Evaluation of Integrals Representing Correlations in XXX Heisenberg Spin Chain

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Abstract

We study XXX Heisenberg spin 1/2 anti-ferromagnet. We evaluate a probability of formation of a ferromagnetic string in the anti-ferromagnetic ground state in thermodynamics limit. We prove that for short strings the probability can be expressed in terms of Riemann zeta function with odd arguments.

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1. Introduction

Riemann zeta function for $Re(s) > 1$ can be defined as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.1}$$

It also can be represented as a product with respect to all prime numbers p

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1} \tag{1.2}$$

It can be analytically continued in the whole complex plane of s. It has only one pole, at $s = 1$. Riemann zeta function satisfies a functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(s\pi/2) \Gamma(1-s) \zeta(1-s) \tag{1.3}$$

It has 'trivial' zeros at $s = -2n$ ($n > 1$ is an integer). The famous Riemann hypothesis [15] states that nontrivial zeros belong to the straight line $Re(s) = 1/2$. Recently Montgomery and Odlyzko conjectured that for large values of imaginary part of s the distribution of zeros can be described by GUE of random matrices. Forrester and Odlyzko related the problem of distribution of zeros to Painleve differential equation and integrable integral operators [21]. Riemann zeta function is useful for study of distribution of prime numbers on the real axis [14]. The values of Riemann zeta function at special points were studied in [17] , [18] . At even values of its argument zeta function can be expressed in terms of powers of π and Bernoulli's numbers

$$\zeta(2n) = (-1)^{n+1} 2^{2n-1} \pi^{2n} B_{2n}/(2n)! \tag{1.4}$$

The values of Riemann zeta function at odd arguments provide infinitely many different irrational numbers [16] . Riemann zeta function plays an important role, not only in pure mathematics but also theoretical physics. Some Feynman diagrams in quantum field theory can be expressed in terms of $\zeta(n)$, see, for example, [1] . In statistical mechanics Riemann zeta function was used for the description of chaotic systems [19]. One can find more information and citation on the following web-cite http://www.maths.ex.ac.uk/ mwatkins/ .

We argue that $\zeta(n)$ is also important for exactly solvable models. The most famous integrable models is the Heisenberg XXX spin chain. This model was first suggested by Heisenberg [3] in 1928 and solved by Bethe [4] in 1931 . Since that time it found multiple applications in solid state physics and statistical mechanics.

The Hamiltonian of the XXX spin chain can be written like this

$$H = \sum_{i=1}^{N} (\sigma_i^x \sigma_{i+1}^x \ + \ \sigma_i^y \sigma_{i+1}^y \ + \ \sigma_i^z \sigma_{i+1}^z \ - 1 \ ) \tag{1.5}$$

Here N is the length of the lattice and $\sigma_i^x, \sigma_i^y, \sigma_i^z$ are Pauli matrices. We consider thermodynamics limit , when N goes to infinity. The sign in front of the Hamiltonian indicates that we are considering the anti-ferromagnetic case. We consider periodic boundary conditions. Notice that this Hamiltonian annihilates the ferromagnetic state [all spins up].

The construction of the anti-ferromagnetic ground state wave function $|AFM>$ can be credited to Hulthén [5]. An important correlation function was defined in [8]. It was called the emptiness formation probability

$$P(n) =< AFM| \prod_{j=1}^{n} P_j |AFM >$$

where $P_j = (1+\sigma_j^z)/2$ is a projector on the state with spin up in jth lattice site. Averaging is over the anti-ferromagnetic ground state. It describes the probability of formation of a ferromagnetic string of the length n in the anti-ferromagnetic background $|AFM>$. In this paper we shall first study short strings (n is small), in the end we shall discuss long distance asymptotics (at finite temperature). The four first values of the emptiness-formation probability look as follows:

$$P(1) \;=\; \frac{1}{2} \;=\; 0.5, \tag{1.6}$$

$$P(2) \;=\; \frac{1}{3}(1 \;-\; \ln 2) \;=\; 0.102284273, \tag{1.7}$$

$$P(3) \;=\; \frac{1}{4} \;-\; \ln 2 \;+\; \frac{3}{8}\,\zeta(3) \;=\; 0.007624158, \tag{1.8}$$

$$\begin{aligned} P(4) \;=\; & \frac{1}{5} \;-\; 2\ln 2 \;+\; \frac{173}{60}\zeta(3) \;-\; \frac{11}{6}\zeta(3)\ln 2 \;-\; \frac{51}{80}\zeta^2(3) \\ & -\; \frac{55}{24}\zeta(5) \;+\; \frac{85}{24}\zeta(5)\ln 2 \;=\; 0.000206270 \end{aligned} \tag{1.9}$$

Let us comment. The value of $P(1)$ is evident from the symmetry, $P(2)$ can be extracted from the explicit expression of the ground state energy [5]. $P(3)$ can be extracted from the results of M.Takahashi [9] on the calculation of the nearest neighbor correlation. It was confirmed in paper [10]. One should also mention independent calculation of $P(3)$ in [11]. One can express $P(3)$ in terms of next to the nearest neighbor correlation

$$<\; S_i^z S_{i+2}^z \;> =\; 2\,P(3) \;-\; 2\,P(2) \;+\; \frac{1}{2}\,P(1) \tag{1.10}$$

The calculation of $P(3)$ and $P(4)$ is discussed in this paper.

The expression above for $P(4)$ is our main result here.

We briefly announced our results in [2], here we provide the detailed derivation. The plan of the paper is as follows. In the next section we discuss a general procedure of the calculation of $P(n)$. We also show how this scheme works for $P(2)$. In the Appendices A and B we describe in detail the calculation of $P(3)$ and $P(4)$ respectively by means of the technique elaborated in the Section 2. The main results are summarized in the conclusion.

2. General discussion of the calculation of $P(n)$

There are several different approaches to investigate $P(n)$:

- **Representation of correlation functions as determinants of Fredholm integral operators** .This approach is based on following steps:

i. Quantum correlation function should be represented as a determinant of a Fredholm integral operators of a special type. We call these operators *integrable* integral operators.

ii. The determinant can be described by completely integrable equation of Painleve type.

iii. Asymptotic of correlation function [and the determinant] can be described by Riemann-Hilbert problem.

This approach was discovered in [23], it is described in detail in the book [6] .

It is interesting to mention that this approach was successfully applied also to matrix models [22]

- **Vertex operator approach** was developed in Kyoto by Foda, Jimbo, Miki, Miwa and Nakayashiki. This approach is based on study of representations of infinite dimensional quantum group $U_q\widehat{SL(2)}$, see [7] .

We shall use the integral representation obtained in [8]

$$P(n) = \int_C \frac{d\lambda_1}{2\pi i\lambda_1} \int_C \frac{d\lambda_2}{2\pi i\lambda_2} \cdots \int_C \frac{d\lambda_n}{2\pi i\lambda_n} \prod_{a=1}^{n} (1+\frac{i}{\lambda_a})^{n-a} (\frac{\pi\lambda_a}{\sinh \pi\lambda_a})^n \prod_{1\le j<k\le n} \frac{\sinh \pi(\lambda_k-\lambda_j)}{\pi(\lambda_k-\lambda_j-i)}. \quad (2.1)$$

The contour C in each integral goes parallel to the real axis with the imaginary part between 0 and $-i$. In the frame of algebraic Bethe Ansatz this formula was derived in [20]. Recently such formula was generalized in paper [12] to the case, where averaging is done over arbitrary Bethe state [with no strings] instead of anti-ferromagnetic state.

Let us describe in general a strategy that may be used for the calculation of $P(n)$. The integral formula (2.1) can be easily represented as follows:

$$P(n) = \prod_{j=1}^{n} \int_C \frac{d\lambda_j}{2\pi i}\, U_n(\lambda_1,\ldots,\lambda_n)\, T_n(\lambda_1,\ldots,\lambda_n) \quad (2.2)$$

where

$$U_n(\lambda_1,\ldots,\lambda_n) \;=\; \pi^{\frac{n(n+1)}{2}} \frac{\prod_{1\le k<j\le n} \sinh \pi(\lambda_j-\lambda_k)}{\prod_{j=1}^{n} \sinh^n \pi\lambda_j} \quad (2.3)$$

and

$$T_n(\lambda_1,\ldots,\lambda_n) \;=\; \frac{\prod_{j=1}^{n} \lambda_j^{j-1}(\lambda_j+i)^{n-j}}{\prod_{1\le k<j\le n}(\lambda_j-\lambda_k-i)} \quad (2.4)$$

First of all, let us note that in principle the contour C can be chosen between 0 and $-i$ arbitrary. Let us denote C_α the contour that goes from $i\alpha-\infty$ to $i\alpha+\infty$. In what follows it will be convenient to choose $\alpha=-1/2$ i.e. to integrate over the contour $C_{-1/2}$.

As appeared we can make a lot of simplifications without taking integrals but using some simple observations and properties of the function in the r.h.s. of (2.2) which has to be integrated.

Let us define a "weak" equality $\sim$. Namely, let us say that two functions $F_n(\lambda_1,\ldots,\lambda_n)$ and $G_n(\lambda_1,\ldots,\lambda_n)$ are "weakly" equivalent

$$F_n(\lambda_1,\ldots,\lambda_n) \;\sim\; G_n(\lambda_1,\ldots,\lambda_n) \quad (2.5)$$

if

$$\prod_{j=1}^{n}\int_{C_{-1/2}}\frac{d\lambda_j}{2\pi i}\,U_n(\lambda_1,\ldots,\lambda_n)\,(F_n(\lambda_1,\ldots,\lambda_n)\;-\;G_n(\lambda_1,\ldots,\lambda_n))\;=\;0. \tag{2.6}$$

Let us also introduce a "canonical" form of the function by the following formula

$$T_n^c(\lambda_1,\ldots,\lambda_n)\;=\;\sum_{j=0}^{[\frac{n}{2}]}\;P_j^{(n)}\prod_{k=1}^{j}\frac{1}{\lambda_{2k}-\lambda_{2k-1}} \tag{2.7}$$

where $P_j^{(n)}$ are some polynomials of the form

$$P_j^{(n)}\equiv P_j^{(n)}(\lambda_1,\lambda_3,\ldots,\lambda_{2j-1}|\lambda_{2j+1},\lambda_{2j+2},\ldots,\lambda_n)\;=$$

$$=\sum_{\substack{0\le i_1,i_3,\ldots,i_{2j-1}\le n-2\\ 0\le i_{2j+1}<i_{2j+2}<\ldots<i_n\le n-1}} C_{i_1,\quad i_3,\,\ldots,\,i_{2j-1}}^{i_{2j+1},i_{2j+2},\ldots,i_n}\;\;\lambda_1^{i_1}\,\lambda_3^{i_3}\,\ldots\,\lambda_{2j-1}^{i_{2j-1}}\,\lambda_{2j+1}^{i_{2j+1}}\,\lambda_{2j+2}^{i_{2j+2}}\,\ldots\,\lambda_n^{i_n} \tag{2.8}$$

where

$$C_{i_1,\quad i_3,\,\ldots,\,i_{2j-1}}^{i_{2j+1},i_{2j+2},\ldots,i_n}=i^{\beta}\hat{C}_{i_1,\quad i_3,\,\ldots,\,i_{2j-1}}^{i_{2j+1},i_{2j+2},\ldots,i_n}$$

with $\beta=0$ or 1 in accordance with the equality

$$\beta+i_1+i_3+\ldots+i_{2j-2}+i_{2j+1}+i_{2j+2}+\ldots+i_n\equiv j+n\mod 2$$

and some rational numbers $\hat{C}_{i_1,\quad i_3,\,\ldots,\,i_{2j-1}}^{i_{2j+1},i_{2j+2},\ldots,i_n}$.

