Uniqueness of Gibbs State for Non-Ideal Gas in $\mathbf{R}^d$: the Case of Multibody Interaction

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Abstract

We study the question of existence and uniqueness of non-ideal gas in $\mathbb{R}^d$ with multi-body interactions among its particles. For each k-tuple of the gas particles, $2 \le k \le m_0 < \infty$, their interaction is represented by a potential function Φ_k of a finite range.

We introduce a *stabilizing* potential function Φ_{k_0}, such that $\Phi(x_1, \dots, x_{k_0})$ grows sufficiently fast, when $\mathrm{diam}\{x_1, \dots, x_{k_0}\}$ shrinks to 0. Our results hold under the assumption that at least one of the potential functions is stabilizing, that causes a sufficiently strong repulsive force due to the stabilizing potential. We prove that *(i)* for any temperature there exists at least one Gibbs field, and *(ii)* there exists exactly one Gibbs filed ξ at low enough temperature, such that $\mathbb{E}e^{\chi|\xi_V|} < \infty$ for any real $\chi > 0$ and any small volume V. The proofs use the criterion of the uniqueness of Gibbs field in non-compact case developed in ([4]), and the technique employed in ([1]) for studying a gas with pair interaction.

Key words and phrases: Gibbs state, Gibbs measure, non-ideal gas in $\mathbb{R}^d$, multi-body interaction, uniqueness, existence, Dobrushin uniqueness condition.

1 Introduction

This work is the second part of our studies of the uniqueness region of non-ideal classical gas. The first part of this work, published in [1], was devoted to the case of pair interactions between gas particles. In the present paper, we study the case of multi-body interactions.

Our main goal is to use the technique of Dobrushin's type for the study of continuous statistical physics models. It is a far different approach with respect to the analysis employed in the works [2] and [3]. Besides, the approach we use gives rather explicit conditions for the uniqueness that is also one of the aims of this work.

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The method we use here, as well as in [1], amounts to checking the conditions of a general theorem from [4]. These conditions split in two groups: 1) the compactness conditions; 2) the contraction conditions. The contraction conditions are close to those from R. Dobrushin's work [5]. The difference is that the Dobrushin's contraction conditions from [5] is normally checked for total spin space while that from [4] is checked here only on a compact sub-space of the whole spin space, although the spin space of a considered model may be non-compact. This difference is an essential advantage of the condition from [4] over that from [5] since it allows us to get uniform estimates that we need for our argument and that would not available, if were verified on a non-compact space. We note that the theorem from [4] that we use has been proved for lattice models. Therefore, to apply it to gas models in $\mathbb{R}^d$, we have to map the studied continuous model to an equivalent lattice model. We do so via partitioning $\mathbb{R}^d$ into small cubes and considering all particle configurations in every cube as the spin space at an appropriate site of $\mathbb{Z}^d$.

In this work, we follow the plan of [1]. However, the verification of the uniqueness conditions for a multi-body interaction case, compared to that for a pair interaction one, requires some extension of the conditions on potential functions, and accordingly, some new computations. The reason of the changes needed to treat the multi-body interaction case is due to the follow simple fact. We recall that the condition we check is physically speaking, a repulsiveness of the particles at small distances. Assume that $\Phi_{m_0}, m_0 > 2$, is the potential function giving the energy of m_0-particle interactions. Assume also that Φ_{m_0} takes negative values, bounded from below. Then m_0 particles attract each other. If there are N particles in a small volume then there exist $C_N^{m_0}$ groups, each having m_0 particles and giving its contribution to the attraction. A repulsive potential is needed. Let Φ_2 be the repulsive potential. Then, there are C_N^2 pairs of the particles in the same volume, each one having repulsive energy. The number C_N^2 of the pairs of particles and the number $C_N^{m_0}$ of the groups of m_0 particles in a volume containing N particles, have different asymptotic as $N \to \infty$; namely, $C_N^2 / C_N^{m_0} \to 0$. To prevent a collapse of infinite number of particles in a bounded volume caused by the predominance of the attraction, the repulsive potential Φ_2 must have a special property that would compensate this difference. Namely, it must grow with approximation of particle positions. Therefore, we introduce a stabilizing potential function Φ_{k_0} which gives the main repulsive energy of the model. We study the case when the multi-bodiness is bounded from above by some integer m_0. As we shall see, in the case $k_0 < m_0$, the stabilization is ensured, when the stabilizing potential Φ_{k_0} has a singularity at the diagonal of $\left(\mathbb{R}^d\right)^{k_0}$ and grows sufficiently fast nearby the diagonal. However, in the case $k_0 = m_0$, it is enough that Φ_{k_0} is large, although finite, at the diagonal.

is it good term?

Our conditions imply also the existence of at least one Gibbs field for any temperature (Theorem 2). It is worth to remark that the existence is proved under the same conditions that provide the uniqueness. This existence result was not proved in [1].

In Section 2, we give the assumptions on the potential functions and define the stabilizing potential function. Section 3 contains the main results of the work. We prove our results in Section 4.

2 The model

The spin space $\mathcal{X}$ is the set of all locally finite point configurations on $\mathbb{R}^d$. The reference measure ν is the non-normalized Poisson measure on $\mathcal{X}$, that gives weight $\frac{|V|^N}{N!}$ to the set of all configurations of $\mathcal{X}$ having N particles in a bounded volume V. The detailed descriptions of the measurable space $(\mathcal{X}, \nu)$ is given in [1], Section 1.1. Next we describe the assumptions on potential functions $\Phi_k(x_1, \dots, x_k) : \mathbb{R}^{dk} \to \mathbb{R}$, $k \geq 2$, determining of the system.

i Finite group: There exists an integer m_0 such that

$$\Phi_k(x_1, \dots, x_k) \equiv 0, \text{ for all } k > m_0,$$

and

$$\inf \Phi_{m_0}(x_1, \dots, x_{m_0}) < 0.$$

ii Finite range: Each potential function is of finite range, i.e., there exists a constant $D > 0$ such that for any $2 \leq k \leq m_0$

$$\Phi_k(x_1, \dots, x_k) = 0, \text{ if } \operatorname{diam}\{x_1, \dots, x_k\} > D.$$

iii Symmetry: $\Phi_k(x_1, \dots, x_k) = \Phi_k(x_{i_1}, \dots, x_{i_k})$, for any k, any $(x_1, \dots, x_k) \in \mathbb{R}^{dk}$, and any permutation $(i_1, \dots, i_k)$ of $(1, \dots, k)$.

iv Translation invariance: $\Phi_k(x_1 + z, \dots, x_k + z) = \Phi_k(x_1, \dots, x_k)$, for any k, any $(x_1, \dots, x_k) \in \mathbb{R}^{dk}$, and any $z \in \mathbb{R}^d$.

v Boundness from below: There exists a constant $M > 0$ such that $\Phi_k(x_1, \dots, x_k) \geq -M$, for all k and all $(x_1, \dots, x_k) \in \mathbb{R}^{dk}$.

vi No hard core: For every k, the function $\Phi_k(x_1, \dots, x_k)$ is continuous at all $(x_1, \dots, x_k) \in \mathbb{R}^{dk}$, except maybe the case when $x_1 = x_2 = \dots = x_k$.

Since the functions Φ_k are continuous out every neighborhood of the diagonals $\{x_1 = ... = x_k\}$ we shall assume that the constant M from **v** is such that for some $\delta_0 > 0$ and all k

$$\sup_{\operatorname{diam}\{x_1, \dots, x_k\} > \delta_0} |\Phi_k(x_1, \dots, x_k)| \leq M.$$

A concrete value of δ_0 will be chosen later, when we introduce the stabilizing potential function.

As usual, the Hamiltonian of a finite configuration $\sigma \in \mathcal{X}$ is

$$H(\sigma) = \sum_{k=2}^{m_0} \sum_{x_1, \dots, x_k \in \sigma} \Phi_k(x_1, \dots, x_k). \tag{1}$$

For two configurations σ and τ having no common particles, let

$$F(\sigma, \tau) = \sum_{k=2}^{m_0} \sum_{\substack{x_1 \cdots x_r \in \sigma \\ y_1 \cdots y_p \in \tau \\ p+r=k,\, p \geq 1,\, r \geq 1}} \Phi_k(x_1, \dots, x_r, y_1, \dots, y_p) \tag{2}$$

be the energy of the interaction between σ and τ.

Let $V \subseteq \mathbb{R}^d$ and $\sigma \in \mathcal{X}$ be such that V is bounded, $\sigma \subset V$, and let $\tau \subset V^c$ then the value of the *conditional* Hamiltonian at σ under the condition τ is

$$H(\sigma \,|\, \tau) = H(\sigma) + F(\sigma, \tau). \tag{3}$$

Observe that $F(\sigma, \tau)$ is finite even though the configuration τ can be infinite because the potentials are of finite range.

Remark 1. Due to assumption **ii**, $F(\sigma, \tau)$ and, consequently, $H(\sigma \,|\, \tau)$ depends on τ through solely its restriction to the D–neighborhood of V. With some abuse of notations, we shall denote this restriction by the same letter τ and $|\tau|$ will denote the number of particles in this neighborhood. Further ∂V denotes the D–neighborhood of V.

