ALGEBRO-GEOMETRIC SOLUTIONS OF THE CAMASSA–HOLM HIERARCHY

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ABSTRACT. We provide a detailed treatment of the Camassa–Holm (CH) hierarchy with special emphasis on its algebro-geometric solutions. In analogy to other completely integrable hierarchies of soliton equations such as the KdV or AKNS hierarchies, the CH hierarchy is recursively constructed by means of basic polynomial formalism invoking a spectral parameter. Moreover, we study Dubrovin-type equations for auxiliary divisors and associated trace formulas, consider the corresponding algebro-geometric initial value problem, and derive the theta function representations of algebro-geometric solutions of the CH hierarchy.

1. INTRODUCTION

Very recently, the Camassa–Holm (CH) equation, also known as the dispersive shallow water equation, as isolated, for instance, in [17] and [18],

$$4u_t - u_{xxt} - 2uu_{xxx} - 4u_x u_{xx} + 24uu_x = 0, \quad (x,t) \in \mathbb{R}^2 \tag{1.1}$$

(chosing a scaling of x, t that's convenient for our purpose), with u representing the fluid velocity in x-direction, received considerable attention. Actually, (1.1) represents the limiting case $\kappa \to 0$ of the general Camassa–Holm equation,

$$4v_t - v_{xxt} - 2vv_{xxx} - 4v_x v_{xx} + 24vv_x + 4\kappa v_x = 0, \quad \kappa \in \mathbb{R},\ (x,t) \in \mathbb{R}^2. \tag{1.2}$$

However, in our formalism the general Cammassa–Holm equation (1.2) just represents a linear combination of the first two equations in the CH hierarchy and hence we consider without loss of generality (1.1) as the first nontrivial element of the Camassa–Holm hierarchy. Alternatively, one can transform

$$v(x,t) \mapsto u(x,t) = v(x - (\kappa/2)t, t) + (\kappa/4) \tag{1.3}$$

and thereby reduce (1.2) to (1.1).

Various aspects of local existence, global existence, and uniqueness of solutions of (1.1) are treated in [27], [24], [25], [30], [51], [52], [57], [58], [60], wave breaking phenomena are discussed in [23], [26], [28]. Soliton-type solutions (called "peakons") were extensively studied due to their unusual non-meromorphic (peak-type) behavior, which features a discontinuity in the x-derivative of u with existing left and right derivatives of opposite sign at the peak. In this context we refer, for instance, to [3], [5], [7], [8], [10], [12], [13], [14], [17], [18]. Integrability aspects such as infinitely many conservation laws, (bi-)Hamiltonian formalism, Bäcklund transformations, infinite dimensional symmetry groups, etc., are discussed, for instance, in [17], [18], [38], [41] (see also [42]), [56]. The general CH equation (1.2)

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is shown to give rise to a geodesic flow of a certain right invariant metric on the Bott-Virasoro group in [53]. In the case $\kappa = 0$, the CH equation (1.1) corresponds to the geodesic flow on the group of orientation preserving diffeomorphisms of the circle. This follows from the Lie-Poisson structure established in [18] and is also remarked upon in [53]. That the equations define a smooth vector field was first observed by Shkoller in the case of periodic [57] and Dirichlet [58] boundary conditions, which directly leads to the corresponding local existence theory. Scattering data and their evolution under the CH flow are determined in [11] and intimate relations with the classical moment problem and the finite Toda lattice are worked out in [12], [13], and [14]. The case of spatially periodic solutions, the corresponding inverse spectral problem, isospectral classes of solutions, and quasi-periodicity of solutions with respect to time are discussed in [20], [21], [22], and [29]. Moreover, algebro-geometric solutions of (1.1) and their properties are studied in [1], [2], [3], [4], [5], [6], [7], [8] (connections as well as differences between the latter references and our own approach to algebro-geometric solutions will be outlined in the following paragraph). Moreover, even though the following very recent developments are not directly related to the principal topic of this paper, they put the CH equation in a broader context: In [34], a basic integrable shallow water equation, originally introduced in [17], is analyzed in detail. It combines the linear dispersion of the KdV equation with the nonlinear/nonlocal dispersion of the CH equation and contains the KdV and CH equations (as well as an equation studied by Fornberg and Whitham [40]) as special limiting cases. Finally, the three-dimensional viscous Camassa-Holm equations, their connection with the Navier-Stokes equations, estimates for the Hausdorff and fractal dimension of the associated global attractor, and turbulence theory according to Kolmogorov, Landau, and Lifshitz, are discussed in [39].

Our own approach to algebro-geometric solutions of the CH hierarchy differs from the ones pursued in [1], [2], [3], [4], [5], [6], [7], [8] in several aspects and we will outline some of the differences next. Following previous treatments of the KdV, AKNS, Toda, and Boussinesq hierarchies and the sine-Gordon and massive Thirring models (cf., e.g., [16], [31], [32], [35], [45], [46], [47], [48], [49], [50]), we develop a systematic polynomial recursion formalism for the CH hierarchy and its algebro-geometric solutions. In contrast to the treatments in [3], [7], and [8], we rely on a zero-curvature approach $U_t - V_x = [V, U]$ (as the compatibility requirement for the system $\Psi_x = U\Psi$, $\Psi_t = V\Psi$) as opposed to their Lax formalism. However, we incorporate important features of the recursion formalism developed in [5] into our zero-curvature approach. Our treatment is comprehensive and self-contained in the sense that it includes Dubrovin-type equations for auxiliary divisors on the associated compact hyperelliptic curve, trace formulas, and theta function representations of solutions, the usual ingredients of such a formalism. Moreover, while [3], [7], [8] focus on solutions of the CH equation itself, we simultaneously derive theta function formulas for solutions of any equation of the CH hierarchy. The key element in our formalism is the solution ϕ of a Riccati-type equation associated with the zero-curvature representation of the CH equation (1.1). Roughly speaking, $\phi = -z\psi_2/\psi_1$, where $\Psi = (\psi_1, \psi_2)^t$ and z denotes a spectral parameter in U and V (cf. (2.42) for more details). ϕ is then used to introduce appropriate auxiliary divisors on the underlying hyperelliptic curve, the Baker-Akhizer vector in the stationary case, etc. Combining ϕ with the polynomial recursion formalism for the

CH hierarchy then leads to Dubrovin-type differential equations and trace formulas for u in terms of auxiliary divisors. Explicit theta function representations for symmetric functions of (projections of) these auxiliary divisors then yield the theta function representations for any algebro-geometric solution u of the CH hierarchy. Here our strategy differs somewhat from that employed in [3], [7], [8] for the CH equation. While the latter references also employ the trace formula for u in terms of (projections of) auxiliary divisors, they subsequently rely on generalized theta functions and generalized Jacobians (going back to investigations of Clebsch and Gordan [19]), whereas we stay within the traditional framework familar from the KdV, AKNS, Toda hierarchies, etc. Finally, we point out a novel feature of our treatment of the CH hierarchy that appears to be with out precedent. In Theorems 3.11 and 4.10 we formulate and solve the algebro-geometric initial value problem for the stationary and time-dependent CH hierarchy, in the following sense. Starting from the initial value problem for auxiliary divisors induced by the Dubrovin-type equations, we define u in terms of the trace formula involving the (projections of) auxiliary divisors and then prove directly that u so defined satisfies the corresponding (stationary, resp., time-dependent) equation of the CH hierarchy.

Without going into further details, we note that our constructions extend in a straightforward manner to a closely related hierarchy of completely integrable nonlinear evolution equations, the Dym hierarchy. For different approaches to algebro-geometric solutions of the latter we refer to [3], [6], [9], [33], and [55].

In Section 2 we develop the basic polynomial recursion formalism that defines the CH hierarchy using a zero-curvature approach. Section 3 then treats the stationary CH hierarchy and its algebro-geometric solutions. The corresponding time-dependent results are the subject of Section 4. Appendix A summarizes the necessary results needed from the theory of compact Riemann surfaces and also serves to establish the notation used throughout this paper. Appendix B contains a few technical results concerning the polynomial recursion formalism and associated high-energy expansions. Finally, Appendix C provides a detailed discussion of elementary symmetric functions associated with Dirichlet divisors and their representations in terms of theta functions associated with the underlying hyperelliptic curve. It contains several core results needed in our derivation of algebro-geometric solutions of the CH hierarchy. The results of this appendix apply to a variety of soliton equations and hence are of independent interest.

2. The CH hierarchy, recursion relations, and hyperelliptic curves

In this section we provide the basic construction of a completely integrable hierarchy of nonlinear evolution equations in which the Camassa–Holm equation, or dispersive shallow water equation, is the first element in the hierarchy (the higher-order CH equations will turn out to be nonlocal with respect to x). We will use a zero-curvature approach and combine it with a polynomial recursion formalism containing a spectral parameter.

Throughout this section we will suppose the following hypothesis.

Hypothesis 2.1. *In the stationary case we assume that*

$$u \in C^\infty(\mathbb{R}),\ \frac{d^m u}{dx^m} \in L^\infty(\mathbb{R}),\ m \in \mathbb{N}_0. \tag{2.1}$$

In the time-dependent case we suppose

$$\begin{aligned}
&u(\,\cdot\,,t)\in C^\infty(\mathbb{R}),\ \frac{\partial^m u}{\partial x^m}(\,\cdot\,,t)\in L^\infty(\mathbb{R}),\ m\in\mathbb{N}_0\,,\,t\in\mathbb{R},\\
&u(x,\,\cdot\,),u_{xx}(x,\,\cdot\,)\in C^1(\mathbb{R}),\ x\in\mathbb{R}.
\end{aligned}\tag{2.2}$$

We start by formulating the basic polynomial setup taken essentially from [5]. One defines $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ recursively by

$$\begin{aligned}
&f_0=1,\\
&f_{\ell,x}=-2\mathcal{G}\big(2(4u-u_{xx})f_{\ell-1,x}+(4u_x-u_{xxx})f_{\ell-1}\big),\quad \ell\in\mathbb{N},
\end{aligned}\tag{2.3}$$

where $\mathcal{G}$ is given by

$$\mathcal{G}\colon L^\infty(\mathbb{R})\to L^\infty(\mathbb{R}),\quad (\mathcal{G}v)(x)=\frac{1}{4}\int_{\mathbb{R}}dy\,e^{-2|x-y|}v(y),\quad x\in\mathbb{R},\ v\in L^\infty(\mathbb{R}).\tag{2.4}$$

One observes that $\mathcal{G}$ is the resolvent of minus the one-dimensional Laplacian at energy parameter equal to -4, that is,

$$\mathcal{G}=\bigg(-\frac{d^2}{dx^2}+4\bigg)^{-1}.\tag{2.5}$$

The first coefficient reads

$$f_1=-2u+c_1,\tag{2.6}$$

where c_1 is an integration constant. Subsequent coefficients are non-local with respect to u. At each level a new integration constant, denoted by c_ℓ, is introduced. Moreover, we introduce coefficients $\{g_\ell\}_{\ell\in\mathbb{N}_0}$ and $\{h_\ell\}_{\ell\in\mathbb{N}_0}$ by

$$g_\ell=f_\ell+\frac{1}{2}f_{\ell,x},\quad \ell\in\mathbb{N}_0\,,\tag{2.7}$$

$$h_\ell=(4u-u_{xx})f_\ell-g_{\ell+1,x},\quad \ell\in\mathbb{N}_0\,.\tag{2.8}$$

Explicitly, one computes

$$\begin{aligned}
f_0&=1,\\
f_1&=-2u+c_1,\\
f_2&=2u^2+2\mathcal{G}\big(u_x^2+8u^2\big)+c_1(-2u)+c_2,\\
g_0&=1,\\
g_1&=-2u-u_x+c_1,\\
g_2&=2u^2+2uu_x+2\mathcal{G}\big(u_x^2+u_xu_{xx}+8uu_x+8u^2\big)\\
&\quad+c_1(-2u-u_x)+c_2,\\
h_0&=4u+2u_x,\\
h_1&=-2u_x^2-4uu_x-8u^2\\
&\quad-2\mathcal{G}\big(u_xu_{xxx}+u_{xx}^2+2u_xu_{xx}+8uu_{xx}+8u_x^2+16uu_x\big)\\
&\quad+c_1(4u+2u_x),\ \text{etc.}
\end{aligned}\tag{2.9}$$

For later use it is convenient also to introduce the corresponding homogeneous coefficients $\hat f_\ell$, $\hat g_\ell$, and $\hat h_\ell$ defined by the vanishing of the integration constants c_k, $k=1,\dots,\ell$,

$$\hat f_0=f_0=1,\quad \hat f_\ell=f_\ell\big|_{c_k=0,\,k=1,\dots,\ell},\quad \ell\in\mathbb{N},\tag{2.10}$$

$$\hat{g}_0 = g_0 = 1, \quad \hat{g}_\ell = g_\ell\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{2.11}$$
$$\hat{h}_0 = h_0 = (4u + 2u_x), \quad \hat{h}_\ell = h_\ell\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N}. \tag{2.12}$$

Hence,

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{f}_k, \quad g_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{g}_k, \quad h_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{h}_k, \quad \ell \in \mathbb{N}_0, \tag{2.13}$$

defining

$$c_0 = 1. \tag{2.14}$$

Next, given Hypothesis 2.1, one introduces the 2×2 matrix

$$U(z,x) = \begin{pmatrix} -1 & 1 \\ z^{-1}(4u(x) - u_{xx}(x)) & 1 \end{pmatrix}, \quad x \in \mathbb{R}, \tag{2.15}$$

and for each $n \in \mathbb{N}_0$ the following 2×2 matrix V_n

$$V_n(z,x) = \begin{pmatrix} -G_n(z,x) & F_n(z,x) \\ z^{-1}H_n(z,x) & G_n(z,x) \end{pmatrix}, \quad n \in \mathbb{N}_0,\, z \in \mathbb{C} \setminus \{0\},\, x \in \mathbb{R}. \tag{2.16}$$

Postulating the zero-curvature condition

$$-V_{n,x}(z,x) + [U(z,x), V_n(z,x)] = 0 \tag{2.17}$$

one finds

$$F_{n,x}(z,x) = 2G_n(z,x) - 2F_n(z,x), \tag{2.18}$$
$$zG_{n,x}(z,x) = (4u(x) - u_{xx}(x))F_n(z,x) - H_n(z,x), \tag{2.19}$$
$$H_{n,x}(z,x) = 2H_n(z,x) - 2(4u(x) - u_{xx}(x))G_n(z,x). \tag{2.20}$$

From (2.18)–(2.20) one infers that

$$\frac{d}{dx}\det(V_n(z,x)) = -\frac{1}{z}\frac{d}{dx}\Big(zG_n(z,x)^2 + F_n(z,x)H_n(z,x)\Big) = 0, \tag{2.21}$$

and hence

$$zG_n(z,x)^2 + F_n(z,x)H_n(z,x) = Q_{2n+1}(z), \tag{2.22}$$

where the polynomial Q_{2n+1} of degree $2n+1$ is x-independent. Actually it turns out that it is more convenient to define

$$R_{2n+2}(z) = zQ_{2n+1}(z) = \prod_{m=0}^{2n+1}(z - E_m), \quad E_0 = 0,\, E_1, \dots, E_{2n+1} \in \mathbb{C} \tag{2.23}$$

so that (2.22) becomes

$$z^2G_n(z,x)^2 + zF_n(z,x)H_n(z,x) = R_{2n+2}(z). \tag{2.24}$$

Next one makes the ansatz that F_n, H_n, and G_n are polynomials of degree n, related to the coefficients f_ℓ, h_ℓ, and g_ℓ by

$$F_n(z,x) = \sum_{\ell=0}^{n} f_{n-\ell}(x)z^\ell, \tag{2.25}$$

$$G_n(z,x) = \sum_{\ell=0}^{n} g_{n-\ell}(x)z^\ell, \tag{2.26}$$

$$H_n(z,x) = \sum_{\ell=0}^{n} h_{n-\ell}(x) z^\ell. \tag{2.27}$$

Insertion of (2.25)–(2.27) into (2.18)–(2.20) then yields the recursion relations (2.3)–(2.4) and (2.7) for f_ℓ and g_ℓ for $\ell = 0, \dots, n$. In the case $n \in \mathbb{N}$ we obtain the recursion (2.8) for h_ℓ for $\ell = 0, \dots, n-1$ and

$$h_n = (4u - u_{xx}) f_n. \tag{2.28}$$

(When $n = 0$ one gets directly $h_0 = (4u - u_{xx})$.) Moreover, taking $z = 0$ in (2.24) yields

$$f_n(x) h_n(x) = \prod_{m=1}^{2n+1} E_m. \tag{2.29}$$

In addition, one finds

$$h_{n,x}(x) - 2h_n(x) + 2(4u(x) - u_{xx}(x)) g_n(x) = 0, \quad n \in \mathbb{N}_0. \tag{2.30}$$

Using the relations (2.28) and (2.7) permits one to write (2.30) as

$$\text{s-CH}_n(u) = (u_{xxx} - 4u_x) f_n - 2(4u - u_{xx}) f_{n,x} = 0, \quad n \in \mathbb{N}_0. \tag{2.31}$$

Varying $n \in \mathbb{N}_0$ in (2.31) then defines the stationary CH hierarchy. We record the first few equations explicitly,

$$\begin{aligned}
\text{s-CH}_0(u) &= u_{xxx} - 4u_x = 0, \\
\text{s-CH}_1(u) &= -2uu_{xxx} - 4u_x u_{xx} + 24uu_x + c_1(u_{xxx} - 4u_x) = 0, \\
\text{s-CH}_2(u) &= 2u^2 u_{xxx} - 8uu_x u_{xx} - 40u^2 u_x + 2(u_{xxx} - 4u_x)\mathcal{G}\big(u_x^2 + 8u^2\big) \\
&\quad - 8(4u - u_{xx})\mathcal{G}\big(u_x u_{xx} + 8uu_x\big) \\
&\quad + c_1(-2uu_{xxx} - 4u_x u_{xx} + 24uu_x) + c_2(u_{xxx} - 4u_x) = 0, \text{ etc.}
\end{aligned} \tag{2.32}$$

By definition, the set of solutions of (2.31), with n ranging in $\mathbb{N}_0$, represents the class of algebro-geometric CH solutions. If u satisfies one of the stationary CH equations in (2.31) for a particular value of n, then it satisfies infinitely many such equations of order higher than n for certain choices of integration constants c_ℓ. At times it will be convenient to abbreviate algebro-geometric stationary CH solutions u simply as CH *potentials.*

For later use we also introduce the corresponding homogeneous polynomials $\widehat{F}_\ell$, $\widehat{G}_\ell$, and $\widehat{H}_\ell$ defined by

$$\widehat{F}_\ell(z) = F_\ell(z)\big|_{c_k=0,\, k=1,\dots,\ell} = \sum_{k=0}^{\ell} \hat{f}_{\ell-k} z^k, \quad \ell = 0, \dots, n, \tag{2.33}$$

$$\widehat{G}_\ell(z) = G_\ell(z)\big|_{c_k=0,\, k=1,\dots,\ell} = \sum_{k=0}^{\ell} \hat{g}_{\ell-k} z^k, \quad \ell = 0, \dots, n, \tag{2.34}$$

$$\widehat{H}_\ell(z) = H_\ell(z)\big|_{c_k=0,\, k=1,\dots,\ell} = \sum_{k=0}^{\ell} \hat{h}_{\ell-k} z^k, \quad \ell = 0, \dots, n-1, \tag{2.35}$$

$$\widehat{H}_n(z) = (4u - u_{xx})\hat{f}_n + \sum_{k=1}^{n} \hat{h}_{n-k} z^k. \tag{2.36}$$

In accordance with our notation introduced in (2.10)–(2.12) and (2.33)–(2.36), the corresponding homogeneous stationary CH equations are then defined by

$$\text{s-}\widehat{\mathrm{CH}}_n(u) = \text{s-CH}_n(u)\big|_{c_\ell=0,\ \ell=1,\dots,n} = 0, \quad n \in \mathbb{N}_0. \tag{2.37}$$

Using equations (2.18)–(2.20) one can also derive individual differential equations for F_n and H_n. Focusing on F_n only, one obtains

$$\begin{aligned} &F_{n,xxx}(z,x) - 4\big(z^{-1}(4u(x) - u_{xx}(x)) + 1\big)F_{n,x}(z,x) \\ &\quad - 2z^{-1}(4u_x(x) - u_{xxx}(x))F_n(z,x) = 0. \end{aligned} \tag{2.38}$$

This is of course consistent with (2.25) and (2.3) (applying $\mathcal{G}^{-1}$ to (2.3)). Multiplying (2.38) with F_n and integrating the result yields

$$F_{n,xx}F_n - 2^{-1}F_{n,x}^2 - 2F_n^2 - 2z^{-1}(4u - u_{xx})F_n^2 = C(z), \tag{2.39}$$

for some $C(z)$, constant with respect to x. Differentiating (2.18), inserting (2.19) into the resulting equation, and comparing with (2.18) and (2.24) then yields

$$C(z) = -2z^{-2}R_{2n+2}(z). \tag{2.40}$$

Thus,

$$\begin{aligned} &-(z^2/2)F_{n,xx}(z,x)F_n(z,x) + (z^2/4)F_{n,x}(z,x)^2 + z^2F_n(z,x)^2 \\ &+ z(4u(x) - u_{xx}(x))F_n(z,x)^2 = R_{2n+2}(z). \end{aligned} \tag{2.41}$$

Next, we turn to the time-dependent CH hierarchy. Introducing a deformation parameter $t_n \in \mathbb{R}$ into u (i.e., replacing $u(x)$ by $u(x,t_n)$), the definitions (2.15), (2.16), and (2.25)–(2.27) of U, V_n, and F_n, G_n, and H_n, respectively, still apply. The corresponding zero-curvature relation reads

$$U_{t_n}(z,x,t_n) - V_{n,x}(z,x,t_n) + [U(z,x,t_n), V_n(z,x,t_n)] = 0, \quad n \in \mathbb{N}_0, \tag{2.42}$$

which results in the following set of equations

$$\begin{aligned} &4u_{t_n}(x,t_n) - u_{xxt_n}(x,t_n) - H_{n,x}(z,x,t_n) + 2H_n(z,x,t_n) \\ &-2(4u(x,t_n) - u_{xx}(x,t_n))G_n(z,x,t_n) = 0, \end{aligned} \tag{2.43}$$

$$F_{n,x}(z,x,t_n) = 2G_n(z,x,t_n) - 2F_n(z,x,t_n), \tag{2.44}$$

$$zG_{n,x}(z,x,t_n) = (4u(x,t_n) - u_{xx}(x,t_n))F_n(z,x,t_n) - H_n(z,x,t_n). \tag{2.45}$$

Inserting the polynomial expressions for F_n, H_n, and G_n into (2.44) and (2.45), respectively, first yields recursion relations (2.3) and (2.7) for f_ℓ and g_ℓ for $\ell = 0,\dots,n$. In the case $n \in \mathbb{N}$ we obtain from (2.43) the recursion for h_ℓ for $\ell = 0,\dots,n-1$ and

$$h_n = (u - u_{xx})f_n. \tag{2.46}$$

(When $n = 0$ one gets directly $h_0 = (u - u_{xx})$.) In addition, one finds

$$\begin{aligned} &(4u_{t_n}(x,t_n) - u_{xxt_n}(x,t_n) - h_{n,x}(x,t_n) + 2h_n(x,t_n) \\ &\quad - 2(4u(x,t_n) - u_{xx}(x,t_n))g_n(x,t_n) = 0, \quad n \in \mathbb{N}_0. \end{aligned} \tag{2.47}$$

Using relations (2.28) and (2.46) permits one to write (2.47) as

$$\mathrm{CH}_n(u) = 4u_{t_n} - u_{xxt_n} + (u_{xxx} - 4u_x)f_n - 2(4u - u_{xx})f_{n,x} = 0, \quad n \in \mathbb{N}_0. \tag{2.48}$$

Varying $n \in \mathbb{N}_0$ in (2.48) then defines the time-dependent CH hierarchy. We record the first few equations explicitly,

$$\begin{aligned}
&\mathrm{CH}_0(u) = 4u_{t_0} - u_{xxt_0} + u_{xxx} - 4u_x = 0, \\
&\mathrm{CH}_1(u) = 4u_{t_1} - u_{xxt_1} - 2uu_{xxx} - 4u_x u_{xx} + 24uu_x + c_1(u_{xxx} - 4u_x) = 0, \\
&\mathrm{CH}_2(u) = 4u_{t_2} - u_{xxt_2} + 2u^2 u_{xxx} - 8uu_x u_{xx} - 40u^2 u_x \\
&\qquad + 2(u_{xxx} - 4u_x)\mathcal{G}\big(u_x^2 + 8u^2\big) - 8(4u - u_{xx})\mathcal{G}\big(u_x u_{xx} + 8uu_x\big) \\
&\qquad + c_1(-2uu_{xxx} - 4u_x u_{xx} + 24uu_x) + c_2(u_{xxx} - 4u_x) = 0, \text{ etc.}
\end{aligned} \tag{2.49}$$

Similarly, one introduces the corresponding homogeneous CH hierarchy by

$$\widehat{\mathrm{CH}}_n(u) = \mathrm{CH}_n(u)\big|_{c_\ell = 0,\, \ell=1,\dots,n} = 0, \quad n \in \mathbb{N}_0. \tag{2.50}$$

Up to an inessential scaling of the (x, t_1) variables, $\widehat{\mathrm{CH}}_1(u) = 0$ represents *the Camassa–Holm* equation as discussed in [17], [18].