This form has some arbitrariness because if we substitute $\lambda_j=x_j-i/2$ where all x_j are real then it is easy to see that the function $\tilde{U}_n(x_1,\ldots,x_n)=U_n(x_1-i/2,\ldots,x_n-i/2)$ transforms when $\{x_1,\ldots,x_n\}\to\{-x_1,\ldots,-x_n\}$ as follows

$$\tilde{U}_n(-x_1,\ldots,-x_n)\;=\;(-1)^n\,\tilde{U}_n(x_1,\ldots,x_n). \tag{2.9}$$

Therefore any function $\tilde{F}_n(x_1,\ldots,x_n)$ that satisfies

$$\tilde{F}_n(-x_1,\ldots,-x_n)=(-1)^{n+1}\,\tilde{F}_n(x_1,\ldots,x_n) \tag{2.10}$$

being integrated makes zero contribution

$$\prod_{j=1}^{n}\int_{-\infty}^{\infty}\frac{dx_j}{2\pi i}\tilde{U}_n(x_1,\ldots,x_n)\,\tilde{F}_n(x_1,\ldots,x_n)\;=\;0.$$

It means that in order to get a nonzero result one should have the function $\tilde{F}_n(x_1,\ldots,x_n)$ of the same parity as of the function $\tilde{U}_n(x_1,\ldots,x_n)$. Then if we re-expand the form (2.7) in terms of variables x_j instead of λ_j we can fix the arbitrariness by imposing some additional constraints, namely,

$$\tilde{P}_j^{(n)}(x_1,x_3,\ldots,x_{2j-1}|x_{2j+1},x_{2j+2},\ldots,x_n)\;=$$

$$=\;P_j^{(n)}(x_1-i/2,x_3-i/2,\ldots,x_{2j-1}-i/2|x_{2j+1}-i/2,x_{2j+2}-i/2,\ldots,x_n-i/2)\;=$$

$$\sum_{\substack{0\le i_1,i_3,\ldots,i_{2j-1}\le n-2\\ 0\le i_{2j+1}<i_{2j+2}<\ldots<i_n\le n-1\\ i_1+i_3+\ldots+i_{2j-2}+i_{2j+1}+i_{2j+2}+\ldots+i_n\equiv j+n\bmod 2}} \tilde{C}_{i_1,\quad i_3,\,\ldots,\,i_{2j-1}}^{i_{2j+1},i_{2j+2},\ldots,i_n}\;\;x_1^{i_1}x_3^{i_3}\ldots x_{2j-1}^{i_{2j-1}}x_{2j+1}^{i_{2j+1}}x_{2j+2}^{i_{2j+2}}\ldots x_n^{i_n}. \tag{2.11}$$

In comparison with the coefficients $C^{i_{2j+1},i_{2j+2},\ldots,i_n}_{i_1,\quad i_3,\,\ldots,\,i_{2j-1}}$ which can be pure imaginary all the coefficients $\tilde{C}^{i_{2j+1},i_{2j+2},\ldots,i_n}_{i_1,\quad i_3,\,\ldots,\,i_{2j-1}}$ are real and rational numbers.

So we can expect that the function

$$\tilde{T}^c_n(x_1,\ldots,x_n) \equiv T^c_n(x_1 - i/2,\ldots,x_n - i/2) \tag{2.12}$$

should satisfy the following property

$$\tilde{T}^c_n(-x_1,\ldots,-x_n) = (-1)^n\ \tilde{T}^c_n(x_1,\ldots,x_n) \tag{2.13}$$

Below the property (2.13) will be implied when we will speak about the "canonical" form (2.7-2.8). Besides, one can note that the function $\tilde{T}^c_n(x_1,\ldots,x_n)$ should be real for real variables x_j.

Our hypothesis is that for any n one can reduce the function T_n defined by (2.4) to the canonical form i.e. there exist polynomials P_j in (2.7) such that

$$T_n(\lambda_1,\ldots,\lambda_n)\ \sim\ T^c_n(\lambda_1,\ldots,\lambda_n). \tag{2.14}$$

Unfortunately, for the moment we do not have a proof of this statement for any n but we will demonstrate below how it works for $n = 2, 3, 4$.

In fact, the problem of the calculation of $P(n)$ given by the integral (2.2) can be reduced to the two steps. The first step corresponds to the obtaining of the "canonical" form for T_n. The second step is the calculation of the integral by means of this "canonical" form.

To do this one needs the following simple facts:

I. Since the function $U_n(\lambda_1,\ldots,\lambda_n)$ is antisymmetric with respect to transposition of any pair of integration variables, say, λ_j and λ_k the following integral

$$\prod_{j=1}^{n}\int_C \frac{d\lambda_j}{2\pi i}\ U(\lambda_1,\ldots,\lambda_n)\ S(\lambda_1,\ldots,\lambda_n)\ =\ 0 \tag{2.15}$$

if the function S is symmetric for at least one pair of λ-s. Therefore for an arbitrary function $F_n(\lambda_1,\ldots,\lambda_n)$ one can transpose any pair of λ-s taking into consideration appearance of additional sign because of the antisymmetry of $U_n(\lambda_1,\ldots,\lambda_n)$. For example, if one transposes λ_j with λ_k one gets

$$F_n(\ldots,\lambda_j,\ldots,\lambda_k,\ldots)\ \sim\ -F_n(\ldots,\lambda_k,\ldots,\lambda_j,\ldots). \tag{2.16}$$

II. The reduction of the power of denominator for T_n is based on two relations which can be checked directly

$$\frac{1}{\lambda_k-\lambda_l-i}\ \frac{1}{\lambda_j-\lambda_l-i}\ \frac{1}{\lambda_j-\lambda_k-i}\ =$$

$$i\ \frac{1}{\lambda_j-\lambda_l-i}\ \frac{1}{\lambda_j-\lambda_k-i}\ +\ i\ \frac{1}{\lambda_k-\lambda_l-i}\ \frac{1}{\lambda_j-\lambda_l-i}\ -\ i\ \frac{1}{\lambda_k-\lambda_l-i}\ \frac{1}{\lambda_j-\lambda_k-i} \tag{2.17}$$

$$\prod_{k=1}^{j-1}\frac{1}{\lambda_j-\lambda_k-i} = \sum_{k=1}^{j-1}\frac{1}{\lambda_j-\lambda_k-i}\quad \prod_{\substack{l\,=\,1\\ l\,\neq\,k}}^{j-1}\quad \frac{1}{\lambda_k-\lambda_l}. \tag{2.18}$$

In Appendices A and B we will show how the reduction can be performed for $n = 3$ and $n = 4$. Unfortunately, so far we have not succeeded in finding a result for general n.

III. The ratio

$$\frac{T_{n+1}(\lambda_1,\ldots,\lambda_{n+1})}{T_n(\lambda_1,\ldots,\lambda_n)} \;=\; \frac{\prod_{j=1}^n(\lambda_j+i)\,\lambda_{n+1}^n}{\prod_{j=1}^n(\lambda_{n+1}-\lambda_j-i)} \tag{2.19}$$

is symmetric with respect to any permutation of $\lambda_1,\ldots,\lambda_n$. Therefore the relation (2.19) allows us to use the result T_n also for derivation of T_{n+1} if this result was obtained by applying the relations (2.16-2.18) from I and II.

IV. **Proposition 1**

Let the function $f(\lambda_1,\ldots,\lambda_n)$ have only poles of the form $1/(\lambda_j-\lambda_k+ia)$ with a an integer i.e. the product $U_n(\lambda_1,\ldots,\lambda_n)f(\lambda_1,\ldots,\lambda_n)$ does not have poles of that kind. Then

$$\lambda_j^m\, f(\ldots,\lambda_j,\ldots) \;\sim\; -(\lambda_j+i)^m\, f(\ldots,\lambda_j+i,\ldots) \tag{2.20}$$

where m is an integer and $m\geq n$.

Proof

Let us suppose that all variables $\lambda_1,\ldots,\lambda_{j-1},\lambda_{j+1},\ldots,\lambda_n$ are fixed. Extracting from $U_n(\lambda_1,\ldots,\lambda_n)$ the function which depends on λ_j one gets

$$\int_{C_{-1/2}}\frac{d\lambda_j}{2\pi i}\frac{\prod_{k\neq j}\sinh\pi(\lambda_j-\lambda_k)}{\sinh^n\pi\lambda_j}\lambda_j^m\; w(\lambda_j) \;=\; -\int_{C_{-3/2}}\frac{d\lambda_j}{2\pi i}\frac{\prod_{k\neq j}\sinh\pi(\lambda_j-\lambda_k)}{\sinh^n\pi\lambda_j}(\lambda_j+i)^m\; w(\lambda_j+i) \;=$$

$$=\; -(\int_{C_{-3/2}}-\int_{C_{-1/2}}+\int_{C_{-1/2}})\frac{d\lambda_j}{2\pi i}\frac{\prod_{k\neq j}\sinh\pi(\lambda_j-\lambda_k)}{\sinh^n\pi\lambda_j}(\lambda_j+i)^m\; w(\lambda_j+i)$$

where $w(\lambda_j)=f(\ldots,\lambda_j,\ldots)$. The first step here was to shift integration variable $\lambda_j\to\lambda_j+i$ and to use the fact that $\sinh\pi(x+i)=-\sinh\pi x$. The two first integrals in the last expression are equal to a contour integral around the point $\lambda_j=-i$ in a complex plane of the variable λ_j. Since, $m\geq n$ the term $(\lambda_j+i)^m$ which is in the numerator and corresponds to a zero of order m compensates the pole from the term $\sinh^n\pi\lambda_j$ in the denominator. Therefore the contribution of those two integrals is zero and we immediately come to the statement (2.20).

One can get two useful corollaries from the proposition 1.

Corollary 1

$$\lambda_j^m\; g(\lambda_1,\ldots,\hat{\lambda}_j,\ldots,\lambda_n) \;\sim\; \frac{(-i)}{2}\sum_{k=0}^{m-1}\lambda_j^k(\lambda_j+i)^{m-1-k}\; g(\lambda_1,\ldots,\hat{\lambda}_j,\ldots,\lambda_n) \tag{2.21}$$

where the function $g(\lambda_1,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)$ does not depend on λ_j and as above it is implied that $m\geq n$.

Proof

The relation (2.21) is easy to derive using the relation $(\lambda_j^m+(\lambda_j+i)^m)g(\lambda_1,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)\sim 0$ or equivalently $\lambda_j^m g(\lambda_1,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)\sim 1/2(\lambda_j^m-(\lambda_j+i)^m)g(\lambda_1,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)$.

