adjacent to $E_{l,\theta}^{-l}$ and I^1 the component of $I \backslash E$ adjacent to $E_{l,\theta}^0$. This yields a partition $\{I^i\}_{-l-1 \leq i \leq 1}$ of I (elements of the partition can be empty). If the leftmost or rightmost nonempty intervals of this partition do not contain a fundamental domain $E_{l,\theta}^{-i}$, $-l-1 \leq -i \leq 1$, join them to the adjacent intervals. Note that if I is partitioned into two elements $\{I^{-i}, I^{-i+1}\}$ neither of which is a fundamental domain, this leaves a choice in coding the resulting interval after I^{-i} or I^{-i+1}. This way an interval I that covers one or more fundamental domains is partitioned into subintervals which are at least as large as a fundamental domain. Given $x \in I^i \subset I$, define

$$\breve{f}_{l,\theta}(x;I) = \begin{cases} f_{l,\theta}(x), & \text{if } i = -1 \text{ or } i = l+1, \\ f_{l,\theta}^{iq+1}(x), & \text{if } 0 \leq i \leq l. \end{cases}$$

Note that as long as I does not cover a fundamental domain in E, $\breve{f}_{l,\theta}(\cdot;I)$ equals some fixed iterate of $f_{l,\theta}$. Also note that $\breve{f}_{l,\theta}(I;I)$ consists of at most two components. We further remark that, if $\breve{f}^j_{l,\theta}$ maps I homeomorphically onto $E^0_{l,\theta}$, then there is a fixed number N of iterates after which $\breve{f}^{j+N}_{l,\theta}(I;I)$ contains c in its interior.

Define maps F_l and $\tilde{F}_l$ by

$$\begin{aligned} F_l(x,\theta) &= (f_{l,\theta}(x),\theta), \\ \tilde{F}_l(x,\theta) &= (\tilde{f}_{l,\theta}(x),\theta). \end{aligned}$$

We would like to consider images $\tilde{F}^i_l(0,\theta)$. As for single maps we come across the difficulty that $\tilde{f}_{l,\theta}$ is discontinuous along the backward orbit of e in E. We consider the set $T = (\theta^* - \epsilon, \theta^* + \epsilon)$. By a fundamental strip we mean a set $\{(E^{-i}_{l,\theta},\theta)\}$, $\theta \in T$. Consider a curve $C = \{x(\theta),\theta)\}$ that projects injectively to $[0,1]$ by the projection $(x,\theta) \to x$. Define C^{-i} to be the intersection of C with the fundamental strip $\{(E^{-i}_{l,\theta},\theta)\}$, $\theta \in T$, and let C^{-l-1} be the connected component of C that is adjacent to C^{-l} and C^1 the connected component adjacent to C^0. This defines a partition $\{C^{-l-1},\dots,C^1\}$ of C with possibly empty elements. If the leftmost or rightmost nonempty element of this partition does not cross a fundamental strip, join it to the adjacent element. This way a partition of a curve C that crosses at least one fundamental strip is obtained all of whose elements cross a fundamental strip. Define

$$\breve{F}_l(x,\theta) = \begin{cases} F_l(x,\theta), & \text{if } (x,\theta) \in C^{-l-1} \cup C^1, \\ F_l^{iq+1}(x,\theta), & \text{if } (x,\theta) \in C^{-i}, 0 \le i \le l. \end{cases} \tag{12}$$

4.3 Inductive constructions

In this section the actual construction of the set Θ_l is described. We make use of the induced maps introduced in Section 4.1 and for the rest we closely follow [Luz00].

Consider θ near θ^*_l for a fixed value of l. Write

$$\tilde{c}_i(l,\theta) = \tilde{f}^i_{l,\theta}(c).$$

Definition 4.2 *For $\delta > 0, \alpha > 0$, we say that $\tilde{f}_{l,\theta}$ satisfies the bounded recurrence condition* $(\mathrm{BR})_n = (\mathrm{BR})_n(\alpha,\delta)$ *if for all positive integers $k \le n$,*

$$\prod_{\substack{\tilde{c}_i(l,\theta)\in(c-\delta,c+\delta),\\ 0\le i\le k}} |\tilde{c}_i(l,\theta) - c| \ge e^{-\alpha k}. \tag{13}$$

We say that $\tilde{f}_{l,\theta}$ satisfies (BR) *if it satisfies* $(\mathrm{BR})_n$ *for all n.*

Similarly, we say that $f_{l,\theta}$ satisfies $(\mathrm{BR})_n$ if it satisfies the equivalent condition on $\{c_i(l,\theta)\}$. It is easy to see that if $\tilde{f}_{l,\theta}$ satisfies $(\mathrm{BR})_n$ then so does $f_{l,\theta}$, and vice versa. The next proposition is the main result of Section 4 and will be shown to imply Theorem A.

Proposition 4.3 *For every $\alpha > 0$ there exists $\delta > 0$, so that there is a set Θ_l of parameter values for which $\tilde{f}_{l,\theta}$, $\theta \in \Theta_l$, satisfies* (BR) *and*

$$\liminf_{\varepsilon \searrow 0} \frac{|\Theta_l \cap (\theta_l^* - \varepsilon, \theta_l^* + \varepsilon)|}{2\varepsilon} \quad > \quad \sigma,$$

for some $\sigma > 0$, uniformly in l.

By Proposition 2.4, $\cup_l g_l(\Theta_l)$ has positive density at $\gamma = 0$. Theorem A is shown by establishing that $f_{l,\theta}$ possesses an absolutely continuous invariant measure for $\theta \in \Theta_l$. This is done more formally in Section 4.9, but the idea is straightforward: one shows that $\tilde{f}_{l,\theta}$, $\theta \in \Theta_l$, is a Collet-Eckmann map (meaning that $|D\tilde{f}^i_\gamma(f_\gamma(c))| \geq K\lambda^i$ for some $K > 0$, $\lambda > 1$). This implies that also $f_{l,\theta}$, $\theta \in \Theta_l$, is a Collet-Eckmann map, albeit with weaker expansion. Collet-Eckmann maps are known to admit absolutely continuous invariant measures.

To prove Proposition 4.3, we adapt the reasoning in [Luz00], where Jakobson's result is proven using a variant of Benedicks-Carleson's proof. Below we will describe the constructions and show key estimates and computations that differ from those in [Luz00]. After some preliminary work, the reasoning will follow [Luz00] closely. We will therefore be able to simply refer to [Luz00] for several of the more technical parts of the proof.

Note that we must consider a sequence of families $\tilde{f}_{l,\theta}$ that tends to the family $\tilde{f}_{\infty,\theta}$ as $l \to \infty$, see Proposition 4.1. It is instructive to compare with [DiaRocVia96] where the analysis leads to the study of a sequence of families $g_{l,\theta}$ of smooth unimodal maps (renormalizations of f), converging to the logistic family as $l \to \infty$. As they remarked, there are uniform lower bounds on the measure of the set of parameter values for which absolutely continuous invariant measures occur, for families from a neighborhood of the logistic family (and so for all large enough values of l). A similar situation was considered in [PumRod97]. Following the reasoning below one can construct a set of parameter values for which $\tilde{f}_{\infty,\theta}$ has bounded recurrence and show that this set has positive measure. In fact, the reasoning below constructs such sets, and uniformly bounds their measures, for families nearby $\tilde{f}_{\infty,\theta}$, in particular for the families $\tilde{f}_{l,\theta}$ with l large. We concentrate on the families $\tilde{f}_{l,\theta}$ and formulate the results in terms of this sequence of families.

We now start the constructions. For a positive integer r, let $I_r = [c + e^{-r}, c + e^{-r+1})$ and $I_{-r} = (c - e^{-r+1}, c - e^{-r}]$. Let ι be a small positive number. Given $\delta > 0$, write $r_\delta = -\ln\delta$ and

$r_{\delta+} = -\iota \ln \delta$. We can suppose that r_δ and $r_{\delta+}$ are integers. Let

$$\begin{aligned} \Delta &= \{c\} \cup \bigcup_{|r| \geq r_\delta + 1} I_r, \\ \Delta^+ &= \{c\} \cup \bigcup_{|r| \geq r_{\delta+} + 1} I_r. \end{aligned}$$

Subdividing each interval I_r into r^2 subintervals, $I_{r,m}$, of equal length provides partitions $\mathcal{I}$ of Δ and $\mathcal{I}^+$ of Δ^+.

Given $x \in \Delta^+$, write $\eta_0 = (c, x)$ (or (x, c)) and let $\eta_j = \breve{f}^j_{l,\theta}(\eta_0; \eta_0)$. Define the *binding period* of x as

$$q_l(x, \theta) \quad = \quad \sup\{m \in \mathbb{N} : |\eta_j| \leq e^{-2\alpha j} \text{ for all } 0 \leq j \leq m - 1\} \tag{14}$$

. Suppose $\tilde{c}_k(l, \theta) \in \Delta^+$ and define the binding period associated with $\tilde{c}_k(l, \theta)$ as

$$p(l, \theta, k) = q_l(\tilde{c}_k(l, \theta), \theta).$$

For a fixed l, let ω be a subinterval of $(\theta_l^* - \epsilon, \theta_l^* + \epsilon)$. Denote

$$\omega_i \quad = \quad \Pi \circ \breve{F}^i_l(c, \omega),$$

where Π is the projection $\Pi(x, \theta) = x$. If ω_k intersects Δ, $0 \leq k \leq n - 1$, then k is called a *return time* for ω. Define the binding period associated with a return time k of a parameter interval ω as

$$p(\omega, k) = \min_{\substack{\theta \in \omega, \\ \tilde{c}_k(l,\theta) \in \Delta^+}} p(l, \theta, k). \tag{15}$$

For each l let $\mathcal{P}_l^{(0)}$ be the trivial partition $\{(\theta_l^* - \epsilon, \theta_l^* + \epsilon)\}$ of the parameter interval $\Theta_l^{(0)} = (\theta_l^* - \epsilon, \theta_l^* + \epsilon)$. Inductively we will define parameter sets $\Theta_l^{(n)}$ and partitions $\mathcal{P}_l^{(n)}$ thereof. In order to define $\mathcal{P}_l^{(n)}$ given $\mathcal{P}_l^{(n-1)}$, we first construct a refinement $\hat{\mathcal{P}}_l^{(n)}$ of $\mathcal{P}_l^{(n-1)}$.

We say that a return of ω at time k is a *bound return* if there is a return time $j < k$ of ω and $k \leq j + p(\omega, j)$. Let $\omega \in \mathcal{P}_l^{(n-1)}$.

Chopping times. We say that n is a chopping time for ω if

1. ω_n intersects Δ^+ in at least three elements of $\mathcal{I}^+$, and
2. ω_n is not a bound return for ω.