The potential function Φ_{k_0} is said to be *stabilizing* if

1 in the case $k_0 < m_0$, let $\delta_1 > 0$ and $A_1 > 0$ be such that

$$\delta_1^{d(m_0-1)} e^{2\left(\frac{D}{\delta_1}+3\right)^d+1} M < \frac{A_1}{2^{m_0-k_0} k_0^{m_0}} \tag{4}$$

then

$$\Phi_{k_0}(x_1, \ldots, x_{k_0}) \geq \frac{A_1}{r^{d(m_0-1)}} \text{ if } \operatorname{diam}\{x_1, \ldots, x_{k_0}\} = r \leq \delta_1. \tag{5}$$

2 in the case $k_0 = m_0$, let $\delta_2 > 0$ and $A_2 > 0$ be such that

$$A_2 \geq m_0^{m_0} M e^{2\left(\frac{D}{\delta_2}+3\right)^d+1} \tag{6}$$

then

$$\Phi_{k_0}(x_1, \ldots, x_{k_0}) \geq A_2 \text{ if } \operatorname{diam}\{x_1, \ldots, x_{k_0}\} \leq \delta_2. \tag{7}$$

Further we shall use the notation

$$\delta_0 = \begin{cases} \delta_1, & \text{if } k_0 < m_0, \\ \delta_2, & \text{if } k_0 = m_0, \end{cases} \tag{8}$$

and

$$A_0 = \begin{cases} A_1, & \text{if } k_0 < m_0, \\ A_2, & \text{if } k_0 = m_0. \end{cases} \tag{9}$$

To construct the specification by the conditional Hamiltonian (3), the integral

$$\int_{\mathcal{X}_V} \exp\{-\beta H(\sigma \,|\, \tau) + \mu|\sigma|\} \nu(d\sigma), \tag{10}$$

must be finite, where $\mathcal{X}_V$ is the configuration set in V. In (10) $V \subseteq \mathbb{R}^d$ is a bounded volume and $\mu \in \mathbb{R}$. As we shall show the existence of the stabilizing potential function provides the finiteness of (10) for finite volumes V.

3 Main Results

Theorem 1. *Let* **i–vi** *be satisfied. If there exists the stabilizing potential function, then for every finite volume V and every boundary condition τ, the integral (10) is finite.*

As a consequence of Theorem 1, we can introduce a Gibbs specification $\{P_{V,\tau}\,;\, V \subseteq \mathbb{R}^d,\ \tau \in \mathcal{X}_{V^c}\}$ such that the density $p_{V,\tau}$ of the measure $P_{V,\tau}$ with the respect to ν is

$$p_{V,\tau}(\sigma) = \frac{\exp\{-\beta H(\sigma \,|\, \tau) + \mu|\sigma|\}}{\int_{\mathcal{X}_V} \exp\{-\beta H(\sigma \,|\, \tau) + \mu|\sigma|\}\nu(d\sigma)}, \tag{11}$$

where $\mu \in \mathbb{R}$.

Theorem 2. *Let* **i–vi** *be satisfied and Φ_{k_0} be the stabilizing potential function for some $k_0 \leq m_0$.*

Then for any $\mu \in \mathbb{R}$ and $\beta > 0$ there exists at least one Gibbs field corresponding to the specification (11).

Theorem 3. *Let the conditions* **i–vi** *be satisfied and Φ_{k_0} be the stabilizing potential function for some $k_0 \leq m_0$. Then for every $\mu \in \mathbb{R}$ there exists $\beta(\mu) > 0$ such that for any β, $0 \leq \beta \leq \beta(\mu)$, there exists the unique random field ξ corresponding to the Gibbs specification (11) and satisfying the condition*

$$\mathrm{E} e^{\chi|\xi_V|} < \infty \tag{12}$$

for every $\chi > 0$ and for $\mathrm{diam}(V) \leq \delta_0$, *where $\xi_V \in \mathcal{X}_V$.*

4 Proofs

4.1 Proof of Theorem 1.

We start off introducing notations that will be used throughout. Let

$$\overline{N} := \begin{cases} k_0(1 - 1/\sqrt[d]{2})^{-d}, & \text{if } k_0 < m_0, \\ m_0, & \text{if } k_0 = m_0. \end{cases} \tag{13}$$

and

$$\rho := \frac{1}{2}\left(\frac{2\sqrt{d}D}{\delta_0} + 3\right)^{-d}. \tag{14}$$

For a bounded volume V and for a configuration $\tau \in \mathcal{X}_{V^c}$ ($|\tau|$ below should be understood in the sense of Remark 1) we define

$$N_\tau := [\rho|\tau|] + 1, \tag{15}$$

and

$$\overline{N}_\tau := \max\{\overline{N}, N_\tau\}. \tag{16}$$

We note that the choice of ρ in (14) will be clear in the Section 4.2.

Lemma 4 presented below will be frequently used in the proofs of Theorem 1, Theorem 2 and Theorem 3.

Lemma 4. *Let potential functions satisfy the assumptions of* **i–vi**, *and let* $k_0 \le m_0$ *be the index of its stabilizing potential function. Then*
(a) *for any finite* $V \subset \mathbb{R}^d$, $\sigma = \emptyset$, *and for any* $\tau \in \mathcal{X}_{V^c}$, *it holds that* $H(\sigma \,|\, \tau) = 0$;
(b) *for any finite* $V \subset \mathbb{R}^d$, *any* $\sigma \in \mathcal{X}_V$ *with* $|\sigma| > 0$, *and any* $\tau \in \mathcal{X}_{V^c}$, *it holds that*

$$H(\sigma \,|\, \tau) \ge -M|\sigma|^{m_0} \exp\{1 + |\tau|/|\sigma|\} \tag{17}$$

(c) *for any* $V \subset \mathbb{R}^d$ *satisfying* $\mathrm{diam}(V) < \delta_0$, *for any* $\tau \in \mathcal{X}_{V^c}$ *and any* $\sigma \in \mathcal{X}_V$ *satisfying* $|\sigma| \ge \overline{N}_\tau$, *it holds that*

$$H(\sigma \,|\, \tau) \ge C_0|\sigma|^{m_0}, \tag{18}$$

where

$$C_0 = \begin{cases} \frac{A_1}{2^{m_0-k_0} k_0^{m_0} \delta_1^{d(m_0-1)}} - Me^{2\left(\frac{2\sqrt{d}D}{\delta_1}+3\right)^d+1}, & \text{if } k_0 < m_0 \\ \frac{A_2}{m_0^{m_0}} - Me^{2\left(\frac{2\sqrt{d}D}{\delta_2}+3\right)^d+1}, & \text{if } k_0 = m_0, \end{cases} \tag{19}$$

The following two lemmas are used in the prove of the lemma 4.

Lemma 5. *For any integers* N *and* k *such that* $N \ge k$ *it holds that*

$$N^k/k! \ge C_N^k \ge N^k/k^k.$$

Proof. The left inequality is obvious. $C_N^k = \frac{N^k}{k^k}$ for $N = k$. Rewrite the right inequality as

$$\left(1 - \frac{1}{N}\right) \dots \left(1 - \frac{k-1}{N}\right) \ge \frac{k!}{k^k}. \tag{20}$$

It is clear that the left hand side of (20) is increasing with N. Since (20) holds for $N = k$ then it holds for $N > k$. □

Lemma 6. *For arbitrary integers* $P \ge 1$ *and* $k \ge 2$ *and real* $R \ge \frac{k^2}{k-1}P$, *let*

$$T(r_1, ..., r_P) = \sum_{i=1}^{P} \mathbf{1}(r_i \ge k) r_i^k$$

be a function of $(r_1, ..., r_P) \in \Delta$, *where* $\Delta = \mathbb{R}_+^P \cap \left\{(r_1, ..., r_P) : \sum_{i=1}^P r_i = R\right\}$ *and* $\mathbf{1}(A)$ *is the indicator of* A. *Then*

$$\inf_{\Delta} T(r_1, ..., r_P) = T\left(\frac{R}{P}, ..., \frac{R}{P}\right) \tag{21}$$

Proof. It is clear that

$$T\left(\frac{R}{P}, ..., \frac{R}{P}\right) = \left(\frac{R}{P}\right)^k P = \inf\left\{T(r_1, ..., r_P) : \sum r_i = R, r_i \ge k \text{ for all } i = 1, ..., P\right\}. \tag{22}$$

Assume next that some of r_i are less than k. Let us for $i = P-p+1, ..., P$ take $r_i < k$, where p is an integer less than P. Then

$$T\left(\frac{R-kp}{P-p}, ..., \frac{R-kp}{P-p}, r_{P-p+1}, ..., r_P\right) = \left(\frac{R-kp}{P-p}\right)^k (P-p) =$$
$$\inf\left\{T(r_1, ..., r_P) : \sum r_i = R, r_i \geq k \text{ for } i \leq P-p, r_i < k \text{ for } i > P-p\right\}. \quad (23)$$

The value in (23) is greater than one in (22) since

$$R \geq \frac{k^2}{k-1} P \geq \frac{kp}{1 - \left(1 - \frac{p}{P}\right)^{1-\frac{1}{k}}}$$

for any $1 \leq p \leq P-1$. □

Proof of Lemma 4.