Corollary 2

$$\frac{\lambda_j^{m-1}}{\lambda_k-\lambda_j}\; g(\lambda_1,\ldots,\hat{\lambda}_k,\ldots,\hat{\lambda}_j,\ldots,\lambda_n) \;\sim$$

$$\sim \frac{i}{m}\Big(\sum_{l=2}^{m}\frac{\left(\begin{array}{c} m \\ l \end{array}\right) i^l \lambda_j^{m-l}}{\lambda_k - l_j} + \sum_{l=0}^{m-1}\lambda_k^l(\lambda_j+i)^{m-1-l}\Big) g(\lambda_1,\ldots,\hat{\lambda}_k,\ldots,\hat{\lambda}_j,\ldots,\lambda_n) \qquad (2.22)$$

where

$$\left(\begin{array}{c} m \\ l \end{array}\right) = \frac{m!}{l!\,(m-l)!}$$

is binomial coefficient and the function $g(\lambda_1,\ldots,\hat{\lambda}_k,\ldots,\hat{\lambda}_j,\ldots,\lambda_n)$ does not depend on λ_k and λ_j and $m \geq n$.

Proof
Using the proposition 1 we get

$$\frac{\lambda_j^m}{\lambda_k-\lambda_j}\, g(\lambda_1,\ldots,\hat{\lambda}_k,\ldots,\hat{\lambda}_j,\ldots,\lambda_n) \sim -\frac{(\lambda_j+i)^m}{\lambda_k-\lambda_j-i}\, g(\lambda_1,\ldots,\hat{\lambda}_k,\ldots,\hat{\lambda}_j,\ldots,\lambda_n) =$$

$$= \Big(-\frac{\lambda_k^m}{\lambda_k-\lambda_j-i} + \frac{\lambda_k^m-(\lambda_j+i)^m}{\lambda_k-\lambda_j-i}\Big)\, g(\lambda_1,\ldots,\hat{\lambda}_k,\ldots,\hat{\lambda}_j,\ldots,\lambda_n) \sim$$

$$\sim \Big(\frac{(\lambda_k+i)^m}{\lambda_k-\lambda_j} + \frac{\lambda_k^m-(\lambda_j+i)^m}{\lambda_k-\lambda_j-i}\Big)\, g(\lambda_1,\ldots,\hat{\lambda}_k,\ldots,\hat{\lambda}_j,\ldots,\lambda_n) \sim$$

$$\sim \Big(\frac{(\lambda_j+i)^m}{\lambda_k-\lambda_j} + \frac{\lambda_k^m-(\lambda_j+i)^m}{\lambda_k-\lambda_j-i}\Big)\, g(\lambda_1,\ldots,\hat{\lambda}_k,\ldots,\hat{\lambda}_j,\ldots,\lambda_n) \sim$$

or

$$\Big(\frac{(\lambda_j+i)^m-\lambda_j^m}{\lambda_k-\lambda_j} + \frac{\lambda_k^m-(\lambda_j+i)^m}{\lambda_k-\lambda_j-i}\Big)\, g(\lambda_1,\ldots,\hat{\lambda}_k,\ldots,\hat{\lambda}_j,\ldots,\lambda_n) \sim 0.$$

Then expanding both numerators according to the formulae

$$(\lambda_j+i)^m - \lambda_j^m = \sum_{l=1}^{m}\left(\begin{array}{c} m \\ l \end{array}\right) i^l \lambda_j^{m-l}$$

$$\lambda_k^m - (\lambda_j+i)^m = (\lambda_k-\lambda_j-i)\sum_{l=0}^{m-1}\lambda_k^l(\lambda_j+i)^{m-1-l}$$

we arrive at the formula (2.22).

With the help of the corollaries 1 and 2 one can effectively reduce the power of the numerator in T_n.

V. For the calculation of integrals we need the following formula

Proposition 2

$$\int_{C_{-1/2}} \frac{d\lambda}{2\pi i}\,\frac{f(\lambda)}{\sinh^n \pi\lambda} = d^{(n)}(\epsilon)_{\epsilon\to 0}\sum_{l=0}^{\infty}(-1)^{ln} f(il+\epsilon) = -d^{(n)}(\epsilon)_{\epsilon\to 0}\sum_{l=1}^{\infty}(-1)^{ln} f(-il+\epsilon) \qquad (2.23)$$

where $f(\lambda)$ is an arbitrary analytic function and a differential operator $d^{(n)}(\epsilon)$ looks as follows

$$d^{(n)}(\epsilon) = \frac{1}{\pi^n(n-1)!}\sum_{l=0}^{n-1}\left(\begin{array}{c} n-1 \\ l \end{array}\right)\Big(\frac{\partial}{\partial\alpha}\Big)_{\alpha\to 0}^{n-1-l}\Big(\sum_{0\leq 2k<n}\frac{(\pi\alpha)^{2k}}{(2k+1)!}\Big)^{-n}\Big(\frac{\partial}{\partial\epsilon}\Big)^{l}. \qquad (2.24)$$

In particular, for $n = 2, 3, 4$

$$d^{(2)}(\epsilon) \;=\; \frac{1}{\pi^2}\frac{\partial}{\partial\epsilon} \tag{2.25}$$

$$d^{(3)}(\epsilon) \;=\; -\frac{1}{2\pi}\,(1 - \frac{1}{\pi^2}\frac{\partial^2}{\partial\epsilon^2}) \tag{2.26}$$

$$d^{(4)}(\epsilon) \;=\; -\frac{2}{3\pi^2}\,(\frac{\partial}{\partial\epsilon} - \frac{1}{4\pi^2}\frac{\partial^3}{\partial\epsilon^3}). \tag{2.27}$$

Proof
To derive the formula (2.23) we should close the contour $C_{-1/2}$ in the complex plane by infinite semi-circles either in the upper half-plane or in the lower half-plane. Then using Cauchy theorem one should calculate contributions corresponding to the poles because of the denominator $\sinh^n \pi\lambda$ which has zeros at the points $\lambda = il$ in the complex plane with $l = 0, 1, \ldots, \infty$ for the contour in the upper half-plane and $l = -1, -2, \ldots, -\infty$ for the contour in the lower half-plane. In latter case one should take into consideration an additional sign because of the clockwise direction of the integration along that contour.

Using the proposition 2 we can get the following
Proposition 3

$$\prod_{j=1}^{n}\int_{C_{-1/2}}\frac{d\lambda_j}{2\pi i}U_n(\lambda_1,\ldots,\lambda_n)\,F_n(\lambda_1,\ldots,\lambda_n) \;=\; D^{(n)}\;\tilde{F}_n(\epsilon_1,\ldots,\epsilon_n) \tag{2.28}$$

where the product $U_n(\lambda_1, \ldots, \lambda_n)F_n(\lambda_1, \ldots, \lambda_n)$ does not have any other poles besides the poles of the denominator $\prod_{j=1}^{n}\sinh^n \pi\lambda_j$ of the function $U_n(\lambda_1, \ldots, \lambda_n)$ and $D^{(n)}$ is the differential operator

$$D^{(n)} = \pi^{\frac{n(n+1)}{2}}\prod_{j=1}^{n} d^{(n)}(\epsilon_j)_{\epsilon_j\to 0}\prod_{0\le k<j\le n}\sinh\pi(\epsilon_j - \epsilon_k) \tag{2.29}$$

$$\tilde{F}_n(\epsilon_1,\ldots,\epsilon_n) \;=\; \sum_{l_1=0}^{\infty}(-1)^{l_1}\ldots\sum_{l_n=0}^{\infty}(-1)^{l_n}F_n(i\,l_1\,+\,\epsilon_1,\ldots,i\,l_n\,+\,\epsilon_n) \tag{2.30}$$

The formula (2.28) can be got by the application of the formula (2.23) to each integral in the l.h.s. of (2.28) and by taking into account the manifest form (2.3) of the function $U_n(\lambda_1, \ldots, \lambda_n)$.

Let us note that each sum $\sum_{l_j=0}^{\infty}(-1)^{l_j}F_n(\ldots, i\lambda_j + \epsilon_j, \ldots)$ in the formula (2.30) can be substituted by $-\sum_{l_j=1}^{\infty}(-1)^{l_j}F_n(\ldots, -i\lambda_j + \epsilon_j, \ldots)$ corresponding to the choice of the contour in the lower half-plane. In what follows we will use this fact depending on a convenience.

Let us consider a special class of functions $\tilde{F}_n(\epsilon_1, \ldots, \epsilon_n)$ that does not have a singularity when $\epsilon_j \to 0$ for $j = 1, \ldots, n$. In this case one can expand $\tilde{F}_n(\epsilon_1, \ldots, \epsilon_n)$ into the infinite series on powers of ϵ-s. We have checked that for $n \le 4$ the differential operator $D^{(n)}$ given by (2.29) when acting on some monomial $\epsilon_1^{i_1}\ldots\epsilon_n^{i_n}$ makes non-zero contribution only for monomial of the form $\epsilon^0_{\sigma(1)}\epsilon^1_{\sigma(2)}\ldots\epsilon^{n-1}_{\sigma(n)}$ where σ is some element of the permutation group S_n of n elements. More precisely, we can write

$$D^{(n)} \;=\; \frac{1}{\prod_{j=1}^{n-1} j!}\prod_{0\le k<j\le n}\Big(\frac{\partial}{\partial\epsilon_k} - \frac{\partial}{\partial\epsilon_j}\Big)_{\vec{\epsilon}\to 0} \tag{2.31}$$

We believe that this relation is valid for any n but this fact is still to be proven.

VI. The last our general comment is about how to proceed if one faces monomial λ_j^a with an integer $a \geq 0$ in the function $F_n(\lambda_1, \ldots, \lambda_n)$, for example, when

$$F_n(\lambda_1, \ldots, \lambda_n) = \lambda_1^a \; G_{n-1}(\lambda_2, \ldots, \lambda_n) \tag{2.32}$$

or if in a sum (2.30) one faces denominators like, for instance, $1/(l_j + l_k + i(\epsilon_j - \epsilon_k))$. In the latter case we shall use an evident integral representation

$$\frac{1}{\alpha} \;=\; \int_0^1 \frac{ds}{s} s^\alpha \tag{2.33}$$

with $Re(\alpha) > 0$. For example, if $l_j + l_k > 0$ then we can use (2.33) to write down the following expansion

$$\frac{1}{l_j + l_k + i(\epsilon_j - \epsilon_k)} \;=\; \int_0^1 \frac{ds}{s} s^{l_j + l_k + i(\epsilon_j - \epsilon_k)} \;=$$

$$= \int_0^1 \frac{ds}{s} s^{l_j + l_k} (1 \;+\; \ln s \, (i(\epsilon_j - \epsilon_k)) \;+\; \frac{1}{2} \ln^2 s \, (i(\epsilon_j - \epsilon_k))^2 \;+\; \ldots) \tag{2.34}$$

Then after using the formulae like (2.34) and taking all summations in (2.30) one can apply differential operator (2.31) to the series in power of ϵ-s. As the last step one should take integrals on auxiliary variables like variable s in (2.34).