We note that our zero-curvature approach is similar (but not identical) to that sketched in [56]. This is in contrast to almost all other treatments of the CH equation where a Lax equation approach appears to be preferred.

Our recursion formalism was introduced under the assumption of a sufficiently smooth function u in Hypothesis 2.1. The actual existence of smooth global solutions of the initial value problem associated with the CH hierarchy (2.49) is a nontrivial issue and various aspects of it are discussed, for instance, in [23], [24], [25], [26], [30], [51], [52], [57], [58], [60].

3. The stationary CH formalism

This section is devoted to a detailed study of the stationary CH equation and its algebro-geometric solutions. Our principal tool will be a combination of the polynomial recursion formalism introduced in Section 2 and a meromorphic function ϕ (the solution of a Riccati-type equation associated with the zero-curvature representation of (1.1)) on a hyperelliptic curve $\mathcal{K}_n$ defined in terms of the polynomial R_{2n+2}.

For major parts of this section we suppose

$$u \in C^\infty(\mathbb{R}), \ \frac{d^m u}{dx^m} \in L^\infty(\mathbb{R}), \ m \in \mathbb{N}_0, \tag{3.1}$$

and assume (2.3), (2.4), (2.7), (2.8), (2.15)–(2.17), (2.23), (2.24), (2.25)–(2.27), (2.28)–(2.31), keeping $n \in \mathbb{N}_0$ fixed.

Returning to (2.24) we infer from (2.26) and (2.9) that $R_{2n+2}(z) = zQ_{2n+1}(z)$ is a monomial of degree $2n+2$ of the form

$$R_{2n+2}(z) = \prod_{m=0}^{2n+1} (z - E_m), \quad E_0 = 0, E_1, \dots, E_{2n+1} \in \mathbb{C}. \tag{3.2}$$

Computing

$$\begin{aligned}
\det(wI_2 - iV_n(z,x)) &= w^2 - \det(V_n(z,x)) \\
&= w^2 + G_n(z,x)^2 + \frac{1}{z}F_n(z,x)H_n(z,x) \\
&= w^2 + \frac{1}{z^2}R_{2n+2}(z),
\end{aligned} \tag{3.3}$$

that is,

$$R_{2n+2}(z) = z^2 G_n(z,x)^2 + zF_n(z,x)H_n(z,x) \tag{3.4}$$

(with I_2 the identity matrix in $\mathbb{C}^2$), we are led to introduce the (possibly singular) hyperelliptic curve $\mathcal{K}_n$ of arithmetic genus n defined by

$$\mathcal{K}_n \colon \mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0. \tag{3.5}$$

In the following we will occasionally impose further constraints on the zeros E_m of R_{2n+2} introduced in (3.2) and assume that

$$E_0 = 0,\ E_1, \dots, E_{2n+1} \in \mathbb{C} \setminus \{0\}. \tag{3.6}$$

We compactify $\mathcal{K}_n$ by adding two points at infinity, P_{∞_+}, P_{∞_-}, with $P_{\infty_+} \neq P_{\infty_-}$, still denoting its projective closure by $\mathcal{K}_n$. Hence $\mathcal{K}_n$ becomes a two-sheeted Riemann surface of arithmetic genus n. Points P on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$ are denoted by $P = (z,y)$, where $y(\,\cdot\,)$ denotes the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z,y) = 0$. For additional facts on $\mathcal{K}_n$ and further notation freely employed throughout this paper, the reader may want to consult Appendix A.

For notational simplicity we will usually tacitly assume that $n \in \mathbb{N}$. (The case $n = 0$ is explicitly treated in Example 3.10.)

In the following the roots of the polynomials F_n and H_n will play a special role and hence we introduce on $\mathbb{C} \times \mathbb{R}$

$$F_n(z,x) = \prod_{j=1}^{n} (z - \mu_j(x)), \quad H_n(z,x) = h_0(x) \prod_{j=1}^{n} (z - \nu_j(x)). \tag{3.7}$$

Moreover, we introduce

$$\hat{\mu}_j(x) = (\mu_j(x), -\mu_j(x) G_n(\mu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \dots, n,\ x \in \mathbb{R}, \tag{3.8}$$

$$\hat{\nu}_j(x) = (\nu_j(x), \nu_j(x) G_n(\nu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \dots, n,\ x \in \mathbb{R}, \tag{3.9}$$

and

$$P_0 = (0,0). \tag{3.10}$$

The branch of $y(\,\cdot\,)$ near $P_{\infty_\pm}$ is fixed according to

$$\lim_{\substack{|z(P)| \to \infty \\ P \to P_{\infty_\pm}}} \frac{y(P)}{z(P) G_n(z(P), x)} = \mp 1. \tag{3.11}$$

Due to assumption (3.1), u is smooth and bounded, and hence $F_n(z,\,\cdot\,)$ and $H_n(z,\,\cdot\,)$ share the same property. Thus, one also concludes

$$\mu_j, \nu_k \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ j,k = 1, \dots, n, \tag{3.12}$$

by (3.7).

Next, define the fundamental meromorphic function $\phi(\,\cdot\,, x)$ on $\mathcal{K}_n$ by

$$\phi(P,x) = \frac{y - zG_n(z,x)}{F_n(z,x)} \tag{3.13}$$

$$= \frac{zH_n(z,x)}{y + zG_n(z,x)}, \quad P = (z,y) \in \mathcal{K}_n,\ x \in \mathbb{R}. \tag{3.14}$$

Assuming (3.6), the divisor $(\phi(\,\cdot\,, x))$ of $\phi(\,\cdot\,, x)$ is given by

$$(\phi(\,\cdot\,, x)) = \mathcal{D}_{P_0 \underline{\hat{\nu}}(x)} - \mathcal{D}_{P_{\infty_+} \underline{\hat{\mu}}(x)}, \tag{3.15}$$

taking into account (3.11). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\},\ \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \sigma^n \mathcal{K}_n. \tag{3.16}$$

Given $\phi(\,\cdot\,, x)$, one defines the associated vector $\Psi(\,\cdot\,, x, x_0)$ on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}$ by

$$\Psi(P, x, x_0) = \begin{pmatrix} \psi_1(P, x, x_0) \\ \psi_2(P, x, x_0) \end{pmatrix}, \quad P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\},\ (x, x_0) \in \mathbb{R}^2, \tag{3.17}$$

where

$$\psi_1(P, x, x_0) = \exp\bigg(-(1/z) \int_{x_0}^{x} dx'\, \phi(P, x') - (x - x_0) \bigg), \tag{3.18}$$

$$\psi_2(P, x, x_0) = -\psi_1(P, x, x_0)\phi(P, x)/z. \tag{3.19}$$

Although Ψ is formally the analog of the stationary Baker-Akhiezer vector of the stationary CH hierarchy when compared to analogous definitions in the context of the KdV or AKNS hierarchies, its actual properties in a neighborhood of its essential singularity will feature characteristic differences to standard Baker-Akhiezer vectors (cf. Remark 3.5). We summarize the fundamental properties of ϕ and Ψ in the following result.

Lemma 3.1. *Suppose* (3.1), *assume the nth stationary CH equation* (2.31) *holds, and let $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}$, $(x, x_0) \in \mathbb{R}^2$. Then ϕ satisfies the Riccati-type equation*

$$\phi_x(P, x) - z^{-1}\phi(P, x)^2 - 2\phi(P, x) + 4u(x) - u_{xx}(x) = 0, \tag{3.20}$$

as well as

$$\phi(P, x)\phi(P^*, x) = -\frac{zH_n(z, x)}{F_n(z, x)}, \tag{3.21}$$

$$\phi(P, x) + \phi(P^*, x) = -2\frac{zG_n(z, x)}{F_n(z, x)}, \tag{3.22}$$

$$\phi(P, x) - \phi(P^*, x) = \frac{2y}{F_n(z, x)}, \tag{3.23}$$

while Ψ fulfills

$$\Psi_x(P, x, x_0) = U(z, x)\Psi(P, x, x_0), \tag{3.24}$$

$$-\,y\Psi(P, x, x_0) = zV_n(z, x)\Psi(P, x, x_0), \tag{3.25}$$

$$\psi_1(P, x, x_0) = \bigg(\frac{F_n(z, x)}{F_n(z, x_0)}\bigg)^{1/2} \exp\bigg(-(y/z) \int_{x_0}^{x} dx' F_n(z, x')^{-1} \bigg), \tag{3.26}$$

$$\psi_1(P, x, x_0)\psi_1(P^*, x, x_0) = \frac{F_n(z, x)}{F_n(z, x_0)}, \tag{3.27}$$

$$\psi_2(P, x, x_0)\psi_2(P^*, x, x_0) = -\frac{H_n(z, x)}{zF_n(z, x_0)}, \tag{3.28}$$

$$\psi_1(P, x, x_0)\psi_2(P^*, x, x_0) + \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) = 2\frac{G_n(z, x)}{F_n(z, x_0)}, \tag{3.29}$$

$$\psi_1(P, x, x_0)\psi_2(P^*, x, x_0) - \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) = \frac{2y}{zF_n(z, x_0)}. \tag{3.30}$$

In addition, as long as the zeros of $F_n(\,\cdot\,, x)$ are all simple for $x \in \Omega$, $\Omega \subseteq \mathbb{R}$ an open interval, $\Psi(\,\cdot\,, x, x_0)$, $x, x_0 \in \Omega$, is meromorphic on $\mathcal{K}_n \setminus \{P_0\}$.

Proof. Equation (3.20) follows using the definition (3.13) of ϕ as well as relations (2.18)–(2.20). The other relations, (3.21)–(3.23), are easy consequences of $y(P^*) = -y(P)$ in addition to (2.18)–(2.20). By (3.17)–(3.19), Ψ is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$ away from the poles $\hat{\mu}_j(x')$ of $\phi(\,\cdot\,,x')$. By (2.18), (3.8), and (3.13),

$$-\frac{1}{z}\phi(P,x') \underset{P\to\hat{\mu}_j(x')}{=} \frac{\partial}{\partial x'}\ln(F_n(z,x')) + O(1) \text{ as } z \to \mu_j(x'), \tag{3.31}$$

and hence ψ_1 is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$ by (3.18) as long as the zeros of $F_n(\,\cdot\,,x)$ are all simple. This follows from (3.18) by restricting P to a sufficiently small neighborhood $\mathcal{U}_j$ of $\{\hat{\mu}_j(x') \in \mathcal{K}_n \mid x' \in \Omega,\, x' \in [x_0,x]\}$ such that $\hat{\mu}_k(x') \notin \mathcal{U}_j$ for all $x' \in [x_0,x]$ and all $k \in \{1,\dots,n\}\setminus\{j\}$. Since ϕ is meromorphic on $\mathcal{K}_n$ by (3.13), ψ_2 is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$ by (3.19). The remaining properties of Ψ can be verified by using the definition (3.17)–(3.19) as well as relations (3.20)–(3.23). In particular, equation (3.26) follows by inserting the definition of ϕ, (3.13), into (3.18), using (2.18). ∎

Next, we derive Dubrovin-type equations for μ_j and ν_j. Since in the remainder of this section we will frequently assume $\mathcal{K}_n$ to be nonsingular, we list all restrictions on $\mathcal{K}_n$ in this case,

$$E_0 = 0,\ E_m \in \mathbb{C}\setminus\{0\},\ E_m \neq E_{m'} \text{ for } m \neq m',\, m,m' = 1,\dots,2n. \tag{3.32}$$

Lemma 3.2. *Suppose* (3.1) *and the nth stationary CH equation* (2.31) *holds subject to the constraint* (3.32) *on an open interval $\widetilde{\Omega}_\mu \subseteq \mathbb{R}$. Moreover, suppose that the zeros μ_j, $j = 1,\dots,n$, of $F_n(\,\cdot\,)$ remain distinct and nonzero on $\widetilde{\Omega}_\mu$. Then $\{\hat{\mu}_j\}_{j=1,\dots,n}$, defined by* (3.8), *satisfies the following first-order system of differential equations*

$$\mu_{j,x}(x) = 2\frac{y(\hat{\mu}_j(x))}{\mu_j(x)}\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}(\mu_j(x)-\mu_\ell(x))^{-1}, \quad j = 1,\dots,n,\ x \in \widetilde{\Omega}_\mu. \tag{3.33}$$

Next, assume $\mathcal{K}_n$ to be nonsingular and introduce the initial condition

$$\{\hat{\mu}_j(x_0)\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{3.34}$$

for some $x_0 \in \mathbb{R}$, where $\mu_j(x_0) \neq 0$, $j = 1,\dots,n$, are assumed to be distinct. Then there exists an open interval $\Omega_\mu \subseteq \mathbb{R}$, with $x_0 \in \Omega_\mu$, such that the initial value problem (3.33), (3.34) *has a unique solution $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1,\dots,n, \tag{3.35}$$

and μ_j, $j = 1,\dots,n$, remain distinct and nonzero on Ω_μ.
For the zeros $\{\nu_j\}_{j=1,\dots,n}$ of $H_n(\,\cdot\,)$ similar statements hold with μ_j and Ω_μ replaced by ν_j and Ω_ν, etc. In particular, $\{\hat{\nu}_j\}_{j=1,\dots,n}$, defined by (3.9), *satisfies the system*

$$\nu_{j,x}(x) = 2\frac{(4u(x)-u_{xx}(x))y(\hat{\nu}_j(x))}{(4u(x)+2u_x(x))\nu_j(x)}\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}(\nu_j(x)-\nu_\ell(x))^{-1}, \tag{3.36}$$
$$j = 1,\dots,n,\ x \in \Omega_\nu.$$

Proof. We only prove equation (3.33) since the proof of (3.36) follows in an identical manner. Inserting $z=\mu_j$ into equation (2.18), one concludes from (3.8),

$$F_{n,x}(\mu_j)=-\mu_{j,x}\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}(\mu_j-\mu_\ell)=2G_n(\mu_j)=-2y(\hat\mu_j)/\mu_j, \tag{3.37}$$

proving (3.33). The smoothness assertion (3.35) is clear as long as $\hat\mu_j$ stays away from the branch points $(E_m,0)$. In case $\hat\mu_j$ hits such a branch point, one can use the local chart around $(E_m,0)$ (with local coordinate $\zeta=\sigma(z-E_m)^{1/2}$, $\sigma\in\{1,-1\}$) to verify (3.35). □

Combining the polynomial approach in Section 2 with (3.7) readily yields trace formulas for the CH invariants. For simplicity we just record the simplest case.

Lemma 3.3. *Suppose* (3.1)*, assume the nth stationary CH equation* (2.31) *holds, and let $x\in\mathbb{R}$. Then*

$$u(x)=\frac{1}{2}\sum_{j=1}^{n}\mu_j(x)-\frac{1}{4}\sum_{m=1}^{2n+1}E_m. \tag{3.38}$$

Proof. Equation (3.38) follows by considering the coefficient of z^{n-1} in F_n in (2.25) which yields

$$u=\frac{1}{2}\sum_{j=1}^{n}\mu_j+\frac{c_1}{2}. \tag{3.39}$$

The constant c_1 can be determined by considering the coefficient of the term z^{2n+1} in (2.24), which results in

$$c_1=-\frac{1}{2}\sum_{m=0}^{2n+1}E_m. \tag{3.40}$$

□

Next we turn to asymptotic properties of ϕ and ψ_j, $j=1,2$.

Lemma 3.4. *Suppose* (3.1)*, assume the nth stationary CH equation* (2.31) *holds, and let $P=(z,y)\in\mathcal{K}_n\setminus\{P_{\infty_+},P_{\infty_-},P_0\}$, $x\in\mathbb{R}$. Then*

$$\phi(P,x)\underset{\zeta\to0}{=}\begin{cases}-2\zeta^{-1}-2u(x)+u_x(x)+O(\zeta), & P\to P_{\infty_+},\\ 2u(x)+u_x(x)+O(\zeta), & P\to P_{\infty_-},\end{cases}\quad \zeta=z^{-1}, \tag{3.41}$$

$$\phi(P,x)\underset{\zeta\to0}{=}\frac{\left(\prod_{m=1}^{2n+1}E_m\right)^{1/2}}{f_n(x)}\zeta+O(\zeta^2),\quad P\to P_0,\quad \zeta=z^{1/2}, \tag{3.42}$$

and

$$\psi_1(P,x,x_0)\underset{\zeta\to0}{=}\exp(\pm(x-x_0))(1+O(\zeta)),\quad P\to P_{\infty_\pm},\quad \zeta=1/z, \tag{3.43}$$

$$\psi_2(P,x,x_0)\underset{\zeta\to0}{=}\exp(\pm(x-x_0))\begin{cases}-2+O(\zeta), & P\to P_{\infty_+},\\ (2u(x)+u_x(x))\zeta+O(\zeta^2), & P\to P_{\infty_-},\end{cases} \tag{3.44}$$
$$\zeta=1/z,$$

$$\psi_1(P,x,x_0) \underset{\zeta\to 0}{=} \exp\bigg(-\frac{1}{\zeta}\int_{x_0}^{x} dx' \frac{\big(\prod_{m=1}^{2n+1} E_m\big)^{1/2}}{f_n(x')} + O(1)\bigg), \quad P \to P_0, \quad \zeta = z^{1/2}, \tag{3.45}$$

$$\psi_2(P,x,x_0) \underset{\zeta\to 0}{=} O(\zeta^{-1}) \exp\bigg(-\frac{1}{\zeta}\int_{x_0}^{x} dx' \frac{\big(\prod_{m=1}^{2n+1} E_m\big)^{1/2}}{f_n(x')} + O(1)\bigg), \quad P \to P_0, \quad \zeta = z^{1/2}. \tag{3.46}$$

Proof. The existence of the asymptotic expansions of ϕ in terms of the appropriate local coordinates $\zeta = 1/z$ near $P_{\infty_\pm}$ and $\zeta = z^{1/2}$ near P_0 is clear from the explicit form of ϕ in (3.13). Insertion of the polynomials F_n, G_n, and H_n into (3.13) then, in principle, yields the explicit expansion coefficients in (3.41) and (3.42). However, a more efficient way to compute these coefficients consists in utilizing the Riccati-type equation (3.20). Indeed, inserting the ansatz

$$\phi \underset{z\to\infty}{=} \phi_1 z + \phi_0 + O(z^{-1}) \tag{3.47}$$

into (3.20) and comparing the leading powers of $1/z$ immediately yields the first line in (3.41). Similarly, the ansatz

$$\phi \underset{z\to\infty}{=} \phi_0 + \phi_1 z^{-1} + O(z^{-2}) \tag{3.48}$$

inserted into (3.20) then yields the second line in (3.41). Finally, the ansatz

$$\phi \underset{z\to 0}{=} \phi_1 z^{1/2} + \phi_2 z + O(z^{3/2}) \tag{3.49}$$

inserted into (3.20) yields (3.42). (3.43)–(3.46) then follow from (3.18), (3.19), (3.41), and (3.42). □

Remark 3.5. We note the unusual fact that P_0, as opposed to $P_{\infty_\pm}$, is the essential singularity of ψ_j, $j = 1, 2$. What makes matters worse is the intricate x-dependence of the leading-order exponential term in ψ_j, $j = 1, 2$, near P_0, as displayed in (3.45), (3.46). This is in sharp contrast to standard Baker-Akhiezer functions that feature a linear behavior with respect to x in connection with their essential singularities of the type $\exp\big(c(x - x_0)\zeta^{-1}\big)$ near $\zeta = 0$.

Introducing

$$\underline{\widehat{B}}_{Q_0} \colon \widehat{\mathcal{K}}_n \setminus \{P_{\infty_+}, P_{\infty_-}\} \to \mathbb{C}^n, \tag{3.50}$$
$$P \mapsto \underline{\widehat{B}}_{Q_0}(P) = \big(\widehat{B}_{Q_0,1}, \dots, \widehat{B}_{Q_0,n}\big) = \bigg(\int_{Q_0}^{P} \eta_2, \dots, \int_{Q_0}^{P} \eta_n, \int_{Q_0}^{P} \widetilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg)$$

and

$$\underline{\widehat{\beta}}_{Q_0} \colon \sigma^n\big(\widehat{\mathcal{K}}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}\big) \to \mathbb{C}^n, \tag{3.51}$$
$$\mathcal{D}_{\underline{Q}} \mapsto \underline{\widehat{\beta}}_{Q_0}(\mathcal{D}_{Q_0}) = \sum_{j=1}^{n} \underline{\widehat{B}}_{Q_0}(Q_j), \quad \underline{Q} = \{Q_1, \dots, Q_n\} \in \sigma^n\widehat{\mathcal{K}}_n \setminus \{P_{\infty_+}, P_{\infty_-}\},$$

choosing identical paths of integration from Q_0 to P in all integrals in (3.50) and (3.51), one obtains the following result, which indicates a characteristic difference between the CH hierarchy and other completely integrable systems such as the KdV and AKNS hierarchies.

Lemma 3.6. *Assume* (3.32) *and suppose that* $\{\hat\mu_j\}_{j=1,\dots,n}$ *satisfies the stationary Dubrovin equations* (3.33) *on an open interval* $\Omega_\mu \subseteq \mathbb{R}$ *such that* μ_j, $j=1,\dots,n$, *remain distinct and nonzero on* Ω_μ. *Introducing the associated divisor* $\mathcal{D}_{\hat{\underline{\mu}}} \in \sigma^n \widehat{\mathcal{K}}_n$, $\hat{\underline{\mu}} = \{\hat\mu_1,\dots,\hat\mu_n\} \in \sigma^n\widehat{\mathcal{K}}_n$, *one computes*

$$\frac{d}{dx}\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}(x)}) = -\frac{2}{\Psi_n(\underline{\mu}(x))}\underline{c}(1), \quad x\in\Omega_\mu. \tag{3.52}$$

In particular, the Abel map does not linearize the divisor $\mathcal{D}_{\hat{\underline{\mu}}(\,\cdot\,)}$ *on* Ω_μ. *In addition,*

$$\frac{d}{dx}\sum_{j=1}^n \int_{Q_0}^{\hat\mu_j(x)} \eta_1 = -\frac{2}{\Psi_n(\underline{\mu}(x))}, \quad x\in\Omega_\mu, \tag{3.53}$$

$$\frac{d}{dx}\underline{\widehat{\beta}}(\mathcal{D}_{\hat{\underline{\mu}}(x)}) = 2(0,\dots,0,1), \quad x\in\Omega_\mu. \tag{3.54}$$

Proof. Let $x\in\Omega_\mu$. Then, using

$$\frac{1}{\mu_j} = \frac{\prod_{\substack{p=1\\p\neq j}}^n \mu_p}{\prod_{m=1}^n \mu_m} = -\frac{\Phi_{n-1}^{(j)}(\underline{\mu})}{\Psi_n(\underline{\mu})}, \quad j=1,\dots,n, \tag{3.55}$$

(cf. (C.3), (C.4)) one obtains

$$\begin{aligned}\frac{d}{dx}\bigg(\sum_{j=1}^n\int_{Q_0}^{\hat\mu_j}\underline{\omega}\bigg) &= \sum_{j=1}^n \mu_{j,x}\sum_{k=1}^n \underline{c}(k)\frac{\mu_j^{k-1}}{y(\hat\mu_j)} = 2\sum_{j=1}^n\sum_{k=1}^n \underline{c}(k)\frac{\mu_j^{k-2}}{\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\mu_j-\mu_\ell)}\\
&= -\frac{2}{\Psi_n(\underline{\mu})}\sum_{j=1}^n\sum_{k=1}^n \underline{c}(k)\frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\mu_j-\mu_\ell)}\Phi_{n-1}^{(j)}(\underline{\mu})\\
&= -\frac{2}{\Psi_n(\underline{\mu})}\sum_{j=1}^n\sum_{k=1}^n \underline{c}(k)(U_n(\underline{\mu}))_{k,j}(U_n(\underline{\mu}))^{-1}_{j,1}\\
&= -\frac{2}{\Psi_n(\underline{\mu})}\sum_{k=1}^n \underline{c}(k)\delta_{k,1} = -\frac{2}{\Psi_n(\underline{\mu})}\underline{c}(1),\end{aligned} \tag{3.56}$$

using (C.14) and (C.15). (3.53) is just a special case of (3.52) and (3.54) follows as in (3.56) using (C.10). ☐

The analogous results hold for the corresponding divisor $\mathcal{D}_{\hat{\underline{\nu}}(x)}$ associated with $\phi(\,\cdot\,,x)$.