Non-chopping times. We say that n is a non-chopping time for ω in all other cases, that is if one or more of the following occurs:

1. $\omega_n \cap \Delta^+ = \emptyset$.
2. ω_n is a bound return of ω.
3. ω_n intersects no more than two elements of the partition $\mathcal{I}^+$ of Δ^+.

In case n is a non-chopping time for ω, we let $\omega \in \hat{\mathcal{P}_l}^{(n)}$. If n is a chopping time for ω we partition ω as follows. Write $\omega_n \cap \Delta^+ = \cup_m \omega_n^m$, so that each ω_n^m fully contains one and at most one element of $\mathcal{I}^+$. If $\omega_n \backslash \Delta^+$ contains an interval of length less then δ^ι, we include this interval in the adjacent interval of $\mathcal{I}^+$. Otherwise an interval of $\omega_n \backslash \Delta^+$ is an element of the partition of ω_n. Write the resulting partition of ω_n as $\omega_n = \cup_m \omega_n^m$. There is a corresponding partition $\cup_m \omega^m$ of ω, given by $\Pi \circ \breve{F}_l^n(c, \omega^m) = \omega_n^m$. Let each element of this partition be an element of $\hat{\mathcal{P}_l}^{(n)}$. Note that an element of $\hat{\mathcal{P}_l}^{(n)}$ partitioning ω need not be connected, but can consist of several intervals. This may happen if ω_j intersects $\tilde{E}_{l,\theta}$ in at least two fundamental domains, for some $j < n$.

Let $\omega \in \mathcal{P}_l^{(n-1)}$ and consider $\nu \in \hat{\mathcal{P}_l}^{(n)}|_\omega$. We speak of a bound, essential or inessential return time or an escape time for ν in the following situations.

bound return time: The interval ν_n intersects Δ and n is a bound return time for ω.

inessential return time: The interval ν_n intersects Δ and n is a non-chopping time for ω that is not a bound return time, but ω_n intersects at most two elements of the partition $\mathcal{I}^+$.

essential return time: The interval ν_n intersects Δ and n is a chopping time for ω.

escape time: The return time n is a chopping time for ω, but ν_n does not intersect Δ. In this case we call ν an escape component of ω.

Any interval $\nu \in \hat{\mathcal{P}_l}^{(n)}$ belongs to a unique nested sequence of intervals

$$\Theta_l^{(0)} \equiv \nu^{(0)} \supseteq \nu^{(1)} \supseteq \ldots \supseteq \nu^{(n-1)} \supseteq \nu^{(n)} \equiv \nu,$$

where $\nu^{(k)} \in \mathcal{P}_l^{(k)}$ for $0 \leq k < n$. If j is a chopping time for $\nu^{(j-1)}$, then $\nu^{(j)}$ is strictly contained in $\nu^{(j-1)}$. Chopping times are either escape times or essential return times.

The return depth of ν at time k is defined if ν_k intersects Δ, as

$$r = \max\{|r|, \ \nu_k \cap I_r \neq \emptyset\}.$$

Define functions $\mathcal{R}^{(n)} : \hat{\mathcal{P}}_l^{(n)} \to \mathbb{N}$ and $\mathcal{E}^{(n)} : \hat{\mathcal{P}}_l^{(n)} \to \mathbb{N}$ which associate to $\nu \in \hat{\mathcal{P}}_l^{(n)}$ the sum of the return depths and the sum of the essential return depths, over the first n iterates $\Pi \circ \breve{F}_l^i(c,\nu)$.

Define

$$\Theta_l^{(n)} = \{\nu \in \hat{\mathcal{P}_l}^{(n)} : \mathcal{E}^{(n)}(\nu) \leq \alpha n/10\}$$

and

$$\mathcal{P}_l^{(n)} = \hat{\mathcal{P}_l}^{(n)}\Big|_{\Theta_l^{(n)}}.$$

The sets

$$\Theta_l = \bigcap_n \Theta_l^{(n)}$$

will be shown to satisfy the stated properties in Proposition 4.3.

4.4 Expansion

Here expansion properties of the maps $f_{l,\theta}$, $\tilde{f}_{l,\theta}$ and $\breve{f}_{l,\theta}$ are discussed. We relate expansion properties of these maps. In Section 4.4.2 we prove a Mañé type result for $\tilde{f}_{l,\theta}$, that is, we show that there is expansion along orbits outside a neighborhood of c.

4.4.1 Expansion for induced maps

The relation between expansion along orbits of $f_{l,\theta}$, $\tilde{f}_{l,\theta}$ and $\breve{f}_{l,\theta}$ is discussed in the next two lemma's.

Lemma 4.4 *If there exist $\tilde{C} > 0$, $\tilde{\lambda} > 1$ such that $|D\tilde{f}_{l,\theta}^n(x)| \geq \tilde{C}\tilde{\lambda}^n$ for all $n > 0$, then there are $C_l > 0$ and $\lambda_l > 1$, so that $|Df_{l,\theta}^n(x)| \geq C_l \lambda_l^n$, for all $n > 0$.*

PROOF. Write $f_{l,\theta}^i(x) = f_{l,\theta}^{k(i)} \circ \tilde{f}_{l,\theta}^{m(i)}(x)$ with $k(i)$ minimal such nonnegative integer. Compute

$$|Df_{l,\theta}^i(x)| \geq \min |Df_{l,\theta}^{k(i)}| \tilde{C}(\tilde{\lambda}^{m(i)/i})^i.$$

Now $k(i) > 0$ implies that the piece of orbit $\tilde{f}_{l,\theta}^{m(i)}(x), \ldots, f_{l,\theta}^{k(i)}(\tilde{f}_{l,\theta}^{m(i)}(x))$ is in $\bar{E}$. Since $c \notin \bar{E}$, the term $|Df_{l,\theta}|$ is bounded below in $\bar{E}$. Further, $k(i)$ is bounded above by $(l+1)q$ since any point in $\bar{E}$ is mapped outside of $\bar{E}$ in $(l+1)q$ or fewer iterations. Therefore the quantity $|Df_{l,\theta}^{k(i)}|$ is bounded from below by a constant D_l. Thus, $|Df^i_{l,\theta}(x)| \geq \tilde{C}D_l(\tilde{\sigma}^{m(i)/i})^i$. We can let $C_l = \tilde{C}D_l$. Since there is a minimum number of iterations of $f_{l,\theta}$ needed for an orbit to enter $\bar{E}$ after leaving I^u, and the number of consecutive iterations in $\bar{E}$ is bounded above by $q(l+1)$, it follows that the fraction $m(i)/i$ is bounded below by a constant $d_l > 0$. Hence, $\tilde{\sigma}^{m(i)/i}$ is strictly larger than some number $\sigma_l > 1$. □

Similarly one derives the following lemma relating expansion of $\breve{f}_{l,\theta}$ to expansion of $\tilde{f}_{l,\theta}$.

Lemma 4.5 *If $|D\breve{f}^n_{l,\theta}(x;I)| \geq \breve{C}\breve{\lambda}^n$ for some $\breve{C} > 0$, $\breve{\lambda} > 1$, then there are constants $\tilde{C} > 0$ and $\tilde{\lambda} > 1$ so that $|D\tilde{f}^n_{l,\theta}(x)| \geq \tilde{C}\tilde{\lambda}^n$. The converse statement holds as well.*

Given a subinterval $I \subset [0,1]$, write $\breve{f}^j_{l,\theta}(x;I) = \tilde{f}^{k(j)}_{l,\theta}(x)$. Then $k(j)/j$ is bounded and bounded away from 0. In fact, if $\tilde{E}_{l,\theta}$ is small, the quotient $k(j)/j$ will be close to 1 for large values of j, since there is then a large number of iterates between any two passages through $\tilde{E}_{l,\theta}$. Observe that for $\tilde{E}_{l,\theta}$ small, $\tilde{\lambda}$ in Lemma 4.5 is close to $\breve{\lambda}$. Write $\tilde{c}_i(l,\theta) = \tilde{f}^i_{l,\theta}(c)$ and $\breve{c}_i(l,\theta) = \breve{f}^i_{l,\theta}(c;I)$. Let $\breve{c}_j(l,\theta) = \tilde{c}_{k(j)}(l,\theta)$. Then the bounded recurrence condition

$$\prod_{\substack{\tilde{c}_i(l,\theta)\in(c-\delta,c+\delta),\\ 0\leq i\leq k}} |\tilde{c}_i(l,\theta)| \geq e^{-\alpha k},$$

where $k = k(j)$, translates into

$$\prod_{\substack{\breve{c}_i(l,\theta)\in(c-\delta,c+\delta),\\ 0\leq i\leq j}} |\breve{c}_i(l,\theta)| \geq e^{-C\alpha j},$$

for a constant C which bounds $k(j)/j$ from below. If $\tilde{E}_{l,\theta}$ is small and j is large, then C is close to 1. Hence, when translating the bounded recurrence condition for $\tilde{f}_{l,\theta}$ into one for $\breve{f}_{l,\theta}$, an almost identical condition is obtained. A similar remark can be made for $\tilde{F}_l$ and $\breve{F}_l$.

4.4.2 Mañé's Theorem for induced maps

The following proposition is central and provides exponential expansion of iterates of $\tilde{f}_{l,\theta}$ that stay away from the critical point c. The estimates are uniform in (l,θ). The proposition is modeled after Theorem III.6.4 in [MelStr93], which treats families of smooth unimodal maps.

Proposition 4.6 *There are constants $\varepsilon, \tilde{C} > 0$ and $\tilde{\lambda} > 1$, and a neighborhood W of c so that for any neighborhood U of c with $U \subset W$ and each large enough integer l, the following holds. For each $|\theta - \theta^*_l| < \varepsilon$, if $\tilde{f}^j_{l,\theta}(x) \notin U$ for $0 \leq j \leq m-1$ and $\tilde{f}^m_{l,\theta}(x) \in W$, then*

$$\left|D\tilde{f}^m_{l,\theta}(x)\right| \geq \tilde{C}\tilde{\lambda}^m.$$

If $\tilde{f}^j_{l,\theta}(x) \notin U$ *for* $0 \leq j \leq m-1$*, but not necessarily* $\tilde{f}^m_{l,\theta} \in W$*, then*

$$\left| D\tilde{f}^m_{l,\theta}(x) \right| \quad \geq \quad \tilde{C}\tilde{\lambda}^m \inf_{j=0,\ldots,m-1} \left| D\tilde{f}_{l,\theta}(\tilde{f}^j_{l,\theta}(x)) \right|.$$

Moreover, if $\tilde{f}^j_{l,\theta}(x) \notin W$ *for* $0 \leq j \leq m$*, then*

$$\left| D\tilde{f}^m_{l,\theta}(x) \right| \quad \geq \quad \tilde{C}\tilde{\lambda}^m.$$

We will make use of Koebe's principle, which we quote here. See [MelStr93] for the proof and additional information. Let $U \subset V$ be two intervals. We say that V contains a δ-scaled neighborhood of U if both components of $V \backslash U$ have at least length $\delta|U|$.