If the function F_n has the form (2.32) then after applying the summation formula (2.30) one faces the series that is formally divergent. Let us consider more attentively how to handle this problem. When extracting the integral over λ_1 in the l.h.s. of (2.28) one gets

$$\int_{C_{-1/2}} \frac{d\lambda_1}{2\pi i} \frac{\sinh \pi(\lambda_2 - \lambda_1) \ldots \sinh \pi(\lambda_n - \lambda_1)}{\sinh^n \pi\lambda_1} \; \lambda_1^a \;= \tag{2.35}$$

$$= \lim_{\delta \to 0^+} \int_{C_{-1/2}} \frac{d\lambda_1}{2\pi i} \frac{\sinh \pi(\lambda_2 - \lambda_1) \ldots \sinh \pi(\lambda_n - \lambda_1)}{\sinh^n \pi\lambda_1} \; \lambda_1^a \; e^{i\lambda_1 \delta} \;=$$

$$= \lim_{\delta \to 0^+} d^{(n)}(\epsilon_1)_{\epsilon_1 \to 0} \sinh \pi(\lambda_2 - \epsilon_1) \ldots \sinh \pi(\lambda_n - \epsilon_1) \sum_{l_1=0}^{\infty} (-1)^{l_1} (il_1 + \epsilon_1)^a \; e^{-l_1\delta + i\epsilon_1 \delta} \;=$$

$$= \lim_{\delta \to 0^+} [d^{(n)}(\epsilon_1)_{\epsilon_1 \to 0} \sinh \pi(\lambda_2 - \epsilon_1) \ldots \sinh \pi(\lambda_n - \epsilon_1) \sum_{l_1=0}^{\infty} (-e^{-\delta})^{l_1} (il_1 + \epsilon_1)^a \;+\; \mathrm{O}(\delta)] \;=$$

$$= d^{(n)}(\epsilon_1)_{\epsilon_1 \to 0} \sinh \pi(\lambda_2 - \epsilon_1) \ldots \sinh \pi(\lambda_n - \epsilon_1) \lim_{\delta \to 0^+} \sum_{l_1=0}^{\infty} (-e^{-\delta})^{l_1} (il_1 + \epsilon_1)^a \tag{2.36}$$

So the integral (2.35) reduces to calculation of the sum

$$\rho(b) \;=\; \lim_{\delta \to 0^+} \sum_{l_1=0}^{\infty} (-e^{-\delta})^{l_1} l_1^b \tag{2.37}$$

with an integer $b \geq 0$. Let us adduce a number of first values of $\rho(b)$

$$\rho(0) = \frac{1}{2}, \quad \rho(1) = -\frac{1}{4}, \quad \rho(2) = 0, \quad \rho(3) = \frac{1}{8}, \quad \rho(4) = 0, \quad \rho(5) = -\frac{1}{4} \tag{2.38}$$

Below for integrals like (2.35) we shall take δ to be zero at once implying the limiting procedure $\lim_{\delta\to 0^+}$ described above.

Now let us illustrate how the whole procedure works for a simple case $P(2)$

$$P(2) \;=\; \pi^3 \int_{C_{-1/2}} \frac{d\lambda_1}{2\pi i} \int_{C_{-1/2}} \frac{d\lambda_2}{2\pi i} \, \frac{\sinh \pi(\lambda_2-\lambda_1)}{\sinh^2 \pi\lambda_1 \sinh^2 \pi\lambda_2} \, T_2(\lambda_1,\lambda_2) \tag{2.39}$$

In this case it is very simple to perform the first step, namely, to get the "canonical" form (2.7) described in the beginning of the Section because we do not need to reduce a power of denominator in this case. Indeed,

$$T_2(\lambda_1,\lambda_2) \;=\; \frac{(\lambda_1+i)\lambda_2}{\lambda_2-\lambda_1-i} \;=\; \lambda_1+i \;+\; \frac{(\lambda_1+i)^2}{\lambda_2-\lambda_1-i} \;\sim\; \lambda_1 \;-\; \frac{\lambda_1^2}{\lambda_2-\lambda_1} \tag{2.40}$$

where we have used the property I and the formula (2.20) from the proposition 1 of the item IV. Then using the formula (2.21) of the corollary 1 for $m=2,3$ one gets

$$-\frac{\lambda_1^2}{\lambda_2-\lambda_1} \;\sim\; -\frac{i}{3}\left(\frac{3i^2\lambda_1+i^3}{\lambda_2-\lambda_1} \;+\; \lambda_2^2 \;+\lambda_2(\lambda_1+i) \;+\; (\lambda_1+i)^2\right) \;\sim$$

$$\sim\; \frac{i\lambda_1}{\lambda_2-\lambda_1} \;-\; \frac{1}{3}\frac{1}{\lambda_2-\lambda_1} \;-\; \frac{i}{3}(i\lambda_1) \;\sim\; \frac{1}{2}\frac{1}{\lambda_2-\lambda_1} \;-\; \frac{1}{3}\frac{1}{\lambda_2-\lambda_1} \;+\; \frac{1}{3}\lambda_1 \;=\; \frac{1}{3}\lambda_1 \;+\; \frac{1}{6}\frac{1}{\lambda_2-\lambda_1}$$

Substituting it to the formula (2.40) we get

$$T_2(\lambda_1,\lambda_2) \;\sim\; T_2^c(\lambda_1,\lambda_2) \tag{2.41}$$

where

$$T_2^c(\lambda_1,\lambda_2) \;=\; \frac{4}{3}\,\lambda_1 \;+\; \frac{1}{6}\,\frac{1}{\lambda_2-\lambda_1} \tag{2.42}$$

and this is the "canonical" form for T_2 i.e. the polynomials $P_0^{(2)}$ and $P_1^{(2)}$ from (2.7-2.8) are equal to $4/3\lambda_1$ and $1/6$ respectively.

Let us take the integral from the first term using the formula (2.30), our comments about the limiting procedure for the integral (2.35) and the result (2.36)

$$J_0^{(2)} \;=\; \pi^3 \int_{C_{-1/2}} \frac{d\lambda_1}{2\pi i} \int_{C_{-1/2}} \frac{d\lambda_2}{2\pi i} \frac{\sinh \pi(\lambda_2-\lambda_1)}{\sinh^2 \pi\lambda_1 \sinh^2 \pi\lambda_2} \frac{4}{3}\,\lambda_1 \;=\; D^{(2)}\frac{4}{3}\sum_{l_1=0}^{\infty}(-1)^{l_1}(il_1+\epsilon_1)\sum_{l_1=0}^{\infty}(-1)^{l_2} \;=$$

$$=\; D^{(2)}\frac{4}{3}(i\rho(1)+\epsilon_1\rho(0))\rho(0) \;=\; D^{(2)}\frac{4}{3}(-\frac{i}{4}+\frac{\epsilon_1}{2})\frac{1}{2} \;=\; (\frac{\partial}{\partial\epsilon_1}-\frac{\partial}{\partial\epsilon_2})_{\epsilon_1,\epsilon_2\to 0}\frac{2}{3}(-\frac{i}{4}+\frac{\epsilon_1}{2}) \;=\; \frac{1}{3} \tag{2.43}$$

where we have used the formulae (2.38) for $\rho(b)$ given by (2.37) implying the limiting procedure as it was explained above.

The second integral is treated as it was described in item VI with the help of the integral representation (2.33)

$$J_1^{(2)} \;=\; \pi^3 \int_{C_{-1/2}} \frac{d\lambda_1}{2\pi i} \int_{C_{-1/2}} \frac{d\lambda_2}{2\pi i} \, \frac{\sinh \pi(\lambda_2-\lambda_1)}{\sinh^2 \pi\lambda_1 \sinh^2 \pi\lambda_2} \, \frac{1}{6} \, \frac{1}{\lambda_2-\lambda_1} \;=$$

$$= D^{(2)}\,\frac{(-1)}{6}\sum_{l_1=1}^{\infty}\sum_{l_2=0}^{\infty}\frac{1}{il_2+il_1+\epsilon_2-\epsilon_1} = D^{(2)}\,\frac{i}{6}\sum_{l_1=1}^{\infty}\sum_{l_2=0}^{\infty}\frac{1}{l_1+l_2+i(\epsilon_1-\epsilon_2)} =$$

$$= D^{(2)}\,\frac{i}{6}\int_0^1\frac{ds}{s}\sum_{l_1=1}^{\infty}\sum_{l_2=0}^{\infty}s^{l_1+l_2+i(\epsilon_1-\epsilon_2)} = D^{(2)}\,\frac{(-i)}{6}\int_0^1 ds\,\frac{s^{i(\epsilon_1-\epsilon_2)}}{(1+s)^2} =$$

$$= (\frac{\partial}{\partial\epsilon_1}-\frac{\partial}{\partial\epsilon_2})_{\epsilon_1,\epsilon_2\to 0}\frac{(-i)}{6}\int_0^1 ds\,\frac{s^{i(\epsilon_1-\epsilon_2)}}{(1+s)^2} = \frac{1}{3}\int_0^1 ds\frac{\ln s}{(1+s)^2} = -\frac{\ln 2}{3} \qquad (2.44)$$

Summing up two answers (2.43) and (2.44) we get the result

$$P(2) = J_0^{(2)} + J_1^{(2)} = \frac{1}{3} - \frac{\ln(2)}{3}. \qquad (2.45)$$

which coincides with the formula (1.7).

In the Appendices A and B we shall derive the formulae (1.8) and (1.9) for $P(3)$ and $P(4)$ respectively. In the end of this Section let us note that both results (1.8) and (1.9) are expressed in terms of the logarithmic function and the Riemann zeta function of odd arguments and do not depend on polylogarithms like, for example, $\mathrm{Li}_4(1/2)$. All coefficients before those functions in (1.6-1.9) are rational. Also they do not contain any powers of π which could be considered as Riemann zeta functions of even arguments, see the formula (1.4) from the Introduction.

Our conjecture is that the final answer for any $P(n)$ will also be expressed in terms of logarithm $\ln 2$ and Riemann zeta functions $\zeta(k)$ with odd integers k and with rational coefficients.

In fact, this conjecture is intimately connected with our hypothesis from the beginning of this Section that the function T_n (2.4) can be reduced to the "canonical" form. Looking at the "canonical" form (2.7) one can conclude that only Riemann zeta functions and their products can enter into the final answer because all the denominators in the r.h.s. of (2.7) are split out. It means that after applying the formula (2.30) the multiple summation can be performed by pairs, say, $\sum_{l_{2k-1}}$ and $\sum_{l_{2k}}$. Each pair of these summations results in some combination of zeta functions.

3. Conclusion

We want to emphasize an interesting connection between integrable and disordered models. In order to describe correlations in integrable models one can use integrable integral operators [23]. On the other hand Tracy and Widom showed that in matrix models the distribution of eigenvalues and level spacing can be described by the integral operators, belonging to the same integrable class [22].

Our current work supports this link between integrable models and chaotic models. Riemann zeta function appears in the description of both kind of models.

Let us repeat that the main result of this paper is the calculation of $P(3)$ and $P(4)$ (1.8-1.9) by means of the multi-integral representation (2.1). The fact that only the logarithm $\ln 2$ and Riemann zeta function with odd arguments participate in the answers for $P(1),\ldots,P(4)$ and with rational coefficients before these functions allows us to suppose that this is the general property of $P(n)$. One could compare the calculation of $P(n)$ with the many-loop calculation of the self-energy diagrams in the renormalizable quantum field theory which can also be expressed in terms of ζ functions of odd arguments [1] .