Item (**a**) is evident.

Item (**b**). Let V, $\sigma \in \mathcal{X}_V$, $|\sigma| \geq 1$, and $\tau \in \mathcal{X}_{V^c}$ be arbitrary. Then, using **v** and Lemma 5 we obtain that for any $k = 2, 3, \ldots, m_0$,

$$\sum_{\substack{\sigma' \subseteq \sigma \\ |\sigma'| = k}} \Phi_k(\sigma') \geq -MC^k_{|\sigma|} \geq -M \frac{|\sigma|^k}{k!}, \quad (24)$$

$$\sum_{\substack{\sigma' \subseteq \sigma, \tau' \subseteq \tau \\ |\sigma'|+|\tau'|=k,\ |\sigma'| \geq 1, |\tau'| \geq 1}} \Phi_k(\sigma' \cup \tau') \geq -M \sum_{1 \leq \ell \leq k-1} C^\ell_{|\sigma|} C^{k-\ell}_{|\tau|} \geq -M \sum_{1 \leq \ell \leq k-1} \frac{|\sigma|^\ell}{\ell!} \frac{|\tau|^{k-\ell}}{(k-\ell)!}. \quad (25)$$

Thus

$$H(\sigma|\tau) \geq -M|\sigma|^{m_0} \left[\sum_{k=2}^{m_0} \frac{1}{|\sigma|^{m_0-k} k!} + \sum_{m=1}^{m_0-1} \left(\frac{|\tau|}{|\sigma|}\right)^m \frac{1}{m!} \sum_{k=m+1}^{m_0} \frac{1}{|\sigma|^{m_0-k}(k-m)!}\right]. \quad (26)$$

Since $|\sigma| > 0$ we obtain from (26)

$$H(\sigma|\tau) \geq -M|\sigma|^{m_0} e \left(1 + \sum_{m=1}^{\infty} (m!)^{-1} (|\tau|/|\sigma|)^m\right) = -M|\sigma|^{m_0} \exp\{1 + |\tau|/|\sigma|\} \quad (27)$$

that proves the assertion (**b**) of the lemma.

We shall prove (**c**) for the volume $V = G$, where G is a cube with its edge size $\delta_0/\sqrt{d}$. It is clear that having the result for G we have it for any volume V that can be surrounded by a cube with its diameter less than or equal to δ_0. We prove (**c**) treating the cases $k_0 < m_0$ and $k_0 = m_0$ separately.

Let $k_0 < m_0$. Let us pick a sequence of integers $P = P_N$ that satisfies the following two relations:

$$\frac{1}{2k_0} N \leq P_N \leq \frac{1}{k_0} N, \quad \text{and} \quad P_N = \ell^d \text{ for some } \ell \in \mathbb{Z}^+. \quad (28)$$

The left hand side of the inequality holds for $N \geq \overline{N} \equiv k_0(1-1/\sqrt[d]{2})^{-d}$ (we could take a smaller value of $\overline{N}$ and enlarge appropriately the coefficient 2 in the denominator of the fraction in the left hand side of (28)). For every σ we consider G represented as a union of $P_{|\sigma|}$ non-intersecting small cubes, each one with the edge length

$$r = \frac{\delta_1}{\sqrt{d} P_{|\sigma|}^{\frac{1}{d}}}. \tag{29}$$

Let the small cubes of the partition of G be enumerated from 1 to $P = P_{|\sigma|}$ and let σ_i denote the sub-configuration of σ in the i-th cube. Then

$$\sum_{\substack{\sigma' \subseteq \sigma \\ |\sigma'| = k_0}} \Phi_{k_0}(\sigma') \geq \sum_{i=1}^{P} \sum_{\substack{\sigma' \subseteq \sigma_i \\ |\sigma'| = k_0}} \Phi_{k_0}(\sigma') \geq \overline{\Phi}_{k_0}(r\sqrt{d}) \sum_{i=1}^{P} C_{|\sigma_i|}^{k_0}, \tag{30}$$

where

$$\overline{\Phi}_{k_0}(r\sqrt{d}) = \inf\left\{ \Phi_{k_0}(x_1, \dots, x_{k_0}) \,:\, \operatorname{diam}\{x_1, \dots, x_{k_0}\} \leq r\sqrt{d} \right\}. \tag{31}$$

We now apply Lemma 6, and after (5), (29) and the left hand side of (28) to obtain that the right hand side of (30) is greater than

$$\overline{\Phi}_{k_0}(r\sqrt{d}) \frac{1}{k_0^{k_0}} \frac{|\sigma|^{k_0}}{P^{k_0}} P \geq \frac{A_1 |\sigma|^{k_0} P^{m_0 - k_0}}{{k_0}^{k_0} \delta_1^{d(m_0-1)}} \geq \frac{A_1 |\sigma|^{m_0}}{2^{m_0-k_0} {k_0}^{m_0} \delta_1^{d(m_0-1)}}. \tag{32}$$

Gathering (24), (25), (30) and (32) and employing the arguments used for (27) give the following relations

$$\begin{aligned} H(\sigma \,|\, \tau) \;&\geq\; \frac{A_1 |\sigma|^{m_0}}{2^{m_0-k_0} k_0^{m_0} \delta_1^{d(m_0-1)}} - M \sum_{k=2}^{m_0} \frac{|\sigma|^k}{k!} - M \sum_{m=1}^{m_0-1} \frac{|\tau|^m}{m!} \sum_{k=m+1}^{m_0} \frac{|\sigma|^{k-m}}{(k-m)!} \\ &\geq\; |\sigma|^{m_0} \left\{ \frac{A_1}{2^{m_0-k_0} k_0^{m_0} \delta_1^{d(m_0-1)}} - M \exp\left\{ 1 + \frac{|\tau|}{|\sigma|} \right\} \right\} \end{aligned} \tag{33}$$

Since $|\sigma| \geq \overline{N}_\tau$ we can use (15), (14) and obvious estimates to conclude that

$$H(\sigma \,|\, \tau) \geq |\sigma|^{m_0} \left(\frac{A_1}{2^{m_0-k_0} k_0^{m_0} \delta_1^{d(m_0-1)}} - M e^{2\left(\frac{2\sqrt{d}D}{\delta_1} + 3\right)^d + 1} \right) = C_0 |\sigma|^{m_0}, \tag{34}$$

where C_0 is defined in (19).

We next prove (**c**) in the case $k_0 = m_0$. The procedure is similar to that employed for the case $k_0 < m_0$. However, now we do not divide the cube G into small ones. Therefore, the counterpart of (30) and (32) are:

$$\sum_{\substack{\sigma' \subseteq \sigma \\ |\sigma'| = m_0}} \Phi_{m_0}(\sigma') \geq \overline{\Phi}_{m_0}(\delta_2)\, C_{|\sigma|}^{m_0} \geq A_2 \frac{|\sigma|^{m_0}}{m_0^{m_0}}, \tag{35}$$

where $\overline{\Phi}$ has been defined in (31). In the first passage of (35), we used that $|\sigma| \geq m_0$, since $|\sigma| \geq \overline{N}$, and in the last one, we used (7) and Lemma 5. The rest of the argument is identical to that for the case $k_0 < m_0$. We obtain

$$H(\sigma \,|\, \tau) \geq |\sigma|^{m_0} \left(\frac{A_2}{m_0^{m_0}} - M e^{2\left(\frac{2\sqrt{d}D}{\delta_2}+3\right)^d+1} \right) = C_0 |\sigma|^{m_0}, \tag{36}$$

(see (19)). □

Remark 2. Now we can prove the existence of the integral (10) for small volumes with diameter less than δ_0. For fixed V with $\operatorname{diam}(V) \leq \delta_0$ and τ on V^c, define $\mathcal{X}_V^N := \{\sigma \in \mathcal{X}_V : |\sigma| = N\}$, $N \in \mathbb{N}$. Then, the partition function (10) is

$$Z_{V,\tau} = \sum_{N=0}^{\infty} \int_{\mathcal{X}_V^N} e^{-\beta H(\sigma \,|\, \tau) + \mu |\sigma|} \nu(d\sigma). \tag{37}$$

Using (**c**) of Lemma 4, each term with $N \geq \overline{N}_\tau$ may be estimated from above by $e^{\mu N - \beta C_0 N^{m_0}} \int_{\mathcal{X}_V^N} \nu(d\sigma) = e^{\mu N - \beta C_0 N^{m_0}} |V|^N / N!$. Thus, (37) is a converging series.

We prove Theorem 1 proving in the next lemma the finiteness of (10) for any bounded volume V. Lemma 4 is essential in this proof.

Lemma 7. *For any bounded volume V and the boundary configuration τ on V^c the following inequality*

$$H(\sigma|\tau) \geq 0 \tag{38}$$

holds for $|\sigma|$ large enough.