The fact that the Abel map does not provide the proper change of variables to linearize the divisor $\mathcal{D}_{\hat{\underline{\mu}}(x)}$ in the CH context is in sharp contrast to standard integrable soliton equations such as the KdV and AKNS hierarchies (cf. also Remark 3.5). The change of variables

$$x \mapsto \tilde{x} = \int^x dx'\,\Psi_n(\underline{\mu}(x'))^{-1} \tag{3.57}$$

linearizes the Abel map $\underline{A}_{Q_0}(\mathcal{D}_{\hat{\underline{\tilde\mu}}(\tilde x)})$, $\tilde\mu_j(\tilde x) = \mu_j(x)$, $j=1,\dots,n$. These facts are well-known and discussed (by different methods) by Constantin and McKean [29], Alber [1], Alber, Camassa, Fedorov, Holm, and Marsden [3], and Alber and Fedorov [7], [8]. The intricate relation between the variables x and $\tilde x$ is detailed in (3.70). Our approach follows a route similar to Novikov's treatment of the Dym equation [55].

Next we turn to representations of ϕ and u in terms of the Riemann theta function associated with $\mathcal{K}_n$, assuming $\mathcal{K}_n$ to be nonsingular. In the following, the notation established in Appendices A–C will be freely employed. In fact, given the preparatory work collected in Appendices A–C, the proof of Theorem 3.7 below will be reduced to a few lines.

We choose a fixed base point Q_0 on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_0\}$. Let $\omega^{(3)}_{P_{\infty_+},P_0}$ be a normal differential of the third kind holomorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_0\}$ with simple poles at P_∞ and P_0 and residues 1 and -1, respectively (cf. (A.22)–(A.27)),

$$\omega^{(3)}_{P_{\infty_+},P_0} = \frac{\prod_{j=1}^n (z-\lambda_j)dz}{y}, \tag{3.58}$$

$$\omega^{(3)}_{P_{\infty_+},P_0} \underset{\zeta\to 0}{=} (\zeta^{-1} + O(1))d\zeta \text{ as } P \to P_{\infty_+}, \tag{3.59}$$

$$\omega^{(3)}_{P_{\infty_+},P_0} \underset{\zeta\to 0}{=} (-\zeta^{-1} + O(1))d\zeta \text{ as } P \to P_0, \tag{3.60}$$

where

$$\zeta = 1/z \text{ for } P \text{ near } P_{\infty_+}, \quad \zeta = \sigma z^{1/2} \text{ for } P \text{ near } P_0,\ \sigma \in \{1,-1\}. \tag{3.61}$$

Moreover,

$$\int_{a_j} \omega^{(3)}_{P_{\infty_+},P_0} = 0, \quad j = 1,\dots,n, \tag{3.62}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{\infty_+},P_0} \underset{\zeta\to 0}{=} \ln(\zeta) + e_0 + O(\zeta) \text{ as } P \to P_{\infty_+}, \tag{3.63}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{\infty_+},P_0} \underset{\zeta\to 0}{=} -\ln(\zeta) + d_0 + O(\zeta) \text{ as } P \to P_0 \tag{3.64}$$

for some constants $e_0, d_0 \in \mathbb{C}$. We also record

$$\underline{A}_{Q_0}(P) - \underline{A}_{Q_0}(P_{\infty_\pm}) \underset{\zeta\to 0}{=} \pm \underline{U}\zeta + O(\zeta^2) \text{ as } P \to P_{\infty_\pm}, \quad \underline{U} = \underline{c}(n). \tag{3.65}$$

In the following it will be convenient to introduce the abbreviations

$$\underline{z}(P,\underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \quad P \in \mathcal{K}_n,\ \underline{Q} = \{Q_1,\dots,Q_n\} \in \sigma^n\mathcal{K}_n, \tag{3.66}$$

and analogously,

$$\hat{\underline{z}}(P,\underline{Q}) = \hat{\underline{\Xi}}_{Q_0} - \hat{\underline{A}}_{Q_0}(P) + \hat{\underline{\alpha}}_{Q_0}(\mathcal{D}_{\underline{Q}}), \quad P \in \hat{\mathcal{K}}_n,\ \underline{Q} = \{Q_1,\dots,Q_n\} \in \sigma^n\hat{\mathcal{K}}_n. \tag{3.67}$$

Theorem 3.7. *Suppose $u \in C^\infty(\Omega)$, $u^{(m)} \in L^\infty(\Omega)$, $m \in \mathbb{N}_0$, and assume the nth stationary CH equation* (2.31) *holds on Ω subject to the constraint* (3.32). *Moreover, let $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_0\}$ and $x \in \Omega$, where $\Omega \subseteq \mathbb{R}$ is an open interval. In addition, suppose that $\mathcal{D}_{\hat{\underline{\mu}}(x)}$, or equivalently, $\mathcal{D}_{\hat{\underline{\nu}}(x)}$, is nonspecial for $x \in \Omega$. Then ϕ and u admit the representations*

$$\phi(P,x) = -2\frac{\theta(\underline{z}(P_{\infty_+},\hat{\underline{\mu}}(x)))\theta(\underline{z}(P,\hat{\underline{\nu}}(x)))}{\theta(\underline{z}(P_{\infty_+},\hat{\underline{\nu}}(x)))\theta(\underline{z}(P,\hat{\underline{\mu}}(x)))}\exp\bigg(-\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_+},P_0} + e_0\bigg), \tag{3.68}$$

$$u(x) = \frac{1}{2}\sum_{j=1}^n \lambda_j - \frac{1}{4}\sum_{m=1}^{2n+1} E_m + \frac{1}{2}\sum_{j=1}^n U_j \frac{\partial}{\partial w_j}\ln\bigg(\frac{\theta(\underline{z}(P_{\infty_+},\hat{\underline{\mu}}(x)) + \underline{w})}{\theta(\underline{z}(P_{\infty_-},\hat{\underline{\mu}}(x)) + \underline{w})}\bigg)\bigg|_{\underline{w}=0}. \tag{3.69}$$

Moreover, let $\widetilde{\Omega} \subseteq \Omega$ be such that μ_j, $j = 1, \dots, n$, are nonvanishing on $\widetilde{\Omega}$, Then, the constraint

$$\begin{aligned} 2(x - x_0) = &-2 \int_{x_0}^{x} \frac{dx'}{\prod_{k=1}^{n} \mu_k(x')} \sum_{j=1}^{n} \bigg(\int_{a_j} \widetilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}} \bigg) c_j(1) \\ &+ \ln \bigg(\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x)))\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}(x_0)))}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}(x)))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x_0)))} \bigg), \quad x, x_0 \in \widetilde{\Omega} \end{aligned} \tag{3.70}$$

holds, with

$$\begin{aligned} \underline{\hat{z}}(P_{\infty_\pm}, \underline{\hat{\mu}}(x))) &= \underline{\widehat{\Xi}}_{Q_0} - \underline{\widehat{A}}_{Q_0}(P_{\infty_\pm}) + \underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) \\ &= \underline{\widehat{\Xi}}_{Q_0} - \underline{\widehat{A}}_{Q_0}(P_{\infty_\pm}) + \underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) - 2 \int_{x_0}^{x} \frac{dx'}{\Psi_n(\underline{\mu}(x'))} \underline{c}(1), \\ &\qquad x \in \widetilde{\Omega}. \end{aligned} \tag{3.71}$$

Proof. First we temporarily assume that

$$\mu_j(x) \neq \mu_{j'}(x),\ \nu_k(x) \neq \nu_{k'}(x) \text{ for } j \neq j',\ k \neq k' \text{ and } x \in \widetilde{\Omega} \subseteq \Omega, \tag{3.72}$$

where $\widetilde{\Omega}$ is open. Since by (3.15), $\mathcal{D}_{P_0 \underline{\hat{\nu}}(x)} \sim \mathcal{D}_{P_{\infty_+} \underline{\hat{\mu}}(x)}$, and $P_{\infty_-} = (P_{\infty_+})^* \notin \{\hat{\mu}_1(x), \dots, \hat{\mu}_n(x)\}$ by hypothesis, one can apply Theorem A.6 to conclude that $\mathcal{D}_{\underline{\hat{\nu}}(x)} \in \sigma^n \mathcal{K}_n$ is nonspecial. This argument is of course symmetric with respect to $\underline{\hat{\mu}}(x)$ and $\underline{\hat{\nu}}(x)$. Thus, $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ is nonspecial if and only if $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ is. The representation (3.68) for ϕ, subject to (3.72), then follows by combining (3.15), (3.41), (3.42), and Theorem A.5 since $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ are nonspecial. The representation (3.69) for u on $\widetilde{\Omega}$ follows from the trace formula (3.38) and (C.46) (taking $k = 1$). By continuity, (3.68) and (3.69) extend from $\widetilde{\Omega}$ to Ω. Assuming $\mu_j \neq 0$, $j = 1, \dots, n$, in addition to (3.72), the constraint (3.70) follows by combining (3.53), (3.54), and (C.45). Equation (3.71) is clear from (3.52). Again the extra assumption (3.72) can be removed by continuity and hence (3.70) and (3.71) extend to $\widetilde{\Omega}$. □

Remark 3.8. While the stationary CH solution u in (3.69) is of course a meromorphic quasi-periodic function with respect to the new variable $\tilde{x}$ in (3.57), u may exhibit a rather intricate behavior with respect to the original variable x. Generically, u has an infinite number of branchpoints of the type

$$u(x) \underset{x \to x_0}{=} O((x - x_0)^{2/3}) \tag{3.73}$$

and

$$\tilde{x} - \tilde{x}_0 \underset{x \to x_0}{=} O((x - x_0)^{1/3}). \tag{3.74}$$

Moreover, real-valued bounded stationary CH solutions fall into two categories and are either smooth quasi-periodic functions in x, or else (3.73) and (3.74) hold at infinitely many points (depending on whether or not $\Psi_n(\underline{\mu})$ is zero-free, cf. (3.57)), as discussed in [3], [7], [8]). We note that (3.70) relates the variables x and $\tilde{x}$.

Remark 3.9. We emphasized in Remark 3.5 that Ψ in (3.17)–(3.19) markedly differs from standard Baker-Akhiezer vectors. Hence one cannot expect the usual theta function representation of ψ_j, $j = 1, 2$, in terms of ratios of theta functions times an exponential term containing a meromorphic differential with a pole at the essential singularity of ψ_j multiplied by $(x-x_0)$. However, combining (C.7) and (C.46), one computes

$$\begin{aligned} F_n(z,x) &= z^n + \sum_{\ell=0}^{n-1} \Psi_{n-\ell}(\underline{\lambda}) z^\ell \\ &= z^n + \sum_{k=1}^{n} \bigg(\Psi_{n+1-k}(\underline{\mu}(x)) \\ &\qquad\qquad - \sum_{j=1}^{n} c_j(k) \frac{\partial}{\partial w_j} \ln\bigg(\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x)) + \underline{w})}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}(x)) + \underline{w})} \bigg)\bigg|_{\underline{w}=0} z^{k-1} \bigg) \\ &= \prod_{j=1}^{n} (z-\lambda_j) - \sum_{j=1}^{n} \sum_{k=1}^{n} c_j(k) \frac{\partial}{\partial w_j} \ln\bigg(\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x)) + \underline{w})}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}(x)) + \underline{w})} \bigg)\bigg|_{\underline{w}=0} z^{k-1}, \end{aligned} \tag{3.75}$$

and hence obtains a theta function representation of ψ_1 upon inserting (3.75) into (3.26). The corresponding theta function representation of ψ_2 is then clear from (3.19) and (3.68).

Next we briefly consider the trivial case $n=0$ excluded in Theorem 3.7.

Example 3.10. Assume $n=0$, $P=(z,y)\in\mathcal{K}_0\setminus\{P_{\infty_+},P_{\infty_-},P_0\}$, and let $(x,x_0)\in\mathbb{R}^2$. Then

$$\begin{aligned} &\mathcal{K}_0 \colon \mathcal{F}_0(z,y) = y^2 - R_2(z) = y^2 - z(z-E_1) = 0, \quad E_0 = 0,\, E_1 \in \mathbb{C}, \\ &u(x) = -E_1/4, \\ &\phi(P,x) = y - z = -\frac{E_1 z}{y+z}, \\ &\psi_1(P,x,x_0) = \exp(-(y/z)(x-x_0)), \\ &\psi_2(P,x,x_0) = (1-(y/z))\exp(-(y/z)(x-x_0)). \end{aligned} \tag{3.76}$$

Actually, the general solution of $\text{s-CH}_0(u) = u_{xxx} - 4u_x = 0$ is given by

$$u(x) = a_1 e^{2x} + a_2 e^{-2x} - (E_1/4), \quad a_j \in \mathbb{C},\ j=1,2. \tag{3.77}$$

However, the requirement $u^{(m)} \in L^\infty(\mathbb{R})$, $m\in\mathbb{N}_0$, according to (3.1), necessitates the choice $a_1 = a_2 = 0$ and hence leads to (3.76). The latter corresponds to the trace formula (3.38) in the special case $n=0$.

Finally, we will show that solvability of the Dubrovin equations (3.33) on $\Omega_\mu \subseteq \mathbb{R}$ in fact implies equation (2.31) on Ω_μ.

Theorem 3.11. *Fix $n\in\mathbb{N}$, assume (3.32), and suppose that $\{\hat{\mu}_j\}_{j=1,\dots,n}$ satisfies the stationary Dubrovin equations (3.33) on an open interval $\Omega_\mu\subseteq\mathbb{R}$ such that μ_j, $j=1,\dots,n$, remain distinct and nonzero on Ω_μ. Then $u\in C^\infty(\Omega_\mu)$ defined by*

$$u(x) = \frac{1}{2}\sum_{j=1}^{n} \mu_j(x) - \frac{1}{4}\sum_{m=1}^{2n+1} E_m \tag{3.78}$$

satisfies the nth stationary CH equation (2.31), that is,

$$\text{s-CH}_n(u) = 0 \ \text{on } \Omega_\mu. \tag{3.79}$$

Proof. Given the solutions $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$, $j = 1, \dots, n$ of (3.33) we introduce

$$F_n(z) = \prod_{j=1}^{n} (z - \mu_j), \tag{3.80}$$

$$G_n(z) = F_n(z) + \frac{1}{2} F_{n,x}(z) \tag{3.81}$$

on $\mathbb{C} \times \Omega_\mu$. The Dubrovin equations imply

$$y(\hat{\mu}_j) = \frac{1}{2} \mu_j \mu_{j,x} \prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} (\mu_j - \mu_\ell) = -\frac{1}{2} \mu_j F_{n,x}(\mu_j) = -\mu_j G_n(\mu_j). \tag{3.82}$$

Thus

$$R_{2n+2}(\mu_j) - \mu_j^2 G_n(\mu_j)^2 = 0, \quad j = 1, \dots, n. \tag{3.83}$$

Furthermore $R_{2n+2}(0) = 0$, and hence there exists a polynomial H_n such that

$$R_{2n+2}(z) - z^2 G_n(z)^2 = z F_n(z) H_n(z). \tag{3.84}$$

Computing the coefficient of the term z^{2n+1} in (3.84) one finds

$$H_n(z) = (4u + 2u_x) z^n + O(z^{n-1}) \text{ as } |z| \to \infty. \tag{3.85}$$

Next, one defines a polynomial P_{n-1} by

$$P_{n-1}(z) = (4u - u_{xx}) F_n(z) - H_n(z) - z G_{n,x}(z). \tag{3.86}$$

Using (3.78), (3.80), (3.81), and (3.85) one infers that indeed P_{n-1} has degree at most $n-1$. Multiplying (3.86) by G_n, and replacing the term $G_n G_{n,x}$ with the result obtained upon differentiating (3.84) with respect to x, yields

$$\begin{aligned} G_n(z) P_{n-1}(z) = {} & F_n(z) \big((4u - u_{xx}) G_n(z) + \frac{1}{2} H_{n,x}(z) \big) \\ & + \big(\frac{1}{2} F_{n,x}(z) - G_n(z) \big) H_n(z), \end{aligned} \tag{3.87}$$

and hence

$$G_n(\mu_j) P_{n-1}(\mu_j) = 0, \quad j = 1, \dots, n \tag{3.88}$$

on Ω_μ. Restricting $x \in \Omega_\mu$ temporarily to $x \in \widetilde{\Omega}_\mu$, where

$$\begin{aligned} \widetilde{\Omega}_\mu & = \{ x \in \Omega_\mu \mid F_{n,x}(\mu_j(x), x) = 2iy(\hat{\mu}_j(x))/\mu_j(x) \neq 0, \, j = 1, \dots, n \} \\ & = \{ x \in \Omega_\mu \mid \mu_j(x) \notin \{E_0, \dots, E_{2n}\}, \, j = 1, \dots, n \} \end{aligned} \tag{3.89}$$

one infers that

$$P_{n-1}(\mu_j) = 0, \quad j = 1, \dots, n \tag{3.90}$$

on $\mathbb{C} \times \widetilde{\Omega}_\mu$. Since $P_{n-1}(z)$ has degree at most $n-1$, one concludes

$$P_{n-1} = 0 \text{ on } \mathbb{C} \times \widetilde{\Omega}_\mu, \tag{3.91}$$

and hence (2.19), that is,

$$z G_{n,x}(z) = (4u - u_{xx}) F_n(z) - H_n(z) \tag{3.92}$$

on $\mathbb{C} \times \widetilde{\Omega}_\mu$. Differentiating (3.84) with respect to x and using equations (3.92) and (3.81) one finds

$$H_{n,x}(z) = 2F_n(z) - 2(4u - u_{xx})G_n(z) \tag{3.93}$$

on $\mathbb{C} \times \widetilde{\Omega}_\mu$. In order to extend these results to Ω_μ one must prove that $\hat{\mu}_j$ does not pause once it hits $(E_m, 0)$. Hence we suppose

$$\mu_{j_0}(x) \to E_{m_0} \text{ as } x \to x_0 \in \Omega_\mu, \tag{3.94}$$

for some $j_0 \in \{1, \dots, n\}$, $m_0 \in \{0, \dots, 2n\}$. Introducing

$$\zeta_{j_0}(x) = \sigma(\mu_{j_0}(x) - E_{m_0})^{1/2}, \quad \sigma \in \{1, -1\}, \quad \mu_{j_0}(x) = E_{m_0} + \zeta_{j_0}(x)^2, \tag{3.95}$$

for some x in an open interval centered around x_0, the Dubrovin equation (3.33) for μ_{j_0} becomes

$$\zeta_{j_0,x}(x) \underset{x \to x_0}{=} -2i\sigma \Bigg(\prod_{\substack{m=0 \\ m \neq m_0}}^{2n} (E_{m_0} - E_m) \Bigg)^{1/2} \prod_{\substack{k=1 \\ k \neq j_0}}^{n} (E_{m_0} - \mu_k(x))\big(1 + O\big(\zeta_{j_0}(x)^2\big)\big). \tag{3.96}$$

Thus, $\hat{\mu}_{j_0}(x)$ does not pause at $(E_{m_0}, 0)$ for x in a small interval centered around x_0, and hence relations (3.91)–(3.93) extend to Ω_μ. We have now established relations (2.18)–(2.20) on $\mathbb{C} \times \Omega_\mu$, and one can now proceed as in Section 2 to obtain (3.79). $\square$

4. The time-dependent CH formalism

In this section we extend the algebro-geometric formalism of Section 3 to the time-dependent CH hierarchy. For most of this section we will assume the following hypothesis.

Hypothesis 4.1. *Suppose that* $u\colon \mathbb{R}^2 \to \mathbb{C}$ *satisfies*

$$\begin{aligned} &u(\,\cdot\,, t) \in C^\infty(\mathbb{R}), \ \frac{\partial^m u}{\partial x^m}(\,\cdot\,, t) \in L^\infty(\mathbb{R}), \ m \in \mathbb{N}_0, t \in \mathbb{R}, \\ &u(x, \,\cdot\,), u_{xx}(x, \,\cdot\,) \in C^1(\mathbb{R}), \ x \in \mathbb{R}. \end{aligned} \tag{4.1}$$

The basic problem in the analysis of algebro-geometric solutions of the CH hierarchy consists in solving the time-dependent rth CH flow with initial data a stationary solution of the nth equation in the hierarchy. More precisely, given $n \in \mathbb{N}_0$, consider a solution $u^{(0)}$ of the nth stationary CH equation s-$\mathrm{CH}_n(u^{(0)}) = 0$ associated with $\mathcal{K}_n$ and a given set of integration constants $\{c_\ell\}_{\ell=1,\dots,n} \subset \mathbb{C}$. Next, let $r \in \mathbb{N}_0$; we intend to construct a solution u of the rth CH flow $\mathrm{CH}_r(u) = 0$ with $u(t_{0,r}) = u^{(0)}$ for some $t_{0,r} \in \mathbb{R}$. To emphasize that the integration constants in the definitions of the stationary and the time-dependent CH equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation $\widetilde{V}_r$, $\widetilde{F}_r$, $\widetilde{G}_r$, $\widetilde{H}_r$, $\tilde{f}_s$, $\tilde{g}_s$, $\tilde{h}_s$, $\tilde{c}_s$, etc., in order to distinguish it from V_n, F_n, G_n, H_n, f_ℓ, g_ℓ, h_ℓ, c_ℓ, etc., in the following. In addition, we will follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual rth CH flow by a separate time variable $t_r \in \mathbb{R}$.

Summing up, we are seeking a solution u of

$$\begin{aligned} &\mathrm{CH}_r(u) = 4u_{t_r} - u_{xxt_r} + (u_{xxx} - 4u_x)\tilde{f}_r - 2(4u - u_{xx})\tilde{f}_{r,x} = 0, \\ &u(x, t_{0,r}) = u^{(0)}(x), \quad x \in \mathbb{R}, \end{aligned} \tag{4.2}$$

$$\text{s-CH}_n(u^{(0)}) = (u_{xxx} - 4u_x)f_n - 2(4u - u_{xx})f_{n,x} = 0, \tag{4.3}$$

for some $t_{0,r} \in \mathbb{R}$, $n, r \in \mathbb{N}_0$, where u satisfies (4.1). Actually, relying on the isospectral property of the CH flows, we will go a step further and assume (4.3) not only at $t_r = t_{0,r}$ but for all $t_r \in \mathbb{R}$. Hence, we start with

$$U_{t_r}(z,x,t_r) - \widetilde{V}_{r,x}(z,x,t_r) + [U(z,x,t_r), \widetilde{V}_r(z,x,t_r)] = 0, \quad (z,x,t_r) \in \mathbb{C} \times \mathbb{R}^2, \tag{4.4}$$

$$-V_{n,x}(z,x,t_r) + [U(z,x,t_r), V_n(z,x,t_r)] = 0, \quad (z,x,t_r) \in \mathbb{C} \times \mathbb{R}^2, \tag{4.5}$$

where (cf. (2.25)–(2.27))

$$\begin{aligned} U(z,x,t_r) &= \begin{pmatrix} -1 & 1 \\ z^{-1}(4u(x,t_r) - u_{xx}(x,t_r)) & 1 \end{pmatrix}, \\ \widetilde{V}_r(z,x,t_r) &= \begin{pmatrix} -\widetilde{G}_r(z,x,t_r) & \widetilde{F}_r(z,x,t_r) \\ z^{-1}\widetilde{H}_r(z,x,t_r) & \widetilde{G}_r(z,x,t_r) \end{pmatrix}, \\ \widetilde{V}_n(z,x,t_r) &= \begin{pmatrix} -G_n(z,x,t_r) & F_n(z,x,t_r) \\ z^{-1}H_n(z,x,t_r) & G_n(z,x,t_r) \end{pmatrix}, \end{aligned} \tag{4.6}$$

and

$$F_n(z,x,t_r) = \sum_{\ell=0}^{n} f_{n-\ell}(x,t_r)z^\ell = \prod_{j=1}^{n}(z - \mu_j(x,t_r)), \tag{4.7}$$

$$G_n(z,x,t_r) = \sum_{\ell=0}^{n} g_{n-\ell}(x,t_r)z^\ell, \tag{4.8}$$

$$H_n(z,x,t_r) = \sum_{\ell=0}^{n} h_{n-\ell}(x,t_r)z^\ell = h_0(x,t_r)\prod_{j=1}^{n}(z - \nu_j(x,t_r)), \tag{4.9}$$

$$h_0(x,t_r) = 4u(x,t_r) + 2u_x(x,t_r), \tag{4.10}$$

$$\widetilde{F}_r(z,x,t_r) = \sum_{s=0}^{r} \tilde{f}_{r-s}(x,t_r)z^s, \tag{4.11}$$

$$\widetilde{G}_r(z,x,t_r) = \sum_{s=0}^{r} \tilde{g}_{r-s}(x,t_r)z^s, \tag{4.12}$$

$$\widetilde{H}_r(z,x,t_r) = \sum_{s=0}^{r} \tilde{h}_{r-s}(x,t_r)z^s, \tag{4.13}$$

$$\tilde{h}_0(x,t_r) = 4u(x,t_r) + 2u_x(x,t_r), \tag{4.14}$$

for fixed $n, r \in \mathbb{N}_0$. Here $f_\ell(x,t_r)$, $\tilde{f}_s(x,t_r)$, $g_\ell(x,t_r)$, $\tilde{g}_s(x,t_r)$, $h_\ell(x,t_r)$, and $\tilde{h}_s(x,t_r)$ for $\ell = 0,\dots,n$, $s = 0,\dots,r$, are defined as in (2.3), (2.7), and (2.8) with $u(x)$ replaced by $u(x,t_r)$, etc., and with appropriate integration constants. Explicitly, (4.4), (4.5) are equivalent to

$$\begin{aligned} &4u_{t_r}(x,t_r) - u_{xxt_r}(x,t_r) - \widetilde{H}_{r,x}(z,x,t_r) + 2\widetilde{H}_r(z,x,t_r) \\ &-2(4u(x,t_r) - u_{xx}(x,t_r))\widetilde{G}_r(z,x,t_r) = 0, \end{aligned} \tag{4.15}$$

$$\widetilde{F}_{r,x}(z,x,t_r) = 2\widetilde{G}_r(z,x,t_r) - 2\widetilde{F}_r(z,x,t_r), \tag{4.16}$$

$$z\widetilde{G}_{r,x}(z,x,t_r) = (4u(x,t_r) - u_{xx}(x,t_r))\widetilde{F}_r(z,x,t_r) - \widetilde{H}_r(z,x,t_r) \tag{4.17}$$

and

$$F_{n,x}(z,x,t_r) = 2G_n(z,x,t_r) - 2F_n(z,x,t_r), \tag{4.18}$$

$$H_{n,x}(z,x,t_r) = 2H_n(z,x,t_r) - 2(4u(x,t_r) - u_{xx}(x,t_r))G_n(z,x,t_r), \tag{4.19}$$

$$zG_{n,x}(z,x,t_r) = (4u(x,t_r) - u_{xx}(x,t_r))F_n(z,x,t_r) - H_n(z,x,t_r), \tag{4.20}$$

First we will assume the existence of a solution of equations (4.15)–(4.20) and derive an explicit formula for u in terms of Riemann theta functions. In addition, we will show in Theorem 4.10 that (4.15)–(4.20) and hence the algebro-geometric initial value problem (4.2), (4.3) has a solution at least locally, that is, for $(x,t_r) \in \Omega$ for some open and connected set $\Omega \subseteq \mathbb{R}^2$.