Unfortunately, so far we have not got even a conjecture for $P(n)$ but we believe that it is not an unsolvable problem. May be already after calculation of $P(5)$ one could guess the right formula for a generic case $P(n)$. It would give an answer to the question discussed in the previous section, namely, the question about the law of decay of $P(n)$ when n tends to infinity.

Also it would be interesting to generalize above results to the XXZ spin chain. Some interesting conjectures were recently invented by Razumov and Stroganov [13] for the special case of the XXZ model with $\Delta = -1/2$. These conjectures would be supported if it were possible to get $P(n)$ from the general integral representation obtained by the RIMS group [7].

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5. Appendix A

Here we discuss in detail the calculation of $P(3)$ performed by means of the general procedure described above.

As was pointed out in the beginning of Section 2 the first step should be a reduction of the function $T_3(\lambda_1, \lambda_2, \lambda_3)$ to the form (2.7) which we have called the "canonical" form. In comparison with the case $P(2)$ here we should reduce the power of the denominator in $T_3(\lambda_1, \lambda_2, \lambda_3)$. To do this we will use the formula (2.17) from the item II. Namely,

$$T_3(\lambda_1, \lambda_2, \lambda_3) \;=\; \frac{(\lambda_1+i)^2(\lambda_2+i)\lambda_2\lambda_3^2}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)(\lambda_3-\lambda_2-i)} \;=\; I_1^{(3)} \;+\; I_2^{(3)} \;+\; I_3^{(3)} \tag{5.1}$$

where

$$I_1^{(3)} \;=\; i\frac{(\lambda_1+i)^2(\lambda_2+i)\lambda_2\lambda_3^2}{(\lambda_3-\lambda_1-i)(\lambda_3-\lambda_2-i)}, \tag{5.2}$$

$$I_2^{(3)} \;=\; i\frac{(\lambda_1+i)^2(\lambda_2+i)\lambda_2\lambda_3^2}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)}, \tag{5.3}$$

$$I_3^{(3)} \;=\; -i\frac{(\lambda_1+i)^2(\lambda_2+i)\lambda_2\lambda_3^2}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_2-i)}, \tag{5.4}$$

Due to the $1 \leftrightarrow 2$ symmetry of the denominator the first term $I_1^{(3)}$ can be simplified as follows

$$I_1^{(3)} \;\sim\; -\frac{(\lambda_1+i)(\lambda_2+i)\lambda_2\lambda_3^2}{(\lambda_3-\lambda_1-i)(\lambda_3-\lambda_2-i)} \;\sim\; \frac{(\lambda_1+i)(\lambda_2+i)\lambda_3^2}{\lambda_3-\lambda_1-i} \;=$$

$$=\; (\lambda_1+i)(\lambda_2+i)(\lambda_1+\lambda_3+i) \;+\; \frac{(\lambda_1+i)^3(\lambda_2+i)}{\lambda_3-\lambda_1-i} \;\sim\; \lambda_1^2\lambda_2 \;-\; \frac{(\lambda_1+i)^3(\lambda_3+i)}{\lambda_2-\lambda_1-i} \tag{5.5}$$

The denominator of the second term (5.3) has the symmetry under the transposition $2 \leftrightarrow 3$. Therefore it can also be simplified

$$I_2^{(3)} \sim -\frac{(\lambda_1+i)^2\lambda_2\lambda_3^2}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)} \sim \frac{(\lambda_1+i)^2\lambda_2\lambda_3}{\lambda_2-\lambda_1-i} =$$

$$= -(\lambda_1+i)^2\lambda_3 - \frac{(\lambda_1+i)^3\lambda_3}{\lambda_2-\lambda_1-i} \sim \lambda_1^2\lambda_2 - \frac{(\lambda_1+i)^3\lambda_3}{\lambda_2-\lambda_1-i} \tag{5.6}$$

The third term (5.4) is treated as follows

$$I_3^{(3)} = -i(\lambda_1+i)^2(\lambda_2+i)(\lambda_3+\lambda_2+i) -$$

$$- i\frac{(\lambda_1+i)^2(\lambda_2+i)^3}{\lambda_3-\lambda_2-i} + i\frac{(\lambda_1+i)^3\lambda_3^2}{\lambda_2-\lambda_1-i} - i\frac{(\lambda_1+i)^3\lambda_3^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_2-i)} \sim$$

$$\sim -\lambda_1^2\lambda_2 - \frac{(\lambda_1+i)^3(\lambda_3+i)^2}{\lambda_2-\lambda_1-i} + i\frac{(\lambda_1+i)^3\lambda_3^2}{\lambda_2-\lambda_1-i} - i\frac{(\lambda_1+i)^3\lambda_3^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_2-i)} \tag{5.7}$$

Now adding up all the three terms together we get

$$T_3(\lambda_1,\lambda_2,\lambda_3) \sim \lambda_1^2\lambda_2 - i\frac{(\lambda_1+i)^3\lambda_3^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_2-i)} \sim -\lambda_2\lambda_3^2 - i\frac{(\lambda_1+i)^3\lambda_3^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_2-i)}. \tag{5.8}$$

Let us note that up to this moment we have used only the symmetry property (2.16) from the item I, the formula (2.17) from the item II and a simple algebra.

Now we would like to use the formula (2.20) of the proposition 1 for $m = 3$ and again apply the transposition formula (2.16)

$$T_3(\lambda_1,\lambda_2,\lambda_3) \sim -\lambda_2\lambda_3^2 - i\frac{\lambda_1^3(\lambda_3+i)^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} \sim$$

$$\sim -\lambda_2\lambda_3^2 + i\frac{(\lambda_1+i)^3\lambda_3^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} = -\lambda_2\lambda_3^2 + i\frac{\lambda_1^3\lambda_3^3+3i\lambda_1^2\lambda_3^3-3\lambda_1\lambda_3^3-i\lambda_3^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} \sim$$

$$\sim -\lambda_2\lambda_3^2 - \frac{3\lambda_1^2\lambda_3^2+3i\lambda_1\lambda_3(\lambda_3+\lambda_2)-\lambda_3^2-\lambda_3\lambda_2-\lambda_2^2}{\lambda_2-\lambda_1} \sim$$

$$\sim -\lambda_2\lambda_3^2 - \frac{3\lambda_1^2\lambda_3^2+3i\lambda_1\lambda_3^2+3i\lambda_1^2\lambda_3-\lambda_3^2-\lambda_3\lambda_1-\lambda_1^2}{\lambda_2-\lambda_1} \tag{5.9}$$

Now we can reduce the power of λ_1 in the numerator of the second term (5.9) by applying the formula (2.22) of the corollary 2 from the item IV. Doing this we finally get the "canonical form" (2.7-2.8) of T_3

$$T_3(\lambda_1,\lambda_2,\lambda_3) \sim T_3^c(\lambda_1,\lambda_2,\lambda_3) = P_0^{(3)} + \frac{P_1^{(3)}}{\lambda_2-\lambda_1} \tag{5.10}$$

where the polynomials $P_0^{(3)}$ and $P_1^{(3)}$ are as follows

$$P_0^{(3)} = -2\lambda_2\lambda_3^2, \quad P_1^{(3)} = \frac{1}{3} - i\lambda_1 - i\lambda_3 - 2\lambda_1\lambda_3 \tag{5.11}$$

Let us note that if we express variables λ_j through the real variables x_j via $\lambda_j = x_j - i/2$ in order to get the polynomials $\tilde{P}_j^{(3)}$ (see the formula (2.11)) we get especially simple formulae, namely,

$$\tilde{P}_0^{(3)} \;=\; -2x_2x_3^2, \quad \tilde{P}_1^{(3)} \;=\; -\frac{1}{6} \;-\; 2x_1x_3. \tag{5.12}$$

So, the function

$$\tilde{T}_3^c(x_1,x_2,x_3) \;=\; \tilde{P}_0^{(3)} \;+\; \frac{\tilde{P}_1^{(3)}}{x_2-x_1} \tag{5.13}$$

is odd i.e.

$$\tilde{T}_3^c(-x_1,-x_2,-x_3) \;=\; -\tilde{T}_3^c(x_1,x_2,x_3)$$

as it should be according to the formula (2.13) from the beginning of Section 2.

Now we are ready to calculate the integral in order to get the result for $P(3)$

$$P(3) \;=\; \prod_{j=1}^{3}\int_{C_{-1/2}} \frac{d\lambda_j}{2\pi i}\, U_3(\lambda_1,\lambda_2,\lambda_3)\, T_3(\lambda_1,\lambda_2,\lambda_3) \;=\; J_0^{(3)} \;+\; J_1^{(3)} \tag{5.14}$$

where

$$J_0^{(3)} = \prod_{j=1}^{3}\int_{C_{-1/2}} \frac{d\lambda_j}{2\pi i}\, U_3(\lambda_1,\lambda_2,\lambda_3)\, P_0^{(3)}, \quad J_1^{(3)} = \prod_{j=1}^{3}\int_{C_{-1/2}} \frac{d\lambda_j}{2\pi i}\, U_3(\lambda_1,\lambda_2,\lambda_3)\, \frac{P_1^{(3)}}{\lambda_2-\lambda_1} \tag{5.15}$$

Using the formulae (5.11),(2.30),(2.31), (2.35-2.38) we get

$$J_0^{(3)} \;=\; \prod_{j=1}^{3}\int_{C_{-1/2}} \frac{d\lambda_j}{2\pi i}\, U_3(\lambda_1,\lambda_2,\lambda_3)\, (-2)\lambda_2\lambda_3^2 \;=$$

$$=\; D^{(3)} \sum_{l_1=0}^{\infty}(-1)^{l_1} \sum_{l_2=0}^{\infty}(-1)^{l_2} \sum_{l_3=0}^{\infty}(-1)^{l_3}(-2)(il_2+\epsilon_2)(il_3+\epsilon_3)^2 \;=$$

$$=\; D^{(3)} \sum_{l_1=0}^{\infty}(-1)^{l_1} \sum_{l_2=0}^{\infty}(-1)^{l_2} \sum_{l_3=0}^{\infty}(-1)^{l_3}(-2)\epsilon_2\epsilon_3^2 \;=\; \frac{1}{2}\prod_{0\le k<j\le 3}\Big(\frac{\partial}{\partial\epsilon_k}-\frac{\partial}{\partial\epsilon_j}\Big)_{\vec{\epsilon}\to 0}(-2)\epsilon_2\epsilon_3^2 \;=\; \frac{1}{4} \tag{5.16}$$