Proof. Represent $\mathbb{R}^d$ as the union of small non-intersected cubes G_t with the diagonal length equal to δ_0. We enumerate the small cubes by points t of $\mathbb{Z}^d$. Let V be a cube in $\mathbb{R}^d$ such that it can be represented as the union of the small cubes G_t, $t \in U \subseteq \mathbb{Z}^d$. We assume also that

$$U = \{t \in \mathbb{Z}^d : \, |t| \leq \ell\},$$

where ℓ is an integer. If σ is a configuration on V then σ_t, $t \in U$, is the subconfiguration localized in $G_t \subset V$. The neighborhood ∂t of $t \in \mathbb{Z}^d$ is

$$\partial t = \{u : \, G_u \cap \partial G_t \neq \emptyset\}, \tag{39}$$

where ∂G_t is taken in the sense of Remark 1. Further we use the notation

$$a = |\partial t|. \tag{40}$$

It follows from (14) that $a\rho \leq \frac{1}{2}$.

Let $L_0 = \max\{N_s : \, s \in U\}$, where $N_s = |\sigma_s|$, and $L_i = L_0 - i$, $i \geq 0$. Assume that $|\sigma| \geq \overline{N}|U|$ then we consider a sequence U_i of subsets in U:

$$U_i = \{t \in U : \, N_t = L_i\}.$$

For every i we enumerate points in U_i by the integers. For $t \in U_i$ let $n_i(t)$ be the number of t. The sequence $U_i^j,\ j \geq 0$, of subsets of U_i is determined as

$$U_i^j = \{t \in U_i :\ n_i(t) > j\}.$$

Clear, $U_i^0 = U_i$. We build next a sequence of configurations $\sigma^{i,j}$ as the following.

$$\sigma^{i,j}_{G_t} = \begin{cases} \emptyset, & \text{if } t \in \cup_{k=0}^{i-1} U_k \cup (U_i \setminus U_i^j) \\ \sigma_{G_t} & \text{otherwise} \end{cases}, \quad \begin{aligned} \sigma^{0,0} &= \sigma, \\ &\dots \\ \sigma^{i,j} &= \cup_{t\in U} \sigma^{i,j}_{G_t}, \\ &\dots \end{aligned}$$

The sequence composition continues until $L_i > \overline{N}$.

The following property of the sequence is important for the estimate of $H(\sigma)$. Recall that $H(\sigma_{G_t}|\sigma_{\partial G_t}) = H(\sigma_{G_t}) + \sum_{u\in\partial t} F(\sigma_t, \sigma_u)$, where ∂t is defined in (39). The property satisfied for any $t \in U_i^j$ is

$$H(\sigma^{i,j}_{G_t}|\sigma^{i,j}_{\partial G_t}) \geq C_0 L_i^{m_0}. \tag{41}$$

This fact follows from Lemma (4)(**c**) since the condition $|\sigma^{i,j}_{G_t}| > [\rho|\sigma^{i,j}_{\partial G_t}|] + 1$ holds if $t \in U_i^j$. Recall that $|\sigma^{i,j}_{G_t}| = L_i > \overline{N}$ if $j = n_i(t)$.

Let $W_i^j = \cup_{k=0}^{i-1} U_k \cup (U_i \setminus U_i^j)$, $W = \cup_i U_i$ and $\overline{U} = \{t :\ N_t \leq \overline{N}\}$. For $T \subseteq U$ let $\widehat{T} = \cup_{s\in T} G_s \subseteq V$. We calculate next $H(\sigma_{\widehat{W}}|\sigma_{\partial\widehat{W}})$. We have

$$\begin{aligned} H(\sigma_{\widehat{W}}|\sigma_{\partial\widehat{W}}) = \sum_{t\in W} H(\sigma_{G_t}) + \sum_{u,s\in W} F(\sigma_{G_u}, \sigma_{G_s}) + \sum_{u\in W, s\in\partial W} F(\sigma_{G_u}, \sigma_{G_s}) = \\ \sum_{i\geq 0} \sideset{}{^o}\sum_{\substack{n_i(t)\geq 1 \\ t\in U_i}} \left[H(\sigma_{G_t}) + \sum_{u\in\partial t\setminus W_i^{n_i(t)}} F(\sigma_{G_t}, \sigma_{G_u}) \right], \end{aligned} \tag{42}$$

where $\sum^o$ means that the sum is taken over t such that $n(t)$ is increasing. Remark now that $\sum_{u\in\partial t\setminus W_i^{n_i(t)}} F(\sigma_{G_t}, \sigma_{G_u}) = \sum_{u\in\partial t} F(\sigma^{i,n_i(t)}_{G_t}, \sigma^{i,n_i(t)}_{G_u})$. Thus

$$H(\sigma_{\widehat{W}}|\sigma_{\partial\widehat{W}}) = \sum_{i\geq 0}\sum_{t\in U_i} \left[H(\sigma^{i,n_i(t)}_{G_t}) + \sum_{u\in\partial t} F(\sigma^{i,n_i(t)}_{G_t}, \sigma^{i,n_i(t)}_{G_u}) \right] \geq C_0 \sum_i L_i^{m_0}|U_i|. \tag{43}$$

Let $U^- = U \setminus (W \cup \partial W)$. If U^- is not empty then $N_t \leq \overline{N}$ for any $t \in U^-$. We enumerate all sites from U^- in arbitrary order and denote $U_k^- = \{t :\ n^-(t) \leq k)\}$, where $n^-(t)$ is a number of t in the enumeration. Using the following relations for any $t, s \in U^-$:

$$\begin{aligned} H(\sigma_{G_t}) &\geq -Mm_0\overline{N}^{m_0} \\ F(\sigma_{G_t}, \sigma_{G_s}) &\geq -Mm_0\overline{N}^{m_0} \end{aligned}$$

we obtain

$$H(\sigma_{\widehat{U}^-}) = \sum_{t\in U^-}^{o} \left[H(\sigma_{G_t}) + \sum_{s\in\partial t\setminus(U^-_{n(t)-1}\cup W)} F(\sigma_{G_t},\sigma_{G_s}) \right] \geq -Mm_0|U^-|\overline{N}^{m_0}(1+a). \quad (44)$$

The total energy of the σ can be estimated now

$$H(\sigma_V) \ = \ H(\sigma_{\widehat{W}}|\sigma_{\partial\widehat{W}}) + H(\sigma_{\widehat{U}^-}) \ \geq \ C_0 \sum_i L_i^{m_0}|U_i| - Mm_0|U^-|\overline{N}^{m_0}(1+a) \quad (45)$$

Now it is clear that $H(\sigma_V) \geq 0$ if $|\sigma|$ is large.

The proof that we have works for the case $\tau = \emptyset$. It is clear that for non-empty $\tau \in \mathcal{X}_{V^c}$ the proof can be done in the similar way. In this case the value of $\overline{N}$ depends on τ, namely, $\overline{N} = \max\{\overline{N}, N_s,\ s \in \partial U\}$. □

4.2 Proof of Theorem 2 and 3

The proofs are based on general theorems for the existence from [5] and for the uniqueness from [4] (the formulation of the latter may be also found in [1]). These theorems work for lattice models, therefore, as in Subsection 4.1 we partition $\mathbb{R}^d$ into cubes $\{G_t,\ t \in \mathbb{Z}^d\}$ such that the diagonal length of each cube is equal to δ_0. All cubes are equivalent in the sense that $G_t = G_0 + t$. The spin space $\mathbf{X}$ is the set of particle configurations in G_0 endowed with a metric such that it becomes a Polish space (see [1]). The models, continuous and lattice, are equivalent. In fact, $\mathbf{X}$ becomes a Polish space if G_0 is a closed cube and so G_t are. Then adjacent cubes have common faces. However, the lattice model with configurations on closed cubes is still equivalent to the original continuous one (see details in [1]). Thus the existence and the uniqueness of the lattice model implies the existence and the uniqueness of the continuous one. Further G_0 means the closed cube.

The neighborhood of a site t is described in (39) (see also (40)).

Further we say '$\mathbf{x}$ is a configuration in G_t' in the sense that $\mathbf{x}$ is an element of $\mathbf{X}$, or, in other words, $\mathbf{x}$ is a configuration in G_0 which is assigned to the site $t \in \mathbb{Z}^d$. We now define the specification of the corresponding lattice model. Let $t \in \mathbb{Z}^d$ and $\overline{\mathbf{y}}$ be a configuration of the lattice model, $\overline{\mathbf{y}} \in \mathbf{X}^{\mathbb{Z}^d\setminus\{t\}}$. The Hamiltonian $H(\mathbf{x}|\overline{\mathbf{y}})$ is then constructed as the energy of the interactions between the particles from $\mathbf{x} \in G_t$ and between the particles from $\mathbf{x}$ and from $\overline{\mathbf{y}}$. Then the distribution density with respect to ν on $\mathbf{X}$, which is, in fact, a distribution on the configuration set in G_t, under the condition $\overline{\mathbf{y}}$ is

$$P_{t,\overline{\mathbf{y}}}(\mathbf{x}) = Z_{\overline{\mathbf{y}}}^{-1} \exp\{-\beta\, H(\mathbf{x}\,|\,\overline{\mathbf{y}}) + \mu|\mathbf{x}|\} \quad (46)$$

(for details, see [1]). It follows from Theorem 1 that we can define a distribution $P_{U,\overline{\mathbf{y}}}$ on $\widehat{U} = \cup_{s\in U} G_s$ for finite $U \subset \mathbb{Z}^d$ under a condition $\overline{\mathbf{y}}$ out of $\widehat{U}$.