One observes that equations (2.17)–(2.40) apply to F_n, G_n, H_n, f_ℓ, g_ℓ, and h_ℓ and (2.3)–(2.10), (2.25)–(2.27), with n replaced by r and c_ℓ replaced by $\tilde{c}_\ell$, apply to $\widetilde{F}_r$, $\widetilde{G}_r$, $\widetilde{H}_r$, $\tilde{f}_\ell$, $\tilde{g}_\ell$, and $\tilde{h}_\ell$. In particular, the fundamental identity (2.24) holds,

$$z^2 G_n(z,x,t_r)^2 + zF_n(z,x,t_r)H_n(z,x,t_r) = R_{2n+2}(z), \tag{4.21}$$

and the hyperelliptic curve $\mathcal{K}_n$ is still given by

$$\mathcal{K}_n\colon \mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0, \quad R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \tag{4.22}$$

assuming (3.6) for the remainder of this section, that is,

$$E_0 = 0,\ E_1, \dots, E_{2n+1} \in \mathbb{C} \setminus \{0\}. \tag{4.23}$$

In analogy to equations (3.8), (3.9) we define

$$\hat{\mu}_j(x,t_r) = (\mu_j(x,t_r), -\mu_j(x,t_r)G_n(\mu_j(x,t_r),x,t_r)) \in \mathcal{K}_n, \quad j = 1,\dots,n,\ (x,t_r) \in \mathbb{R}^2, \tag{4.24}$$

$$\hat{\nu}_j(x,t_r) = (\nu_j(x,t_r), \nu_j(x,t_r)G_n(\nu_j(x,t_r),x,t_r)) \in \mathcal{K}_n, \quad j = 1,\dots,n,\ (x,t_r) \in \mathbb{R}^2. \tag{4.25}$$

As in Section 3, the regularity assumptions (4.1) on u imply analogous regularity properties of F_n, H_n, μ_j, and ν_k.

Next, one defines the meromorphic function $\phi(\,\cdot\,,x,t_r)$ on $\mathcal{K}_n$ by

$$\phi(P,x,t_r) = \frac{y - zG_n(z,x,t_r)}{F_n(z,x,t_r)} \tag{4.26}$$

$$= \frac{zH_n(z,x,t_r)}{y + zG_n(z,x,t_r)}, \quad P = (z,y) \in \mathcal{K}_n \setminus \{P_{\infty_\pm}\},\ (x,t_r) \in \mathbb{R}^2. \tag{4.27}$$

Assuming (4.23), the divisor $(\phi(\,\cdot\,,x,t_r))$ of $\phi(\,\cdot\,,x,t_r)$ reads

$$(\phi(\,\cdot\,,x,t_r)) = \mathcal{D}_{P_0\underline{\hat{\nu}}(x,t_r)} - \mathcal{D}_{P_{\infty_+}\underline{\hat{\mu}}(x,t_r)}, \tag{4.28}$$

with

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\},\ \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \sigma^n\mathcal{K}_n. \tag{4.29}$$

The corresponding time-dependent vector Ψ,

$$\Psi(P,x,x_0,t_r,t_{0,r}) = \begin{pmatrix}\psi_1(P,x,x_0,t_r,t_{0,r})\\ \psi_2(P,x,x_0,t_r,t_{0,r})\end{pmatrix}, \quad P \in \mathcal{K}_n \setminus \{P_{\infty_\pm}\},\ (x,x_0,t_r,t_{0,r}) \in \mathbb{R}^4 \tag{4.30}$$

is defined by

$$\psi_1(P,x,x_0,t_r,t_{0,r}) = \exp\bigg(-\int_{t_{0,r}}^{t_r} ds\,\big((1/z)\widetilde F_r(z,x_0,s)\phi(P,x_0,s)+\widetilde G_r(z,x_0,s)\big) \\ -(1/z)\int_{x_0}^{x} dx'\,\phi(P,x',t_r)-(x-x_0)\bigg), \tag{4.31}$$

$$\psi_2(P,x,x_0,t_r,t_{0,r}) = -\psi_1(P,x,x_0,t_r,t_{0,r})\phi(P,x,t_r)/z. \tag{4.32}$$

The properties of ϕ can now be summarized as follows.

Lemma 4.2. *Assume Hypothesis* 4.1 *and* (4.4), (4.5). *Moreover, let* $P=(z,y)\in\mathcal{K}_n\setminus\{P_{\infty_+},P_{\infty_-},P_0\}$ *and* $(x,t_r)\in\mathbb{R}^2$. *Then* ϕ *satisfies*

$$\phi_x(P,x,t_r)-z^{-1}\phi(P,x,t_r)^2-2\phi(P,x,t_r)+4u(x,t_r)-u_{xx}(x,t_r)=0, \tag{4.33}$$

$$\begin{aligned}\phi_{t_r}(P,x,t_r) &= (4u(x,t_r)-u_{xx}(x,t_r))\widetilde F_r(z,x,t_r) \\ &\quad -\widetilde H_r(z,x,t_r)+2(\widetilde F_r(z,x,t_r)\phi(P,x,t_r))_x \qquad (4.34)\\ &= (1/z)\widetilde F_r(z,x,t_r)\phi(P,x,t_r)^2+2\widetilde G_r(z,x,t_r)\phi(P,x,t_r) \\ &\quad -\widetilde H_r(z,x,t_r), \qquad (4.35)\end{aligned}$$

$$\phi(P,x,t_r)\phi(P^*,x,t_r) = -\frac{zH_n(z,x,t_r)}{F_n(z,x,t_r)}, \tag{4.36}$$

$$\phi(P,x,t_r)+\phi(P^*,x,t_r) = -2\frac{zG_n(z,x,t_r)}{F_n(z,x,t_r)}, \tag{4.37}$$

$$\phi(P,x,t_r)-\phi(P^*,x,t_r) = \frac{2y}{F_n(z,x,t_r)}. \tag{4.38}$$

Proof. Equations (4.33) and (4.36)–(4.38) are proved as in Lemma 3.1. To prove (4.35) one first observes that

$$\big(\partial_x-((1/z)\phi+1)\big)\big(\phi_{t_r}-(1/z)\widetilde F_r\phi^2-2\widetilde G_r\phi+\widetilde H_r\big)=0 \tag{4.39}$$

using (4.33) and relations (4.15)–(4.17) repeatedly. Thus,

$$\phi_{t_r}-\frac{1}{z}\widetilde F_r\phi^2-2\widetilde G_r\phi+\widetilde H_r = C\exp\bigg(\int^x dx'\,((1/z)\phi+1)\bigg), \tag{4.40}$$

where the left-hand side is meromorphic in a neighborhood of P_{∞_-}, while the right-hand side is meromorphic near P_{∞_-} only if $C=0$. This proves (4.35).

Using (4.16) and (4.33) one obtains

$$(4u-u_{xx})\widetilde F_r+2(\widetilde F_r\phi)_x = 2\widetilde G_r\phi+(1/z)\phi^2\widetilde F_r. \tag{4.41}$$

Combining this result with (4.35) one concludes that (4.34) holds. ☐

Using relations (4.18)–(4.20) and (4.15)–(4.17), we next determine the time evolution of F_n, G_n, and H_n.

Lemma 4.3. *Assume Hypothesis* 4.1 *and* (4.4), (4.5). *In addition, let* $(z,x,t_r)\in\mathbb{C}\times\mathbb{R}^2$. *Then*

$$F_{n,t_r}(z,x,t_r) = 2(G_n(z,x,t_r)\widetilde F_r(z,x,t_r)-F_n(z,x,t_r)\widetilde G_r(z,x,t_r)), \tag{4.42}$$

$$zG_{n,t_r}(z,x,t_r) = F_n(z,x,t_r)\widetilde H_r(z,x,t_r)-H_n(z,x,t_r)\widetilde F_r(z,x,t_r), \tag{4.43}$$

$$H_{n,t_r}(z,x,t_r) = 2(H_n(z,x,t_r)\widetilde G_r(z,x,t_r)-G_n(z,x,t_r)\widetilde H_r(z,x,t_r)). \tag{4.44}$$

Equations (4.42)–(4.44) *are equivalent to*

$$V_{n,t_r}(z,x,t_r) = [\widetilde{V}_r(z,x,t_r), V_n(z,x,t_r)]. \tag{4.45}$$

Proof. We prove (4.42) by using (4.38) which shows that

$$(\phi(P) - \phi(P^*))_{t_r} = -2\frac{yF_{n,t_r}}{F_n^2}. \tag{4.46}$$

However, the left-hand side of (4.46) also equals

$$\phi(P)_{t_r} - \phi(P^*)_{t_r} = \frac{4y}{F_n^2}(\widetilde{G}_r F_n - \widetilde{F}_r G_n), \tag{4.47}$$

using (4.35), (4.37), and (4.38). Combining (4.46) and (4.47) proves (4.42). Similarly, to prove (4.43), we use (4.37) to write

$$(\phi(P) + \phi(P^*))_{t_r} = -\frac{2z}{F_n^2}\big(G_{n,t_r}F_n - G_n F_{n,t_r}\big). \tag{4.48}$$

Here the left-hand side can be expressed as

$$\phi(P)_{t_r} + \phi(P^*)_{t_r} = 2\frac{zG_n}{F_n^2}F_{n,t_r} + \frac{2}{F_n}(\widetilde{F}_r H_n - \widetilde{H}_r F_n), \tag{4.49}$$

using (4.35), (4.36), and (4.37). Combining (4.48) and (4.49), using (4.42), proves (4.43). Finally, (4.44) follows by differentiating (2.24), that is, $(zG_n)^2 + zF_nH_n = R_{2n+2}$, with respect to t_r, and using (4.42) and (4.43). ◻

Lemmas 4.2 and 4.3 permit one to characterize Ψ.

Lemma 4.4. *Assume Hypothesis* 4.1 *and* (4.4), (4.5). *Moreover, let* $P = (z,y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}$ *and* $(x, x_0, t_r, t_{0,r}) \in \mathbb{R}^4$. *Then the Baker–Akhiezer vector* Ψ *satisfies*

$$\Psi_x(P,x,x_0,t_r,t_{0,r}) = U(z,x,t_r)\Psi(P,x,x_0,t_r,t_{0,r}), \tag{4.50}$$

$$-y\Psi(P,x,x_0,t_r,t_{0,r}) = zV_n(z,x,t_r)\Psi(P,x,x_0,t_r,t_{0,r}), \tag{4.51}$$

$$\Psi_{t_r}(P,x,x_0,t_r,t_{0,r}) = \widetilde{V}_r(z,x,t_r)\Psi(P,x,x_0,t_r,t_{0,r}), \tag{4.52}$$

$$\begin{aligned}\psi_1(P,x,x_0,t_r,t_{0,r}) &= \left(\frac{F_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}\right)^{1/2} \times \\ &\times \exp\bigg(-(y/z)\int_{t_{0,r}}^{t_r} ds\,\widetilde{F}_r(z,x_0,s)F_n(z,x_0,s)^{-1} - (y/z)\int_{x_0}^{x} dx' F_n(z,x',t_r)^{-1}\bigg),\end{aligned} \tag{4.53}$$

$$\psi_1(P,x,x_0,t_r,t_{0,r})\psi_1(P^*,x,x_0,t_r,t_{0,r}) = \frac{F_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}, \tag{4.54}$$

$$\psi_2(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) = -\frac{H_n(z,x,t_r)}{zF_n(z,x_0,t_{0,r})}, \tag{4.55}$$

$$\begin{aligned}&\psi_1(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) \\ &\quad + \psi_1(P^*,x,x_0,t_r,t_{0,r})\psi_2(P,x,x_0,t_r,t_{0,r}) = 2\frac{G_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})},\end{aligned} \tag{4.56}$$

$$\begin{aligned}&\psi_1(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) \\ &\quad - \psi_1(P^*,x,x_0,t_r,t_{0,r})\psi_2(P,x,x_0,t_r,t_{0,r}) = \frac{2y}{zF_n(z,x_0,t_{0,r})}.\end{aligned} \tag{4.57}$$

In addition, as long as the zeros of $F_n(\,\cdot\,,x,t_r)$ are all simple for $(x,t_r),(x_0,t_{0,r})\in\Omega$, $\Omega\subseteq\mathbb{R}^2$ open and connected, $\Psi(\,\cdot\,,x,x_0,t_r,t_{0,r})$, $(x,t_r),(x_0,t_{0,r})\in\Omega$, is meromorphic on $\mathcal{K}_n\setminus\{P_0,P_{\infty_\pm}\}$.

Proof. By (4.31), $\psi_1(\,\cdot\,,x,x_0,t_r,t_{0,r})$ is meromorphic on $\mathcal{K}_n\setminus\{P_{\infty_\pm}\}$ away from the poles $\hat\mu_j(x_0,s)$ of $\phi(\,\cdot\,,x_0,s)$ and $\hat\mu_k(x',t_r)$ of $\phi(\,\cdot\,,x',t_r)$. That $\psi_1(\,\cdot\,,x,x_0,t_r,t_{0,r})$ is meromorphic on $\mathcal{K}_n\setminus\{P_{\infty_\pm}\}$ if $F_n(\,\cdot\,,x,t_r)$ has only simple zeros is a consequence of (cf. (3.31))

$$-\frac{1}{z}\phi(P,x',t_r)\underset{P\to\hat\mu_j(x',t_r)}{=}\frac{\partial}{\partial x'}\ln\big(F_n(z,x',t_r)\big)+O(1)\text{ as }z\to\mu_j(x',t_r),\tag{4.58}$$

and

$$-\frac{1}{z}\widetilde F_r(z,x_0,s)\phi(P,x_0,s)\underset{P\to\hat\mu_j(x_0,s)}{=}\frac{\partial}{\partial s}\ln\big(F_n(z,x_0,s)\big)+O(1)\text{ as }z\to\mu_j(x_0,s),\tag{4.59}$$

using (4.24), (4.26), and (4.42). This follows from (4.31) by restricting P to a sufficiently small neighborhood $\mathcal{U}_j(x_0)$ of $\{\hat\mu_j(x_0,s)\in\mathcal{K}_n\mid(x_0,s)\in\Omega,\ s\in[t_{0,r},t_r]\}$ such that $\hat\mu_k(x_0,s)\notin\mathcal{U}_j(x_0)$ for all $s\in[t_{0,r},t_r]$ and all $k\in\{1,\dots,n\}\setminus\{j\}$ and by simultaneously restricting P to a sufficiently small neighborhood $\mathcal{U}_j(t_r)$ of $\{\hat\mu_j(x',t_r)\in\mathcal{K}_n\mid(x',t_r)\in\Omega,\ x'\in[x_0,x]\}$ such that $\hat\mu_k(x',t_r)\notin\mathcal{U}_j(t_r)$ for all $x'\in[x_0,x]$ and all $k\in\{1,\dots,n\}\setminus\{j\}$. By (4.32) and the fact that ϕ is meromorphic on $\mathcal{K}_n$ one concludes that ψ_2 is meromorphic on $\mathcal{K}_n\setminus\{P_{\infty_\pm}\}$ as well. Relations (4.50) and (4.51) follow as in Lemma 3.1, while the time evolution (4.52) is a consequence of the definition of Ψ in (4.31), (4.32) as well as (4.35), rewriting

$$(1/z)\phi_{t_r}=\big((1/z)2\phi\widetilde F_r+\widetilde G_r\big)_x,\tag{4.60}$$

using (4.17) and (4.34). To prove (4.53) we recall the definition (4.31), that is,

$$\begin{aligned}\psi_1(P,x,x_0,t_r,t_{0,r})&=\exp\bigg(-(x-x_0)-(1/z)\int_{x_0}^x dx'\,\phi(P,x',t_r)\\&\quad-\int_{t_{0,r}}^{t_r}ds\,\big((1/z)\widetilde F_r(z,x_0,s)\phi(P,x_0,s)+\widetilde G_r(z,x_0,s)\big)\bigg)\\&=\bigg(\frac{F_n(z,x,t_r)}{F_n(z,x_0,t_r)}\bigg)^{1/2}\exp\bigg(-(y/z)\int_{x_0}^x dx'F_n(z,x',t_r)^{-1}\\&\quad-\int_{t_{0,r}}^{t_r}ds\,\big((1/z)\widetilde F_r(z,x_0,s)\phi(P,x_0,s)+\widetilde G_r(z,x_0,s)\big)\bigg),\end{aligned}\tag{4.61}$$

using the calculation leading to (3.26). Equations (4.26) and (4.42) show that

$$\frac{1}{z}\widetilde F_r(z,x_0,s)\phi(P,x_0,s)+\widetilde G_r(z,x_0,s)=\frac{y}{z}\,\frac{\widetilde F_r(z,x_0,s)}{F_n(z,x_0,s)}-\frac{1}{2}\,\frac{F_{n,t_r}(z,x_0,s)}{F_n(z,x_0,s)},\tag{4.62}$$

which inserted into (4.61) yields (4.53). Evaluating (4.53) at the points P and P^* and multiplying the resulting expressions yields (4.54). The remaining statements are direct consequences of (4.37)–(4.38) and (4.53). ☐

Next, we turn to the time evolution of the quantities μ_j and ν_j assuming (3.32), that is,

$$E_0=0,\ E_m\in\mathbb{C}\setminus\{0\},\ E_m\neq E_{m'}\text{ for }m\neq m',\ m,m'=1,\dots,2n.\tag{4.63}$$

Lemma 4.5. *Assume Hypothesis 4.1, (4.63), and (4.4), (4.5) on an open and connected set $\widetilde{\Omega}_\mu \subseteq \mathbb{R}^2$. Moreover, suppose that the zeros μ_j, $j = 1, \dots, n$, of $F_n(\,\cdot\,)$ remain distinct and nonzero on $\widetilde{\Omega}_\mu$. Then $\{\hat\mu_j\}_{j=1,\dots,n}$, defined by (4.24), satisfies the following first-order system of differential equations*

$$\mu_{j,x}(x,t_r) = 2\mu_j(x,t_r)^{-1} y(\hat\mu_j(x,t_r)) \prod_{\substack{\ell=1\\ \ell\neq j}}^n (\mu_j(x,t_r) - \mu_\ell(x,t_r))^{-1}, \tag{4.64}$$

$$\begin{aligned} \mu_{j,t_r}(x,t_r) &= 2\widetilde{F}_r(\mu_j(x,t_r),x,t_r) \times \\ &\quad \times \mu_j(x,t_r)^{-1} y(\hat\mu_j(x,t_r)) \prod_{\substack{\ell=1\\ \ell\neq j}}^n (\mu_j(x,t_r) - \mu_\ell(x,t_r))^{-1}, \\ &\qquad j = 1, \dots, n,\ (x,t_r) \in \widetilde{\Omega}_\mu. \end{aligned} \tag{4.65}$$

Next, assume $\mathcal{K}_n$ to be nonsingular and introduce the initial condition

$$\{\hat\mu_j(x_0,t_{0,r})\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{4.66}$$

for some $(x_0,t_{0,r}) \in \mathbb{R}^2$, where $\mu_j(x_0,t_{0,r}) \neq 0$, $j = 1, \dots, n$, are assumed to be distinct and $\widetilde{F}_r(\mu_j(x_0,t_{0,r}),x_0,t_{0,r}) \neq 0$, $j = 1, \dots, n$. Then there exists an open and connected set $\Omega_\mu \subseteq \mathbb{R}^2$, with $(x_0,t_{0,r}) \in \Omega_\mu$, such that the initial value problem (4.64)–(4.66) has a unique solution $\{\hat\mu_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ satisfying

$$\hat\mu_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{4.67}$$

and μ_j, $j = 1, \dots, n$, remain distinct and nonzero on Ω_μ with $\widetilde{F}_r(\mu_j) \neq 0$ on Ω_μ, $j = 1, \dots, n$.
For the zeros $\{\nu_j\}_{j=1,\dots,n}$ of $H_n(\,\cdot\,)$ similar statements hold with μ_j and Ω_μ replaced by ν_j and Ω_ν, etc. In particular, $\{\hat\nu_j\}_{j=1,\dots,n}$, defined by (4.25), satisfies the system

$$\begin{aligned} \nu_{j,x}(x,t_r) &= 2(4u(x,t_r) - u_{xx}(x,t_r))(4u(x,t_r) + 2u_x(x,t_r))^{-1} \times \\ &\quad \times \nu_j(x,t_r)^{-1} y(\hat\nu_j(x,t_r)) \prod_{\substack{\ell=1\\ \ell\neq j}}^n (\nu_j(x,t_r) - \nu_\ell(x,t_r))^{-1}, \end{aligned} \tag{4.68}$$

$$\begin{aligned} \nu_{j,t_r}(x,t_r) &= 2\widetilde{H}_r(\nu_j(x,t_r),x,t_r)(4u(x,t_r) + 2u_x(x,t_r))^{-1} \times \\ &\quad \times \nu_j(x,t_r)^{-1} y(\hat\nu_j(x,t_r)) \prod_{\substack{\ell=1\\ \ell\neq j}}^n (\nu_j(x,t_r) - \nu_\ell(x,t_r))^{-1}, \\ &\qquad j = 1, \dots, n,\ (x,t_r) \in \Omega_\nu. \end{aligned} \tag{4.69}$$

Proof. It suffices to prove (4.65) since the argument for (4.69) is analogous and that for (4.64) and (4.68) has been given in the proof of Lemma 3.2. Inserting $z = \mu_j(x,t_r)$ into (4.42), observing (4.24), yields

$$F_{n,t_r}(\mu_j) = -\mu_{j,t_r} \prod_{\substack{\ell=1\\ \ell\neq j}}^n (\mu_j - \mu_\ell) = 2\widetilde{F}_r(\mu_j) G_n(\mu_j) = -2\frac{\widetilde{F}_r(\mu_j)}{\mu_j} y(\hat\mu_j). \tag{4.70}$$

The rest is analogous to the proof of Lemma 3.2. ∎

Next we note the following trace formula, the t_r-dependent analog of (3.38).

Lemma 4.6. *Assume Hypothesis* 4.1, (4.4), (4.5), *and let* $(x,t_r)\in\mathbb{R}^2$. *Then*

$$u(x,t_r)=\frac{1}{2}\sum_{j=1}^{n}\mu_j(x,t_r)-\frac{1}{4}\sum_{m=1}^{2n+1}E_m. \tag{4.71}$$

We also record the asymptotic properties of ϕ, the analogs of (3.41) and (3.42).

Lemma 4.7. *Assume Hypothesis* 4.1, (4.4), (4.5), *and let* $P=(z,y)\in\mathcal{K}_n\setminus\{P_{\infty_+},P_{\infty_-},P_0\}$, $(x,t_r)\in\mathbb{R}^2$. *Then*

$$\phi(P,x,t_r)\underset{\zeta\to0}{=}\begin{cases}-2\zeta^{-1}-2u(x,t_r)+u_x(x,t_r)+O(\zeta), & P\to P_{\infty_+},\\ 2u(x,t_r)+u_x(x,t_r)+O(\zeta), & P\to P_{\infty_-},\end{cases}\quad \zeta=z^{-1}, \tag{4.72}$$

$$\phi(P,x,t_r)\underset{\zeta\to0}{=}\frac{\Big(\prod_{m=1}^{2n+1}E_m\Big)^{1/2}}{f_n(x,t_r)}\zeta+O(\zeta^2),\quad P\to P_0,\quad \zeta=z^{1/2}. \tag{4.73}$$

Since the proofs of Lemmas 4.6 and 4.7 are identical to the corresponding stationary results in Lemmas 3.3 and 3.4 we omit the corresponding details.