Definition 4.7 *The distortion of a diffeomorphism* g *on an interval* I *is defined as*

$$\sup_{x,y \in I} \frac{|Dg(x)|}{|Dg(y)|}.$$

Theorem 4.8 [Koebe principle] *Let* f *have negative Schwarzian derivative and let* $\mathcal{U}$ *be a neighborhood of* f *in* C^3*. Then for each* $\delta > 0$*, there exists* $K < \infty$ *so that* $g \in \mathcal{U}$*, if* $I \subset J$ *are intervals,* $g^n|_J$ *is a diffeomorphism and* $g^n(J)$ *contains a* δ*-scaled neighborhood of* $g^n(I)$ *then the distortion of* $g^n|_I$ *is bounded by* K*.*

A useful property of maps with negative Schwarzian derivative is the following principle, see [MelStr93].

Theorem 4.9 [Minimum principle] *Let* f *be a map with negative Schwarzian derivative on a closed interval* $I = [a, b]$*. If* Df *does not vanish on* I*, then*

$$|Df(x)| \quad \geq \quad \min\{|Df(a)|, |Df(b)|\}$$

for all $x \in I$*.*

The following lemma is similar to Theorem III.6.2 in [MelStr93]. So is its proof.

Lemma 4.10 *There are constants* $K > 0$*,* $0 < \rho < 1$*, so that for all large enough* l *the following holds. Let* I_m *be a maximal interval with* $\tilde{f}^m_{l,\theta^*_l}|_{I_m}$ *a homeomorphism. Then*

$$|I_m| \quad \leq \quad K\rho^m.$$

PROOF. Denote $\tilde{\mathcal{O}}(c) = \{\tilde{f}^i_{l,\theta_l^*}(c)\}_{i\geq 0}$. The number of elements in $\tilde{\mathcal{O}}(c)$ is fixed. There is also a minimum distance between any two points in $\tilde{\mathcal{O}}(c)$, uniformly in l. Thus we may let W be a neighborhood of c such that $\tilde{\mathcal{O}}(c) \cap W = \emptyset$ for all l. Similarly, $\tilde{\mathcal{O}}(e) = \{\tilde{f}^i_{l,\theta_l^*}(e)\}_{i\geq 0}$ is a finite set.

Let J_m be a maximal interval on which $\tilde{f}^m_{l,\theta_l^*}$ is a homeomorphism, but $\tilde{f}^{m+1}_{l,\theta_l^*}$ not. Let $\{J^i_{m+1}\}$ be the subintervals of J_m on which $\tilde{f}^{m+1}_{l,\theta_l^*}$ is a homeomorphism. The boundary points of $\tilde{f}^m_{l,\theta_l^*}(J_m)$ are contained in $\tilde{\mathcal{O}}(c) \cup \tilde{\mathcal{O}}(e)$. Since $\tilde{\mathcal{O}}(c)$ and $\tilde{\mathcal{O}}(e)$ finite, all intervals $\tilde{f}^m_{l,\theta_l^*}(J_m), \tilde{f}^{m+1}_{l,\theta_l^*}(J^i_{m+1})$ have lengths which are bounded below uniformly in l. Applying the Koebe principle (to $\tilde{f}_{l,\theta_l^*}$) one checks that there is a constant $\tau < 1$ with

$$\frac{\left|J^i_{m+1}\right|}{|J_m|} \leq \tau.$$

Further, since $\tilde{\mathcal{O}}(c)$ and $\tilde{\mathcal{O}}(e)$ are each finite it is clear that $\tilde{f}^{m+k}_{l,\theta_l^*}\Big|_{J^i_{m+1}}$ is not a homeomorphism for some uniformly bounded k. The result follows and it is clear that the constants can be chosen uniformly in l. □

The next proposition discusses expansion properties of $\tilde{f}_{l,\theta_l^*}$, for $\theta = \theta_l^*$.

Proposition 4.11 *For any small enough neighborhood W of c, there are constants $\tilde{C} > 0$ and $\tilde{\lambda} > 1$, so that the following holds for all l sufficiently large. If $\tilde{f}^j_{l,\theta_l^*}(x) \notin W$ for $0 \leq j \leq m-1$, then*

$$\left|D\tilde{f}^m_{l,\theta_l^*}(x)\right| \geq \tilde{C}\tilde{\lambda}^m.$$

If $\tilde{f}^m_{l,\theta_l^*}(x) \in W$, then

$$\left|D\tilde{f}^m_{l,\theta_l^*}(x)\right| \geq \tilde{C}\tilde{\lambda}^m.$$

Without any condition,

$$\left|D\tilde{f}^m_{l,\theta_l^*}(x)\right| \geq \tilde{C}\tilde{\lambda}^m \inf_{j=0,\ldots,m-1} \left|D\tilde{f}_{l,\theta_l^*}(\tilde{f}^j_{l,\theta_l^*}(x))\right|.$$

PROOF. Let W be a neighborhood of c, small enough so that $\tilde{f}^i_{l,\theta_l^*}(c) \cap W = \emptyset$ for $i > 0$.

We claim that there are $\tilde{C} > 0$, $\tilde{\lambda} > 1$ so that for all large enough integers l, if $\tilde{f}^j_{l,\theta}(x) \notin W$ for $0 \leq j \leq m-1$, then $|D\tilde{f}^m_{l,\theta_l^*}(x)| \geq \tilde{C}\tilde{\lambda}^m$. It suffices to show that there exists M with $|D\tilde{f}^M_{l,\theta_l^*}(x)| > 1$ (compare the proof of Theorem III.3.3 in [MelStr93]). Assume there exist points $x_m \in [0,1]$ with $\tilde{f}^j_{l,\theta_l^*}(x_m) \notin W$, $0 \leq j \leq m-1$, and $|D\tilde{f}^m_{l,\theta_l^*}(x_m)| \leq 1$. Since $\tilde{f}_{l,\theta_l^*}$ has negative Schwarzian derivative we have that on one side of x_m, $|D\tilde{f}^m_{l,\theta_l^*}(x)| \leq 1$. Let H_m be the maximal interval bounded by x_m so that $\breve{f}^m_{l,\theta_l^*}(\cdot;H_m)$ is a homeomorphism on H_m and $\left|D\breve{f}^m_{l,\theta_l^*}(\cdot;H_m)\right| \leq 1$ on H_m.

This implies that $|\breve{f}^m_{l,\theta^*_l}(H_m;H_m)| \leq |H_m|$. Let y_m be the other boundary point of H_m. Then either $\breve{f}^{k(m)}_{l,\theta^*_l}(y_m;H_m) = c$ or $\breve{f}^{k(m)}_{l,\theta^*_l}(H_m;H_m) = e$, for some $k(m)$. In both cases, $|\breve{f}^{k(m)}_{l,\theta^*_l}(H_m;H_m)|$ is bounded away from 0. By lemma 4.10, $|H_m| \leq \tilde{C}\rho^m$ is small as m is big. Since $|\breve{f}^m_{l,\theta^*_l}(H_m;H_m)| \leq |H_m|$, also $\breve{f}^m_{l,\theta^*_l}(H_m;H_m)$ is small for large m. The interval $\breve{f}^{k(m)}_{l,\theta^*_l}(H_m;H_m)$ has length bounded away from 0, say $|\breve{f}^{k(m)}_{l,\theta^*_l}(H_m;H_m)| \geq \delta$. It follows that $m - k(m)$ tends to ∞ as $m \to \infty$. In fact, if D denotes the minimum of $|D\breve{f}_{l,\theta^*_l}(\cdot;\cdot)|$ over $[0,1]\backslash W$, then

$$m - k(m) \quad \geq \quad \ln(\tilde{C}\rho^m/\delta)/\ln D. \tag{16}$$

Lemma 4.10 yields that $|\breve{f}^{k(m)}_{l,\theta^*_l}(H_m;H_m)| \leq \tilde{C}\rho^{m-k(m)}$, so that

$$m - k(m) \quad \leq \quad \ln(\delta/\tilde{C})/\ln\rho. \tag{17}$$

Now (16) and (17) contradict each other, proving the claim. Observe that M and hence the constants $\tilde{C}$ and $\tilde{\lambda}$ can be chosen uniformly in l. This proves the first estimate.

Next, suppose $\tilde{f}_{l,\theta^*_l}(x) \in W$. Let H_m be the maximal interval containing x such that $\breve{f}^m_{l,\theta}(\cdot;H_m)$ is a homeomorphism on H_m. Because the orbit of c is finite, the interval $\breve{f}^m_{l,\theta}(H_m;H_m)$ extends a positive distance away from $\breve{f}^m_{l,\theta}(x;H_m)$ to both sides. Koebe's principle implies $|D\breve{f}^j_{\eta_l}(x;H_m)| \geq \tilde{C}\tilde{\lambda}^j$ for some $\tilde{C} > 0, \tilde{\lambda} > 1$ which gives the second estimate.

Finally, not assuming any condition, split the iterates $x, \ldots, \tilde{f}^m_{l,\theta}(x)$ into a part that ends in W, one iterate starting in W, and a part that stays outside W. Combining the first two estimates for the first and last part, proves the last estimate. $\square$

Proof of proposition 4.6. As in the proof of Theorem III.6.4 in [MelStr93]. $\square$

4.5 Parameter dependence

Proposition 4.12 *There are constants $C, \varepsilon, K > 0$ so that for all large enough l the following holds. If $|\theta - \theta^*_l| < \varepsilon$, $k > K$ and*

$$|(\tilde{f}^k_{l,\theta})'(f_{l,\theta}(c))| \quad \geq \quad e^{\tilde{\lambda}k},$$

then for all $k > K$,

$$\frac{1}{C} \leq \frac{|\frac{d}{d\theta}\tilde{f}^k_{l,\theta}(c)|}{|(\tilde{f}^{k-1}_{l,\theta})'(f_{l,\theta}(c))|} \leq C.$$

PROOF. Let $x(l,\theta)$ be the continuation of $\tilde f_{l,\theta_l^*}(c)$ for θ near θ_l^* given by $\tilde f_{l,\theta}^{k-1}(x(l,\theta)) = y_{l,\theta}^*$ (where y_{l,θ^*}^* is the hyperbolic periodic point from the definition of $\theta_l(\gamma_l^*) = \theta_l^*$). Then Proposition 2.5 implies that

$$\frac{1}{K} \le \left| \frac{d}{d\theta}(x(l,\theta) - \tilde f_{l,\theta}(c)) \right| \le K, \tag{18}$$

for some constant $K > 1$. Writing $\tilde f_{l,\theta}^{k-1}(\tilde f_{l,\theta}(c)) = \tilde f_{l,\theta}^{k-1}(x(l,\theta) + \tilde f_{l,\theta}(c) - x(l,\theta))$, the chain rule gives

$$\frac{d}{d\theta}\tilde f_{l,\theta}^{k-1}(\tilde f_{l,\theta}(c))\Big|_{\theta=\theta_l^*} \;=\; \frac{d}{d\theta}\tilde f_{l,\theta}^{k-1}(x)\Big|_{\theta=\theta_l^*, x=x(l,\theta)} + (\tilde f_{l,\theta_l^*}^{k-1})'(\tilde f_{l,\theta_l^*}(c))\frac{d}{d\theta}(\tilde f_{l,\theta}(c) - x(l,\theta))\Big|_{\theta=\theta_l^*}.$$

Now $\frac{d}{d\theta}\tilde f_{l,\theta}^{k-1}(x_l(l,\theta))\Big|_{\theta=\theta_l^*} \equiv \frac{d}{d\theta}y_{l,\theta}^*$ is arbitrarily small for l sufficiently large. By (18) and the exponential growth of $(\tilde f_{l,\theta_l^*}^{k-1})'(\tilde f_{l,\theta_l^*}(c))$, the statement of the proposition holds for $\theta = \theta_l^*$.