To calculate the second term $J_1^{(3)}$ we should also use integral representations (2.33-2.34)

$$J_1^{(3)} \;=\; \prod_{j=1}^{3}\int_{C_{-1/2}} \frac{d\lambda_j}{2\pi i}\, U_3(\lambda_1,\lambda_2,\lambda_3)\, \frac{\frac{1}{3} \;-\; i\lambda_1 \;-\; i\lambda_3 \;-\; 2\lambda_1\lambda_3}{\lambda_2-\lambda_1} \;=$$

$$= -D^{(3)} \sum_{l_1=1}^{\infty}(-1)^{l_1} \sum_{l_2=0}^{\infty}(-1)^{l_2} \sum_{l_3=0}^{\infty}(-1)^{l_3} \frac{\frac{1}{3} \;-\; i(-il_1+\epsilon_1) \;-\; i(il_3+\epsilon_3) \;-\; 2(-il_1+\epsilon_1)(il_3+\epsilon_3)}{il_2+il_1+\epsilon_2-\epsilon_1} =$$

$$=\; i\, D^{(3)} \sum_{l_1=1}^{\infty}(-1)^{l_1} \sum_{l_2=0}^{\infty}(-1)^{l_2} \sum_{l_3=0}^{\infty}(-1)^{l_3} \frac{\frac{1}{3} \;-\; l_1 \;+\; l_3 \;-i\epsilon_1 \;-\; i\epsilon_3 \;-\; 2(l_1+i\epsilon_1)(l_3-i\epsilon_3)}{l_1+l_2+i(\epsilon_1-\epsilon_2)} \;=$$

$$=\; i\, D^{(3)} \sum_{l_1=1}^{\infty}(-1)^{l_1} \sum_{l_2=0}^{\infty}(-1)^{l_2} \frac{-\frac{1}{12} \;-\; i\epsilon_3 \;+ l_1 i\epsilon_3 \;-\; \epsilon_1\epsilon_3}{l_1+l_2+i(\epsilon_1-\epsilon_2)} \;=$$

$$= \; i\, D^{(3)} \int_0^1 \frac{ds}{s} s^{i(\epsilon_1-\epsilon_2)} \sum_{l_1=1}^{\infty}\sum_{l_2=0}^{\infty} (-\frac{1}{12} \; - \; i\epsilon_3 \; + l_1 i\epsilon_3 \; - \; \epsilon_1\epsilon_3)(-s)^{l_1+l_2} \; =$$

$$= \; i\, D^{(3)} \int_0^1 \frac{ds}{s} s^{i(\epsilon_1-\epsilon_2)} \Big(\frac{(-s)}{(1+s)^2} (-\frac{1}{12} \; - \; i\epsilon_3 \; - \; \epsilon_1\epsilon_3) \; + \; \frac{(-s)}{(1+s)^3} i\epsilon_3 \Big) \; =$$

$$= \; \frac{i}{2} \prod_{0\le k<j\le 3} \Big(\frac{\partial}{\partial\epsilon_k} - \frac{\partial}{\partial\epsilon_j} \Big)_{\vec{\epsilon}\to 0} \int_0^1 \frac{ds}{(1+s)^2} \Big(\frac{1}{12} \, (-3\epsilon_1^2\epsilon_2 \; + \; 3\epsilon_1\epsilon_2^2) \, (-i) \, \frac{\ln^3 s}{6} \; + \; i \ln s \; \epsilon_1^2\epsilon_3 \Big) \; =$$

$$= \; \int_0^1 ds \frac{\ln s}{(1+s)^2} \; - \; \frac{1}{12} \; \int_0^1 ds \frac{\ln^3 s}{(1+s)^2} \; = \; -\ln 2 \; + \; \frac{3}{8}\zeta(3) \tag{5.17}$$

Summing up $J_0^{(3)}$ and $J_1^{(3)}$ we get the final answer (1.8)

$$P(3) \; = \; J_0^{(3)} \; + \; J_1^{(3)} \; = \; \frac{1}{4} \; - \; \ln 2 \; + \; \frac{3}{8}\zeta(3). \tag{5.18}$$

6. Appendix B

Here we will calculate $P(4)$. This case is much more complicated than the previous ones. But using our general technique we will try to simplify our discussion as much as possible.

As above the first step of our scheme is to get the "canonical" form for the function $T_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

But before start doing this let us list some useful formulae which can be derived from the proposition 1 and corollary 2 in the case $n = 4$

$$\frac{\lambda_1^3}{\lambda_2 - \lambda_1} \; \sim \; \frac{-\frac{3}{2} i\lambda_1^2 \; + \; \lambda_1 \; + \; \frac{i}{4}}{\lambda_2 - \lambda_1} \; + \; \frac{1}{2}\lambda_2^2 \; + \; \frac{i}{2}\lambda_2 \tag{6.1}$$

$$\frac{\lambda_1^4}{\lambda_2 - \lambda_1} \; \sim \; \frac{-\lambda_1^2 \; - \; i\lambda_1 \; + \; \frac{3}{10}}{\lambda_2 - \lambda_1} \; + \; \frac{3}{5}\lambda_2^3 \; + \; \frac{1}{5}\lambda_1\lambda_2^2 \; + \; \frac{2}{5}\lambda_2 \tag{6.2}$$

$$\frac{\lambda_1^5}{\lambda_2 - \lambda_1} \; \sim \; \frac{-\frac{\lambda_1}{6} \; - \; \frac{i}{12}}{\lambda_2 - \lambda_1} \; + \; \frac{1}{3}\lambda_1\lambda_2^3 \; - \; \frac{4}{3} i\lambda_2^3 \; + \; \frac{13}{6}\lambda_2^2 \; + \; \frac{4}{3} i\lambda_2 \; - \; \frac{1}{3} \tag{6.3}$$

$$\frac{\lambda_1^6}{\lambda_2 - \lambda_1} \; \sim \; \frac{-\frac{1}{2}\lambda_1^2 \; - \; \frac{1}{2} i\lambda_1 \; - \; \frac{1}{7}}{\lambda_2 - \lambda_1} \; +$$

$$+ \; \frac{1}{7}\lambda_1^2\lambda_2^3 \; - \; \frac{6}{7} i\lambda_1\lambda_2^3 \; + \; \frac{2}{7}\lambda_2^3 \; + \; \frac{10}{7}\lambda_1\lambda_2^2 \; - \; \frac{25}{14} i\lambda_2^2 + \frac{31}{14}\lambda_2 \tag{6.4}$$

$$\frac{(\lambda_1 + i)^3}{\lambda_2 - \lambda_1} \; \sim \; \frac{\frac{3}{2} i\lambda_1^2 \; - \; 2\lambda_1 \; - \; \frac{3}{4} i}{\lambda_2 - \lambda_1} \; + \; \frac{1}{2}\lambda_2^2 \; + \; \frac{i}{2}\lambda_2 \tag{6.5}$$

$$\frac{(\lambda_1 + i)^4}{\lambda_2 - \lambda_1} \; \sim \; \frac{-\lambda_1^2 \; - \; i\lambda_1 \; + \frac{3}{10}}{\lambda_2 - \lambda_1} \; + \; \frac{3}{5}\lambda_2^3 \; + \; \frac{1}{5}\lambda_1\lambda_2^2 \; + \; 2i\lambda_2^2 \; - \; \frac{8}{5}\lambda_2 \tag{6.6}$$

$$\frac{(\lambda_1 + i)^5}{\lambda_2 - \lambda_1} \; \sim \; \frac{-\frac{\lambda_1}{6} \; - \; \frac{i}{12}}{\lambda_2 - \lambda_1} \; + \; \frac{1}{3}\lambda_1\lambda_2^3 \; + \; \frac{5}{3} i\lambda_2^3 \; + \; i\lambda_1\lambda_2^2 \; - \; \frac{17}{6}\lambda_2^2 \; - \; \frac{5}{3} i\lambda_2 \; - \; \frac{1}{3} \tag{6.7}$$

Here we have omitted an arbitrary function $g(\lambda_3, \lambda_4)$ that can be multiplied on the r.h.s. and l.h.s. of (6.1-6.7) as in the formula (2.22).

Now we can start our derivation. Fortunately, due to the formula (2.19) of the observation III which in this case looks as follows

$$\frac{T_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}{T_3(\lambda_1, \lambda_2, \lambda_3)} \;=\; \frac{(\lambda_1 + i)(\lambda_2 + i)(\lambda_3 + i)\lambda_4^3}{(\lambda_4 - \lambda_1 - i)(\lambda_4 - \lambda_2 - i)(\lambda_4 - \lambda_3 - i)} \tag{6.8}$$

and has a symmetry under any permutation of the variables $\lambda_1, \lambda_2, \lambda_3$ we can use the result (5.8) that was obtained by means of the symmetry and simple algebra. So, after the application of the formulae (5.8) and (6.8) we can start with the following expression

$$\begin{aligned} T_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \;\sim\; & -\frac{(\lambda_1 + i)(\lambda_2 + i)\lambda_2(\lambda_3 + i)\lambda_3^2\lambda_4^3}{(\lambda_4 - \lambda_1 - i)(\lambda_4 - \lambda_2 - i)(\lambda_4 - \lambda_3 - i)} - \\ & - i\frac{(\lambda_1 + i)^4(\lambda_2 + i)(\lambda_3 + i)\lambda_3^3\lambda_4^3}{(\lambda_2 - \lambda_1 - i)(\lambda_3 - \lambda_2 - i)(\lambda_4 - \lambda_1 - i)(\lambda_4 - \lambda_2 - i)(\lambda_4 - \lambda_3 - i)} \end{aligned} \tag{6.9}$$

The denominator of the first term in the r.h.s. of (6.9) is symmetric under permutation of $\lambda_1, \lambda_2, \lambda_3$. Therefore it can be simplified as follows

$$\begin{aligned} -\frac{(\lambda_1 + i)(\lambda_2 + i)\lambda_2(\lambda_3 + i)\lambda_3^2\lambda_4^3}{(\lambda_4 - \lambda_1 - i)(\lambda_4 - \lambda_2 - i)(\lambda_4 - \lambda_3 - i)} \;\sim\; & \frac{(\lambda_1 + i)(\lambda_2 + i)\lambda_2(\lambda_3 + i)(\lambda_3 + \lambda_4 + i)\lambda_4^3}{(\lambda_4 - \lambda_1 - i)(\lambda_4 - \lambda_2 - i)} \;\sim \\ \sim\; -\frac{(\lambda_1 + i)(\lambda_2 + i)(\lambda_3 + i)(\lambda_3 + \lambda_4 + i)\lambda_4^3}{\lambda_4 - \lambda_1 - i} \;\sim\; & -\frac{(\lambda_1 + i)(\lambda_2 + i)(\lambda_3 + i)\lambda_3\lambda_4^3}{\lambda_4 - \lambda_1 - i} \;= \\ = \;(\lambda_2 + i)(\lambda_3 + i)\lambda_3\lambda_4^3 \;-\; \frac{(\lambda_2 + i)(\lambda_3 + i)\lambda_3\lambda_4^4}{\lambda_4 - \lambda_1 - i} \;\sim\; & \lambda_2\lambda_3^2\lambda_4^3 \;-\; \frac{\lambda_1^4(\lambda_3 + i)(\lambda_4 + i)\lambda_4}{\lambda_2 - \lambda_1 + i} \;\sim \\ \sim\; \lambda_2\lambda_3^2\lambda_4^3 \;-\; \frac{\lambda_1^4(\lambda_3\lambda_4^2 + i\lambda_4^2 - \lambda_4)}{\lambda_2 - \lambda_1 + i} \;\sim\; & \lambda_2\lambda_3^2\lambda_4^3 \;+\; \frac{(\lambda_1 + i)^4(\lambda_3\lambda_4^2 + i\lambda_4^2 - \lambda_4)}{\lambda_2 - \lambda_1} \;\sim \\ \sim\; \frac{8}{5}\lambda_2\lambda_3^2\lambda_4^3 \;-\; & \frac{(\lambda_1^2 \;+\; i\lambda_1 \;-\; \frac{3}{10})\,(\lambda_3\lambda_4^2 + i\lambda_4^2 - \lambda_4)}{\lambda_2 - \lambda_1} \end{aligned} \tag{6.10}$$

The latter formula was obtained with the help of the formula (6.6). In fact, it is nothing but the "canonical" form (2.7) for the first term in (6.9).