Next, recalling the definition of $\widetilde{F}_r(\mu_j)$ introduced in (C.27) and also the definition of $\widehat{\underline{B}}_{Q_0}$ and $\widehat{\underline{\beta}}_{Q_0}$ in (3.50) and (3.51), respectively, we now state the analog of Lemma 3.6, thereby underscoring the marked differences between the CH hierarchy and other completely integrable systems such as the KdV and AKNS hierarchies.

Lemma 4.8. *Assume* (4.63) *and suppose that* $\{\hat\mu_j\}_{j=1,\dots,n}$ *satisfies the Dubrovin equations* (4.64), (4.65) *on an open set* $\Omega_\mu\subseteq\mathbb{R}^2$ *such that* μ_j, $j=1,\dots,n$, *remain distinct and nonzero on* Ω_μ *and that* $\widetilde{F}_r(\mu_j)\neq0$ *on* Ω_μ, $j=1,\dots,n$. *Introducing the associated divisor* $\mathcal{D}_{\underline{\hat\mu}}\in\sigma^n\widehat{\mathcal{K}}_n$, $\underline{\hat\mu}=\{\hat\mu_1,\dots,\hat\mu_n\}\in\sigma^n\widehat{\mathcal{K}}_n$, *one computes*

$$\frac{\partial}{\partial x}\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x,t_r)})=-\frac{2}{\Psi_n(\underline{\mu}(x,t_r))}\underline{c}(1),\quad (x,t_r)\in\Omega_\mu, \tag{4.74}$$

$$\begin{aligned}\frac{\partial}{\partial t_r}\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x,t_r)})=&-\frac{2}{\Psi_n(\underline{\mu}(x,t_r))}\bigg(\sum_{k=0}^{r\wedge n}d_{r,k}(\underline{E})\Psi_k(\underline{\mu}(x,t_r))\bigg)\underline{c}(1)\\ &+2\bigg(\sum_{\ell=1\vee(n+1-r)}^{n}d_{r,n+1-\ell}(\underline{E})\underline{c}(\ell)\bigg),\quad (x,t_r)\in\Omega_\mu.\end{aligned} \tag{4.75}$$

In particular, the Abel map does not linearize the divisor $\mathcal{D}_{\underline{\hat\mu}(\,\cdot\,,\,\cdot\,)}$ *on* Ω_μ. *In addition,*

$$\frac{\partial}{\partial x}\sum_{j=1}^{n}\int_{Q_0}^{\hat\mu_j(x,t_r)}\eta_1=-\frac{2}{\Psi_n(\underline{\mu}(x,t_r))},\quad (x,t_r)\in\Omega_\mu, \tag{4.76}$$

$$\frac{\partial}{\partial x}\widehat{\underline{\beta}}(\mathcal{D}_{\underline{\hat\mu}(x,t_r)})=2(0,\dots,0,1),\quad (x,t_r)\in\Omega_\mu, \tag{4.77}$$

$$\frac{\partial}{\partial t_r}\sum_{j=1}^{n}\int_{Q_0}^{\hat\mu_j(x,t_r)}\eta_1=-\frac{2}{\Psi_n(\underline{\mu}(x,t_r))}\sum_{k=0}^{r\wedge n}d_{r,k}(\underline{E})\Psi_k(\underline{\mu}(x,t_r))+2d_{r,n}\delta_{n,r\wedge n},\quad (x,t_r)\in\Omega_\mu, \tag{4.78}$$

$$\frac{\partial}{\partial t_r}\widehat{\underline{\beta}}(\mathcal{D}_{\underline{\hat\mu}(x,t_r)})=2\bigg(\sum_{s=0}^{r}\tilde c_{r-s}c_{s+1-n}(\underline{E}),\dots,\sum_{s=0}^{r}\tilde c_{r-s}c_{s+1}(\underline{E}),\sum_{s=0}^{r}\tilde c_{r-s}c_s(\underline{E})\bigg),$$

$$c_{-\ell}(\underline{E}) = 0,\ \ell \in \mathbb{N},\ (x,t_r) \in \Omega_\mu. \tag{4.79}$$

Proof. Let $(x,t_r) \in \Omega_\mu$. Since (4.74), (4.76), and (4.77) are proved as in in the stationary context of Lemma 3.6, we focus on the proofs of (4.75), (4.78), and (4.79). Then, using (4.65), (3.55), (C.11), and (C.9), (C.14), and (C.15) one obtains

$$\begin{aligned}
\frac{\partial}{\partial t_r}\bigg(\sum_{j=1}^n \int_{Q_0}^{\hat\mu_j} \underline{\omega}\bigg) &= \sum_{j=1}^n \mu_{j,t_r} \sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-1}}{y(\hat\mu_j)} = 2\sum_{j=1}^n\sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}^n (\mu_j-\mu_\ell)} \frac{\widetilde F_r(\mu_j)}{\mu_j} \\
&= 2\sum_{j=1}^n\sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}^n (\mu_j-\mu_\ell)} \bigg(-\sum_{m=0}^{r\wedge n} d_{r,m}(\underline{E})\Psi_m(\underline{\mu}) \frac{\Phi_{n-1}^{(j)}(\underline{\mu})}{\Psi_n(\underline{\mu})} \\
&\qquad\qquad + \sum_{m=1}^{r\wedge n} d_{r,m}(\underline{E})\Phi_{m-1}^{(j)}(\underline{\mu})\bigg) \\
&= -2\sum_{m=0}^{r\wedge n} d_{r,m}(\underline{E}) \frac{\Psi_m(\underline{\mu})}{\Psi_n(\underline{\mu})} \sum_{k=1}^n\sum_{j=1}^n \underline{c}(k)(U_n(\underline{\mu}))_{k,j}(U_n(\underline{\mu}))^{-1}_{j,1} \\
&\quad + 2\sum_{m=1}^{r\wedge n} d_{r,m}(\underline{E}) \sum_{k=1}^n\sum_{j=1}^n \underline{c}(k)(U_n(\underline{\mu}))_{k,j}(U_n(\underline{\mu}))^{-1}_{j,n-m+1} \\
&= -\frac{2}{\Psi_n(\underline{\mu})}\sum_{m=0}^{r\wedge n} d_{r,m}(\underline{E})\Psi_m(\underline{\mu})\underline{c}(1) + 2\sum_{m=1}^{r\wedge n} d_{r.m}(\underline{E})\underline{c}(n-m+1) \\
&= -\frac{2}{\Psi_n(\underline{\mu})}\sum_{m=0}^{r\wedge n} d_{r,m}(\underline{E})\Psi_m(\underline{\mu})\underline{c}(1) + 2\sum_{m=1\vee(n+1-r)}^{n} d_{r,n+1-m}(\underline{E})\underline{c}(m).
\end{aligned} \tag{4.80}$$

Equation (4.78) is just a special case of (4.75) and (4.79) follows as in (4.80) using again (C.9). ☐

The analogous results hold for the corresponding divisor $\mathcal{D}_{\underline{\hat\nu}(x,t_r)}$ associated with $\phi(\,\cdot\,,x,t_r)$.

The fact that the Abel map does not effect a linearization of the divisor $\mathcal{D}_{\underline{\hat\mu}(x,t_r)}$ in the CH context is well-known and discussed (using different approaches) by Constantin and McKean [29], Alber, Camassa, Fedorov, Holm, and Marsden [3], Alber and Fedorov [7], [8]. A change of the variable t_1 in analogy to that in (3.57) in the stationary context, which avoids the use of a meromorphic differential (cf. (3.50), (3.51)) and linearizes the Abel map when considering the CH_1 flow, is discussed in [1]. That change of variables corresponds to the case $r=1$ in (4.83).

Next we turn to one of the principal results of this section, the representations of ϕ and u in terms of the Riemann theta function associated with $\mathcal{K}_n$, assuming $\mathcal{K}_n$ to be nonsingular. Recalling (3.58)–(3.67), the analog of Theorem 3.7 in the stationary case then reads as follows.

Theorem 4.9. *Suppose Hypothesis* 4.1 *and* (4.2), (4.3) *on* Ω *subject to the constraint* (4.63). *In addition, let* $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_0\}$ *and* $(x,t_r),(x_0,t_{0,r}) \in \Omega$, *where* $\Omega \subseteq \mathbb{R}^2$ *is open and connected. Moreover, suppose that* $\mathcal{D}_{\underline{\hat\mu}(x,t_r)}$, *or equivalently,*

$\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$, *is nonspecial for* $(x,t_r)\in\Omega$. *Then* ϕ *and* u *admit the representations*

$$\phi(P,x,t_r) = -2\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t_r)))\theta(\underline{z}(P,\underline{\hat{\nu}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t_r)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))}\exp\bigg(-\int_{Q_0}^P \omega^{(3)}_{P_{\infty_+},P_0}+e_0\bigg), \tag{4.81}$$

$$\begin{aligned} u(x,t_r) &= \frac{1}{2}\sum_{j=1}^n \lambda_j - \frac{1}{4}\sum_{m=1}^{2n+1} E_m \\ &\quad + \frac{1}{2}\sum_{j=1}^n U_j \frac{\partial}{\partial w_j}\ln\bigg(\frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t_r))+\underline{w}\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t_r))+\underline{w}\big)}\bigg)\bigg|_{\underline{w}=0}. \end{aligned} \tag{4.82}$$

Moreover, let $\widetilde{\Omega}\subseteq\Omega$ *be such that* μ_j, $j=1,\dots,n$, *are nonvanishing on* $\widetilde{\Omega}$*, Then, the constraint*

$$\begin{aligned} &2(x-x_0)+2(t_r-t_{0,r})\sum_{s=0}^r \tilde{c}_{r-s}c_s(\underline{E}) \\ &= \bigg(-2\int_{x_0}^x \frac{dx'}{\prod_{k=1}^n \mu_k(x',t_r)} \\ &\quad -2\sum_{k=0}^{r\wedge n} d_{r,k}(\underline{E})\int_{t_{0,r}}^{t_r}\frac{\Psi_k(\underline{\mu}(x_0,t'))}{\Psi_n(\underline{\mu}(x_0,t'))}dt'\bigg)\sum_{j=1}^n\bigg(\int_{a_j}\widetilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg)c_j(1) \\ &\quad +2(t_r-t_{0,r})\sum_{\ell=1\vee(n+1-\ell)}^n d_{r,n+1-\ell}(\underline{E})\sum_{j=1}^n\bigg(\int_{a_j}\widetilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg)c_j(\ell) \\ &\quad +\ln\bigg(\frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t_r))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0,t_{0,r}))\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t_r))\big)\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0,t_{0,r}))\big)}\bigg), \\ &\qquad\qquad (x,t_r),(x_0,t_{0,r})\in\widetilde{\Omega} \end{aligned} \tag{4.83}$$

holds, with

$$\begin{aligned} \underline{\hat{z}}(P_{\infty_\pm},\underline{\hat{\mu}}(x,t_r))) &= \underline{\widehat{\Xi}}_{Q_0}-\underline{\widehat{A}}_{Q_0}(P_{\infty_\pm})+\underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) \\ &= \underline{\widehat{\Xi}}_{Q_0}-\underline{\widehat{A}}_{Q_0}(P_{\infty_\pm})+\underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_r)}) \\ &\quad -2\bigg(\int_{x_0}^x\frac{dx'}{\Psi_n(\underline{\mu}(x',t_r))}\bigg)\underline{c}(1) \\ &= \underline{\widehat{\Xi}}_{Q_0}-\underline{\widehat{A}}_{Q_0}(P_{\infty_\pm})+\underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_{0,r})}) \\ &\quad -2\bigg(\sum_{k=0}^{r\wedge n}d_{r,k}(\underline{E})\int_{t_{0,r}}^{t_r}\frac{\Psi_k(\underline{\mu}(x,t'))}{\Psi_n(\underline{\mu}(x,t'))}dt'\bigg)\underline{c}(1) \\ &\quad +2(t_r-t_{0,r})\bigg(\sum_{\ell=1\vee(n+1-r)}^n d_{r,n+1-\ell}(\underline{E})\underline{c}(\ell)\bigg), \\ &\qquad\qquad (x,t_r),(x_0,t_{0,r})\in\widetilde{\Omega}. \end{aligned} \tag{4.84, 4.85}$$

Proof. First, let $\widetilde{\widetilde{\Omega}}\subseteq\Omega$ be defined by requiring that μ_j, $j=1,\dots,n$, are distinct and nonvanishing on $\widetilde{\widetilde{\Omega}}$ and $\widetilde{F}_r(\mu_j)\neq 0$ on $\widetilde{\widetilde{\Omega}}$, $j=1,\dots,n$. The representation (4.81) for ϕ on $\widetilde{\widetilde{\Omega}}$ then follows by combining (4.28), (4.72), (4.73), and Theorem

A.5 since $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ are simultaneously nonspecial as discussed in the proof of Theorem 3.7. The representation (4.82) for u on $\widetilde{\widetilde{\Omega}}$ follows from the trace formula (4.71) and (C.46) (taking $k = 1$). By continuity, (4.81) and (4.82) extend from $\widetilde{\widetilde{\Omega}}$ to Ω. The constraint (4.83) then holds on $\widetilde{\widetilde{\Omega}}$ by combining (4.76)–(4.79), and (C.45). Equations (4.84) and (4.85) are clear from (4.74) and (4.75). Again by continuity, (4.83)–(4.85) extend from $\widetilde{\widetilde{\Omega}}$ to $\widetilde{\Omega}$. □

As discussed by Alber, Camassa, Fedorov, Holm, and Marsden [3], Alber and Fedorov [7], [8], the algebro-geometric CH solution u in (4.82) is not meromorphic with respect to x, t_r, in general. In more geometrical terms, the CH_r flows evolve on a nonlinear subvariety (corresponding to the constraint (4.83)) of a generalized Jacobian, topologically given by $J(\mathcal{K}_n) \times \mathbb{C}^*$ ($\mathbb{C}^* = \mathbb{C} \setminus \{0\}$). For discussions of generalized Jacobians in this context we refer, for instance, to [37], [43], [44]. Smooth (i.e., C^1 with respect to t_1 and C^3 and hence C^∞ with respect to x) spatially periodic CH_1 solutions u are quasi-periodic in t_1 as shown by Constantin [22].

Without going into details we mention that our approach extends in a straightforward manner to the Dym-type equation,

$$v_{xxt} + 2vv_{xxx} + 4v_x v_{xx} - 4\kappa v = 0, \quad \kappa \in \mathbb{R}, \ (x,t) \in \mathbb{R}^2. \tag{4.86}$$

The corresponding zero-curvature formalism leads to a trace formula analogous to (4.71) (cf. [3], [7], [8]). One needs to replace the polynomial $R_{2n+2}(z)$ by $R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m)$, which results in a branch point P_∞ at infinity, and replaces the (non-normalized) differential $\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}$ of the third kind by the (non-normalized) differential $\tilde{\omega}^{(2)}_{P_\infty} = z^n dz/y$ of the second kind, etc. This approach (applied to the Dym equation $4\rho_t = \rho^3 \rho_{xxx}$, related to (4.86) by proper variable transformations) was first realized by Novikov [55] and inspired our treatment of the CH hierarchy.

Expressing $\widetilde{F}_r$ in terms of $\Psi_k(\underline{\mu})$ and hence in terms of the theta function associated with $\mathcal{K}_n$, one can use (4.53) to derive a theta function representation of ψ_j, $j = 1, 2$, in analogy to the stationary case discussed in Remark 3.9. We omit further details.

Up to this point we assumed Hypothesis 4.1 together with the basic equations (4.4) and (4.5). Next, we will show that solvability of the Dubrovin equations (4.64) and (4.65) on $\Omega_\mu \subseteq \mathbb{R}^2$ in fact implies equations (4.4) and (4.5) on Ω_μ and hence solves the algebro-geometric initial value problem (4.2), (4.3) on Ω_μ. In this context we recall the definition of $\widetilde{F}_r(\mu_j)$ in terms of $\mu_1, \dots, \mu_n$, introduced in (C.27), (C.24),

$$\widetilde{F}_r(\mu_j) = \sum_{k=0}^{r\wedge n} \tilde{d}_{r,k}(\underline{E}) \Phi_k^{(j)}(\underline{\mu}), \quad r \in \mathbb{N}_0, \ \tilde{c}_0 = 1, \tag{4.87}$$

$$\tilde{d}_{r,k}(\underline{E}) = \sum_{s=0}^{r-k} \tilde{c}_{r-k-s} c_s(\underline{E}), \quad k = 0, \dots, r \wedge n, \tag{4.88}$$

in terms of a given set of integration constants $\{\tilde{c}_1, \dots, \tilde{c}_r\} \subset \mathbb{C}$.

Theorem 4.10. *Fix $n \in \mathbb{N}$ and assume (4.63). Suppose that $\{\hat{\mu}_j\}_{j=1,\dots,n}$ satisfies the Dubrovin equations (4.64), (4.65) on an open and connected set $\Omega_\mu \subseteq \mathbb{R}^2$,* *with* $\widetilde{F}_r(\mu_j)$ *in* (4.65) *expressed in terms of* μ_k, $k = 1,\dots,n$, *by* (4.87), (4.88). *Moreover, assume that* μ_j, $j = 1,\dots,n$, *remain distinct and nonzero on* Ω_μ *and that* $\widetilde{F}_r(\mu_j) \neq 0$ *on* Ω_μ, $j = 1,\dots,n$. *Then* $u \in C^\infty(\Omega_\mu)$ *defined by*

$$u(x,t_r) = \frac{1}{2}\sum_{j=1}^{n} \mu_j(x,t_r) - \frac{1}{4}\sum_{m=1}^{2n+1} E_m, \tag{4.89}$$

satisfies the r*th CH equation* (4.2), *that is,*

$$\mathrm{CH}_r(u) = 0 \text{ on } \Omega_\mu, \tag{4.90}$$

with initial values satisfying the n*th stationary CH equation* (4.3).

Proof. Given solutions $\hat\mu_j = (\mu_j, y(\hat\mu_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$, $j = 1,\dots,n$ of (4.64) and (4.65), we define polynomials F_n, G_n, and H_n on Ω_μ as in the stationary case, cf. Theorem 3.11, with properties

$$F_n(z) = \prod_{j=1}^{n}(z-\mu_j), \tag{4.91}$$

$$G_n(z) = F_n(z) + \frac{1}{2}F_{n,x}(z), \tag{4.92}$$

$$zG_{n,x}(z) = (4u - u_{xx})F_n(z) - H_n(z), \tag{4.93}$$

$$H_{n,x}(z) = 2H_n(z) - 2(4u-u_{xx})G_n(z), \tag{4.94}$$

$$R_{2n+2}(z) = z^2 G_n(z)^2 + zF_n(z)H_n(z), \tag{4.95}$$

treating t_r as a parameter. Define polynomials $\widetilde{G}_r$ and $\widetilde{H}_r$ by

$$\widetilde{G}_r(z) = \widetilde{F}_r(z) + \frac{1}{2}\widetilde{F}_r(z), \tag{4.96}$$

$$\widetilde{H}_r(z) = (4u - u_{xx})\widetilde{F}_r(z) - z\widetilde{G}_{r,x}(z), \tag{4.97}$$

respectively. We claim that

$$F_{n,t_r}(z) = 2\big(G_n(z)\widetilde{F}_r(z) - F_n(z)\widetilde{G}_r(z)\big). \tag{4.98}$$

To prove (4.98) we compute from (4.64) and (4.65) that

$$F_{n,t_r}(z) = -F_n(z)\sum_{j=1}^{n}\widetilde{F}_r(\mu_j)\mu_{j,x}(z-\mu_j)^{-1}, \tag{4.99}$$

$$F_{n,x}(z) = -F_n(z)\sum_{j=1}^{n}\mu_{j,x}(z-\mu_j)^{-1}. \tag{4.100}$$

Using (4.92) and (4.96) we see that (4.98) is equivalent to

$$\widetilde{F}_{r,x}(z) = \sum_{j=1}^{n}\big(\widetilde{F}_r(z) - \widetilde{F}_r(\mu_j)\big)\mu_{j,x}(z-\mu_j)^{-1}. \tag{4.101}$$

Equation (4.101) is proved in Lemma C.5. This in turn proves (4.98). Next, taking the derivative of (4.98) with respect to x and inserting (4.93) and (4.92), yields

$$\begin{aligned} F_{n,t_rx}(z) = 2\bigg(&\frac{1}{z}(4u-u_{xx})F_n(z) + G_n(z)\widetilde{F}_{r,x}(z) \\ &- 2(G_n(z) - F_n(z))\widetilde{G}_r(z) - F_n(z)\widetilde{G}_{r,x}(z)\bigg). \end{aligned} \tag{4.102}$$

On the other hand, by differentiating (4.92) with respect to t_r, using (4.98) we obtain

$$F_{n,t_r x}(z) = 2\big(G_{n,t_r}(z) - 2(G_n(z)\widetilde{F}_r(z) - F_n(z)\widetilde{G}_r(z)\big). \tag{4.103}$$

Combining (4.102) and (4.103) we conclude

$$zG_{n,t_r}(z) = F_n(z)\widetilde{H}_r(z) - \widetilde{F}_r(z)H_n(z). \tag{4.104}$$

Next, we take the derivative of (4.95) with respect to t_r and use the expressions (4.98) and (4.104) for F_{n,t_r} and G_{n,t_r}, respectively, to obtain

$$H_{n,t_r}(z) = 2\big(\widetilde{G}_r(z)H_n(z) - G_n(z)\widetilde{H}_r(z)\big). \tag{4.105}$$

Finally, we compute G_{n,xt_r} in two different ways. Differentiating (4.104) with respect to x, using (4.92), (4.96), and (4.94), one finds

$$\begin{aligned} zG_{n,xt_r}(z) &= \widetilde{H}_{r,x}(z)F_n(z) + 2(G_n(z)\widetilde{H}_r(z) - \widetilde{G}_r(z)H_n(z)) \\ &\quad + 2(4u - u_{xx})G_n(z)\widetilde{F}_r(z) - 2F_n(z)\widetilde{H}_r(z). \end{aligned} \tag{4.106}$$

Differentiating (4.93) with respect to t_r, using (4.98) and (4.105), results in

$$\begin{aligned} zG_{n,xt_r}(z) &= (u_{t_r} - u_{xxt_r})F_n(z) - 2(\widetilde{G}_r(z)H_n(z) - G_n(z)\widetilde{H}_r(z)) \\ &\quad + 2(4u - u_{xx})(G_n(z)\widetilde{F}_r(z) - F_n(z)\widetilde{G}_r(z)). \end{aligned} \tag{4.107}$$

Combining (4.106) and (4.107) one concludes

$$u_{t_r} - u_{xxt_r} = \widetilde{H}_r(z) + 2(4u - u_{xx})\widetilde{G}_r(z) - \widetilde{H}_r(z) \tag{4.108}$$

which is equivalent to (4.90). □

Appendix A. Hyperelliptic curves and their theta functions

We provide a brief summary of some of the fundamental properties and notations needed from the theory of hyperelliptic curves. More details can be found in some of the standard textbooks [36] and [54], as well as monographs dedicated to integrable systems such as [15, Ch. 2], [46, App. A–C].