4.2.1 Proof of Theorem 2

In the first step we check the *compactness* condition that will be used to establish both the existence and the uniqueness. Actually the compactness condition for the uniqueness from [4] is slightly stronger than that for the existence from [5]. In the next lemma we prove the stronger version of the condition.

Lemma 8. *For any $\chi > 0$, $t \in \mathbb{Z}^d$ and $\overline{\mathbf{y}} \in \mathbf{X}^{\mathbb{Z}^d \setminus \{t\}}$, there exist $C > 0$ and $c_u \geq 0, u \in \partial t$, such that*

$$\mathrm{E}_{\overline{\mathbf{y}}} e^{\chi|\mathbf{x}|} = \int_{\mathbf{X}} e^{\chi|\mathbf{x}|} P_{t,\overline{\mathbf{y}}}(d\mathbf{x}) \leq C + \sum_{u \in \partial t} c_u e^{\chi|\mathbf{y}(u)|} \tag{47}$$

and $\sum_{u \in \partial t} c_u$ is small enough.

Remark 3. In order to prove the existence it is sufficient that (47) holds with $\sum_{u \in \partial t} c_u < 1$. However for the needs of the proof of uniqueness, the sum in the above inequality must be less than a certain number determined by some specific characteristics of the potential functions (see [4]). Of course, this number is less than 1.

Proof. We represent the space $\mathbf{X} = \cup_{N=0}^{\infty} \mathbf{X}^N$ as the union of the subspaces $\mathbf{X}^N$ such that configurations $\mathbf{x} \in \mathbf{X}^N$ have the fixed particle number: $|\mathbf{x}| = N$. Then,

$$\begin{aligned}
\mathrm{E}_{\overline{\mathbf{y}}} e^{\chi|\mathbf{x}|} &= \sum_{N=0}^{\infty} e^{\chi N} \Pr(|\mathbf{x}| = N) \\
&= \left(\sum_{N=0}^{N_{\overline{\mathbf{y}}}} + \sum_{N=N_{\overline{\mathbf{y}}}+1}^{\overline{N}} + \sum_{N=\overline{N}+1}^{\infty} \right) e^{\chi N} \Pr(|\mathbf{x}| = N) \\
&=: J_1 + J_2 + J_3
\end{aligned} \tag{48}$$

with the convention that $J_2 = 0$, if $\overline{N} \leq N_{\overline{\mathbf{y}}}$.

We shall show that

$$J_1 \leq C_1 + \sum_{u \in \partial t} c_u e^{\chi|\overline{\mathbf{y}}(u)|}, \quad J_2 \leq C_1', \quad J_3 \leq C_1'', \tag{49}$$

and $C_1, \quad C_1', \quad C_1''$ are independent on $\overline{\mathbf{y}}$. Recall that $|\overline{\mathbf{y}}|$ denotes the number of particles of $\overline{\mathbf{y}}$ outside of G_t and inside $\cup_{r \in G_t} B_D(r)$.

First, we shall estimate J_1. Since $\chi > 0$ the sum J_1 is less than $e^{\chi N_{\overline{\mathbf{y}}}} \leq e^{\chi\rho|\overline{\mathbf{y}}|+\chi}$ (see (15)). Applying then the obvious inequality

$$\prod_{i=1}^{n} x_i \leq \frac{1}{n} \sum_{i=1}^{n} x_i^n, \quad n \in \mathbb{N}, \quad x_1, \ldots, x_n \geq 0 \tag{50}$$

to $e^{\chi\rho|\overline{\mathbf{y}}|} = \Pi_{i=1}^{n} e^{\chi\rho|\mathbf{y}(u)|}$, where $\overline{\mathbf{y}}(u)$ is the part of $\overline{\mathbf{y}}$ in G_u, $u \in \partial t$, we obtain

$$e^{\chi} \prod_{u \in \partial t} e^{\chi\rho|\overline{\mathbf{y}}(u)|} \leq \frac{e^{\chi}}{a} \sum_{u \in \partial t} e^{\chi\rho a|\overline{\mathbf{y}}(u)|} \leq \frac{e^{\chi}}{2a} \sum_{u \in \partial t} e^{2T} + \frac{e^{\chi}}{2a} \sum_{u \in \partial t} e^{-2T} e^{2\chi\rho a|\overline{\mathbf{y}}(u)|} \tag{51}$$

for any $T>0$. Because of the convexity of the function e^x, we obtain

$$J_1 \le e^{2T+\chi}/2 + e^{-2T+\chi}(1-2\rho a)/2 + \sum_{u\in\partial t} e^{\chi|\overline{\mathbf{y}}(u)|}\, e^{-2T+\chi} 2\rho a. \tag{52}$$

The crucial point in the previous estimates is that $1-2\rho a \ge 0$ since

$$a \le \left(\frac{2D\sqrt{d}}{\delta_0}+3\right)^d = \frac{1}{2\rho}, \tag{53}$$

(see (14)).

Denoting $C_1 := e^{2T+\chi}/2 + e^{-2T+\chi}(1-2\rho a)/2$ and $c = c_u = e^{-2T+\chi}2\rho a$, we obtain the first inequality of (49).

Note that both C_1 and c_u do not depend on β, and that we can make $\sum c_u = ca$ as small as desired by choosing T appropriately large.

To estimate J_2 we use (**b**) of Lemma 4It gives that

$$J_2 \le \sum_{N=N_{\overline{\mathbf{y}}}+1}^{\overline{N}} e^{(\chi+\mu)N} \exp\left\{\beta M N^{m_0} e^{1+\frac{|\tau|}{N}}\right\} \frac{\delta_0^{dN}}{N!} \le$$

$$\exp\left\{(\chi+\mu)\overline{N} + \beta M \overline{N}^{m_0} e^{1+\frac{1}{\rho}} + \delta_0^d\right\} := C_1'.$$

We assumed above that $N_{\overline{\mathbf{y}}} < \overline{N}$, otherwise $C_1' = J_2 = 0$.

Finally, let us estimate J_3. For $|\mathbf{x}| > \overline{N}$ we can apply Lemma 4, (**c**), to conclude that $H(\mathbf{x}\,|\,\overline{\mathbf{y}}) \ge 0$. Hence, we have that

$$\begin{aligned} J_3 &\le \sum_{N=\overline{N}+1}^{\infty} \int_{\mathbf{X}^N} e^{-\beta H(\mathbf{x}\,|\,\overline{\mathbf{y}})+(\chi+\mu)|\mathbf{x}|}\nu(d\mathbf{x}) \le \sum_{N\ge\overline{N}+1} e^{(\mu+\chi)N}\frac{(\delta_0^d)^N}{N!} \\ &\le \exp\{\delta_0^d e^{\mu+\chi}\} - 1 =: C_1''. \end{aligned} \tag{54}$$

Remark that C_1 and C_1'' do not depend on β. However, C_1' depends on β such that it is not decreasing with β. We take $C = C_1 + C_1' + C_1''$. □

The Dobrushin's existence condition (see [5]) rests on the lemma just proved and on continuity of the specification with respect to boundary configurations that will be established below.

Lemma 9. *Let $\overline{\mathbf{y}}$ be a boundary configuration on $\mathbb{Z}^d\backslash\{t\}$ and let $(\overline{\mathbf{y}}_n)$ be a sequence of configurations on $\mathbb{Z}^d\backslash\{t\}$ that converges to $\overline{\mathbf{y}}$. Then the measures $P_{t,\overline{\mathbf{y}}_n}$ weakly converge to the measure $P_{t,\overline{\mathbf{y}}}$.*

Proof. Because the potential functions are translation invariant, it is sufficient to treat solely the case $t=0$. Therefore we omit the subscript t.