By the chain rule,

$$\begin{aligned}
(\tilde f_{l,\theta}^{k-1})'(\tilde f_{l,\theta}(c)) &= \tilde f_{l,\theta}'(\tilde f_{l,\theta}^{k-1}(c))(\tilde f_{l,\theta}^{k-2})'(\tilde f_{l,\theta}(c)),\\
\frac{d}{d\theta}\tilde f_{l,\theta}^{k}(c) &= \tilde f_{l,\theta}'(\tilde f_{l,\theta}^{k-1}(c))\frac{d}{d\theta}\tilde f_{l,\theta}^{k-1}(c) + \frac{\partial}{\partial\theta}\tilde f_{l,\theta}(\tilde f_{l,\theta}^{k-1}(c)).
\end{aligned}$$

It follows that

$$\begin{aligned}
\left| \frac{\frac{d}{d\theta}\tilde f_{l,\theta}^{k}(c)}{(\tilde f_{l,\theta}^{k-1})'(\tilde f_{l,\theta}(c))} - \frac{\frac{d}{d\theta}\tilde f_{l,\theta}^{k-1}(c)}{(\tilde f_{l,\theta}^{k-2})'(\tilde f_{l,\theta}(c))} \right| &= \left| \frac{\tilde f_{l,\theta}'(\tilde f_{l,\theta}^{k-1}(c))\frac{d}{d\theta}\tilde f_{l,\theta}^{k-1}(c) + \frac{\partial}{\partial\theta}\tilde f_{l,\theta}(\tilde f_{l,\theta}^{k-1}(c))}{\tilde f_{l,\theta}'(\tilde f_{l,\theta}^{k-1}(c))(\tilde f_{l,\theta}^{k-2})'(\tilde f_{l,\theta}(c))} - \frac{\frac{d}{d\theta}\tilde f_{l,\theta}^{k-1}(c)}{(\tilde f_{l,\theta}^{k-2})'(\tilde f_{l,\theta}(c))} \right|\\
&= \left| \frac{\frac{\partial}{\partial\theta}\tilde f_{l,\theta}(\tilde f_{l,\theta}^{k-1}(c))}{(\tilde f_{l,\theta}^{k-1})'(\tilde f_{l,\theta}(c))} \right|.
\end{aligned}$$

By assumption, $|(\tilde f_{l,\theta}^{k})'(\tilde f_{l,\theta}(c))| \ge e^{\tilde\lambda k}$ for $k > K$. For each positive integer K, there are constants C, ε so that

$$\frac{1}{C} \le \left| \frac{\frac{d}{d\theta}\tilde f_{l,\theta}^{K}(c)\Big|_{\theta=\theta_l^*}}{(\tilde f_{l,\theta_l^*}^{K-1})'(\tilde f_{l,\theta_l^*}(c))} \right| \le C,$$

if $|\theta - \theta_l^*| \le \varepsilon$. Hence,

$$\left| \frac{\frac{d}{d\theta}\tilde f_{l,\theta}^{k}(c)}{(\tilde f_{l,\theta}^{k-1})'(\tilde f_{l,\theta}(c))} - \frac{\frac{d}{d\theta}\tilde f_{l,\theta}^{K}(c)}{(\tilde f_{l,\theta}^{K-1})'(\tilde f_{l,\theta}(c))} \right| \;\le\; C_K,$$

for $|\theta - \theta_l^*| \le \varepsilon$. The constant C_K is small if K is large. The proposition follows. $\square$

4.6 Binding

The next proposition is also used in Section 5 to show the existence of absolutely continuous invariant measures of $f_{l,\theta}$ for $\theta \in \Theta$. The binding period of $\omega \in \mathcal{P}_l^{(n)}$ is defined by (14), (15).

Proposition 4.13 *There exist constants $\tilde{\lambda}$ and $\beta < 1$ such that the following holds. Let $0 \le k \le n-1$, $\omega \in \mathcal{P}_l^{(n)}$ and suppose that k is an essential return time for ω with return depth r. Then the binding period $p = p(\omega, k)$ satisfies $p \le 2r$. We have*

$$\left| D\tilde{f}_{l,\theta}^{p+1}(\tilde{c}_k(l,\theta)) \right| \;\; \ge \;\; e^{\frac{1}{6}\tilde{\lambda}(p+1)}.$$

Furthermore,

$$|\omega_{k+p+1}| \;\; \ge \;\; |\omega_k|^{\beta}.$$

Recall that $\eta_0 = (c, \tilde{c}_k(l,\theta))$, or $(\tilde{c}_k(l,\theta), c)$, and $\eta_j = \breve{f}_{l,\theta}^j(\eta_0; \eta_0)$. The next lemma implies bounded distortion of iterates of $\breve{f}_\gamma$ on η_0 during the binding period. The lemma is an ingredient for the proof of Proposition 4.13, as in [Luz00]. The proof of the lemma differs from [Luz00]; distortion estimates for a passage through $\tilde{E}_\gamma$ have to be treated separately. The remainder of the proof of Proposition 4.13 is as in [Luz00].

Lemma 4.14 *For $y_0, z_0 \in \eta_0$,*

$$\left| \frac{D\breve{f}_{l,\theta}^i(y_0; \eta_0)}{D\breve{f}_{l,\theta}^i(z_0; \eta_0)} \right| \;\; \le \;\; K,$$

for $0 \le i \le p$ and for some $K > 0$.

PROOF. Write $y_j = \breve{f}_{l,\theta}^j(y_0; \eta_0)$ and $z_j = \breve{f}_{l,\theta}^j(z_0; \eta_0)$. By the chain rule,

$$\begin{aligned} \left| \frac{D\breve{f}_{l,\theta}^i(y_0; \eta_0)}{D\breve{f}_{l,\theta}^i(z_0; \eta_0)} \right| &= \left| \prod_{j=0}^{i-1} \frac{D\breve{f}_{l,\theta}(y_j; \eta_0)}{D\breve{f}_{l,\theta}(z_j; \eta_0)} \right| \\ &= \left| \prod_{j=0}^{i-1} \left(1 + \frac{D\breve{f}_{l,\theta}(y_j; \eta_0) - D\breve{f}_{l,\theta}(z_j; \eta_0)}{D\breve{f}_{l,\theta}(z_j; \eta_0)} \right) \right|. \end{aligned}$$

If η_j lies outside $\tilde{E}_{l,\theta}$, then $|D^2\breve{f}_{l,\theta}(\cdot; \eta_0)|$ is bounded by a constant $C > 0$ and

$$|D\breve{f}_{l,\theta}(y_j; \eta_0) - D\breve{f}_{l,\theta}(z_j; \eta_0)| \le C|\eta_j|.$$

If $z_j, y_j \in \tilde{E}_{l,\theta}$, we can write $z_j = \phi^s_{l,\theta}(y_j)$. Thus $\breve{f}_{l,\theta}(z_j;\eta_0) = \phi^{-s}_{l,\theta} \circ \breve{f}_{l,\theta}(\phi^s_{l,\theta}(y_j);\eta_0)$. Using this, it follows that again $|D\breve{f}_{l,\theta}(y_j;\eta_0) - D\breve{f}_{l,\theta}(z_j;\eta_0)| \leq C|\eta_j|$ for some $C > 0$. Hence,

$$
\begin{aligned}
\left| \frac{D\breve{f}^i_{l,\theta}(y_0;\eta_0)}{D\breve{f}^i_{l,\theta}(z_0;\eta_0)} \right| &\leq e^{\ln \prod_{j=0}^{i-1} \left(1 + C\left|\frac{\eta_j}{D\breve{f}_{l,\theta}(z_j;\eta_0)}\right|\right)} \\
&= e^{\sum_{j=0}^{i-1} \ln \left(1 + C\left|\frac{\eta_j}{D\breve{f}_{l,\theta}(z_j;\eta_0)}\right|\right)} \\
&\leq e^{C\sum_{j=0}^{i-1} \left|\frac{\eta_j}{D\breve{f}_{l,\theta}(z_j;\eta_0)}\right|}.
\end{aligned}
$$

We proceed to estimate $\sum_{j=0}^{i-1} \left|\frac{\eta_j}{D\breve{f}_{l,\theta}(z_j;\eta_0)}\right|$. By the definition of binding period,

$$|\eta_j| \leq e^{-2\alpha j}. \tag{19}$$

The bounded recurrence assumption implies that $|\tilde{c}_k - c| \geq e^{-\alpha k}$. Write $\breve{f}^j_{l,\theta}(c;\eta_0) = \tilde{c}_{k(j)}$. If $\tilde{E}_{l,\theta}$ is small enough, we have $k(j) \leq \frac{4}{3}j$ (see Section 4.4.1). We may hence assume that $|\breve{f}^j_{l,\theta}(c;\eta_0) - c| \geq e^{-\frac{4}{3}\alpha j}$. It follows from this and (19) that $|z_j - c| \geq e^{-\frac{4}{3}\alpha j} - e^{-2\alpha j} = e^{-\frac{4}{3}\alpha j}(1 - e^{-\frac{2}{3}\alpha j})$. Further $|D\breve{f}_{l,\theta}(z_j;\eta_0)| \geq C|z_j - c|$ for some $C > 0$, so that

$$|D\breve{f}_{l,\theta}(z_j;\eta_0)| \geq Ce^{-\frac{4}{3}\alpha j}, \tag{20}$$

for some $C > 0$. Combining (19) and (20) shows that $\sum_{j=0}^{i-1} \left|\frac{\eta_j}{D\breve{f}_{l,\theta}(z_j;\eta_0)}\right|$ is bounded, thus proving the lemma. We remark that the distortion bound is close to 1 if $|\theta - \theta^*_l|$ is small. This follows from the observation that $\breve{f}^j_{l,\theta}(\eta_0;\eta_0)$ is outside a neighborhood of c, where it undergoes exponential expansion, for a large number of iterates. □

4.7 Induction

The main results from the previous sections are Propositions 4.6, 4.12, 4.13. These results have their counterparts in proofs of the work of Benedicks and Carleson. From this point on, we can follow [Luz00] closely. For completeness we sketch the remaining steps leading to the proof of Proposition 4.3 in the next two sections.