Fix $n \in \mathbb{N}$. The hyperelliptic curve $\mathcal{K}_n$ of genus n used in Sections 3 and 4 is defined by

$$\mathcal{K}_n \colon \mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0, \quad R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \tag{A.1}$$

$$\{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{C}, \quad E_m \neq E_n \text{ for } m \neq n,\, m,n = 0,\dots,2n+1. \tag{A.2}$$

The curve (A.2) is compactified by adding the points P_{∞_+} and P_{∞_-}, $P_{\infty_+} \neq P_{\infty_-}$, at infinity. One then introduces an appropriate set of $n+1$ nonintersecting cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$. We denote

$$\mathcal{C} = \bigcup_{j=1}^{n+1} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \tag{A.3}$$

Define the cut plane

$$\Pi = \mathbb{C} \setminus \mathcal{C}, \tag{A.4}$$

and introduce the holomorphic function

$$R_{2n+2}(\,\cdot\,)^{1/2}\colon \Pi \to \mathbb{C}, \quad z \mapsto \bigg(\prod_{m=0}^{2n+1}(z-E_m)\bigg)^{1/2} \tag{A.5}$$

on Π with an appropriate choice of the square root branch in (A.5). Define

$$\mathcal{M}_n = \{(z, \sigma R_{2n+2}(z)^{1/2}) \mid z \in \mathbb{C},\ \sigma \in \{1,-1\}\} \cup \{P_{\infty_+}, P_{\infty_-}\} \tag{A.6}$$

by extending $R_{2n+2}(\,\cdot\,)^{1/2}$ to $\mathcal{C}$. The hyperelliptic curve $\mathcal{K}_n$ is then the set $\mathcal{M}_n$ with its natural complex structure obtained upon gluing the two sheets of $\mathcal{M}_n$ crosswise along the cuts. The set of branch points $\mathcal{B}(\mathcal{K}_n)$ of $\mathcal{K}_n$ is given by

$$\mathcal{B}(\mathcal{K}_n) = \{(E_m, 0)\}_{m=0,\dots,2n+1} \tag{A.7}$$

and finite points P on $\mathcal{K}_n$ are denoted by $P=(z,y)$, where $y(P)$ denotes the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0$. Local coordinates near $P_0 = (z_0, y_0) \in \mathcal{K}_n \setminus \{\mathcal{B}(\mathcal{K}_n) \cup \{P_{\infty_+}, P_{\infty_-}\}\}$ are given by $\zeta_{P_0} = z - z_0$, near $P_{\infty_\pm}$ by $\zeta_{P_{\infty_\pm}} = 1/z$, and near branch points $(E_{m_0}, 0) \in \mathcal{B}(\mathcal{K}_n)$ by $\zeta_{(E_{m_0},0)} = (z - E_{m_0})^{1/2}$. The Riemann surface $\mathcal{K}_n$ defined in this manner has topological genus n. Moreover, we introduce the holomorphic sheet exchange map (involution)

$$*\colon \mathcal{K}_n \to \mathcal{K}_n, \quad P=(z,y) \mapsto P^* = (z,-y),\ P_{\infty_\pm} \mapsto P^*_{\infty_\pm} = P_{\infty_\mp} \tag{A.8}$$

One verifies that dz/y is a holomorphic differential on $\mathcal{K}_n$ with zeros of order $n-1$ at $P_{\infty_\pm}$ and hence

$$\eta_j = \frac{z^{j-1}dz}{y}, \quad j = 1, \dots, n \tag{A.9}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_n$. Introducing the invertible matrix C in $\mathbb{C}^n$,

$$C = (C_{j,k})_{j,k=1,\dots,n}, \quad C_{j,k} = \int_{a_k} \eta_j, \tag{A.10}$$

$$\underline{c}(k) = (c_1(k), \dots, c_n(k)), \quad c_j(k) = C^{-1}_{j,k}, \quad j,k = 1,\dots,n, \tag{A.11}$$

the corresponding basis of normalized holomorphic differentials ω_j, $j=1,\dots,n$ on $\mathcal{K}_n$ is given by

$$\omega_j = \sum_{\ell=1}^n c_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j,k=1,\dots,n. \tag{A.12}$$

Here $\{a_j, b_j\}_{j=1,\dots,n}$ is a homology basis for $\mathcal{K}_n$ with intersection matrix of the cycles satisfying

$$a_j \circ b_k = \delta_{j,k}, \quad j,k=1,\dots,n. \tag{A.13}$$

Near $P_{\infty_\pm}$ one infers

$$\begin{aligned}\underline{\omega} = (\omega_1, \dots, \omega_n) &= \pm\bigg(\sum_{j=1}^n \frac{\underline{c}(j)\zeta^{n-j}}{\big(\prod_{m=0}^{2n+1}(1-E_m\zeta)\big)^{1/2}}\bigg)d\zeta \\ &\underset{\zeta\to 0}{=} \pm\bigg(\underline{c}(n) + \bigg(\frac{1}{2}\underline{c}(n)\sum_{m=0}^{2n+1}E_m + \underline{c}(n-1)\bigg)\zeta + O(\zeta^2)\bigg)d\zeta \text{ as } P \to P_{\infty_\pm}, \\ &\qquad \zeta = 1/z, \end{aligned} \tag{A.14}$$

and

$$y(P) \underset{\zeta\to 0}{=} \mp\bigg(1-\frac{1}{2}\bigg(\sum_{m=0}^{2n+1} E_m\bigg)\zeta + O(\zeta^2)\bigg)\zeta^{-n-1} \text{ as } P\to P_{\infty_\pm}. \tag{A.15}$$

Similarly, near P_0 one computes

$$\underline{\omega} \underset{\zeta\to 0}{=} -2i\big(\widetilde{Q}^{-1/2}\underline{c}(1) + O(\zeta^2)\big)d\zeta \text{ as } P\to P_0, \tag{A.16}$$

$$\widetilde{Q}^{1/2} = \bigg(\prod_{m=1}^{2n+1} E_m\bigg)^{1/2}, \quad \zeta = \sigma z^{1/2},\ \sigma\in\{1,-1\},$$

using

$$y(P) \underset{\zeta\to 0}{=} i\widetilde{Q}^{1/2}\zeta + O(\zeta^3) \text{ as } P\to P_0, \quad \zeta=\sigma z^{1/2},\ \sigma\in\{1,-1\}, \tag{A.17}$$

with the sign of $\widetilde{Q}^{1/2}$ determined by the compatibility of charts.

Associated with the homology basis $\{a_j, b_j\}_{j=1,\dots,n}$ we also recall the canonical dissection of $\mathcal{K}_n$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_n$ of the fundamental polygon $\partial\widehat{\mathcal{K}}_n$ given by

$$\partial\widehat{\mathcal{K}}_n = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n^{-1} b_n^{-1}. \tag{A.18}$$

Let $\mathcal{M}(\mathcal{K}_n)$ and $\mathcal{M}^1(\mathcal{K}_n)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on $\mathcal{K}_n$. The residue of a meromorphic differential $\nu\in\mathcal{M}^1(\mathcal{K}_n)$ at a point $Q\in\mathcal{K}_n$ is defined by

$$\operatorname{res}_Q(\nu) = \frac{1}{2\pi i}\int_{\gamma_Q}\nu, \tag{A.19}$$

where γ_Q is a counterclockwise oriented smooth simple closed contour encircling Q but no other pole of ν. Holomorphic differentials are also called Abelian differentials of the first kind (dfk). Abelian differentials of the second kind (dsk) $\omega^{(2)}\in\mathcal{M}^1(\mathcal{K}_n)$ are characterized by the property that all their residues vanish. They will usually be normalized by demanding that all their a-periods vanish, that is,

$$\int_{a_j}\omega^{(2)} = 0, \quad j=1,\dots,n. \tag{A.20}$$

If $\omega^{(2)}_{P_1,n}$ is a dsk on $\mathcal{K}_n$ whose only pole is $P_1\in\widehat{\mathcal{K}}_n$ with principal part $\zeta^{-n-2}\,d\zeta$, $n\in\mathbb{N}_0$ near P_1 and $\omega_j = \big(\sum_{m=0}^{\infty} d_{j,m}(P_1)\zeta^m\big)\,d\zeta$ near P_1, then

$$\frac{1}{2\pi i}\int_{b_j}\omega^{(2)}_{P_1,m} = \frac{d_{j,m}(P_1)}{m+1}, \quad m=0,1,\dots \tag{A.21}$$

Any meromorphic differential $\omega^{(3)}$ on $\mathcal{K}_n$ not of the first or second kind is said to be of the third kind (dtk). A dtk $\omega^{(3)}\in\mathcal{M}^1(\mathcal{K}_n)$ is usually normalized by the vanishing of its a-periods, that is,

$$\int_{a_j}\omega^{(3)} = 0, \quad j=1,\dots,n. \tag{A.22}$$

A normal dtk $\omega^{(3)}_{P_1,P_2}$ associated with two points P_1, $P_2\in\widehat{\mathcal{K}}_n$, $P_1\neq P_2$ by definition has simple poles at P_j with residues $(-1)^{j+1}$, $j=1,2$ and vanishing a-periods. If

$\omega^{(3)}_{P,Q}$ is a normal dtk associated with $P, Q \in \widehat{\mathcal{K}}_n$, holomorphic on $\mathcal{K}_n \setminus \{P, Q\}$, then

$$\frac{1}{2\pi i}\int_{b_j} \omega^{(3)}_{P,Q} = \int_Q^P \omega_j, \quad j = 1, \dots, n, \tag{A.23}$$

where the path from Q to P lies in $\widehat{\mathcal{K}}_n$ (i.e., does not touch any of the cycles a_j, b_j). Explicitly, one obtains

$$\omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = \frac{z^n dz}{y} + \sum_{j=1}^n \gamma_j\omega_j = \frac{\prod_{j=1}^n (z-\lambda_j)\, dz}{y}, \tag{A.24}$$

$$\omega^{(3)}_{P_1,P_{\infty_+}} = \frac{1}{2}\frac{y+y_1}{z-z_1}\frac{dz}{y} - \frac{\prod_{j=1}^n (z-\tilde\lambda_j)\, dz}{2y}, \tag{A.25}$$

$$\omega^{(3)}_{P_1,P_{\infty_-}} = \frac{1}{2}\frac{y+y_1}{z-z_1}\frac{dz}{y} + \frac{\prod_{j=1}^n (z-\lambda'_j)\, dz}{2y}, \tag{A.26}$$

$$\omega^{(3)}_{P_1,P_2} = \left(\frac{y+y_1}{z-z_1} - \frac{y+y_2}{z-z_2}\right)\frac{dz}{2y} + \lambda''_n \frac{\prod_{j=1}^{n-1} (z-\lambda''_j) dz}{y}, \tag{A.27}$$
$$P_1, P_2 \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\},$$

where $\gamma_j, \lambda_j, \tilde\lambda_j, \lambda'_j, \lambda''_j \in \mathbb{C}$, $j = 1, \dots, n$, are uniquely determined by the requirement of vanishing a-periods and we abbreviated $P_j = (z_j, y_j)$, $j = 1, 2$. (If $n = 0$, we use the standard convention that the product over an empty index set is replaced by 1.)

We shall always assume (without loss of generality) that all poles of dsk's and dtk's on $\mathcal{K}_n$ lie on $\widehat{\mathcal{K}}_n$ (i.e., not on $\partial\widehat{\mathcal{K}}_n$).

Define the matrix $\tau = (\tau_{j,\ell})_{j,\ell=1,\dots,n}$ by

$$\tau_{j,\ell} = \int_{b_j} \omega_\ell, \quad j, \ell = 1, \dots, n. \tag{A.28}$$

Then

$$\operatorname{Im}(\tau) > 0, \quad \text{and} \quad \tau_{j,\ell} = \tau_{\ell,j}, \quad j, \ell = 1, \dots, n. \tag{A.29}$$

Associated with τ one introduces the period lattice

$$L_n = \{\underline{z} \in \mathbb{C}^n \mid \underline{z} = \underline{m} + \tau\underline{n},\ \underline{m}, \underline{n} \in \mathbb{Z}^n\} \tag{A.30}$$

and the Riemann theta function associated with $\mathcal{K}_n$ and the given homology basis $\{a_j, b_j\}_{j=1,\dots,n}$,

$$\theta(\underline{z}) = \sum_{\underline{n}\in\mathbb{Z}^n} \exp\big(2\pi i(\underline{n}, \underline{z}) + \pi i(\underline{n}, \tau\underline{n})\big), \quad \underline{z} \in \mathbb{C}^n, \tag{A.31}$$

where $(\underline{u}, \underline{v}) = \sum_{j=1}^n \overline{u}_j v_j$ denotes the scalar product in $\mathbb{C}^n$. It has the fundamental properties

$$\theta(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n) = \theta(\underline{z}), \tag{A.32}$$

$$\theta(\underline{z} + \underline{m} + \tau\underline{n}) = \exp\big(-2\pi i(\underline{n}, \underline{z}) - \pi i(\underline{n}, \tau\underline{n})\big)\theta(\underline{z}), \quad \underline{m}, \underline{n} \in \mathbb{Z}^n. \tag{A.33}$$

Next, fix a base point $Q_0 \in \mathcal{K}_n \setminus \{P_{0_\pm}, P_{\infty_\pm}\}$, denote by $J(\mathcal{K}_n) = \mathbb{C}^n/L_n$ the Jacobi variety of $\mathcal{K}_n$, and define the Abel map $\underline{A}_{Q_0}$ by

$$\underline{A}_{Q_0}\colon \mathcal{K}_n \to J(\mathcal{K}_n), \quad \underline{A}_{Q_0}(P) = \bigg(\int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_n\bigg) \pmod{L_n}, \quad P \in \mathcal{K}_n. \tag{A.34}$$

Similarly, we introduce

$$\underline{\alpha}_{Q_0}\colon \operatorname{Div}(\mathcal{K}_n) \to J(\mathcal{K}_n), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P\in\mathcal{K}_n} \mathcal{D}(P)\underline{A}_{Q_0}(P), \tag{A.35}$$

where $\operatorname{Div}(\mathcal{K}_n)$ denotes the set of divisors on $\mathcal{K}_n$. Here $\mathcal{D}\colon \mathcal{K}_n \to \mathbb{Z}$ is called a divisor on $\mathcal{K}_n$ if $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_n$. (In the main body of this paper we will choose Q_0 to be one of the branch points, i.e., $Q_0 \in \mathcal{B}(\mathcal{K}_n)$, and for simplicity we will always choose the same path of integration from Q_0 to P in all Abelian integrals.) For subsequent use in Remark A.7 we also introduce

$$\begin{aligned} &\underline{\widehat{A}}_{Q_0}\colon \widehat{\mathcal{K}}_n \to \mathbb{C}^n, \\ &\qquad P \mapsto \underline{\widehat{A}}_{Q_0}(P) = \big(\widehat{A}_{Q_0,1}(P), \dots, \widehat{A}_{Q_0,n}(P)\big) = \bigg(\int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_n\bigg) \end{aligned} \tag{A.36}$$

and

$$\underline{\widehat{\alpha}}_{Q_0}\colon \operatorname{Div}(\widehat{\mathcal{K}}_n) \to \mathbb{C}^n, \quad \mathcal{D} \mapsto \underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}) = \sum_{P\in\widehat{\mathcal{K}}_n} \mathcal{D}(P)\underline{\widehat{A}}_{Q_0}(P). \tag{A.37}$$

In connection with divisors on $\mathcal{K}_n$ we shall employ the following (additive) notation,

$$\begin{aligned} &\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}, \\ &\underline{Q} = \{Q_1, \dots, Q_m\} \in \sigma^m\mathcal{K}_n, \quad Q_0 \in \mathcal{K}_n,\ m \in \mathbb{N}, \end{aligned} \tag{A.38}$$

where for any $Q \in \mathcal{K}_n$,

$$\mathcal{D}_Q\colon \mathcal{K}_n \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_n \setminus \{Q\}, \end{cases} \tag{A.39}$$

and $\sigma^m\mathcal{K}_n$ denotes the mth symmetric product of $\mathcal{K}_n$. In particular, $\sigma^m\mathcal{K}_n$ can be identified with the set of nonnegative divisors $0 \le \mathcal{D} \in \operatorname{Div}(\mathcal{K}_n)$ of degree $m \in \mathbb{N}$.

For $f \in \mathcal{M}(\mathcal{K}_n) \setminus \{0\}$, $\omega \in \mathcal{M}^1(\mathcal{K}_n) \setminus \{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E} \in \operatorname{Div}(\mathcal{K}_n)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_n) \setminus \{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \operatorname{Div}(\mathcal{K}_n) \mid \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0,\ \deg((\omega)) = 2(n-1),\ f \in \mathcal{M}(\mathcal{K}_n) \setminus \{0\},\ \omega \in \mathcal{M}^1(\mathcal{K}_n) \setminus \{0\}, \tag{A.40}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P\in\mathcal{K}_n} \mathcal{D}(P)$. It is customary to call (f) (respectively, (ω)) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_n) \mid f = 0 \text{ or } (f) \ge \mathcal{D}\},\ r(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}(\mathcal{D}), \tag{A.41}$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_n) \mid \omega = 0 \text{ or } (\omega) \ge \mathcal{D}\},\ i(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}^1(\mathcal{D}), \tag{A.42}$$

($i(\mathcal{D})$ the index of speciality of $\mathcal{D}$) one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental facts.

Theorem A.1. *Let $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_n)$, $\omega \in \mathcal{M}^1(\mathcal{K}_n) \setminus \{0\}$. Then*
$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad n \in \mathbb{N}_0. \tag{A.43}$$
The Riemann-Roch theorem reads
$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - n + 1, \quad n \in \mathbb{N}_0. \tag{A.44}$$
By Abel's theorem, $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_n)$, $n \in \mathbb{N}$, is principal if and only if
$$\deg(\mathcal{D}) = 0 \text{ and } \underline{\alpha}_{Q_0}(\mathcal{D}) = \underline{0}. \tag{A.45}$$
Finally, assume $n \in \mathbb{N}$. Then $\underline{\alpha}_{Q_0} : \operatorname{Div}(\mathcal{K}_n) \to J(\mathcal{K}_n)$ is surjective (Jacobi's inversion theorem).

Next we introduce
$$\underline{W}_0 = \{0\} \subset J(\mathcal{K}_n), \quad \underline{W}_m = \underline{\alpha}_{Q_0}(\sigma^m \mathcal{K}_n), \; m \in \mathbb{N} \tag{A.46}$$
and note that while $\sigma^m \mathcal{K}_n \not\subset \sigma^n \mathcal{K}_n$ for $m < n$, one has $\underline{W}_m \subseteq \underline{W}_n$ for $m < n$. Thus $\underline{W}_m = J(\mathcal{K}_n)$ for $m \geq n$ by Jacobi's inversion theorem.

Denote by $\underline{\Xi}_{Q_0} = (\Xi_{Q_{0,1}}, \dots, \Xi_{Q_{0,n}})$ the vector of Riemann constants,
$$\Xi_{Q_{0,j}} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{n} \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \dots, n. \tag{A.47}$$

Theorem A.2. *The set $\underline{W}_{n-1} + \underline{\Xi}_{Q_0} \subset J(\mathcal{K}_n)$ is the complete set of zeros of θ on $J(\mathcal{K}_n)$, that is,*
$$\theta(X) = 0 \text{ if and only if } X \in \underline{W}_{n-1} + \underline{\Xi}_{Q_0} \tag{A.48}$$
(i.e., if and only if $X = (\underline{\alpha}_{Q_0}(\mathcal{D}) + \underline{\Xi}_{Q_0}) \pmod{L_n}$ for some $\mathcal{D} \in \sigma^{n-1}\mathcal{K}_n$). The set $\underline{W}_{n-1} + \underline{\Xi}_{Q_0}$ has complex dimension $n-1$.

Theorem A.3. *Let $\mathcal{D}_{\underline{Q}} \in \sigma^n \mathcal{K}_n$, $\underline{Q} = \{Q_1, \dots, Q_n\}$. Then*
$$1 \leq i(\mathcal{D}_{\underline{Q}}) = s \tag{A.49}$$
if and only if there are s pairs of the type $\{P, P^*\} \subseteq \{Q_1, \dots, Q_n\}$ (this includes, of course, branch points for which $P = P^*$). Obviously, one has $s \leq n/2$.

Remark A.4. While $\theta(\underline{z})$ is well-defined (in fact, entire) for $\underline{z} \in \mathbb{C}^n$, it is not well-defined on $J(\mathcal{K}_n) = \mathbb{C}^n / L_n$ because of (A.33). Nevertheless, θ is a "multiplicative function" on $J(\mathcal{K}_n)$ since the multipliers in (A.33) cannot vanish. In particular, if $\underline{z}_1 = \underline{z}_2 \pmod{L_n}$, then $\theta(\underline{z}_1) = 0$ if and only if $\theta(\underline{z}_2) = 0$. Hence it is meaningful to state that θ vanishes at points of $J(\mathcal{K}_n)$. Since the Abel map $\underline{A}_{Q_0}$ maps $\mathcal{K}_n$ into $J(\mathcal{K}_n)$, the function $\theta(\underline{A}_{Q_0}(P) - \underline{\xi})$ for $\underline{\xi} \in \mathbb{C}^n$, becomes a multiplicative function on $\mathcal{K}_n$. Again it makes sense to say that $\theta(\underline{A}_{Q_0}(\,\cdot\,) - \underline{\xi})$ vanishes at points of $\mathcal{K}_n$.

Theorem A.5. *Let $\underline{Q} = \{Q_1, \dots, Q_n\} \in \sigma^n \mathcal{K}_n$ and assume $\mathcal{D}_{\underline{Q}}$ to be nonspecial, that is, $i(\mathcal{D}_{\underline{Q}}) = 0$. Then*
$$\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \alpha_{Q_0}(\mathcal{D}_{\underline{Q}})) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_n\}. \tag{A.50}$$

Theorem A.6. *Suppose $\mathcal{D}_{\underline{\hat{\mu}}} \in \sigma^n\mathcal{K}_n$ is nonspecial, $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$, and $\hat{\mu}_{n+1} \in \mathcal{K}_n$ with $\hat{\mu}_{n+1}^* \notin \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$. Let $\{\hat{\lambda}_1, \dots, \hat{\lambda}_{n+1}\} \subset \mathcal{K}_n$ with $\mathcal{D}_{\underline{\hat{\lambda}}\hat{\lambda}_{n+1}} \sim \mathcal{D}_{\underline{\hat{\mu}}\hat{\mu}_{n+1}}$ (i.e., $\mathcal{D}_{\underline{\hat{\lambda}}\hat{\lambda}_{n+1}} \in [\mathcal{D}_{\underline{\hat{\mu}}\hat{\mu}_{n+1}}]$). Then any n points $\hat{\nu}_j \in \{\hat{\lambda}_1, \dots, \hat{\lambda}_{n+1}\}$, $j = 1, \dots, n$ define a nonspecial divisor $\mathcal{D}_{\underline{\hat{\nu}}} \in \sigma^n\mathcal{K}_n$, $\underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\}$.*

Proof. Since $i(\mathcal{D}_P) = 0$ for all $P \in \mathcal{K}_1$, there is nothing to prove in the special case $n = 1$. Hence we assume $n \geq 2$. Let $Q_0 \in \mathcal{B}(\mathcal{K}_n)$ be a fixed branch point of $\mathcal{K}_n$ and suppose that $\mathcal{D}_{\underline{\hat{\nu}}}$ is special. Then by Theorem A.3 there is a pair $\{\hat{\nu}, \hat{\nu}^*\} \subset \{\hat{\nu}_1, \dots, \hat{\nu}_n\}$ such that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\underline{\hat{\nu}}}}), \tag{A.51}$$

where $\underline{\underline{\hat{\nu}}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \backslash \{\hat{\nu}, \hat{\nu}^*\} \in \sigma^{n-2}\mathcal{K}_n$. Let $\hat{\nu}_{n+1} \in \{\hat{\lambda}_1, \dots, \hat{\lambda}_{n+1}\} \backslash \{\hat{\nu}_1, \dots, \hat{\nu}_n\}$ so that $\{\hat{\nu}_1, \dots, \hat{\nu}_{n+1}\} = \{\hat{\lambda}_1, \dots, \hat{\lambda}_{n+1}\} \subset \sigma^{n+1}\mathcal{K}_n$. Then

$$\begin{aligned} \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\underline{\hat{\nu}}}\hat{\nu}_{n+1}}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}\hat{\nu}_{n+1}}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}\hat{\lambda}_{n+1}}) \\ &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}\hat{\mu}_{n+1}}) = -\underline{A}_{Q_0}(\hat{\mu}_{n+1}^*) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}), \end{aligned} \tag{A.52}$$

and hence by Theorem A.2 and (A.52),

$$0 = \theta(\underline{\Xi}_{Q_0} + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\underline{\hat{\nu}}}\hat{\nu}_{n+1}})) = \theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(\hat{\mu}_{n+1}^*) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}})). \tag{A.53}$$

Since by hypothesis $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial and $\hat{\mu}_{n+1}^* \notin \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$, (A.53) contradicts Theorem A.5. Thus, $\mathcal{D}_{\underline{\hat{\nu}}}$ is nonspecial. ☐

Remark A.7. In Sections 3 and 4 we frequently deal with theta function expressions of the type

$$\phi(P) = \frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_1))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_2))} \exp\bigg(\int_{Q_0}^P \omega_{Q_1,Q_2}^{(3)}\bigg), \quad P \in \mathcal{K}_n \tag{A.54}$$

and

$$\psi(P) = \frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_1))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_2))} \exp\bigg(-c\int_{Q_0}^P \Omega^{(2)}\bigg), \quad P \in \mathcal{K}_n, \tag{A.55}$$

where $\mathcal{D}_j \in \sigma^n\mathcal{K}_n$, $j = 1, 2$, are nonspecial positive divisors of degree n, $c \in \mathbb{C}$ is a constant, $Q_j \in \mathcal{K}_n \setminus \{P_{\infty_1}, \dots, P_{\infty_N}\}$, where $\{P_{\infty_1}, \dots, P_{\infty_N}\}$, $N \in \mathbb{N}$, denotes the set of points of $\mathcal{K}_n$ at infinity, $\omega_{Q_1,Q_2}^{(3)}$ is a normal differential of the third kind, and $\Omega^{(2)}$ a normalized differential of the second kind with singularities contained in $\{P_{\infty_1}, \dots, P_{\infty_N}\}$. In particular, one has

$$\int_{a_j} \omega_{Q_1,Q_2}^{(3)} = \int_{a_j} \Omega^{(2)} = 0, \quad j = 1, \dots, n. \tag{A.56}$$

Even though we agree to always choose identical paths of integration from P_0 to P in all Abelian integrals (A.54) and (A.55), this is not sufficient to render ϕ and ψ single-valued on $\mathcal{K}_n$. To achieve single-valuedness, one needs to replace $\mathcal{K}_n$ by its simply connected canonical dissection $\widehat{\mathcal{K}}_n$ and then replace $\underline{A}_{Q_0}$, $\underline{\alpha}_{Q_0}$ in (A.54) and (A.55), with $\underline{\widehat{A}}_{Q_0}$, $\underline{\widehat{\alpha}}_{Q_0}$ as introduced in (A.36) and (A.37). In particular, one regards a_j, b_j, $j = 1, \dots, n$, as curves (being a part of $\partial\widehat{\mathcal{K}}_n$, cf. (A.18)) and not as homology classes. Moreover, to render ϕ single-valued on $\widehat{\mathcal{K}}_n$ one needs to assume in addition that

$$\underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_1) - \underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_2) = 0 \tag{A.57}$$

(as opposed to merely $\underline{\alpha}_{Q_0}(\mathcal{D}_1)-\underline{\alpha}_{Q_0}(\mathcal{D}_2)=0 \pmod{L_n}$). Similarly, in connection with ψ, one introduces the vector of b-periods $\underline{U}^{(2)}$ of $\Omega^{(2)}$ by

$$\underline{U}^{(2)}=(U_1^{(2)},\dots,U_g^{(2)}),\quad U_j^{(2)}=\frac{1}{2\pi i}\int_{b_j}\Omega^{(2)},\quad j=1,\dots,n, \tag{A.58}$$

and then renders ψ single-valued on $\widehat{\mathcal{K}}_n$ by requiring

$$\widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_1)-\widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_2)=c\,\underline{U}^{(2)} \tag{A.59}$$

(as opposed to merely $\underline{\alpha}_{Q_0}(\mathcal{D}_1)-\underline{\alpha}_{Q_0}(\mathcal{D}_2)=c\,\underline{U}^{(2)} \pmod{L_n}$). These statements easily follow from (A.23) and (A.33) in the case of ϕ and simply from (A.33) in the case of ψ. In fact, by (A.33),

$$\widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_1+\mathcal{D}_{Q_1})-\widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_2+\mathcal{D}_{Q_2})\in\mathbb{Z}^n, \tag{A.60}$$

respectively,

$$\widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_1)-\widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_2)-c\,\underline{U}^{(2)}\in\mathbb{Z}^n, \tag{A.61}$$

suffice to guarantee single-valuedness of ϕ, respectively, ψ on $\widehat{\mathcal{K}}_n$. Without the replacement of $\underline{A}_{Q_0}$ and $\underline{\alpha}_{Q_0}$ by $\widehat{\underline{A}}_{Q_0}$ and $\widehat{\underline{\alpha}}_{Q_0}$ in (A.54) and (A.55) and without the assumptions (A.57) and (A.59) (or (A.60) and (A.61)), ϕ and ψ are multiplicative (multi-valued) functions on $\mathcal{K}_n$, and then most effectively discussed by introducing the notion of characters on $\mathcal{K}_n$ (cf. [36, Sect. III.9]). For simplicity, we decided to avoid the latter possibility and throughout this paper will tacitly always assume (A.57) and (A.59) without particularly emphasizing this convention each time it is used.