Now let us treat the second term. Using the formulae (2.17) and (2.18) we can write it down as follows

$$\begin{aligned} -i\frac{(\lambda_1 + i)^4(\lambda_2 + i)(\lambda_3 + i)\lambda_3^3\lambda_4^3}{(\lambda_2 - \lambda_1 - i)(\lambda_3 - \lambda_2 - i)(\lambda_4 - \lambda_1 - i)(\lambda_4 - \lambda_2 - i)(\lambda_4 - \lambda_3 - i)} \;= \\ = \;-i(\lambda_1 + i)^4(\lambda_2 + i)(\lambda_3 + i)\lambda_3^3\lambda_4^3 \Big\{ \\ \frac{1}{2}\frac{1}{(\lambda_3 - \lambda_2 - i)(\lambda_4 - \lambda_1 - i)(\lambda_4 - \lambda_3 - i)} \;-\; \frac{1}{2}\frac{1}{(\lambda_2 - \lambda_1 - i)(\lambda_4 - \lambda_1 - i)(\lambda_4 - \lambda_3 - i)} \;- \\ -\;\frac{1}{2}\frac{1}{(\lambda_2 - \lambda_1 - i)(\lambda_3 - \lambda_2 - i)(\lambda_4 - \lambda_1 - i)} \;-\; \frac{1}{2}\frac{1}{(\lambda_2 - \lambda_1 - i)(\lambda_3 - \lambda_2 - i)(\lambda_4 - \lambda_3 - i)} \;- \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1-i)} - \frac{1}{(\lambda_3-\lambda_2-i)(\lambda_4-\lambda_1-i)(\lambda_4-\lambda_2-i)} - \\
&- \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_3-i)} + \frac{1}{(\lambda_2-\lambda_1-i)(\lambda_4-\lambda_2-i)(\lambda_4-\lambda_3-i)} + \\
&+ \frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2-i)} + \frac{1}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_2-i)(\lambda_4-\lambda_2-i)} \Big\}
\end{aligned}
$$

Let us enumerate all ten terms as they enter here and make some appropriate transformations for each of them

$$
I_1^{(4)} = -\frac{i(\lambda_1+i)^4(\lambda_2+i)(\lambda_3+i)\lambda_3^3\lambda_4^3}{2(\lambda_3-\lambda_2-i)(\lambda_4-\lambda_1-i)(\lambda_4-\lambda_3-i)} \underset{\substack{1\to 4\\ 2\to 2\\ 3\to 1\\ 4\to 3}}{\sim} -\frac{i(\lambda_1+i)\lambda_1^3(\lambda_2+i)\lambda_3^3(\lambda_4+i)^4}{2(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)} \tag{6.11}
$$

$$
I_2^{(4)} = \frac{i}{2}\frac{(\lambda_1+i)^4(\lambda_2+i)(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1-i)(\lambda_4-\lambda_1-i)(\lambda_4-\lambda_3-i)} \underset{3\leftrightarrow 4}{\sim} \frac{i}{2}\frac{(\lambda_1+i)^4(\lambda_2+i)\lambda_3^3(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)} \tag{6.12}
$$

$$
I_3^{(4)} = \frac{i}{2}\frac{(\lambda_1+i)^4(\lambda_2+i)(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_2-i)(\lambda_4-\lambda_1-i)} \underset{\substack{1\to 1\\ 2\to 3\\ 3\to 4\\ 4\to 2}}{\sim} \frac{i}{2}\frac{(\lambda_1+i)^4\lambda_2^3(\lambda_3+i)(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3-i)} \tag{6.13}
$$

$$
I_4^{(4)} = \frac{i}{2}\frac{(\lambda_1+i)^4(\lambda_2+i)(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_2-i)(\lambda_4-\lambda_3-i)} \underset{1\leftrightarrow 2}{\sim} \frac{i}{2}\frac{(\lambda_1+i)(\lambda_2+i)^4(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3-i)} \tag{6.14}
$$

$$
I_5^{(4)} = i\frac{(\lambda_1+i)^4(\lambda_2+i)(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1-i)} \tag{6.15}
$$

$$
I_6^{(4)} = i\frac{(\lambda_1+i)^4(\lambda_2+i)(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_3-\lambda_2-i)(\lambda_4-\lambda_1-i)(\lambda_4-\lambda_2-i)} \underset{\substack{1\to 4\\ 2\to 1\\ 3\to 2\\ 4\to 3}}{\sim} i\frac{(\lambda_1+i)(\lambda_2+i)\lambda_2^3\lambda_3^3(\lambda_4+i)^4}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)} \tag{6.16}
$$

$$
I_7^{(4)} = i\frac{(\lambda_1+i)^4(\lambda_2+i)(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_3-i)} \underset{1\leftrightarrow 3}{\sim} -i\frac{(\lambda_1+i)\lambda_1^3(\lambda_2+i)(\lambda_3+i)^4\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1-i)} \tag{6.17}
$$

$$
I_8^{(4)} = -\frac{i(\lambda_1+i)^4(\lambda_2+i)(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1-i)(\lambda_4-\lambda_2-i)(\lambda_4-\lambda_3-i)} \underset{\substack{1\leftrightarrow 2\\ 3\leftrightarrow 4}}{\sim} -\frac{i(\lambda_1+i)(\lambda_2+i)^4\lambda_3^3(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)} \tag{6.18}
$$

$$I_9^{(4)} \;=\; -i\frac{(\lambda_1+i)^4(\lambda_2+i)(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2-i)} \quad \underset{\substack{1\to 3\\ 2\to 1\\ 3\to 2\\ 4\to 4}}{\sim} \quad i\frac{(\lambda_1+i)(\lambda_2+i)\lambda_2^3(\lambda_3+i)^4\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1-i)} \tag{6.19}$$

$$I_{10}^{(4)} = -\frac{i(\lambda_1+i)^4(\lambda_2+i)(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_2-i)(\lambda_4-\lambda_2-i)} \quad \underset{1\leftrightarrow 2}{\sim} \quad -\frac{i(\lambda_1+i)(\lambda_2+i)^4(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_1-i)} \tag{6.20}$$

Let us note that after these transformations $I_j^{(4)}$ have two kinds of the denominators, namely, $I_5^{(4)}, I_7^{(4)}, I_9^{(4)}, I_{10}^{(4)}$ have denominators of the form

$$\frac{1}{(\lambda_2-\lambda_1-ia)(\lambda_3-\lambda_1-ib)(\lambda_4-\lambda_1-ic)}$$

with a set of integers a, b, c, while the denominators of the rest of them are of the form

$$\frac{1}{(\lambda_2-\lambda_1-ia')(\lambda_3-\lambda_1-ib')(\lambda_4-\lambda_3-ic')}$$

with some other set of integers a', b', c'. Moreover, some of the terms $I_j^{(4)}$ have just coinciding denominators like, for example, $I_2^{(4)}$ and $I_6^{(4)}$. Nevertheless sometimes it will be more convenient to treat them separately.

Let us start with the first group which is easier to treat. Since, the denominator of $I_5^{(4)}$ has the $2 \leftrightarrow 3$ symmetry we can simplify it as follows

$$I_5^{(4)} \;\sim\; i\frac{(\lambda_1+i)^4(\lambda_2+i)(\lambda_3+i)(\lambda_3^2+\lambda_3\lambda_1+\lambda_1^2)\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_1-i)} \tag{6.21}$$

where we have used the trivial identity

$$\lambda_3^3 \;=\; (\lambda_3-\lambda_1)(\lambda_3^2+\lambda_3\lambda_1+\lambda_1^2) \;+\; \lambda_1^3 \tag{6.22}$$

and the fact that the second term in the r.h.s. of (6.22) does not contribute into $I_5^{(4)}$.

Then summing up (6.17) and (6.19) we get

$$I_7^{(4)} \;+\; I_9^{(4)} \;\sim\; i\frac{(\lambda_1+i)(\lambda_2+i)(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)(\lambda_3+i)^4\lambda_4^3}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1-i)} \quad \underset{1\leftrightarrow 2}{\sim}$$

$$\sim\; -i\frac{(\lambda_1+i)(\lambda_2+i)^4(\lambda_3+i)(\lambda_1^2+\lambda_1\lambda_3+\lambda_3^2)\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_1-i)} \tag{6.23}$$

Now adding the r.h.s. of (6.21) to (6.23) we get

$$I_5^{(4)} \;+\; I_7^{(4)} \;+\; I_9^{(4)} \;\sim$$

$$\sim\; -i\frac{(\lambda_1+i)(\lambda_2+i)(\lambda_3+i)((\lambda_1+i)^2+(\lambda_1+i)(\lambda_2+i)+(\lambda_2+i)^2)(\lambda_3^2+\lambda_3\lambda_1+\lambda_1^2)\lambda_4^3}{\lambda_4-\lambda_1-i} \;\sim$$

$$\sim\ 3\,\frac{(\lambda_1+i)(\lambda_2+i)(\lambda_3+i)(\lambda_1+\lambda_2+i)(\lambda_3^2+\lambda_3\lambda_1+\lambda_1^2)\lambda_4^3}{\lambda_4-\lambda_1-i}\ =$$

$$=\ -3\,(\lambda_2+i)(\lambda_3+i)(\lambda_1+\lambda_2+i)(\lambda_3^2+\lambda_3\lambda_1+\lambda_1^2)\lambda_4^3\ +$$

$$+\ 3\,\frac{(\lambda_2+i)(\lambda_3+i)(\lambda_1+\lambda_2+i)(\lambda_3^2+\lambda_3\lambda_1+\lambda_1^2)\lambda_4^4}{\lambda_4-\lambda_1-i}\ \sim$$

$$\sim\ 3\,\frac{(\lambda_2+i)(\lambda_3+i)(\lambda_1+\lambda_2+i)(\lambda_3^2+\lambda_3\lambda_1+\lambda_1^2)\lambda_4^4}{\lambda_4-\lambda_1-i}\ \sim$$

$$\sim\ 3\,\frac{(\lambda_1+i)(\lambda_2+i)(\lambda_3+i)^2\lambda_3\lambda_4^4}{\lambda_4-\lambda_1-i}\ +\ 3\,\frac{(\lambda_2+i)(\lambda_3+i)\lambda_3(\lambda_2\lambda_3+1)\lambda_4^4}{\lambda_4-\lambda_1-i}\ \sim$$

$$\sim\ -3\lambda_2\lambda_3^2\lambda_4^3\ +\ 3\,\frac{(\lambda_2+i)(\lambda_3+i)^2\lambda_3\lambda_4^5}{\lambda_4-\lambda_1-i}\ +\ 3\,\frac{(\lambda_2+i)(\lambda_3+i)\lambda_3(\lambda_2\lambda_3+1)\lambda_4^4}{\lambda_4-\lambda_1-i}\ \sim$$