The weak convergence means that for any bounded continuous function ϕ on $\mathbf{X}$ the following convergence

$$\int_{\mathbf{X}} \phi(\mathbf{x}) P_{\overline{\mathbf{y}}_n}(d\mathbf{x}) \to \int_{\mathbf{X}} \phi(\mathbf{x}) P_{\overline{\mathbf{y}}}(d\mathbf{x}), \text{ as } \overline{\mathbf{y}}_n \to \overline{\mathbf{y}} \tag{55}$$

holds. The convergence $\overline{\mathbf{y}}_n \to \overline{\mathbf{y}}$ means that $\overline{\mathbf{y}}_n(u) \to \overline{\mathbf{y}}(u)$ for every $u \in \partial 0$. The convergence $\overline{\mathbf{y}}_n(u) \to \overline{\mathbf{y}}(u)$ is understood in the sense of the metric on $\mathbf{X}$. According to this metric, $\overline{\mathbf{y}}_n(u) \to \overline{\mathbf{y}}(u)$ implies the following. Let $B_\varepsilon(y)$, $y \in \overline{\mathbf{y}}(u)$, be a ε–ball around y such that no particles of $\overline{\mathbf{y}}(u)$ belong to $B_\varepsilon(y) \setminus \{y\}$. Then there exists a sequence $y_n \in \overline{\mathbf{y}}_n(u)$ such that for n large only y_n of $\overline{\mathbf{y}}_n(u)$ belongs to $B_\varepsilon(y)$. It follows from this fact and the continuity of the potential functions out of diagonals that

$$H(\mathbf{x} \,|\, \overline{\mathbf{y}}_n) \to H(\mathbf{x} \,|\, \overline{\mathbf{y}}),$$

almost everywhere with respect to Lesbegue measure in $|G_0|^{|\mathbf{x}|}$. Hence,

$$\int_{\mathbf{X}^N} \left| e^{-\beta H(\mathbf{x} \,|\, \overline{\mathbf{y}}_n)} - e^{-\beta H(\mathbf{x} \,|\, \overline{\mathbf{y}})} \right| \nu(d\mathbf{x}) \to 0, \text{ for any } N \text{ as } \mathbf{y}_n \to \mathbf{y}. \tag{56}$$

Recall that $\mathbf{x} \in \mathbf{X}^N$ if $|\mathbf{x}| = N$. It is not difficult to obtain the inequality

$$\begin{aligned} R_n(\phi) &= \left| \int_{\mathbf{X}} \phi(\mathbf{x}) P_{\overline{\mathbf{y}}_n}(d\mathbf{x}) - \int_{\mathbf{X}} \phi(\mathbf{x}) P_{\overline{\mathbf{y}}}(d\mathbf{x}) \right| \\ &\leq \sum_{N,N'} e^{\mu(N+N')} \int_{\mathbf{X}^N} \int_{\mathbf{X}^{N'}} |\phi(\mathbf{x})| \cdot \left| e^{-\beta H(\mathbf{x}' \,|\, \overline{\mathbf{y}}) - \beta H(\mathbf{x} \,|\, \overline{\mathbf{y}}_n)} - e^{-\beta H(\mathbf{x}' \,|\, \overline{\mathbf{y}}_n) - \beta H(\mathbf{x} \,|\, \overline{\mathbf{y}})} \right| \nu(d\mathbf{x}) \nu(d\mathbf{x}') \end{aligned} \tag{57}$$

Let $\varepsilon > 0$ be small then we can find N_ε such that the sum tail of the right hand side of (57) is less than or equal to ε:

$$\sum_{N \geq N_\varepsilon} \sum_{N' \geq N_\varepsilon} \leq \varepsilon.$$

Taking the limit along n in (57) and using (56), we obtain

$$\lim_{n \to \infty} R_n(\phi) \leq \varepsilon \qquad \square$$

Proof of Theorem 2 is now a direct consequence of Lemmas 8 and 9. $\square$.

4.2.2 Proof of Theorem 3

In Lemma 8, we proved the compactness condition of the theorem from [4]. We next verify the *contraction condition* (see [1]) that in our case, amounts to the following requirement: If we take two different boundary conditions $\overline{\mathbf{y}}_1$ and $\overline{\mathbf{y}}_2$ at a site $t \in \mathbb{Z}^d$, then the variation distance $V(P_{t,\overline{\mathbf{y}}_1}, P_{t,\overline{\mathbf{y}}_2})$ between two distributions on $\mathbf{X}_t$ corresponding to $\overline{\mathbf{y}}_1$ and $\overline{\mathbf{y}}_2$ must be less than the discrete distance between $\overline{\mathbf{y}}_1$ and $\overline{\mathbf{y}}_2$. This is the general Dobrushin condition of uniqueness, which is here formulated for the case of variation distance between measures. In our approach based on [4], we verify this condition only for a compact set of boundary configurations $\overline{\mathbf{y}}$. According to [4] this compact set must be large enough. We consider the compact sets of the type $\{\sigma : e^{\chi|\sigma|} \leq K\}$. The compact set of this kind is large if K is large. There is no any estimate of K in [4]. Therefore we check the contraction condition for any K. It gives us also the contraction condition with a large compact set as it is necessary.

Lemma 10. *For any positive integer K_0 there exists $\beta(K_0) > 0$ such that for any $\beta \le \beta(K_0)$, any $t \in \mathbb{Z}^d$, and any two configurations $\overline{\mathbf{y}}_1$ and $\overline{\mathbf{y}}_2$ from $\mathbf{X}^{\mathbb{Z}^d \setminus \{t\}}$ satisfying*

$$\max_{u \in \partial t} \{|\overline{\mathbf{y}}_1(u)|, |\overline{\mathbf{y}}_2(u)|\} \le K_0, \tag{58}$$

the following inequality holds:

$$V(P_{t,\overline{\mathbf{y}}_1}, P_{t,\overline{\mathbf{y}}_2}) < \frac{1}{a} \sum_{u \in \partial t} \mathbf{1}_{\{\overline{\mathbf{y}}_1(u) \neq \overline{\mathbf{y}}_2(u)\}}. \tag{59}$$

Proof. As in the lemma 9 we perform all our calculations for $t = 0 \in \mathbb{Z}^d$ and omit t from all subscripts in this proof.

The following estimate for the variational distance may be obtained straightforwardly. For any $\overline{\mathbf{y}}_1, \overline{\mathbf{y}}_2 \in \mathbf{X}^{\mathbb{Z}^d \setminus \{0\}}$,

$$\begin{aligned}
&V(P_{\overline{\mathbf{y}}_1}, P_{\overline{\mathbf{y}}_2}) = \frac{1}{2} \sum_{N=0}^{\infty} \int_{\mathbf{X}^N} |p_{\overline{\mathbf{y}}_1}(\mathbf{x}) - p_{\overline{\mathbf{y}}_2}(\mathbf{x})| \, \nu(d\mathbf{x}) \\
&\le \frac{1}{2} \sum_{N=0}^{\infty} e^{\mu N} \int_{\mathbf{X}^N} \left| \frac{e^{-\beta H(\mathbf{x}|\overline{\mathbf{y}}_1)}}{Z_{\overline{\mathbf{y}}_1}} - \frac{e^{-\beta H(\mathbf{x}|\overline{\mathbf{y}}_2)}}{Z_{\overline{\mathbf{y}}_2}} \right| \nu(\mathrm{d}\mathbf{x}) \\
&\le \sum_{N=1}^{\infty} e^{\mu N} \int_{\mathbf{X}^N} \left| e^{-\beta H(\mathbf{x}|\overline{\mathbf{y}}_1)} - e^{-\beta H(\mathbf{x}|\overline{\mathbf{y}}_2)} \right| \nu(\mathrm{d}\mathbf{x}) \\
&+ \frac{1}{2} \sum_{N=1}^{\infty} \sum_{N'=1}^{\infty} e^{\mu(N+N')} \int_{\mathbf{X}^N} \int_{\mathbf{X}^{N'}} \left| e^{-\beta H(\mathbf{x}|\overline{\mathbf{y}}_1) - \beta H(\mathbf{x}'|\overline{\mathbf{y}}_2)} - e^{-\beta H(\mathbf{x}|\overline{\mathbf{y}}_2) - \beta H(\mathbf{x}'|\overline{\mathbf{y}}_1)} \right| \times \\
&\qquad \times \nu(d\mathbf{x}) \, \nu(d\mathbf{x}').
\end{aligned} \tag{60}$$

where, to obtain the last inequality, we used that $Z_{\overline{\mathbf{y}}} \ge 1$ for any $\overline{\mathbf{y}}$.

First, we shall prove the lemma in the case

A There exists $u_0 \in \partial 0$ such that the particle configuration $\overline{\mathbf{y}}_2(u_0)$ is exactly the particle configuration $\overline{\mathbf{y}}_1(u_0)$ plus one "extra" particle whose coordinate will be denoted by y_0, and $\overline{\mathbf{y}}_1(u) = \overline{\mathbf{y}}_2(u)$ for all $u \in \partial 0, u \neq u_0$.