In the inductive constructions, the following two propositions are shown to hold. The proofs are as in [Luz00], relying on Propositions 4.6, 4.12, and 4.13.

Proposition 4.15 [Bounded recurrence] *Each point in* $\Theta^{(n)}_l$ *satisfies* (BR)$_n$.

One shows that in fact $\mathcal{R}^{(k)}_l(\theta) \leq 5\mathcal{E}^{(k)}_l(\theta)$, $0 \leq k \leq n$, if $\theta \in \Theta^{(n)}_l$. That is, a substantial proportion of the returns are chopping times. By assumption, $\mathcal{E}^{(k)}(\gamma) \leq \alpha k/10$. Hence, $\mathcal{R}^{(k)}(\gamma) \leq \alpha k/2$. As in [Luz00] one shows that this bound implies (BR)$_n$.

Proposition 4.16 [Bounded distortion] *Restricted to a connected component of $\omega \in \mathcal{P}_l^{(n)}$, the map $\tilde{c}_j$ is a diffeomorphism with uniformly bounded distortion for all $j \leq \nu+p+1$ where $\nu \leq n$ is the last essential or inessential return time of ω and p is the associated binding period. If $n > \nu+p+1$ then the same statement holds for all $j \leq n$ restricted to any subinterval $\bar{\omega}$ such that $\bar{\omega}_j \subset \Delta^+$.*

4.8 Combinatorial estimates and measure bounds

For the proof of Proposition 4.3, combinatorial properties of $\mathcal{P}_l^{(n)}$ are studied. Escape times play a central role in this study. The combinatorial properties described next are used at the end of the section to prove Proposition 4.3.

To each $\omega \in \hat{\mathcal{P}}_l^{(n)}$ is associated a sequence $0 = \eta_0 < \eta_1 < \ldots < \eta_s \leq n$, $s = s(\omega)$ of escape times and a corresponding sequence of escaping components $\omega \subset \omega^{(\eta_s)} \subset \ldots \subset \omega^{(\eta_0)}$ with $\omega^{(\eta_i)} \in \mathcal{P}_l^{(\eta_i)}$. Let $\omega_i^* = \omega^{(\eta_i)}$ for $1 \leq i \leq s$ and $\omega_i^* = \omega$ for $s+1 \leq i \leq n$. This defines ω_i^* for each $0 \leq i \leq n$. Observe that for $\omega, \nu \in \mathcal{P}_l^{(n)}$ and $0 \leq i \leq n$, the sets ω_i^* and ν_i^* are either disjoint or coincide. Define

$$Q_l^{(i)} = \bigcup_{\omega \in \hat{\mathcal{P}}_l{}^{(n)}} \omega_i^*$$

and let

$$\mathcal{Q}_l^{(i)} = \{\omega_i^*\}$$

be the natural partition of $Q_l^{(i)}$ into sets of the form ω_i^*. Observe that $\Theta_l^{(n-1)} = Q_l^{(n)} \subset \ldots \subset Q_l^{(0)} = \Theta_l^{(0)}$ and $\mathcal{Q}_l^{(n)} = \hat{\mathcal{P}}_l^{(n)}$. For $\omega = \omega_i^* \in \mathcal{Q}_l^{(i)}$, $0 \leq i \leq n-1$, let

$$Q_l^{(i+1)}(\omega) = \left\{\omega' = \omega_{i+1}^* \in \mathcal{Q}_l^{(i+1)};\ \ \omega' \subset \omega\right\}.$$

Denote by $\mathcal{Q}_l^{(i+1)}(\omega)$ the partition $\mathcal{Q}_l^{(i+1)}$ restricted to ω. Define $\mathcal{D}_l^{(i+1)} : Q_l^{(i+1)}(\omega) \to \mathbb{N}$ by

$$\mathcal{D}_l^{(i+1)}(\omega') = \mathcal{E}_l^{(\eta_{i+1})}(\omega') - \mathcal{E}_l^{(\eta_i)}(\omega') \quad \text{for} 0 \leq i \leq s,$$

and $\mathcal{D}_l^{(i+1)} = 0$ for $i > s$. Let

$$Q_l^{(i+1)}(\omega, R) = \left\{\omega' \in Q_l^{(i+1)};\ \ \omega' \subset \omega, \mathcal{D}_l^{(i)}|_{\omega'} = R\right\}.$$

Proposition 4.17

$$\sum_{\omega' \in \mathcal{Q}_l^{(i+1)}(\omega, R)} |\omega'| \leq e^{-\eta R}|\omega|,$$

for some $\eta > 0$.

PROOF. One can take $-\eta = 10\frac{\alpha}{\tilde{\lambda}} - 1$, which is negative for α small enough. The proof divides into two parts. One bounds the cardinality of $\mathcal{Q}_l^{(i)}(\omega, R)$ by $e^{\beta R}$ and one shows that for any $\omega \in \mathcal{Q}_l^{(i)}$, $0 \le i \le n-1$, and $\hat{\omega} \in \mathcal{Q}_l^{(i+1)}(\omega, R)$, one has $|\hat{\omega}| \le e^{(9\beta-1)R}|\omega|$. Combining the two statements proves the proposition. For the proofs one can follow [Luz00]. $\square$

Lemma 4.18

$$\int_{\Theta_l^{(n-1)}} e^{\mathcal{E}^{(n)}/2} = \sum_{\omega \in \mathcal{Q}_l^{(n)}} e^{\mathcal{E}^{(n)}/2}|\omega| \le e^{3n/r_\delta}|\Theta_l^{(0)}|.$$

PROOF. The equality follows immediately from the definitions. For the inequality, let $0 \le i \le n$, $\omega \in \mathcal{Q}_l^{(i)}$ and write

$$\sum_{\omega' \in \mathcal{Q}_l^{(i+1)}(\omega)} e^{\mathcal{D}_l^{(i)}(\omega')/2}|\omega'| \quad = \quad \sum_{\omega' \in \mathcal{Q}_l^{(i+1)}(\omega,0)} |\omega'| + \sum_{R \ge r_\delta} e^{R/2} \sum_{\omega' \in \mathcal{Q}_l^{(i+1)}(\omega,R)} |\omega'|.$$

Proposition 4.17 (with the remark from its proof that $-\eta = 10\beta - 1$ with $\beta = \alpha/\tilde{\lambda}$) implies

$$\sum_{\omega' \in \mathcal{Q}_l^{(i+1)}(\omega)} e^{\mathcal{D}_l^{(i)}(\omega')/2}|\omega'| + \sum_{R \ge r_\delta} e^{R/2} \sum_{\omega' \in \mathcal{Q}_l^{(i+1)}(\omega,R)} |\omega'| \quad \le \quad \left(1 + \sum_{R \ge r_\delta} e^{(10\beta - \frac{1}{2})R}\right)|\omega|,$$

so that

$$\begin{aligned} \sum_{\omega' \in \mathcal{Q}_l^{(i+1)}(\omega)} e^{\mathcal{D}_l^{(i)}(\omega')/2}|\omega'| \quad &\le \quad (1 + e^{-r_\delta/3})|\omega| \\ &\le \quad e^{3/r_\delta}|\omega|, \end{aligned} \tag{21}$$

assuming that β has been chosen small enough and r_δ large enough. Since $\mathcal{E}^{(n)} = \mathcal{D}_l^{(0)} + \ldots + \mathcal{D}_l^{(n-1)}$ and $\mathcal{D}_l^{(i)}$ is constant on elements of $\mathcal{Q}_l^{(i)}$, we have

$$\sum_{\omega \in \mathcal{Q}_l^{(n)}} e^{\mathcal{E}^{(n)}(\omega)/2}|\omega| =$$

$$\sum_{\omega_1^*} \in \mathcal{Q}_l^{(1)}(\omega_2^*) e^{\mathcal{D}_l^{(0)}(\omega_1^*)/2} \cdots \sum_{\omega_{n-1}^* \in \mathcal{Q}_l^{(n-1)}(\omega_n^*)} e^{\mathcal{D}_l^{(n-2)}(\omega_{n-1}^*)/2} \sum_{\omega = \omega_n^* \in \mathcal{Q}_l^{(n)}} e^{\mathcal{D}_l^{(n-1)}(\omega_n^*)/2}|\omega|.$$

Applying (21) repeatedly gives

$$\sum_{\omega \in \mathcal{Q}_l^{(n)}} e^{\mathcal{E}^{(n)}(\omega)/2}|\omega| \quad \le \quad e^{3n/r_\delta}|\Theta_l^{(0)}|.$$

$\square$

Observe

$$\left|\Theta_l^{(n-1)} \backslash \Theta_l^{(n)}\right| = \left|\{\omega \in \mathcal{Q}_l^{(n)};\ e^{\mathcal{E}(n)/2} \geq e^{\alpha n/20}\}\right|.$$

Chebyshev's inequality and Lemma 4.18 yield

$$\begin{aligned} \left|\Theta_l^{(n-1)} \backslash \Theta_l^{(n)}\right| &\leq e^{-\alpha n/20} \int_{\Theta_l^{(n-1)}} e^{\mathcal{E}(n)/2} \\ &\leq e^{\left(\frac{3}{r_\delta} - \frac{\alpha}{20}\right)n} |\Theta_l^{(0)}| \\ &\leq e^{-\alpha n/30} |\Theta_l^{(0)}|, \end{aligned}$$

if r_δ is large enough. This implies

$$|\Theta_l^{(n)}| \geq |\Theta_l^{(n-1)}| - e^{-\alpha n/30} |\Theta_l^{(0)}|.$$

Write $\Theta_l^{(0)} = (\theta_l^* - \epsilon, \theta_l^* + \epsilon)$. For ϵ small, there exists N so that $\Theta_l^{(j)} = \Theta_l^{(j+1)}$ for all $j \leq N$. Hence

$$|\Theta_l^{(n)}| \geq \left(1 - \sum_{i=N}^{n} e^{-\alpha i/30}\right) |\Theta_l^{(0)}|.$$

Noting that $|\Theta_l^{(0)}| = 2\epsilon$, for all l, it follows that a uniform lower bound for $m(\Theta_l)$ exists. This concludes the proof of Proposition 4.3. Since N goes to ∞ as $\epsilon \to 0$, we have also shown that θ_l^* is a Lebesgue density point of Θ_l.