Appendix B. High-Energy Expansions

In this appendix we study the relationship between the homogeneous coefficients $\hat f_\ell$ and nonhomogeneous coefficients f_ℓ of the polynomial F_n, discuss the high-energy expansion of F_n/y, and use it to derive a nonlinear recursion relation for $\hat f_\ell$, $\ell\in\mathbb{N}_0$.

Let

$$\{E_m\}_{m=0,\dots,2n+1}\ \text{ for some }\ n\in\mathbb{N}_0 \tag{B.1}$$

$$\text{and }\ \eta\in\mathbb{C}\ \text{ such that }\ |\eta|<\min\{|E_0|^{-1},\dots,|E_{2n+1}|^{-1}\}. \tag{B.2}$$

Then

$$\bigg(\prod_{m=0}^{2n+1}\big(1-E_m\eta\big)\bigg)^{-1/2}=\sum_{k=0}^{\infty}\hat c_k(\underline{E})\eta^k, \tag{B.3}$$

where

$$\begin{aligned}
&\hat c_0(\underline{E})=1,\\
&\hat c_k(\underline{E})=\sum_{\substack{j_0,\dots,j_{2n+1}=0\\ j_0+\cdots+j_{2n+1}=k}}^{k}\frac{(2j_0-1)!!\cdots(2j_{2n+1}-1)!!}{2^k j_0!\cdots j_{2n+1}!}E_0^{j_0}\cdots E_{2n+1}^{j_N},\ k\in\mathbb{N}.
\end{aligned}\tag{B.4}$$

The first few coefficients explicitly read

$$\hat{c}_0(\underline{E}) = 1,\ \hat{c}_1(\underline{E}) = \frac{1}{2}\sum_{m=0}^{2n+1} E_m,\ \hat{c}_2(\underline{E}) = \frac{1}{4}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{2n+1} E_{m_1}E_{m_2} + \frac{3}{8}\sum_{m=0}^{2n+1} E_m^2, \quad \text{etc.} \tag{B.5}$$

Similarly,

$$\bigg(\prod_{m=0}^{2n+1}\big(1-E_m\eta\big)\bigg)^{1/2} = \sum_{k=0}^{\infty} c_k(\underline{E})\eta^k, \tag{B.6}$$

where

$$c_0(\underline{E}) = 1,$$
$$c_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_{2n+1}=0\\ j_0+\dots+j_{2n+1}=k}}^{k} \frac{(2j_0-3)!!\cdots(2j_{2n+1}-3)!!}{2^k j_0!\cdots j_{2n+1}!} E_0^{j_0}\cdots E_{2n+1}^{j_{2n+1}},\ k\in\mathbb{N}. \tag{B.7}$$

The first few coefficients explicitly are given by

$$c_0(\underline{E}) = 1,\ c_1(\underline{E}) = -\frac{1}{2}\sum_{m=0}^{2n+1} E_m,\ c_2(\underline{E}) = \frac{1}{4}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{2n+1} E_{m_1}E_{m_2} - \frac{1}{8}\sum_{m=0}^{2n+1} E_m^2, \quad \text{etc.} \tag{B.8}$$

Here we used the abreviations

$$(2q-1)!! = 1\cdot 3\cdots(2q-1), \quad q\in\mathbb{N}, \quad (-3)!! = -1, \quad (-1)!! = 1. \tag{B.9}$$

Theorem B.1. *Assume* $u \in C^\infty(\mathbb{R})$, $d^m u/dx^m \in L^\infty(\mathbb{R})$, $m \in \mathbb{N}_0$, $\text{s-CH}_n(u) = 0$ *and suppose* $P = (z,y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$. *Then* F_n/y *has the following convergent expansion as* $P \to P_{\infty_\pm}$,

$$\frac{F_n(z)}{y} \underset{\zeta\to 0}{=} \mp\sum_{\ell=0}^{\infty} \hat{f}_\ell\, \zeta^{\ell+1}, \tag{B.10}$$

with $\zeta = 1/z$ *the local coordinate near* $P_{\infty_\pm}$ *described in Appendix* A *and* $\hat{f}_\ell$ *the homogeneous coefficients* f_ℓ *in* (2.10). *In particular,* $\hat{f}_\ell$ *can be computed from the nonlinear recursion relation*

$$\hat{f}_0 = 1, \quad \hat{f}_1 = -2u,$$
$$\hat{f}_{\ell+1} = \mathcal{G}\bigg(\sum_{k=1}^{\ell}\big(\hat{f}_{\ell+1-k,xx}\hat{f}_k - \tfrac{1}{2}\hat{f}_{\ell+1-k,x}\hat{f}_{k,x} - 2\hat{f}_{\ell+1-k}\hat{f}_k\big)$$
$$+ 2(u_{xx}-4u)\sum_{k=0}^{\ell}\hat{f}_{\ell-k}\hat{f}_k\bigg), \quad \ell\in\mathbb{N}, \tag{B.11}$$

assuming

$$\hat{f}_\ell \in L^\infty(\mathbb{R}), \quad \ell \in \mathbb{N}. \tag{B.12}$$

Moreover, one infers for the E_m*-dependence of the integration constants* c_ℓ, $\ell = 0,\dots,n$, *in* F_n,

$$c_\ell = c_\ell(\underline{E}), \quad \ell = 0,\dots,n \tag{B.13}$$

and

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E})\hat f_k, \quad \ell = 0,\dots,n, \tag{B.14}$$

$$\hat f_\ell = \sum_{k=0}^{\ell} \hat c_{\ell-k}(\underline{E}) f_k, \quad \ell = 0,\dots,n. \tag{B.15}$$

Proof. Dividing F_n by $R_{2n+1}^{1/2}$ (temporarily fixing the branch of $R_{2n+1}(z)^{1/2}$ as z^{n+1} near infinity), one obtains

$$\frac{F_n(z)}{R_{2n+1}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat c_k(\underline{E}) z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n} f_\ell z^{-\ell-1}\bigg) = \sum_{\ell=0}^{\infty} \check f_\ell z^{-\ell-1} \tag{B.16}$$

for some coefficients $\check f_\ell$ to be determined next. Dividing (2.41) by R_{2n+1}, and inserting the expansion (B.16) into the resulting equation then yields the recursion relation (B.11) (with $\hat f_\ell$ replaced by $\check f_\ell$). More precisely, for $\check f_1$ one originally obtains the relation

$$-\check f_{1,xx} + 4\check f_1 = 2(u_{xx} - 4u), \text{ that is, } \Big(-\frac{d^2}{dx^2} + 4\Big)\Big(\check f_1 + 2u\Big) = 0. \tag{B.17}$$

Thus,

$$\check f_1(x) = -2u(x) + a_1 e^{2x} + b_1 e^{-2x} \tag{B.18}$$

for some $a_1, b_1 \in \mathbb{C}$, and hence the requirement $\check f_1 \in L^\infty(\mathbb{R})$ then yields $a_1 = b_1 = 0$. The open sign of $\check f_0$ has been chosen such that $\check f_0 = \hat f_0 = 1$. For $\ell \geq 2$ one obtains similarly

$$\begin{aligned} -\check f_{\ell+1,xx} + 4\check f_{\ell+1} = \bigg(&\sum_{k=1}^{\ell} \big(\check f_{\ell+1-k,xx}\check f_k - \tfrac12 \check f_{\ell+1-k,x}\check f_{k,x} - 2\check f_{\ell+1-k}\check f_k\big) \\ &+ 2(u_{xx} - 4u)\sum_{k=0}^{\ell} \check f_{\ell-k}\check f_k\bigg), \quad \ell \geq 1, \end{aligned} \tag{B.19}$$

and hence,

$$\begin{aligned} \check f_{\ell+1} = \mathcal{G}\bigg(&\sum_{k=1}^{\ell} \big(\check f_{\ell+1-k,xx}\check f_k - \tfrac12 \check f_{\ell+1-k,x}\check f_{k,x} - 2\check f_{\ell+1-k}\check f_k\big) \\ &+ 2(u_{xx} - 4u)\sum_{k=0}^{\ell} \check f_{\ell-k}\check f_k\bigg) + a_{\ell+1}e^{2x} + b_{\ell+1}e^{-2x}, \quad \ell \geq 1 \end{aligned} \tag{B.20}$$

for some $a_{\ell+1}, b_{\ell+1} \in \mathbb{C}$. Again the requirement $\check f_{\ell+1} \in L^\infty(\mathbb{R})$ then yields $a_{\ell+1} = b_{\ell+1} = 0$, $\ell \geq 1$. Introducing $\hat f_\ell$ by (2.10) with $c_k = 0$, $k \in \mathbb{N}$, and $\check f_\ell$ by (B.11), a straightforward computation shows that

$$\begin{aligned} \check f_{\ell,x} = \mathcal{G}\bigg(&\sum_{k=1}^{\ell-1} \big(f_{\ell-k,xxx} - 4f_{\ell-k,x}\big) f_k - \sum_{k=0}^{\ell-1} 2\big(-2(u_{xx} - 4u) f_{\ell-k-1,x} \\ &+ (4u_x - u_{xxx}) f_{\ell-k-1}\big)\bigg) f_k \end{aligned}$$

$$\begin{aligned} &= \mathcal{G}\bigg(- \sum_{k=1}^{\ell-1} \mathcal{G}^{-1} f_{\ell-k,x} f_k + \sum_{k=0}^{\ell-1} \big(\mathcal{G}^{-1} f_{\ell-k,x}\big) f_k \bigg) \\ &= \hat{f}_{\ell,x}, \quad \ell \in \mathbb{N}. \end{aligned} \tag{B.21}$$

Hence,

$$\check{f}_\ell = \hat{f}_\ell + d_\ell, \quad \ell \in \mathbb{N} \tag{B.22}$$

for some constants $d_\ell \in \mathbb{C}$, $\ell \in \mathbb{N}$. Since $d_0 = d_1 = 0$ by inspection, we next proceed by induction on ℓ and suppose that

$$d_k = 0 \text{ and hence } \check{f}_k = \hat{f}_k \text{ for } k = 0, \dots, \ell. \tag{B.23}$$

Thus, (B.11) and (B.22) imply

$$\check{f}_{\ell+1} = \mathcal{G}\{\dots\} = \hat{f}_{\ell+1} + d_{\ell+1}, \tag{B.24}$$

where $\{\dots\}$ denotes the expression on the right-hand side of (B.11) in terms of $\check{f}_k = \hat{f}_k$, $k = 0, \dots, \ell$. Hence,

$$\{\dots\} - \hat{f}_{\ell+1} + \alpha_{\ell+1} e^{2x} + \beta_{\ell+1} e^{-2x} = \mathcal{G}^{-1} d_{\ell+1} = 4 d_{\ell+1} \tag{B.25}$$

for some constants $\alpha_{\ell+1}, \beta_{\ell+1} \in \mathbb{C}$. Since $\{\dots\} - \hat{f}_{\ell+1} \in L^\infty(\mathbb{R})$, one concludes once more that $\alpha_{\ell+1} = \beta_{\ell+1} = 0$. Moreover, since $\{\dots\} - \hat{f}_{\ell+1}$ contains no constants by construction, one concludes $d_{\ell+1} = 0$ and hence

$$\check{f}_\ell = \hat{f}_\ell \text{ for all } \ell \in \mathbb{N}_0. \tag{B.26}$$

Thus, we proved

$$\frac{F_n(z)}{R_{2n+1}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k} \bigg) \bigg(\sum_{\ell=0}^{n} f_\ell z^{-\ell-1} \bigg) = \sum_{\ell=0}^{\infty} \hat{f}_\ell z^{-\ell-1} \tag{B.27}$$

and hence (B.10). A comparison of coefficients in (B.27) then proves (B.15). Next, multiplying (B.3) and (B.6), a comparison of coefficients of z^{-k} yields

$$\sum_{\ell=0}^{k} \hat{c}_{k-\ell}(\underline{E}) c_\ell(\underline{E}) = \delta_{k,0}, \quad k \in \mathbb{N}_0. \tag{B.28}$$

Thus, one computes

$$\begin{aligned} &\sum_{m=0}^{\ell} c_{\ell-m}(\underline{E}) \hat{f}_m = \sum_{m=0}^{\ell} \sum_{k=0}^{m} c_{\ell-m}(\underline{E}) \hat{c}_{m-k}(\underline{E}) f_k = \sum_{k=0}^{\ell} \sum_{p=k}^{\ell} c_{\ell-p}(\underline{E}) \hat{c}_{p-k}(\underline{E}) f_k \\ &= \sum_{k=0}^{\ell} \bigg(\sum_{m=0}^{\ell-k} c_{\ell-k-m}(\underline{E}) \hat{c}_m(\underline{E}) \bigg) f_k = f_\ell, \quad \ell = 0, \dots, n, \end{aligned} \tag{B.29}$$

applying (B.28). Hence one obtains (B.14) and thus (B.13). ∎

Appendix C. Symmetric Functions and their Theta Function Representations

In this appendix we consider Dubrovin-type equations for auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}}$ of degree n on $\mathcal{K}_n$ and study in detail elementary symmetric functions associated with the projections μ_j of $\hat{\mu}_j$, $j = 1, \dots, n$. In addition to various applications of Lagrange interpolation formulas we derive explicit theta function representations of elementary symmetric functions of μ_j, $j = 1, \dots, n$. While some of the material of this appendix is classical, some parts are taken from [45] (cf. also [46, App. F and G]), and [55]. Proofs are only presented for results that do not appear to belong to the standard arsenal of the literature on hierarchies of soliton equations. Our principal results on theta function representations derived in Sections 3 and 4 are based on Theorem C.6. The results of this appendix apply to a variety of soliton equations and hence are of independent interest.

Assuming $n \in \mathbb{N}$ to be fixed and introducing

$$\mathcal{S}_k = \{\underline{\ell} = (\ell_1, \dots, \ell_k) \in \mathbb{N}^k \mid \ell_1 < \cdots < \ell_k \leq n\}, \quad 1 \leq k \leq n, \tag{C.1}$$

$$\mathcal{I}_k^{(j)} = \{\underline{\ell} = (\ell_1, \dots, \ell_k) \in \mathcal{S}_k \mid \ell_m \neq j\}, \quad 1 \leq k \leq n-1, \tag{C.2}$$

one defines

$$\Psi_0(\underline{\mu}) = 1, \quad \Psi_k(\underline{\mu}) = (-1)^k \sum_{\underline{\ell} \in \mathcal{S}_k} \mu_{\ell_1} \cdots \mu_{\ell_k}, \quad 1 \leq k \leq n, \tag{C.3}$$

$$\Phi_0^{(j)}(\underline{\mu}) = 1, \quad \Phi_k^{(j)}(\underline{\mu}) = (-1)^k \sum_{\underline{\ell} \in \mathcal{I}_k^{(j)}} \mu_{\ell_1} \cdots \mu_{\ell_k}, \quad 1 \leq k \leq n-1, \quad \Phi_n^{(j)}(\underline{\mu}) = 0, \tag{C.4}$$

where $\underline{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$. Explicitly, one verifies

$$\Psi_1(\underline{\mu}) = -\sum_{\ell=1}^n \mu_\ell, \quad \Psi_2(\underline{\mu}) = \sum_{\substack{\ell_1, \ell_2 = 1 \\ \ell_1 < \ell_2}}^n \mu_{\ell_1} \mu_{\ell_2}, \text{ etc.}, \tag{C.5}$$

$$\Phi_1^{(j)}(\underline{\mu}) = -\sum_{\substack{\ell=1 \\ \ell \neq j}}^n \mu_\ell, \quad \Phi_2^{(j)}(\underline{\mu}) = \sum_{\substack{\ell_1, \ell_2 = 1 \\ \ell_1, \ell_2 \neq j \\ \ell_1 < \ell_2}}^n \mu_{\ell_1} \mu_{\ell_2}, \text{ etc.} \tag{C.6}$$

Introducing

$$F_n(z) = \prod_{j=1}^n (z - \mu_j) = \sum_{\ell=0}^n f_{n-\ell} z^\ell = \sum_{\ell=0}^n \Psi_{n-\ell}(\underline{\mu}) z^\ell, \tag{C.7}$$

one infers ($F_n'(z) = \partial F_n(z)/\partial z$)

$$F_n'(\mu_k) = \prod_{\substack{j=1 \\ j \neq k}}^n (\mu_k - \mu_j). \tag{C.8}$$

The general form of Lagrange's interpolation theorem (cf., e.g., [46, App. F], [59, App. E]) then reads as follows.

Theorem C.1. *Assume that $\mu_1,\dots,\mu_n$ are n distinct complex numbers. Then*

$$\sum_{j=1}^{n} \frac{\mu_j^{m-1}}{F_n'(\mu_j)} \Phi_k^{(j)}(\underline{\mu}) = \delta_{m,n-k} - \Psi_{k+1}(\underline{\mu})\delta_{m,n+1}, \tag{C.9}$$
$$m=1,\dots,n+1, \quad k=0,\dots,n-1.$$

The simplest Lagrange interpolation formula reads in the case $k=0$,

$$\sum_{j=1}^{n} \frac{\mu_j^{m-1}}{F_n'(\mu_j)} = \delta_{m,n}, \quad m=1,\dots,n. \tag{C.10}$$

For use in the main text we also recall the following results.

Lemma C.2 ([45], [46, App. F]). *Assume that $\mu_1,\dots,\mu_n$ are n distinct complex numbers. Then*

$$(i) \quad \Psi_{k+1}(\underline{\mu}) + \mu_j \Phi_k^{(j)}(\underline{\mu}) = \Phi_{k+1}^{(j)}(\underline{\mu}), \quad k=0,\dots,n-1,\ j=1,\dots,n. \tag{C.11}$$

$$(ii) \quad \sum_{\ell=0}^{k} \Psi_{k-\ell}(\underline{\mu})\mu_j^{\ell} = \Phi_k^{(j)}(\underline{\mu}), \quad k=0,\dots,n,\ j=1,\dots,n. \tag{C.12}$$

$$(iii) \quad \sum_{\ell=0}^{k-1} \Phi_{k-1-\ell}^{(j)}(\underline{\mu}) z^{\ell} = \frac{1}{z-\mu_j}\bigg(\sum_{\ell=0}^{k} \Psi_{k-\ell}(\underline{\mu}) z^{\ell} - \Phi_k^{(j)}(\underline{\mu})\bigg), \tag{C.13}$$
$$k=0,\dots,n,\ j=1,\dots,n.$$

Next, assuming $\mu_j \neq \mu_{j'}$ for $j \neq j'$, introduce the $n\times n$ matrix $U_n(\underline{\mu})$ by

$$U_1(\underline{\mu}) = 1, \quad U_n(\underline{\mu}) = \left(\frac{\mu_k^{j-1}}{\prod_{\substack{m=1\\ m\neq k}}^{n}(\mu_k-\mu_m)}\right)_{j,k=1}^{n}, \tag{C.14}$$

where $\underline{\mu} = (\mu_1,\dots,\mu_n) \in \mathbb{C}^n$.

Lemma C.3 ([45], [46, App. G]). *Suppose $\mu_j \in \mathbb{C}$, $j=1,\dots,n$, are n distinct complex numbers. Then*

$$U_n(\underline{\mu})^{-1} = \left(\Phi_{n-k}^{(j)}(\underline{\mu})\right)_{j,k=1}^{n}. \tag{C.15}$$

Next, we express f_ℓ, $F_n(\mu_j)$, and $\widetilde{F}_r(\mu_j)$ in terms of elementary symmetric functions of $\mu_1,\dots,\mu_n$. We start with the homogeneous expressions denoted by $\hat{f}_\ell$ and $\widehat{F}_r(\mu_j)$, where $c_k=0$, $k=0,\dots,\ell$ and $\tilde{c}_s=0$, $s=1,\dots,r$. Let $\hat{c}_k(\underline{E})$, $k\in\mathbb{N}_0$, be defined as in (B.4) and suppose $r\in\mathbb{N}_0$. Then, combining (B.15) and (C.7) one infers

$$\hat{f}_\ell = \sum_{k=0}^{\ell} \hat{c}_{\ell-k}(\underline{E})\Psi_k(\underline{\mu}), \quad \ell=0,\dots,n. \tag{C.16}$$

Next, we turn to $\widehat{F}_r(\mu_j)$.

Lemma C.4. *Let $r\in\mathbb{N}_0$. Then*[1],

$$\widehat{F}_r(\mu_j) = \sum_{s=(r-n)\vee 0}^{r} \hat{c}_s(\underline{E})\Phi_{r-s}^{(j)}(\underline{\mu}). \tag{C.17}$$

[1] $m\vee n = \max\{m,n\}$.