Let

$$\Psi(\mathbf{x}) := H(\mathbf{x}|\overline{\mathbf{y}}_2) - H(\mathbf{x}|\overline{\mathbf{y}}_1). \tag{61}$$

Plugging (61) in (60), we get that

$$\begin{aligned}
V\left(P_{\overline{\mathbf{y}}_1}, P_{\overline{\mathbf{y}}_2}\right) \;\le\; & \sum_{N=1}^{\infty} e^{\mu N} \int_{\mathbf{X}^N} e^{-\beta H(\mathbf{x}|\overline{\mathbf{y}}_1)} \left|1 - \exp\{-\beta \Psi(\mathbf{x})\}\right| \, \nu(\mathrm{d}\mathbf{x}) \\
+ \; & \sum_{N=1}^{\infty} \sum_{N'=1}^{\infty} e^{\mu(N+N')} \int_{\mathbf{X}^N} e^{-\beta H(\mathbf{x}|\overline{\mathbf{y}}_1)} \, \nu(\mathrm{d}\mathbf{x}) \\
& \times \int_{\mathbf{X}^{N'}} e^{-\beta H(\mathbf{x}'|\overline{\mathbf{y}}_1)} \left|1 - \exp\{-\beta \Psi(\mathbf{x}')\}\right| \nu(d\mathbf{x}').
\end{aligned} \tag{62}$$

Introducing

$$Y_{\overline{\mathbf{y}}_1}(y_0) = \sum_{N'=1}^{\infty} \int_{\mathbf{X}^{N'}} e^{-\beta H(\mathbf{x}'|\overline{\mathbf{y}}_1)+\mu N'} \left|1-\exp\{-\beta\Psi(\mathbf{x}')\}\right| \nu(d\mathbf{x}'), \tag{63}$$

we thus have from (62) that

$$V(P_{\overline{\mathbf{y}}_1}, P_{\overline{\mathbf{y}}_2}) \leq Y_{\overline{\mathbf{y}}_1}(y_0) Z_{\overline{\mathbf{y}}_1}. \tag{64}$$

Our programme is as follows: we shall found estimates on $Z_{\overline{\mathbf{y}}_1}$ and $Y_{\mathbf{y}_1}(y_0)$ that will show that, as $\beta \to 0$, $Z_{\overline{\mathbf{y}}_1}$ remains bounded while $Y_{\overline{\mathbf{y}}_1} \to 0$. These facts together with (64) will imply the lemma assertion.

We now start an argument that will estimate $Z_{\overline{\mathbf{y}}_1}$. We note that $|\overline{\mathbf{y}}_1| \leq aK_0$, according to (58). This inequality, together with the definitions of $N_{\overline{\mathbf{y}}_1}$ (see (15)) and the estimate (53) imply the inequality $N_{\overline{\mathbf{y}}_1} \leq K_0$ that will be used below. We represent the partition function in the following form:

$$Z_{\overline{\mathbf{y}}_1} = 1 + \sum_{N=1}^{\overline{N}_{\overline{\mathbf{y}}_1}} \int_{\mathbf{X}^N} e^{-\beta H(\mathbf{x}\,|\,\overline{\mathbf{y}}_1)+\mu|\mathbf{x}|} \nu(d\mathbf{x}) + \sum_{N=\overline{N}_{\overline{\mathbf{y}}_1}+1}^{\infty} \int_{\mathbf{X}^N} e^{-\beta H(\mathbf{x}\,|\,\overline{\mathbf{y}}_1)+\mu|\mathbf{x}|} \nu(d\mathbf{x}). \tag{65}$$

Due to Lemma 4, $H(\mathbf{x}\,|\,\overline{\mathbf{y}}_1) \geq -MN^{m_0}e^{1+|\overline{\mathbf{y}}_1|/N}$ in each term of the first sum, while $H(\mathbf{x}\,|\,\overline{\mathbf{y}}_1) \geq 0$ in each term of the second sum. We plug in these estimates in (65) and obtain that

$$Z_{\mathbf{y}_1} \;\; \leq \;\; 1 + \sum_{N=1}^{\max\{\overline{N},K_0\}} \exp\{\beta M N^{m_0} e^{|\partial 0|K_0+1} + \mu N\} \frac{\delta_0^{dN}}{N!} + \left(\exp\{\delta_0^d e^{\mu}\} - 1\right) \tag{66}$$

We now start an argument that will estimate $Y_{\overline{\mathbf{y}}_1}(y_0)$ from above. It consists of two steps.

At the first step, we estimate $\int_{\mathbf{X}^N} |1-\exp\{-\beta\Psi(\mathbf{x})\}|\nu(d\mathbf{x})$. Let L denote the number of particles of $\overline{\mathbf{y}}_1$ in $\partial 0$ and let y_0 be as determined in **A**. We define a set of multi-indices R as the collection of triples

$$r = (k,\, \{i_1, i_2, \ldots, i_{k_1}\},\, \{j_1, \ldots, j_{k_2}\}), \tag{67}$$

where the first element of each triple is an integer $k \in \{2, 3, \ldots, m_0\}$, the second one is a subset $\{i_1, i_2, \ldots, i_{k_1}\}$ of $\{1, \ldots, N\}$, and the third element is a subset $\{j_1, \ldots, j_{k_2}\}$ of $\{1, \ldots, L\}$ that satisfy the conditions $k_1 \geq 1$, $k_2 \geq 0$, and $k_1 + k_2 = k - 1$. We shall use the notation:

$$\Phi_k((\mathbf{x}\overline{\mathbf{y}})_r) = \Phi_k(x_{i_1}, \ldots, x_{i_{k_1}}, y_0, y_{j_1}, \ldots, y_{j_{k_2}}),$$

where r as in (67), $x_{i_1}, ..., x_{i_{k_1}} \in \mathbf{x}$ and $y_{j_1}, ..., y_{j_{k_2}} \in \cup_{u\in\partial 0}\overline{\mathbf{y}}_1(u) \setminus \{y_0\}$. Then (see (61))

$$\Psi(\mathbf{x}) = \sum_{r\in R} \Phi_k((\mathbf{x}\overline{\mathbf{y}})_r), \tag{68}$$

and

$$\int_{\mathbf{X}^N} |1-\exp\{-\beta\Psi(\mathbf{x})\}|\,\nu(d\mathbf{x}) = \frac{1}{N!}\int_{G_0^N}\left|1-\exp\left\{-\beta\sum_{r\in R}\Phi_k((\mathbf{x}\overline{\mathbf{y}})_r)\right\}\right| dx_1\dots dx_N. \quad (69)$$

Using the inequality

$$\left|1-\exp\left\{a-\sum_{i=1}^{n}a_i\right\}\right| \le |1-\exp\{a\}| + \sum_{i=1}^{n}|1-\exp\{-a_i\}|, \quad (70)$$

that holds for any $n\in\mathbb{N}$, and any $a, a_1, \dots, a_r \ge 0$, we get that the integral in the r.h.s. of (69) is bounded from above by the following expression:

$$\begin{aligned}&\int_{G_0^N}\left|1-\exp\left\{-\beta{\textstyle\sum_{r\in R}}\Phi_k((\mathbf{x}\overline{\mathbf{y}})_r)\mathbb{1}_{\{\Phi_k((\mathbf{x}\overline{\mathbf{y}})_r)<0\}}(x_1,\dots,x_N)\right\}\right| dx_1\dots dx_N\\ &+\sum_{r\in R}\int_{G_0^N}\left|1-\exp\left\{-\beta\Phi_k((\mathbf{x}\overline{\mathbf{y}})_r)\mathbb{1}_{\{\Phi_k((\mathbf{x}\overline{\mathbf{y}})_r)\ge 0\}}(x_1,\dots,x_N)\right\}\right| dx_1\dots dx_N\\ &=: I_1+I_2, \end{aligned}\quad (71)$$

The following inequality is obvious

$$\begin{aligned}I_1 \le -\beta\int_{G_0^N}\sum_{r\in R}\Phi_k((\mathbf{x}\overline{\mathbf{y}})_r)\mathbb{1}_{\{\Phi_k((\mathbf{x}\overline{\mathbf{y}})_r)<0\}}(x_1,\dots,x_N)\times\\ \exp\left\{-\beta{\textstyle\sum_{r\in R}}\Phi_k((\mathbf{x}\overline{\mathbf{y}})_r)\mathbb{1}_{\{\Phi_k((\mathbf{x}\overline{\mathbf{y}})_r)<0\}}(x_1,\dots,x_N)\right\} dx_1\dots dx_N\end{aligned}$$

Combining this estimate with the lemma 4 (**b**) we obtain that

$$I_1 \le \beta M m_0(N+L)^{m_0-1}\exp\{\beta M m_0(N+L)^{m_0-1}\}(\delta_0/\sqrt{d})^{dN}. \quad (72)$$

We next estimate I_2. The assumption **v** on the potential functions guarantees that when $u_1,\dots,u_{k_1}\in G_0$ and $v_1,\dots,v_{k_2}\in\overline{\mathbb{R}^d\setminus G_0}$ (the bar means the closure) then $\Phi_k(u_1,...,u_{k_1},v_1,...,v_{k_2})$ may take an infinite value, only if $u_1=\ldots=u_{k_1}=v_1=\ldots=v_{k_2}$. A necessary condition for that to occur, is that all u's belong to the boundary of G_0. Hence, the function

$$\begin{aligned}&F_k(u_1,\dots,u_{k_1}) :=\\ &\sup_{v_1,\dots,v_{k_2}}\left\{\left|1-e^{-\beta\Phi_k(u_1,\dots,u_{k_1},v_1,\dots,v_{k_2})\mathbb{1}_{\{\Phi_k(u_1,\dots,u_{k_1},v_1,\dots,v_{k_2})\ge 0\}}(u_1,\dots,u_{k_1})}\right|\right\}\end{aligned}\quad (73)$$

converges to 0, as $\beta\to 0$, almost surely with respect to Lebesgue measure on ${G_0}^{k_1}$. Since for any $r\in R$,

$$\begin{aligned}&\int_{G_0^N}\left|1-\exp\left\{-\beta\Phi_k((\mathbf{x}\overline{\mathbf{y}})_r)\mathbb{1}_{\{\Phi_k((\mathbf{x}\overline{\mathbf{y}})_r)\ge 0\}}(x_1,\dots,x_N)\right\}\right| dx_1\dots dx_N\\ &\le\int_{G_0^{k_1}}F_k(x_1,\dots,x_{k_1})dx_1\dots dx_{k_1}\int_{G_0^{N-k_1}}dx_{k_1+1}\dots dx_N,\end{aligned}\quad (74)$$

we have that for any $\varepsilon > 0$ there exists $\beta_\varepsilon > 0$ such that for all $\beta \leq \beta_\varepsilon$

$$I_2 \leq m_0(N+L)^{m_0-1}\varepsilon \, \max\{1; (\delta_0/\sqrt{d})^{dN}\}. \tag{75}$$