4.9 Proof of Theorem A

In the previous sections it was proved that $\tilde{f}_{l,\theta}$ has bounded recurrence for a set of parameter values $\theta \in \Theta_l$ with measure bounded from below, uniformly in l. By Proposition 2.4, this implies that $\Omega = \cup_l g_l(\Theta_l)$ has positive measure and has positive density at $\gamma = 0$.

The following proposition implies that $f_{l,\theta}$, $\theta \in \Theta_l$, has exponential expansion along the orbit of $f_{l,\theta}(c)$. The proof is as in [Luz00].

Proposition 4.19 *If θ is close to θ_l^* and $\tilde{f}_{l,\theta}$ satisfies* $(\mathrm{BR})_k$*, then*

$$\left|D\tilde{f}_{l,\theta}^{k+1}(f_{l,\theta}(c))\right| \geq \tilde{C} e^{\tilde{\lambda}(k+1)},$$

for some $\tilde{C} > 0$, $\tilde{\lambda} > 1$.

Combining Proposition 4.19 with Lemma 4.4, gives

Proposition 4.20 *For each $\gamma \in \Omega$, there are $\tilde{C} > 0$, $\lambda > 1$, so that*

$$\left| Df_\gamma^k(f_\gamma(c)) \right| \quad \geq \quad \tilde{C} e^{\lambda k}.$$

Thus f_γ is a Collet-Eckmann map if $\gamma \in \Omega$. Collet-Eckmann maps are known to admit absolutely continuous invariant measures, see Theorem V.4.6 in [MelStr93]. This concludes the proof of Theorem A, except for the conclusion that $\mathrm{supp}(\nu_\gamma) = [f_\gamma^2(c), f_\gamma(c)]$, which we postpone until the next section.

5 Intermittency

In this section we study intermittent time series of $\tilde{f}_\gamma$ at parameter values γ for which f_γ admits an absolutely continuous invariant measure. A proof of Theorem B is in Section 5.1. For the proof one needs to know the (average) length of the relaminarization phase. The relaminarization phase is studied in Section 5.2.

5.1 Invariant measures and intermittency

In the previous section we constructed a set Ω of parameter values with positive density at $\gamma = 0$, so that f_γ has bounded recurrence (see Definition 4.2) for $\gamma \in \Omega$. By Proposition 4.20, the bounded recurrence condition implies the Collet-Eckmann condition stating that $|Df_\gamma^n(f_\gamma(c))| \geq C\lambda^n$ for some $C > 0$, $\lambda > 1$. It is known that a map f_γ satisfying the Collet-Eckmann condition, possesses an absolutely continuous invariant measure, see [MelStr93]. Because we need bounds on the density of the invariant measures in our discussion of intermittency, we give an alternative way to produce invariant measures following [You92]. (See also [Ryc88] and [RycSor92])

Proposition 5.1 *For $\gamma \in \Omega$, $\tilde{f}_\gamma$ possesses an absolutely continuous invariant measure $\tilde{\nu}_\gamma$. There is a constant $K > 0$ not depending on γ so that for any Borel set A,*

$$\tilde{\nu}_\gamma(A) \quad \leq \quad K\sqrt{m(A)}.$$

The support of $\tilde{\nu}_\gamma$ equals $[f_\gamma^2(c), f_\gamma(c)]$.

Proof. By Proposition 4.13, the binding period $p(x)$ defined near c satisfies

(i) $p(x) \leq -\tilde{C} \ln |x - c|$ for some $\tilde{C} > 0$ not depending on γ,

(ii) $|D\tilde{f}_\gamma^j(f_\gamma(x))| \geq \tilde{C}\tilde{\lambda}^j$ for some $\tilde{C} > 0$, $\tilde{\lambda} > 1$ and for all $0 \leq j < p$,

(iii) $|D\tilde{f}_\gamma^p(x)| \geq \sigma^p$ for some $\sigma > 1$.

Outside the domain of definition of p, let $p = 1$. Define the return map R_γ on $[0, 1]$ by

$$R_\gamma(x) \quad = \quad \tilde{f}_\gamma^{p(x)}$$

For any $K > 0$ and all $\gamma \in \Omega$ we may assume that $p|_\Delta$ is bounded from below by K, by taking Δ small enough. By Proposition 4.6, if $x, \ldots, \tilde{f}_\gamma^{m-1}(x) \notin \Delta$ and $\tilde{f}_\gamma^m(x) \in \Delta$, then $|D\tilde{f}_\gamma^m(x)| \geq \tilde{C}\tilde{\lambda}^m$ for some $\tilde{C} > 0, \tilde{\lambda} > 1$. As stated in Proposition 4.6, the constant $\tilde{C}$ does not depend on Δ. By taking Δ small, $\tilde{C}\tilde{\lambda}^K > 1$. It follows that some power of R_γ is expanding. The first part of the proposition follows as in the proof of Theorem 1 in [You92]. We sketch the necessary arguments to make clear that the constant K does not depend on γ. Details are left to the reader. The measure $\tilde{\nu}_\gamma$ is constructed by finding its density as a fixed point of a Perron-Frobenius operator P_γ. Define

$$P_\gamma \phi(x) = \sum_{y \in \tilde{f}_\gamma^{-1}(x)} \frac{1}{(\tilde{f}_\gamma)'(x)} \phi(y).$$

Note that

$$P_\gamma^n \phi(x) = \sum_{y \in \tilde{f}_\gamma^{-n}(x)} \frac{1}{(\tilde{f}_\gamma^n)'(x)} \phi(y).$$

Let $g_n : [0, 1] \to \mathbb{R}$ be the function $g_n(x) = 1/(\tilde{f}_\gamma^n)'(x)$ where $\tilde{f}_\gamma^n$ is continuous and $g_n(x) = 0$ elsewhere. One shows that the variation of g_n is bounded, uniformly in n and γ. This relies on the negative Schwarzian derivative (see [You92]) and the analysis of the local saddle-node bifurcation in Section 2 (see Proposition 2.1). By induction the variation of g_n is bounded. Using the uniform bound on the variation of g_n, one bounds the variation of $P_\gamma^n \phi$ for a density ϕ with bounded variation. It follows that P_γ has a fixed point with uniformly bounded variation.

If ζ_γ denotes the measure whose density is the fixed point of P_γ, then $\tilde{\nu}_\gamma$ is obtained by pushing forward ζ_γ,

$$\tilde{\nu}_\gamma(A) = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \zeta_\gamma(\tilde{f}_\gamma^{-j}(A) \cap B_k),$$

where B_k is the set on which $R_\gamma = \tilde{f}_\gamma^k$. The uniform bound for $\tilde{\nu}_\gamma$ follows from the properties of R_γ, as in [You92], see also [HomYou00].

That the support of $\tilde{\nu}_\gamma$ equals all of $[f_\gamma^2(c), f_\gamma(c)]$ follows as in Theorem 2 in [You92]. It relies on the fact that, if $\gamma \in \Omega$, then for each interval $I \subset [0,1]$, there exists n so that $[f_\gamma^2(c), f_\gamma(c)] \subset f_\gamma^n(I)$. The necessary topological expansion at $\gamma = 0$ is guaranteed by Lemma 3.1. $\square$

The invariant measure ν_γ for f_γ is constructed by pushing forward $\tilde{\nu}_\gamma$. Recall that $\tilde{E}_\gamma = \cup_{i=0}^{l} E^{-i}$ with $E^{-i} = (f|_E)^{-iq}(I_\gamma^u)$. Let

$$\bar{\nu}_\gamma(A) \quad = \quad \tilde{\nu}_\gamma(A \cap ([0,1] \backslash \tilde{E}_\gamma)) + \sum_{k=0}^{l} \sum_{i=0}^{kq-1} \tilde{\nu}_\gamma(f_\gamma^{-i}(A) \cap E^{-k}).$$

This measure is obviously finite and can thus be rescaled to a probability measure ν_γ.

Lemma 5.2 *The measure ν_γ, $\gamma \in \Omega$, is an absolutely continuous invariant probability measure for f_γ.*

Proof. To prove that ν_γ is invariant for f_γ, recall that $\tilde{f}_\gamma|_{E^{-j}} = f_\gamma^{jq}$ and $\tilde{f}_\gamma$ equals f_γ outside $\tilde{E}_\gamma$. Compute

$$\begin{aligned}
\bar{\nu}_\gamma(f_\gamma^{-1}(A)) \quad &= \quad \tilde{\nu}_\gamma(f_\gamma^{-1}(A) \cap ([0,1] \backslash \tilde{E}_\gamma)) + \sum_{k=0}^{l} \sum_{i=1}^{kq-1} \tilde{\nu}_\gamma(f_\gamma^{-i}(A) \cap E^{-k}) + \\
& \quad\quad \sum_{k=0}^{l} \tilde{\nu}_\gamma(f_\gamma^{-kq}(A) \cap E^{-k}) \\
&= \quad \tilde{\nu}_\gamma(\tilde{f}_\gamma^{-1}(A) \cap ([0,1] \backslash \tilde{E}_\gamma)) + \sum_{k=0}^{l} \sum_{i=1}^{kq-1} \tilde{\nu}_\gamma(f_\gamma^{-i}(A) \cap E^{-k}) + \\
& \quad\quad \sum_{k=0}^{l} \tilde{\nu}_\gamma(\tilde{f}_\gamma^{-1}(A) \cap E^{-k}).
\end{aligned} \tag{22}$$

By $\tilde{f}_\gamma$ invariance of $\tilde{\nu}_\gamma$, $\tilde{\nu}_\gamma(\tilde{f}_\gamma^{-1}(A) \cap ([0,1] \backslash \tilde{E}_\gamma))$ equals $\tilde{\nu}_\gamma(A \cap ([0,1] \backslash \tilde{E}_\gamma))$ and $\sum_{k=0}^{l} \tilde{\nu}_\gamma(\tilde{f}_\gamma^{-1}(A) \cap E^{-k})$ equals $\sum_{k=0}^{l} \tilde{\nu}_\gamma(A \cap E^{-k})$. The right hand side of (22) therefore is $\bar{\nu}_\gamma(A)$, showing f_γ-invariance of $\bar{\nu}_\gamma$ and thus of ν_γ. $\square$

The following proposition serves several purposes, it is used to provide estimates for the average length of laminar and relaminarization phases. Its proof will be postponed to Section 5.2.