Proof. By definition

$$\widehat{F}_r(z) = \sum_{\ell=0}^{r} \hat{f}_{r-\ell} z^\ell = \sum_{\ell=0}^{r} z^\ell \sum_{m=0}^{(r-\ell)\wedge n} \Psi_m(\underline{\mu}) \hat{c}_{r-\ell-m}(\underline{E}). \tag{C.18}$$

Consider first the case $r \leq n$. Then

$$\widehat{F}_r(z) = \sum_{s=0}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_{r-\ell-s}(\underline{\mu}) z^\ell \tag{C.19}$$

and hence

$$\widehat{F}_r(\mu_j) = \sum_{s=0}^{r} \hat{c}_s(\underline{E}) \Phi_{r-s}^{(j)}(\underline{\mu}), \tag{C.20}$$

using (C.12). In the case where $r \geq n+1$ we find (applying (C.7))

$$\begin{aligned}
\widehat{F}_r(z) &= \sum_{m=0}^{n} \Psi_m(\underline{\mu}) \sum_{s=0}^{r-m} z^{r-m-s} \hat{c}_s(\underline{E}) \\
&= \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) \bigg(\sum_{\ell=0}^{n} \Psi_\ell(\underline{\mu}) z^{n-\ell} \bigg) z^{r-n-s} + \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_\ell(\underline{\mu}) z^{r-s-\ell} \\
&= F_n(z) \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} + \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_\ell(\underline{\mu}) z^{r-s-\ell} \qquad (C.21)\\
&= F_n(z) \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} + \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell}(\underline{\mu}) z^\ell.
\end{aligned}$$

Hence

$$\widehat{F}_r(\mu_j) = \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \Phi_{r-s}^{(j)}(\underline{\mu}), \tag{C.22}$$

using (C.12) again. □

Introducing

$$d_{\ell,k}(\underline{E}) = \sum_{m=0}^{\ell-k} c_{\ell-k-m}(\underline{E}) \hat{c}_m(\underline{E}), \quad k = 0, \dots, \ell, \ \ell = 0, \dots, n, \tag{C.23}$$

$$\tilde{d}_{r,k}(\underline{E}) = \sum_{s=0}^{r-k} \tilde{c}_{r-k-s} \hat{c}_s(\underline{E}), \quad k = 0, \dots, r \wedge n, \tag{C.24}$$

for a given set of constants $\{\tilde{c}_s\}_{s=1,\dots,r} \subset \mathbb{C}$, the corresponding nonhomogeneous quantities f_ℓ, $F_n(\mu_j)$, and $\widetilde{F}_r(\mu_j)$ are then given by[2]

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \hat{f}_k = \sum_{k=0}^{\ell} d_{\ell,k}(\underline{E}) \Psi_k(\underline{\mu}), \quad \ell = 0, \dots, n, \tag{C.25}$$

$$F_n(\mu_j) = \sum_{\ell=0}^{n} c_{n-\ell}(\underline{E}) \widehat{F}_\ell(\mu_j) = \sum_{\ell-0}^{n} d_{n,\ell}(\underline{E}) \Phi_\ell^{(j)}(\underline{\mu}), \quad c_0 = 1, \tag{C.26}$$

[2] $m \wedge n = \min\{m, n\}$.

$$\widetilde{F}_r(\mu_j) = \sum_{s=0}^{r} \tilde{c}_{r-s} \widehat{F}_s(\mu_j) = \sum_{k=0}^{r\wedge n} \tilde{d}_{r,k}(\underline{E}) \Phi_k^{(j)}(\underline{\mu}), \quad r \in \mathbb{N}_0, \; \tilde{c}_0 = 1, \tag{C.27}$$

using (B.13) and (B.14). Here $c_k(\underline{E})$, $k \in \mathbb{N}_0$, is defined by (B.7).

Next, we prove a result needed in the proof of Theorem 4.10.

Lemma C.5. *Suppose $r \in \mathbb{N}_0$, $(x, t_r) \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{R}^2$ is open and connected, and assume $\mu_j \neq \mu_{j'}$ on Ω_μ for $j \neq j'$, $j, j' = 1, \dots, n$. Then,*

$$\widetilde{F}_{r,x}(z, x, t_r) = \sum_{j=1}^{n} \big(\widetilde{F}_r(\mu_j(x, t_r), x, t_r) - \widetilde{F}_r(z, x, t_r)\big) \frac{\mu_{j,x}(x, t_r)}{(z - \mu_j(x, t_r))}. \tag{C.28}$$

Proof. It suffices to prove (C.28) for the homogeneous case where $\widetilde{F}_r$ is replaced by $\widehat{F}_r$. Using

$$\Psi_{k,x}(\underline{\mu}) = -\sum_{j=1}^{n} \mu_{j,x} \Phi_{k-1}^{(j)}(\underline{\mu}), \quad k = 0, \dots, n, \tag{C.29}$$

with the convention

$$\Phi_{-1}^{(j)}(\underline{\mu}) = 0, \quad j = 1, \dots, n, \tag{C.30}$$

one computes for $r \leq n$,

$$\begin{aligned}
\widehat{F}_{r,x}(z) &= \sum_{s=0}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell,x}(\underline{\mu}) z^\ell \\
&= -\sum_{j=1}^{n} \mu_{j,x} \sum_{s=0}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Phi_{r-s-\ell-1}^{(j)}(\underline{\mu}) z^\ell \\
&= \sum_{j=1}^{n} \mu_{j,x} (z - \mu_j)^{-1} \sum_{s=0}^{r} \hat{c}_s(\underline{E}) \bigg(\Phi_{r-s}^{(j)}(\underline{\mu}) - \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell}(\underline{\mu}) z^\ell \bigg) \\
&= \sum_{j=1}^{n} \big(\widehat{F}_r(\mu_j) - \widehat{F}_r(z)\big) \mu_{j,x} (z - \mu_j)^{-1},
\end{aligned} \tag{C.31}$$

applying (C.13), (C.19), and (C.20). For $r \geq n+1$ one obtains from (C.13), (C.21), and (C.22),

$$\begin{aligned}
\widehat{F}_{r,x}(z) &= F_{n,x}(z) \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} + \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell,x}(\underline{\mu}) z^\ell \\
&= -F_n(z) \sum_{j=1}^{n} \mu_{j,x} (z - \mu_j)^{-1} \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} \\
&\quad - \sum_{j=1}^{n} \mu_{j,x} \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Phi_{r-s-\ell-1}^{(j)}(\underline{\mu}) z^\ell \\
&= -F_n(z) \sum_{j=1}^{n} \mu_{j,x} (z - \mu_j)^{-1} \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} \\
&\quad + \sum_{j=1}^{n} \mu_{j,x} (z - \mu_j)^{-1} \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \bigg(\Phi_{r-s}^{(j)}(\underline{\mu}) - \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell}(\underline{\mu}) z^\ell \bigg)
\end{aligned}$$

$$= \sum_{j=1}^{n} \big(\widehat{F}_r(\mu_j) - \widehat{F}_r(\underline{\mu})\big)\mu_{j,x}(z-\mu_j)^{-1}. \tag{C.32}$$

$\square$

Next we turn to a detailed discussion of elementary symmetric functions of $\{\mu_1, \dots, \mu_n\}$. Given the nonsingular hyperelliptic curve $\mathcal{K}_n$ in (A.1), (A.2), we introduce the first-order Dubrovin-type system

$$\frac{\partial \mu_j(\underline{v})}{\partial v_k} = \Phi^{(j)}_{n-k}(\underline{\mu}(\underline{v}))\frac{y(\hat{\mu}_j(\underline{v}))}{\prod_{\substack{m=1\\ m\neq j}}^n(\mu_j(\underline{v})-\mu_m(\underline{v}))}, \quad j,k=1,\dots,n, \qquad \underline{v}=(v_1,\dots,v_n)\in\mathcal{V}, \tag{C.33}$$

with initial conditions

$$\{\hat{\mu}_j(\underline{v}_0)\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{C.34}$$

for some $\underline{v}_0 \in \mathcal{V}$, where $\mathcal{V} \subseteq \mathbb{C}^n$ is an open connected set such that μ_j remain distinct on $\mathcal{V}$, $\mu_j \neq \mu_{j'}$ for $j\neq j'$, $j,j'=1,\dots,n$, and $\Phi^{(j)}_{n-k}(\underline{\mu})\neq 0$, $j,k=1,\dots,n$. One then obtains, using (C.33) and (C.9),

$$\begin{aligned}
\frac{\partial}{\partial v_k}\sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(\underline{v})}\frac{z^{k-1}}{y} &= \sum_{j=1}^n \frac{\mu_j(\underline{v})^{k-1}}{y(\hat{\mu}_j(\underline{v}))}\frac{\partial \mu_j(\underline{v})}{\partial v_k} \\
&= \sum_{j=1}^n \frac{\mu_j(\underline{v})^{k-1}}{y(\hat{\mu}_j(\underline{v}))}\Phi^{(j)}_{n-k}(\underline{\mu}(\underline{v}))\frac{y(\hat{\mu}_j(\underline{v}))}{\prod_{\substack{m=1\\ m\neq j}}^n(\mu_j(\underline{v})-\mu_m(\underline{v}))} \\
&= \sum_{j=1}^n \Phi^{(j)}_{n-k}(\underline{\mu}(\underline{v}))\frac{\mu_j(\underline{v})^{k-1}}{\prod_{\substack{m=1\\ m\neq j}}^n(\mu_j(\underline{v})-\mu_m(\underline{v}))} = 1,
\end{aligned} \tag{C.35}$$

implying

$$\sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(\underline{v})}\frac{z^{k-1}dz}{y} - \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(\underline{v}_0)}\frac{z^{k-1}dz}{y} = (\underline{v})_k - (\underline{v}_0)_k, \qquad k=1,\dots,n, \quad \underline{v},\underline{v}_0\in\mathcal{V}. \tag{C.36}$$

Moreover, introducing

$$v_{n+1}(\underline{v}) = \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(\underline{v})}\frac{z^n dz}{y}, \tag{C.37}$$

one then computes as in (C.35)

$$\frac{\partial v_{n+1}(\underline{v})}{\partial v_k} = -\Psi_{n+1-k}(\underline{\mu}(\underline{v})), \quad k=1,\dots,n, \tag{C.38}$$

using

$$\sum_{\ell=1}^n \Phi^{(\ell)}_{n-p}(\underline{\mu})\frac{\mu_\ell^n}{\prod_{\substack{q=1\\ q\neq \ell}}^n(\mu_\ell-\mu_q)} = -\Psi_{n+1-p}(\underline{\mu}), \quad p=1,\dots,n \tag{C.39}$$

(cf. (C.9)). Thus, one concludes

$$\prod_{j=1}^n(z-\mu_j(\underline{v})) = \sum_{\ell=0}^n \Psi_{n-\ell}(\underline{\mu}(\underline{v}))z^\ell = z^n - \sum_{k=1}^n \frac{\partial v_{n+1}(\underline{v})}{\partial v_k}z^{k-1}, \quad \underline{v}\in\mathcal{V}, \tag{C.40}$$

whenever $\underline{\mu}$ satisfies (C.33).

In order to derive theta function representations of the elementary symmetric functions $\Psi_k(\underline{\mu})$ of $\mu_1(\underline{v}), \dots, \mu_n(\underline{v})$, $k = 1, \dots, n$ we recall that $\mathcal{K}_n$ corresponds to the curve $y^2 = \prod_{m=0}^{2n+2}(z - E_m)$ with pairwise distinct $E_m \in \mathbb{C}$, $m = 0, \dots, 2n+2$ (cf. (A.1) and (A.2)). Using the notation established in Appendix A, $v_{n+1}(\underline{v})$ can be written as

$$v_{n+1}(\underline{v}) = \sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j(\underline{v})} \frac{z^n dz}{y} = \sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j(\underline{v})} \widetilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}}, \tag{C.41}$$

where

$$\widetilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}} = z^n dz / y \tag{C.42}$$

represents a differential of the third kind with simple poles at P_{∞_+} and P_{∞_-} and corresponding residues $+1$ and -1, respectively. This differential is not normalized, that is, the a-periods of $\widetilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}}$ are not all vanishing. We also introduce the notation

$$\underline{z}(P, \underline{\mu}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\mu}}), \quad P \in \mathcal{K}_n,\ \underline{\mu} = \{\mu_1, \dots, \mu_n\} \in \sigma^n \mathcal{K}_n \tag{C.43}$$

in connection with $\mathcal{K}_n$, and similarly,

$$\underline{\hat{z}}(P, \underline{\mu}) = \widehat{\underline{\Xi}}_{Q_0} - \widehat{\underline{A}}_{Q_0}(P) + \widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\mu}}), \quad P \in \mathcal{K}_n,\ \underline{\mu} = \{\mu_1, \dots, \mu_n\} \in \sigma^n \widehat{\mathcal{K}}_n \tag{C.44}$$

in connection with $\widehat{\mathcal{K}}_n$. Moreover, we conveniently choose $Q_0 \in \partial \widehat{\mathcal{K}}_n$ (e.g., the initial point of the curve $a_1 \subset \partial \widehat{\mathcal{K}}_n$).

Theorem C.6. *Suppose $\mathcal{D}_{\underline{\hat{\mu}}} \in \sigma^n \widehat{\mathcal{K}}_n$ is nonspecial, $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\} \in \sigma^n \widehat{\mathcal{K}}_n$. Then,*

$$\begin{aligned}\sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j} \widetilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}} = \sum_{j=1}^{n} \bigg(\int_{a_j} \widetilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}} \bigg) \bigg(\sum_{k=1}^{n} \int_{Q_0}^{\hat{\mu}_k} \omega_j - \sum_{k=1}^{n} \int_{a_k} \big(\widehat{\underline{A}}_{Q_0}\big)_j \omega_k \bigg) \\ + \ln \bigg(\frac{\theta(\underline{\hat{z}}(P_{\infty_+}, \underline{\hat{\mu}}))}{\theta(\underline{\hat{z}}(P_{\infty_-}, \underline{\hat{\mu}}))} \bigg)\end{aligned} \tag{C.45}$$

and

$$\Psi_{n+1-k}(\underline{\mu}) = \Psi_{n+1-k}(\underline{\lambda}) - \sum_{j=1}^{n} c_j(k) \frac{\partial}{\partial w_j} \ln \bigg(\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}) + \underline{w})}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}) + \underline{w})} \bigg) \bigg|_{\underline{w}=0}, \quad k = 1, \dots, n, \tag{C.46}$$

with $\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ introduced in (A.24).

Proof. Let $\mathcal{D}_{\underline{\hat{\mu}}} \in \sigma^n \widehat{\mathcal{K}}_n$ be a nonspecial divisor on $\widehat{\mathcal{K}}_n$, $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\} \in \sigma^n \widehat{\mathcal{K}}_n$. Introducing

$$\widetilde{\Omega}^{(3)}(P) = \int_{Q_0}^{P} \widetilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}}, \quad P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}, \tag{C.47}$$

we can render $\widetilde{\Omega}^{(3)}(\,\cdot\,)$ single-valued on

$$\widehat{\widehat{\mathcal{K}}}_n = \widehat{\mathcal{K}}_n \setminus \Sigma, \tag{C.48}$$

where Σ denotes the union of cuts

$$\Sigma = \Sigma(P_{\infty_+}) \cup \Sigma(P_{\infty_-}), \quad \Sigma(P_{\infty_+}) \cap \Sigma(P_{\infty_-}) = \{Q_0\}, \tag{C.49}$$

with $\Sigma(P_{\infty_+})$ (resp., $\Sigma(P_{\infty_-})$) a cut connecting Q_0 and P_{∞_+} (resp., P_{∞_-}) through the open interior $\widehat{\mathcal{K}}_n$ (i.e., avoiding all curves $a_j, b_j, a_j^{-1}, b_j^{-1}$, $j=1,\dots,n$, with the exception of the point $Q_0 \in \partial\widehat{\mathcal{K}}_n$), avoiding the points $\hat{\mu}_j$, $j=1,\dots,n$. The left and right side of the cut $\Sigma(P_{\infty_\pm})$ is denoted by $\Sigma(P_{\infty_\pm})_\ell$ and $\Sigma(P_{\infty_\pm})_r$. The oriented boundary $\partial\widehat{\widehat{\mathcal{K}}}_n$ of $\widehat{\widehat{\mathcal{K}}}_n$, in obvious notation, is then given by

$$\partial\widehat{\widehat{\mathcal{K}}}_n = \Sigma(P_{\infty_+})_\ell \cup \Sigma(P_{\infty_+})_r \cup \Sigma(P_{\infty_-})_\ell \cup \Sigma(P_{\infty_-})_r \cup \partial\widehat{\mathcal{K}}_n, \tag{C.50}$$

that is, it consists of $\partial\widehat{\mathcal{K}}_n$ together with the piece from Q_0 to P_{∞_+} along the left side of the cut $\Sigma(P_{\infty_+})$ and then back to Q_0 along the right side of $\Sigma(P_{\infty_+})$, plus the corresponding pieces from Q_0 to P_{∞_-} and back to Q_0 along the cut $\Sigma(P_{\infty_-})$, preserving orientation. Introducing the meromorphic differential,

$$\nu = d\ln(\theta(\underline{z}(\,\cdot\,,\underline{\mu})), \tag{C.51}$$

the residue theorem applied to $\widetilde{\Omega}^{(3)}\nu$ yields

$$\begin{aligned}\int_{\partial\widehat{\mathcal{K}}_n} \widetilde{\Omega}^{(3)}\nu = \sum_{j=1}^n \bigg(\Big(\int_{a_j} \widetilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}\Big)\Big(\int_{b_j}\nu\Big) - \Big(\int_{b_j} \widetilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}\Big)\Big(\int_{a_j}\nu\Big)\bigg)\\ + \int_{\Sigma} \widetilde{\Omega}^{(3)}\nu = 2\pi i \sum_{P\in\widehat{\widehat{\mathcal{K}}}_n} \operatorname*{res}_P\big(\widetilde{\Omega}^{(3)}\nu\big).\end{aligned} \tag{C.52}$$

Investigating separately the items occurring in (C.52) then yields the following facts:

$$\sum_{P\in\widehat{\mathcal{K}}_n} \operatorname*{res}_P\big(\widetilde{\Omega}^{(3)}\nu\big) = \sum_{j=1}^n \widetilde{\Omega}^{(3)}(\hat{\mu}_j) = \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j} \widetilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}, \tag{C.53}$$

$$\int_{a_j}\nu = 0, \quad j=1,\dots,n, \tag{C.54}$$

$$\int_{b_j}\nu = 2\pi i\big(\big(\widehat{\underline{\Xi}}_{Q_0}\big)_j - \big(\underline{\widehat{A}}_{Q_0}(R(a_j))\big)_j + \big(\underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\mu}})\big)_j\big) - i\pi\tau_{j,j}, \quad j=1,\dots,n, \tag{C.55}$$

applying (A.33) in (C.54) and (C.55). Here $R(a_j)$ denotes the end point of $a_j \subset \partial\widehat{\mathcal{K}}_n$, $j=1,\dots,n$. In addition, the cut Σ produces the contribution

$$\int_{\Sigma} \widetilde{\Omega}^{(3)}\nu = 2\pi i\bigg(\int_{Q_0}^{P_{\infty_+}}\nu - \int_{Q_0}^{P_{\infty_-}}\nu\bigg) = 2\pi i\int_{P_{\infty_-}}^{P_{\infty_+}}\nu = 2\pi i\ln\bigg(\frac{\theta(\underline{\hat{z}}(P_{\infty_+},\underline{\mu}))}{\theta(\underline{\hat{z}}(P_{\infty_-},\underline{\mu}))}\bigg), \tag{C.56}$$

since (by an application of the residue theorem)

$$\widetilde{\Omega}^{(3)}(\hat{\mu}_\ell) - \widetilde{\Omega}^{(3)}(\hat{\mu}_r) = \pm 2\pi i, \quad \hat{\mu}_\ell \in \Sigma(P_{\infty_\pm})_\ell,\ \hat{\mu}_r \in \Sigma(P_{\infty_\pm})_r, \tag{C.57}$$

where $\hat{\mu}_\ell \in \Sigma(P_{\infty_\pm})_\ell$ and $\hat{\mu} \in \Sigma(P_{\infty_\pm})_r$ are on opposite sides of the cut $\Sigma(P_{\infty_\pm})$. Recalling the well-known results,

$$\big(\underline{\widehat{A}}_{Q_0}(R(a_j))\big)_j = \frac{1}{2} + \int_{a_j} \big(\underline{\widehat{A}}_{Q_0}\big)_j\omega_j, \quad j=1,\dots,n, \tag{C.58}$$

$$\big(\widehat{\underline{\Xi}}_{Q_0}\big)_j = \frac{1}{2}(1+\tau_{j,j}) - \sum_{\substack{k=1\\k\neq j}}^n \int_{a_k} \big(\widehat{\underline{A}}_{Q_0}\big)_j \omega_k, \quad j=1,\dots,n, \tag{C.59}$$

equations (C.52)–(C.59) imply

$$\begin{aligned}\sum_{j=1}^n \int_{Q_0}^{\hat\mu_j} \widetilde\omega^{(3)}_{P_{\infty_+},P_{\infty_-}} &= \sum_{j=1}^n \bigg(\int_{a_j} \widetilde\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg)\bigg(\sum_{k=1}^n \int_{Q_0}^{\hat\mu_k} \omega_j - \sum_{k=1}^n \int_{a_k} \big(\widehat{\underline{A}}_{Q_0}\big)_j \omega_k\bigg)\\ &\quad + \ln\bigg(\frac{\theta(\underline{\hat z}(P_{\infty_+},\underline{\mu}))}{\theta(\underline{\hat z}(P_{\infty_-},\underline{\mu}))}\bigg).\end{aligned} \tag{C.60}$$

This proves (C.45).

In the following we will apply (C.60) to $\hat\mu_j$, $j=1,\dots,n$ satisfying the first-order system (C.33), (C.34) on some open connected set $\mathcal{V}$ such that μ_j, $j=1,\dots,n$, remain distinct on $\mathcal{V}$ and $\Phi^{(j)}_{n-k}(\underline{\mu}) \neq 0$ on $\mathcal{V}$, $j,k=1,\dots,n$. Using (A.12), (C.35), and (C.9) one computes

$$\begin{aligned}\frac{\partial}{\partial v_k}\big(\underline{\widehat\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(\underline{v})})\big)_j &= \frac{\partial}{\partial v_k}\sum_{\ell=1}^n \int_{Q_0}^{\hat\mu_\ell(\underline{v})} \omega_j = \sum_{\ell,m=1}^n \frac{\partial}{\partial v_k}\int_{Q_0}^{\hat\mu_\ell(\underline{v})} c_j(m)\eta_m\\ &= \sum_{\ell,m=1}^n c_j(m)\frac{\mu_\ell(\underline{v})^{m-1}}{y(\hat\mu_\ell(\underline{v}))}\frac{\partial}{\partial v_k}\mu_\ell(\underline{v})\\ &= \sum_{\ell,m=1}^n c_j(m)\Phi^{(\ell)}_{n-k}\frac{\mu_\ell(\underline{v})^{m-1}}{\prod_{\substack{\ell'=1\\ \ell'\neq\ell}}^n(\mu_\ell(\underline{v})-\mu_{\ell'}(\underline{v}))}\\ &= c_j(k), \quad \underline{v}\in\mathcal{V}.\end{aligned} \tag{C.61}$$

Thus, (C.40) and (C.61) imply

$$\begin{aligned}\Psi_{n+1-k}(\underline{\mu}(\underline{v})) &= -\frac{\partial v_{n+1}(\underline{v})}{\partial v_k} = -\sum_{j=1}^n c_j(k)\bigg(\int_{a_j}\widetilde\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg)\\ &\quad -\sum_{j=1}^n c_j(k)\frac{\partial}{\partial w_j}\ln\bigg(\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat\mu}(\underline{v}))+\underline{w})}{\theta(\underline{z}(P_{\infty_-},\underline{\hat\mu}(\underline{v}))+\underline{w})}\bigg)\bigg|_{\underline{w}=0},\\ &\qquad\qquad \underline{v}\in\mathcal{V},\ k=1,\dots,n.\end{aligned} \tag{C.62}$$

We replaced $\underline{\hat z}$ by $\underline{z}$ to arrive at (C.62) using properties (A.33) of θ. If $\hat\mu_j$, $j=1,\dots,n$, are distinct and $\Phi^{(j)}_{n-k}(\underline{\mu}) \neq 0$, $j,k=1,\dots,n$, we can choose $\hat\mu_j(\underline{v}_0)=\hat\mu_j$, $j=1,\dots,n$, and obtain (C.46). The general case where $\mathcal{D}_{\underline{\hat\mu}}$ is nonspecial, then follows from (C.62) by continuity, choosing $\mathcal{V}$ such that there exists a sequence $\underline{v}_p\in\mathcal{V}$ with $\underline{\hat\mu}(\underline{v}_p)\to\underline{\hat\mu}$ as $p\to\infty$. Finally, invoking the normal differential of the third kind in (A.24), $\omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = \prod_{j=1}^n(z-\lambda_j)dz/y$, corresponding to $\widetilde\omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = z^n dz/y$, a simple computation, combining (A.9), (A.10), (A.11), (A.24), and the normalization $\int_{a_j}\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}=0$, $j=1,\dots,n$, yields

$$\sum_{j=1}^n c_j(k)\bigg(\int_{a_j}\widetilde\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg) = \Psi_{n+1-k}(\underline{\lambda}), \quad k=1,\dots,n. \tag{C.63}$$

Equations (C.62) and (C.63) complete the proof of (C.46). □

Formulas (C.40), (C.45), and (C.46) (without explicit proofs and without the explicit form of the constant terms on the right-hand sides of (C.45) and (C.46)) have been used in [55] in the course of deriving algebro-geometric solutions of the Dym equation. Our approach based on the Dubrovin-type system (C.33) appears to be new. It can easily be adapted to the case of KdV-type hyperelliptic curves branched at infinity (cf. [46, App. B]). Since solutions of $1+1$-dimensional soliton equations typically can be expressed in terms of trace formulas involving elementary symmetric functions of (projections of) auxiliary divisors, results of the type of (C.46) are of general interest in this context.

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