We now combine (72) and (75) and obtain that for all $\beta \leq \beta_\varepsilon$, the following estimate is valid:

$$\begin{aligned}
&\int_{\mathbf{X}^N} |1 - \exp\{-\beta\Psi(\mathbf{x})\}| \, \nu(\mathrm{d}\mathbf{x}) \\
&\leq \frac{1}{N!}\Big[\beta M m_0(N+L)^{m_0-1} \exp\{\beta M m_0(N+L)^{m_0-1}\}(\delta_0/\sqrt{d})^{dN} \\
&\qquad\qquad + m_0(N+L)^{m_0-1}\varepsilon \, \max\{1; (\delta_0/\sqrt{d})^{dN}\}\Big] \\
&\leq (\beta+\varepsilon)\frac{1}{N!} \max\{1; (\delta_0/\sqrt{d})^{dN}\} M m_0(N + a|K_0)^{m_0-1} \\
&\qquad\qquad \times \exp\{\beta M m_0(N + aK_0)^{m_0-1}\},
\end{aligned} \tag{76}$$

where we have used that $L \equiv |\overline{\mathbf{y}}_1| \leq aK_0$, as it follows from (58).

For the second step of the argument that estimates $Y_{\overline{\mathbf{y}}_1}(y_0)$ we define

$$N^* := \min\{N \,:\, N \geq \overline{N},\, N \geq K_0, \text{ and } C_0 N^{m_0} - M m_0(N + |\partial 0|K_0)^{m_0-1} > N\}. \tag{77}$$

Then (see (63))

$$Y_{\overline{\mathbf{y}}_1}(y_0) = \left(\sum_{N=1}^{N^*} + \sum_{N=N^*+1}^{\infty}\right) \int_{\mathbf{X}_N} e^{-\beta H(\mathbf{x}|\overline{\mathbf{y}}_1) + \mu|\mathbf{x}|} \, |1 - \exp\{-\beta\Psi(\mathbf{x})\}| \, \nu(d\mathbf{x}). \tag{78}$$

We shall estimate $-H(\mathbf{x}\,|\,\overline{\mathbf{y}}_1)$ from above by $M|\mathbf{x}|^{m_0} e^{|\overline{\mathbf{y}}_1|+1}$, when $|\mathbf{x}| \leq N^*$, and by $-C_0|\mathbf{x}|^{m_0}$, when $|\mathbf{x}| > N^*$. The first estimate holds directly by Lemma 4, (**b**). The second estimate follows from Lemma 4, (**c**) since $N^* \geq K_0 \geq N_{\overline{\mathbf{y}}_1}$. Plugging the estimates in (78), using (76), and substituting $|\overline{\mathbf{y}}_1|$ by its maximal possible value aK_0, we conclude that

$$\begin{aligned}
Y_{\overline{\mathbf{y}}_1}(y_0) \;\; &\leq \;\; (\beta+\varepsilon) M m_0 \\
\times \Bigg[&\sum_{N=1}^{N^*} (\delta_0/\sqrt{d})^{dN} \frac{(N + aK_0)^{m_0-1}}{N!} \\
&\qquad \times \exp\left\{\beta N^{m_0} e^{aK_0+1} + \mu N + \beta M m_0 (N + aK_0)^{m_0-1}\right\} \\
+ &\sum_{N=N^*+1}^{\infty} (\delta_0/\sqrt{d})^{dN} \frac{(N + aK_0)^{m_0-1}}{N!} \\
&\qquad \times \exp\left\{-\beta C_0 N^{m_0} + \mu N + \beta M m_0 (N + aK_0)^{m_0-1}\right\}\Bigg].
\end{aligned} \tag{79}$$

It follows from the above expression and the fact that $\varepsilon \to 0$ as $\beta \to 0$ that $Y_{\overline{\mathbf{y}}_1}(y_0) \to 0$ as $\beta \to 0$. Thus there exists $\beta(K_0)$ such that

$$V(P_{\overline{\mathbf{y}}_1}, P_{\overline{\mathbf{y}}_2}) < \frac{1}{2a}, \text{ for any } \beta \leq \beta(K_0), \text{ when } \overline{\mathbf{y}}_1, \overline{\mathbf{y}}_2 \text{ satisfy } \mathbf{A}. \tag{80}$$

Let now $\overline{\mathbf{y}}_1$ and $\overline{\mathbf{y}}_2$ be two arbitrary configurations that satisfy the assumption (58) of Lemma 10. Then, there is a sequence $\overline{\mathbf{y}}^1, \overline{\mathbf{y}}^2, \ldots, \overline{\mathbf{y}}^j$ of at most $2K_0$ configurations such that each one satisfies (58), and $\overline{\mathbf{y}}^i$ and $\overline{\mathbf{y}}^{i+1}$ differ at solely one particle (in the sense of **A**). Using the triangle inequality and the conclusion (80) we complete the proof of Lemma 10. $\square$

Proof of Theorem 3. Lemmas 8 and 10 allow us to check the uniqueness conditions of [4]. For any $\chi > 0$ and any $\mu \in \mathbb{R}$ there exists $\beta(\widehat{K}, \chi, \mu) > 0$ such that for $\beta \le \beta(\widehat{K}, \chi, \mu)$ there exists only Gibbs field ξ corresponding to the specification $P_{V,\tau}$ and $\mathrm{E}e^{\chi|\xi_V|} < \infty$ if $\mathrm{diam}(V) < \delta_0$. According to the uniqueness theorem from [4] the value of $\widehat{K}$ depends on D, δ_0 and C (see (47)). We recall that C depends on β. Thus $\widehat{K}$ depends on β. However, C can only increase with β (see the proof of the lemma 8). Hence for any interval $[0, \beta_0]$ we can choose a unique $C(\beta_0)$ such that Lemma 8 holds for $\beta \le \beta_0$ with the same $C \equiv C(\beta_0)$. Then $\widehat{K} \equiv \widehat{K}(\beta_0)$ is a constant when $\beta \in [0, \beta_0]$. Finally, we obtain the uniqueness for any $\beta \le \min\{\beta_0, \beta(\widehat{K}(\beta_0), \chi, \mu)\}$. Moreover we obtain the uniqueness for

$$\beta \le \beta(\chi, \mu) = \max_{\beta_0} \min\{\beta_0, \beta(\widehat{K}(\beta_0), \chi, \mu)\}. \tag{81}$$

Consider next two different positive numbers χ_1 and χ_2. Let $\chi_1 < \chi_2$. It follows from the previous considerations that for $\beta \le \min\{\beta(\chi_1, \mu), \beta(\chi_2, \mu)\}$ there exists an unique Gibbs field ξ corresponding to the specification. On other hand there are Gibbs fields ξ^1 and ξ^2 which are unique in the regions $\beta \le \beta(\chi_1, \mu)$ and $\beta \le \beta(\chi_2, \mu)$ accordingly. It is now clear that $\xi = \xi^1 = \xi^2$ because of the uniqueness. Hence ξ is unique in the region $\beta \le \max\{\beta(\chi_1, \mu), \beta(\chi_2, \mu)\}$. It is obvious that ξ is unique in the region

$$\beta \le \beta(\mu) = \sup_{\chi}\{\beta(\chi, \mu)\}. \quad \square \tag{82}$$

5 Conclusion Remarks

1. The stabilizing function Φ_{k_0} has a singularity the order of which is $d(m_0-1)$ if $k_0 < m_0$. The order of the singularity is independent on k_0. However, the greater k_0 is the weaker the singularity is, because the dimension of the domain of Φ_{k_0} increases with k_0.
2. We require in this work that the stabilizing potential Φ_{k_0} has its singularity near diagonal, if $k_0 < m_0$. It means that $\Phi_{k_0}(x_1, ..., x_{k_0})$ takes large values if the co-ordinates $x_1, ..., x_{k_0}$ of all k_0 particles pertain to a small volume of $\mathbb{R}^d$, that is, when $\mathrm{diam}\{x_1, ..., x_{k_0}\}$ is small. However, our technics permits to extend our results to the cases when Φ_{k_0} takes large values, when l_0, $l_0 < k_0$, variables of $\{x_1, ..., x_{k_0}\}$ pertain to a small volume.
3. We can extend our results to the case when there is a finite number of potential functions $\Phi_{m_0+1}, \Phi_{m_0+2}, ...$ all taking non-negative values. This extension can be done with the same technics we use.

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