Proposition 5.3 *Let V be a small neighborhood of c. For $x \in [0,1]$, let $\tilde{L}_V$ be the number of iterations under $\tilde{f}_\gamma$ required for x to enter V. Then there is $L > 0$ so that for all $\gamma \in \Omega$,*

$$\int_{[0,1]} \tilde{L}_V d\tilde{\nu}_\gamma / \tilde{\nu}_\gamma([0,1] \backslash V) \quad \leq \quad L.$$

Corresponding statements for the study of the boundary crisis bifurcation are contained in [HomYou00]. Note that V need not contain $\tilde{E}_\gamma$; a similar statement where $\tilde{f}_\gamma$ is replaced by f_γ is therefore untrue. However, Proposition 5.3 has the following corollary which deals with f_γ. Paraphrasing, it shows that a typical (with respect to the invariant measure ν_γ) point $x \in [0,1]$ needs a bounded number of iterates to enter $\tilde{E}_\gamma$.

Proposition 5.4 *For $x \in [0,1]$, let $L_{\tilde{E}_\gamma}$ be the number of iterations under f_γ required for x to enter $\tilde{E}_\gamma$. Then there is $L > 0$ so that for all $\gamma \in \Omega$,*

$$\int_{[0,1]} L_{\tilde{E}_\gamma} d\nu_\gamma / \nu_\gamma([0,1] \backslash \tilde{E}_\gamma) \quad \leq \quad L.$$

Proof. At $\gamma = 0$, f_0 is renormalizable: there is an interval V containing c in its interior and a in its boundary, so that $f_0^q(V) \subset V$. By slightly extending V, we may assume that it contains $\tilde{E}_\gamma$ for small values of γ. The result follows from Proposition 5.3 by noting that outside $\tilde{E}_\gamma$, $\tilde{f}_\gamma$ equals f_γ. □

We will now show how Theorem B is proved by combining Propositions 5.1, 5.3 and 5.4.

Proof of Theorem B. Let Ω be as constructed in Section 4 and take $\gamma \in \Omega$. Let ν_γ be the absolutely continuous invariant measure for f_γ, obtained in Lemma 5.2. The measure ν_γ is ergodic, so that by Birkhoff's Ergodic Theorem,

$$\lim_{m \to \infty} \sum_{i=0}^{m-1} 1_A(f_\gamma^i(x)) \quad = \quad \nu_\gamma(A), \tag{23}$$

for any Borel set $A \subset [0,1]$ and almost all $x \in [0,1]$. It follows that also

$$\lim_{m \to \infty} \sum_{i=0}^{m-1} 1_A(\tilde{f}_\gamma^i(x)) \quad = \quad \tilde{\nu}_\gamma(A). \tag{24}$$

Hence, for almost all $x \in [0,1]$, the distribution with which points in the orbit $\{\tilde{f}_\gamma^i(x)\}$ are in $\tilde{E}_\gamma$ is given by $\tilde{\nu}_\gamma$. Let I be a compact interval in $[d,a)$. If V is a neighborhood of c, then by (24) and applying Proposition 5.3, the measure $\tilde{\nu}_\gamma(V)$ of V is bounded from below by a positive constant, uniformly in γ. Observe that by invariance of $\tilde{\nu}_\gamma$, $\tilde{\nu}_\gamma(\tilde{f}_\gamma^k(V)) \geq \tilde{\nu}_\gamma(V)$, $k \geq 0$. Therefore, the measure $\tilde{\nu}_\gamma(I)$ of I is bounded from below by a positive constant, uniformly in γ. An easy computation shows that for any compact interval I inside $[d,a)$, the number of iterates needed for a point $x \in I$ to leave $\tilde{E}_\gamma$ is bounded from below by $K/\sqrt{\gamma}$ for some $K > 0$. It follows that the average duration of orbit pieces of f_γ in $\tilde{E}_\gamma$ is bounded from below by $K/\sqrt{\gamma}$ for some $K > 0$. Combining this with Proposition 5.4 proves the upper bound on $\chi_{\bar{E}}(\gamma)$ in Theorem B. The lower bound is a trivial consequence of the fact that there is always a positive number of iterates between two laminar phases and that the maximum number of consecutive iterations in $\tilde{E}$ is bounded above by $K'/\sqrt{\gamma}$ for some constant K'.

The argument to show that ν_γ converges weakly to ν_0 is similar. By definition of $\bar{\nu}_\gamma$,

$$\bar{\nu}_\gamma(E^{-j}) = \sum_{k=j}^{l} \tilde{\nu}_\gamma(E^{-k}).$$

Reasoning as above one shows that $\tilde{\nu}_\gamma(E^{-j})$, $0 \le j \le l$, is bounded from below, uniformly in γ. It follows that $\bar{\nu}_\gamma(\tilde{E}_\gamma)$ gets arbitrarily large as $\gamma \to 0$ (because then $l \to \infty$). Since $\bar{\nu}_\gamma([0,1]\backslash\tilde{E}_\gamma) = \tilde{\nu}_\gamma([0,1]\backslash\tilde{E}_\gamma)$ is obviously bounded, $\nu_\gamma(\tilde{E}_\gamma) \to 1$ as $\gamma \to 0$. Because this holds for any neighborhood $\tilde{E}_\gamma$, this shows that $\nu_\gamma \rightharpoonup \nu_0$ as $\gamma \to 0$. □

5.2 Relaminarization

Purpose of this section is to indicate a proof of Proposition 5.3. To introduce the reasoning, we start with an alternative proof for Proposition 5.4, which does not derive it as a corollary to Proposition 5.3.

PROOF OF PROPOSITION 5.4. We claim that $\tilde{\nu}_\gamma(([0,1]\backslash\tilde{E}_\gamma)$ is bounded away from 0, uniformly in γ. To establish the claim, note that $\tilde{f}^{-1}([0,1]\backslash\tilde{E}_\gamma)$ and $[0,1]\backslash\tilde{E}_\gamma$ together cover $[0,1]$. By $\tilde{f}_\gamma$-invariance of $\tilde{\nu}_\gamma$, the measures of both sets are the same, and, since they add up to at least 1, are bounded from below by $\frac{1}{2}$. The normalized measure $\nu_\gamma/\nu_\gamma([0,1]\backslash\tilde{E}_\gamma)$ equals $\tilde{\nu}_\gamma/\tilde{\nu}_\gamma([0,1]\backslash\tilde{E}_\gamma)$ outside $\tilde{E}_\gamma$. Therefore, applying Proposition 5.1, it follows that a uniform bound of the form

$$\nu_\gamma(A)/\nu_\gamma([0,1]\backslash\tilde{E}_\gamma) \quad \le \quad K\sqrt{m(A)} \tag{25}$$

holds for Borel sets $A \subset ([0,1]\backslash\tilde{E}_\gamma)$.

At $\gamma = 0$, f_0 is renormalizable: there is an interval V containing c in its interior and a in its boundary, so that $f_0^q(V) \subset V$. By slightly extending V, we may assume that it contains $\tilde{E}_\gamma$ for small values of γ. Write $\bar{V} = V \cup f_\gamma(V) \cup \cdots \cup f_\gamma^{k-1}(V)$. Write O for the union of $\bar{V}$ and those intervals in $f_\gamma^{-q}(\bar{V})$ that contain a critical point for f_γ^q, see Figure 4. Because any point in $O \setminus \tilde{E}_\gamma$ either enters $\tilde{E}_\gamma$ in a uniformly bounded number of iterates (q), it suffices to prove the proposition with $\tilde{E}_\gamma$ replaced by O.

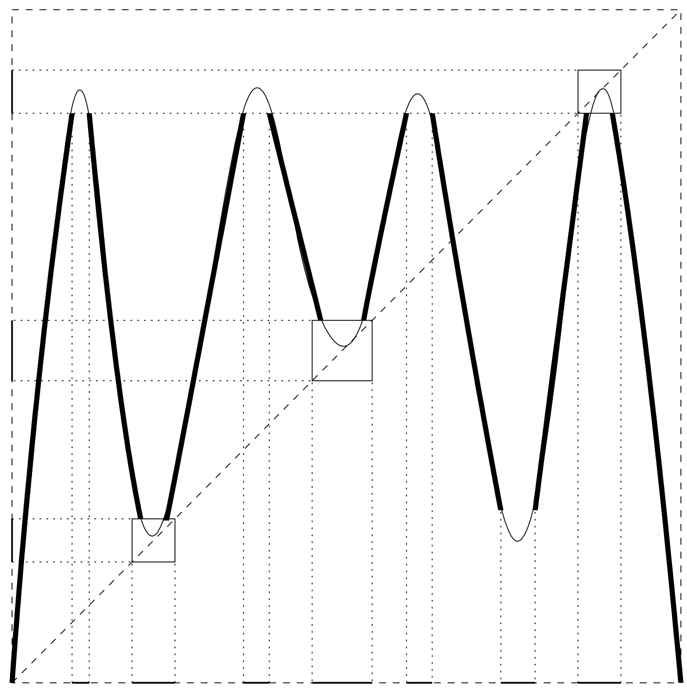

Figure 4: This figure illustrates the proof of Proposition 5.4. It shows the third iterate of a unimodal map at or near a saddle-node bifurcation. Projected on the left side of the box are the intervals that make up $\bar{N}$. Projected on the bottom one finds O.

Of course, also for Borel sets $A \subset [0,1] \backslash O$ a uniform bound of the form $\nu_\gamma(A)/\nu_\gamma([0,1]\backslash O) \leq K\sqrt{m(A)}$ holds. Let $h_\gamma : [0,1]\backslash O \to O$ denote the restriction of f_γ to $[0,1]\backslash O$. By [PiaYor79], h_γ possesses a conditionally invariant measure ζ_γ; ζ_γ is characterized by

$$\zeta_\gamma(A) = \zeta_\gamma(h_\gamma^{-1}(A))/\zeta_\gamma(h_\gamma^{-1}([0,1]\backslash O)$$

for Borel sets A. Moreover, ζ_γ is absolutely continuous with respect to Lebesgue measure and has density bounded and bounded away from 0 on its support, uniformly in γ. Say

$$\frac{1}{C} m(A) \leq \zeta_\gamma(A)$$

for Borel sets A. Write $(h_\gamma)_* \zeta_\gamma = \beta_\gamma \zeta_\gamma$ for some $0 < \beta_\gamma < 1$. Since $m(O)$ is bounded from below, β_γ is bounded away from 1. Applying (25),

$$\begin{aligned} \nu_\gamma(h_\gamma^{-n}([0,1])) &\leq K\sqrt{m(h_\gamma^{-n}([0,1]))} \\ &\leq K\sqrt{C}\sqrt{\zeta_\gamma(h_\gamma^{-n}([0,1]))} \\ &= K\sqrt{C}\sqrt{\beta_\gamma}^n. \end{aligned}$$

It follows that

$$\int_{[0,1]} L_O d\nu_\gamma = \sum_{i\geq 1} \nu_\gamma(h_\gamma^{-i}([0,1]))$$

is bounded uniformly in γ. $\square$

Proof of Proposition 5.3. Let N be the interval containing c so that $f_0^q(N) = N$. We may assume that $V \subset N$ and $V \cap \tilde{E}_\gamma = \emptyset$. Given V, let $\bar{V} = \cup_{i=0}^{q-1} f_\gamma^{-i}(V) \cap f_\gamma^{q-i}(N)$ be the union of the interval in $\bar{N}$ that are mapped onto V by iterates of f_γ. Write O for the union of $\bar{V}$ and those intervals in $f_\gamma^{-q}(\bar{V})$ that contain a critical point for f_γ^q. Restricted to $\bar{V}$, f_γ equals $\tilde{f}_\gamma$. Include into O the set of points $\tilde{f}_\gamma^{-q}(\tilde{f}_\gamma(D))$, where D is the set of discontinuities of $\tilde{f}_\gamma$. Write $\tilde{h}_\gamma$ for the restriction of $\tilde{f}_\gamma$ to $[0,1]\backslash O$. By Proposition 4.6, some iterate of $\tilde{h}_\gamma$ is expanding. Observe that the number of branches of $\tilde{h}_\gamma$ is constant for $\gamma \in [\gamma_{l+1}, \gamma_l]$, but increases with l since the branches of $\tilde{f}_\gamma\Big|_{\tilde{E}_\gamma}$ are included. For fixed l, Proposition 5.3 follows for $\gamma \in [\gamma_{l+1}, \gamma_l] \cap \Omega$ as in the above indicated alternative proof of Proposition 4. To obtain uniform bounds in l, we must investigate properties of conditionally invariant measures for $\tilde{h}_\gamma$. Following [PiaYor79], there is a conditionally invariant measure $\tilde{\zeta}_\gamma$ for $\tilde{h}_\gamma$; $\tilde{\zeta}_\gamma$ is characterized by

$$\tilde{\zeta}_\gamma(A) = \tilde{\zeta}_\gamma(\tilde{h}_\gamma^{-1}(A))/\tilde{\zeta}_\gamma(\tilde{h}_\gamma^{-1}([0,1]\backslash O).$$

The proof of Proposition 5.3 is identical to the above proof of Proposition 5.4, once Lemma 5.5 is proved. $\square$

Lemma 5.5 *Let $\tilde{h}_\gamma$ be as in the above proof of Proposition 5.3. The density of the conditionally invariant measure $\tilde{\zeta}_\gamma$ for $\tilde{h}_\gamma$ is bounded away from 0 uniformly in γ.*

Proof. By Proposition 4.6, an iterate $\tilde{h}_\gamma^N$ of $\tilde{h}_\gamma$ is an expansion. Hence, the existence [PiaYor79] of the conditionally invariant measure follows from [PiaYor79]. To derive bounds on its density, we must examine the existence proof.

The conditionally invariant measure $\tilde{\zeta}_\gamma$ is constructed by finding its density as a fixed point of a Perron-Frobenius operator. Write $\mathcal{C}([0,1])$ for the set of positive continuous functions g on $[0,1]$ with $\int_{[0,1]} gdm = 1$. Near a point $x \in [0,1]$, $\tilde{h}_\gamma$ has a number of inverse functions ψ_i. Define a Perron-Frobenius operator P_γ on $\mathcal{C}([0,1])$ by

$$P_\gamma g(x) = \sum_i |\psi_i'(x)| g \circ \psi_i(x) / \int_{\tilde{h}_\gamma^{-1}([0,1])} gdm$$

A fixed point of P_γ is the density of a conditionally invariant measure. It is shown in [PiaYor79] that P_γ possesses a unique fixed point ξ_γ and that for each $g \in \mathcal{C}([0,1])$,

$$\lim_{n\to\infty} P_\gamma^n g = \xi_\gamma. \tag{26}$$

The fixed point ξ_γ is the density of the measure $\tilde{\zeta}_\gamma$.

For a Lipschitz density g, let the regularity of g be given by

$$\text{Reg } g = \sup\{|g'(x)|/g(x);\ \ x \in [0,1], g'(x) \text{ is defined}, g(x) > 0\}.$$

We will show that

$$\limsup_{n\to\infty} \text{Reg } P_\gamma^n g \le \rho, \tag{27}$$

for some ρ independent of g. We remark that from [PiaYor79] one concludes that such a bound holds when γ is restricted to an interval $[\gamma_{l+1}, \gamma_l]$ (see also [LasYor81]). Their arguments do not imply (27), the difficulty being the growth of the number of branches of $\tilde{h}_\gamma$ as $\gamma \to 0$.

To evaluate Reg $P_\gamma g$, compute

$$\begin{aligned}
\frac{|(P_\gamma g)'(x)|}{P_\gamma g(x)} &= \frac{|\sum_i [\psi_i'(x) g \circ \psi_i(x)]'|}{\sum_i \psi_i'(x) g \circ \psi_i(x)} \\
&\le \frac{|\sum_i \psi_i''(x) g \circ \psi_i(x)|}{\sum_i \psi_i'(x) g \circ \psi_i(x)} + \frac{|\sum_i \psi_i'(x) g' \circ \psi_i(x) \psi_i'(x)|}{\sum_i \psi_i'(x) g \circ \psi_i(x)} \\
&\le \max_i \frac{|\psi_i''(x)|}{\psi_i'(x)} + \max_i |\phi_i'(x)| \frac{g' \circ \psi_i(x)}{g \circ \psi_i(x)}.
\end{aligned}$$

In the last step, we used that $|\sum a_i / \sum b_i| \le \max_i |a_i/b_i|$ for numbers a_i and positive numbers b_i. We claim that $\max_i \frac{|\psi_i''(x)|}{\psi_i'(x)}$ is bounded, uniformly in γ. This is clear for x outside $\tilde{E}_\gamma$, where ψ_i' is bounded away from 0 and ψ_i'' is bounded. Take $x \in \tilde{E}_\gamma$ and consider the inverse branch ψ_i provided by the flow ϕ_γ^t of the adapted vector field ϕ_γ (see Proposition 2.1). Thus $\frac{d}{dt}\phi_\gamma^t(x_0) = \phi_\gamma(\phi_\gamma^t(x_0))$. To bound $|\psi_i''|/\psi_i'$ we consider the flow ϕ_γ^t for negative t. Let $J_\gamma^t = (\phi_\gamma^t)''/(\phi_\gamma^t)'$. Then J_γ^t is a solution of

$$\frac{d}{dt} J_\gamma^t(x_0) = \phi_\gamma'' \Big|_{\phi_\gamma^t(x_0)} (\phi_\gamma^t)'(x_0). \tag{28}$$

For γ small, ϕ_γ'' is close to ϕ_0'', which is nonzero by assumption. Further, $(\phi_\gamma^t)'(x_0)$ is a solution of $\frac{d}{dt}(\phi_\gamma^t)'(x_0) = \phi_\gamma' \Big|_{\phi_\gamma^t(x_0)} (\phi_\gamma^t)'(x_0)$. Using this one can show that J_γ^t is bounded, uniformly in γ. We sketch a possible reasoning. One may assume that $\phi_\gamma(x) = x^2 + \gamma + \mathcal{O}(x^2\gamma) + \mathcal{O}(x^3)$. Let $\phi_{\gamma,a}(x) = ax^2 + \gamma$ and denote the corresponding flow by $\phi_{\gamma,a}^t$. For arguments sake, take $x_0 < 0$.

Then, for $t < 0$, $\phi^t_{\gamma,a^+}(x_0) \leq \phi^t_\gamma(x_0) \leq \phi^t_{\gamma,a^-}(x_0)$ for some $a_- < a_+$ close to 1. Solving for $\phi^t_{\gamma,a}$ shows that

$$\frac{1}{\sqrt{a}}\arctan\left(\sqrt{\frac{a}{\gamma}}\phi^t_{\gamma,a}(x_0)\right) \quad = \quad \sqrt{\gamma}t + \frac{1}{\sqrt{a}}\arctan\left(\sqrt{\frac{a}{\gamma}}x_0\right). \tag{29}$$

Note that $\frac{d}{dt}(\phi^t_{\gamma,a})'(x_0) = 2a\phi^t_{\gamma,a}(x_0)(\phi^t_{\gamma,a})'(x_0)$. It follows that

$$(\phi^t_{\gamma,a^-})'(x_0) \leq (\phi^t_\gamma)'(x_0) \leq (\phi^t_{\gamma,a^+})'(x_0).$$

Differentiating (29) yields

$$(\phi^t_{\gamma,a})'(x_0) = \frac{a(\phi^t_{\gamma,a}(x_0))^2 + \gamma}{ax_0^2 + \gamma}.$$

These bounds and (28) prove that J^t_γ is bounded, uniformly in γ.

Replacing $\tilde{h}_\gamma$ by $\tilde{h}^N_\gamma$, which is an expanding map, the above reasoning shows that

$$\text{Reg } P^N_\gamma g \leq M + \frac{1}{\lambda}\text{Reg } g$$

for constants $M > 0$ and $\lambda > 1$. Now (27) follows by iterating the bound on the regularity of $P^N_\gamma g$.

We can now basically follow the proof of Proposition 7 in [PiaYor79] to conclude the lemma. Take $g \in \mathcal{C}([0,1])$ with Reg $g \leq \rho$. Then, for $x \in [0,1]$,

$$P^n_\gamma g(x) = \sum_i |\psi''_i(x)| g \circ \psi_i(x),$$

where ψ_i are the inverse branches of $\tilde{h}^n_\gamma$ near x. Since $\int_{[0,1]} gdm = 1$, there is a component A_j of $[0,1]\backslash O$ on which $\sup_{A_j} g \geq 1$. We may also assume that $\sup_{A_j} g \geq \frac{1}{2}$ on a component A_j with $m(A_j) \geq \beta$ for some constant $\beta > 0$. Then $\inf_{A_j} g \geq \frac{1}{2}e^{-\beta\rho}$.

If n is large enough, $[f^2_\gamma(c), f_\gamma(c)] \subset \tilde{h}^n_\gamma(A_j)$ for all small γ. Hence, for each $x \in [f^2_\gamma(c), f_\gamma(c)]$, there exists i_0 with $\psi_{i_0}(x) \in A_j$. This gives

$$\begin{aligned} P^n_\gamma g(x) \quad &\geq \quad |\psi''_{i_0}(x)| g \circ \psi_{i_0}(x), \\ &\geq \quad |\psi''_{i_0}(x)| e^{-\rho\beta}. \end{aligned}$$

Since $m(A_j) \geq \beta$, we may take i_0 so that $|\psi''_{i_0}(x)|$ is bounded from below. Therefore, $P^n_\gamma g(x) \geq d$ for some $d > 0$ which is independent of g and γ. By (26), this proves the lemma. $\square$

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