$$\sim\ -3\lambda_2\lambda_3^2\lambda_4^3\ +\ 3\,\frac{(\lambda_2+i)(\lambda_3+i)^2\lambda_3(\lambda_4+i)^5}{\lambda_4-\lambda_1}\ +\ 3\,\frac{(\lambda_2+i)(\lambda_3+i)\lambda_3(\lambda_2\lambda_3+1)(\lambda_4+i)^4}{\lambda_4-\lambda_1}\ \underset{\substack{1\to 2\\2\to 3\\3\to 4\\4\to 1}}{\sim}$$

$$\sim\ -3\lambda_2\lambda_3^2\lambda_4^3\ -\ 3\,\frac{(\lambda_1+i)^5(\lambda_3+i)(\lambda_4+i)^2\lambda_4}{\lambda_2-\lambda_1}\ -\ 3\,\frac{(\lambda_1+i)^4(\lambda_3+i)(\lambda_4+i)\lambda_4(\lambda_3\lambda_4+1)}{\lambda_2-\lambda_1}\ \sim$$

$$\sim\ -\frac{17}{10}\lambda_2\lambda_3^2\lambda_4^3\ +\ \frac{(\frac{\lambda_1}{2}+\frac{i}{4})(\lambda_3+i)(\lambda_4+i)^2\lambda_4}{\lambda_2-\lambda_1}\ +$$

$$+\ \frac{(3\lambda_1^2+3i\lambda_1-\frac{9}{10})(\lambda_3+i)(\lambda_4+i)\lambda_4(\lambda_3\lambda_4+1)}{\lambda_2-\lambda_1}\ \sim\ -\frac{17}{10}\lambda_2\lambda_3^2\lambda_4^3\ +$$

$$+\ \frac{(\frac{\lambda_1}{2}+\frac{i}{4})(\lambda_3\lambda_4^3+i\lambda_4^3+2i\lambda_3\lambda_4^2-2\lambda_4^2-i\lambda_4)\ +\ (3\lambda_1^2+3i\lambda_1-\frac{9}{10})(\lambda_3^2\lambda_4^3+i\lambda_3\lambda_4^3+i\lambda_4^2-\lambda_4)}{\lambda_2-\lambda_1} \tag{6.24}$$

In fact, it is the "canonical" form. To get this expression we have used symmetries, the formula (2.20) for $m=4$ and $m=5$ of the proposition 1 and relations (6.6) and (6.7).

Now let us treat $I_{10}^{(4)}$

$$I_{10}^{(4)}\ =\ -i\frac{(\lambda_1+i)(\lambda_2+i)^4(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_1-i)}\ \sim\ -i\frac{(\lambda_1+i)(\lambda_2+i)^4\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_4-\lambda_1-i)}\ =$$

$$=\ i\frac{(\lambda_2+i)^4\lambda_3^3\lambda_4^3}{\lambda_2-\lambda_1+i}\ -\ i\frac{(\lambda_2+i)^4\lambda_3^3\lambda_4^4}{(\lambda_2-\lambda_1+i)(\lambda_4-\lambda_1-i)}\ \sim\ -i\frac{(\lambda_2+i)^4\lambda_3^3\lambda_4^4}{(\lambda_2-\lambda_1+i)(\lambda_4-\lambda_1-i)}\ \sim$$

$$\sim\ -i\frac{\lambda_2^4\lambda_3^3(\lambda_4+i)^4}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_1)}\ =\ i\frac{\lambda_3^3(\lambda_2^4\lambda_4^4+4i\lambda_2^3\lambda_4^4-6\lambda_2^2\lambda_4^4-4i\lambda_2\lambda_4^4+\lambda_4^4)}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_1)}\ \sim$$

$$\sim\ i\frac{\lambda_3^3(4i\lambda_2^3\lambda_4^3-6\lambda_2^2\lambda_4^2(\lambda_4+\lambda_1)-4i\lambda_2\lambda_4(\lambda_4^2+\lambda_4\lambda_1+\lambda_1^2)+\lambda_4^3+\lambda_4^2\lambda_1+\lambda_4\lambda_1^2+\lambda_1^3)}{\lambda_2-\lambda_1}\ \sim$$

$$\sim\ i\frac{\lambda_3^3(-6\lambda_1\lambda_2^2\lambda_4^2-4i\lambda_1\lambda_2\lambda_4^2-4i\lambda_1^2\lambda_2\lambda_4+\lambda_1\lambda_4^2+\lambda_1^2\lambda_4+\lambda_1^3)}{\lambda_2-\lambda_1}\ =$$

$$= i\lambda_3^3(-6\lambda_1^2\lambda_4^2 - 4i\lambda_1\lambda_4^2 - 4i\lambda_1^2\lambda_4) + i\frac{\lambda_3^3(-6\lambda_1^3\lambda_4^2 - 4i\lambda_1^2\lambda_4^2 - 4i\lambda_1^3\lambda_4 + \lambda_1\lambda_4^2 + \lambda_1^2\lambda_4 + \lambda_1^3)}{\lambda_2 - \lambda_1} \sim$$

$$\sim i\frac{\lambda_3^3(\lambda_4^2(5i\lambda_1^2 - 5\lambda_1 - \frac{3}{2}i) + \lambda_4(-5\lambda_1^2 + 4i\lambda_1 + 1) - \frac{3}{2}i\lambda_1^2 + \lambda_1 + \frac{i}{4})}{\lambda_2 - \lambda_1} +$$

$$+ \lambda_3^3(-3)\lambda_4^2 i\lambda_2 + i\lambda_3^3(-4)i\lambda_4\frac{1}{2}\lambda_2^2 \sim$$

$$\sim -\lambda_2\lambda_3^2\lambda_4^3 + \frac{(5\lambda_1^2 + 5i\lambda_1 - \frac{3}{2})\lambda_3^2\lambda_4^3 + (5i\lambda_1^2 - 4\lambda_1 - i)\lambda_3\lambda_4^3 + (-\frac{3}{2}\lambda_1^2 - i\lambda_1 + \frac{1}{4})\lambda_4^3}{\lambda_2 - \lambda_1} \tag{6.25}$$

Here the symmetry, the formula (2.20) for $m = 4$ and the relation (6.1).

Now we shall treat the other six terms $I_1^{(4)}, I_2^{(4)}, I_3^{(4)}, I_4^{(4)}, I_6^{(4)}$ and $I_8^{(4)}$ given by the expressions (6.11),(6.12),(6.13),(6.14), (6.16) and (6.18) respectively. Here we shall proceed in two steps. As a first step we will reduce all these six terms to a following form

$$I_j^{(4)} \sim \frac{A_j}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_3)} + \frac{B_j}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \frac{C_j}{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)} + \frac{D_j}{\lambda_2 - \lambda_1} \tag{6.26}$$

where $j = 1, 2, 3, 4, 6, 8$ and A_j, B_j, C_j, D_j are some polynomials. Then we shall sum up all the six results and get the "canonical" form for the sum obtained.

Let us start with the expression (6.11) for the term $I_1^{(4)}$

$$I_1^{(4)} = -\frac{i}{2}\frac{(\lambda_1 + i)\lambda_1^3(\lambda_2 + i)\lambda_3^3(\lambda_4 + i)^4}{(\lambda_2 - \lambda_1 + i)(\lambda_3 - \lambda_1 - i)(\lambda_4 - \lambda_3 + i)} =$$

$$= -\frac{i}{2}\frac{(\lambda_1^2 + \lambda_1(\lambda_2 + i) + (\lambda_2 + i)^2)(\lambda_2 + i)\lambda_3^3(\lambda_4 + i)^4}{\lambda_4 - \lambda_3 + i} +$$

$$+ \frac{i}{2}\frac{(\lambda_1^2 + \lambda_1(\lambda_2 + i) + (\lambda_2 + i)^2)(\lambda_2 + i)\lambda_3^4(\lambda_4 + i)^4}{(\lambda_3 - \lambda_1 - i)(\lambda_4 - \lambda_3 + i)} + \frac{i}{2}\frac{(\lambda_2 + i)^4\lambda_3^3(\lambda_4 + i)^3}{\lambda_2 - \lambda_1 + i} -$$

$$- \frac{i}{2}\frac{(\lambda_2 + i)^4\lambda_3^4(\lambda_4 + i)^3}{(\lambda_2 - \lambda_1 - i)(\lambda_3 - \lambda_1 + i)} - \frac{i}{2}\frac{(\lambda_1 + i)(\lambda_2 + i)^4\lambda_3^4(\lambda_4 + i)^3}{(\lambda_2 - \lambda_1 + i)(\lambda_3 - \lambda_1 - i)(\lambda_4 - \lambda_3 + i)} \sim$$

$$\sim \frac{i}{2}\frac{(\lambda_1^2 + \lambda_1(\lambda_2 + i) + (\lambda_2 + i)^2)(\lambda_2 + i)\lambda_3^3\lambda_4^4}{\lambda_4 - \lambda_3} -$$

$$- \frac{i}{2}\frac{(\lambda_1^2 + \lambda_1(\lambda_2 + i) + (\lambda_2 + i)^2)(\lambda_2 + i)(\lambda_3 + i)^4(\lambda_4 + i)^4}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_3)} - \frac{i}{2}\frac{\lambda_2^4\lambda_3^3(\lambda_4 + i)^3}{\lambda_2 - \lambda_1} -$$

$$- \frac{i}{2}\frac{\lambda_2^4(\lambda_3 + i)^4(\lambda_4 + i)^3}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} - \frac{i}{2}\frac{(\lambda_1 + i)\lambda_2^4(\lambda_3 + i)^4(\lambda_4 + i)^3}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_3)} \sim$$

$$\sim \frac{i}{2}\frac{\lambda_1^3\lambda_2^4(\lambda_3^2 + \lambda_3(\lambda_4 + i) + (\lambda_4 + i)^2)(\lambda_4 + i)}{\lambda_2 - \lambda_1} +$$

$$+ \frac{i}{2}\frac{(\lambda_1 + i)^4(\lambda_2 + i)^4(\lambda_3^2 + \lambda_3(\lambda_4 + i) + (\lambda_4 + i)^2)(\lambda_4 + i)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} - \frac{i}{2}\frac{\lambda_2^4\lambda_3^3(\lambda_4 + i)^3}{\lambda_2 - \lambda_1} -$$

$$- \frac{i}{2}\frac{\lambda_2^4(\lambda_3 + i)^4(\lambda_4 + i)^3}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} - \frac{i}{2}\frac{(\lambda_1 + i)\lambda_2^4(\lambda_3 + i)^4(\lambda_4 + i)^3}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_3)} \tag{6.27}$$

where we have used symmetries and the formula (2.20) with $m = 4$.

For the next term $I_2^{(4)}$ we start with the expression (6.12)

$$I_2^{(4)} \sim \frac{i}{2}\frac{(\lambda_1+i)^4(\lambda_2+i)\lambda_3^3(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)} =$$

$$= -\frac{i}{2}\frac{(\lambda_1+i)^4(\lambda_2+i)(\lambda_3^2+\lambda_3(\lambda_4+i)+(\lambda_4+i)^2)(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)} +$$

$$+ \frac{i}{2}\frac{(\lambda_1+i)^4(\lambda_2+i)(\lambda_4+i)^4\lambda_4^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)} \sim$$

$$\sim \frac{i}{2}\frac{\lambda_1^4(\lambda_2+i)(\lambda_3^2+\lambda_3(\lambda_4+i)+(\lambda_4+i)^2)(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{i}{2}\frac{\lambda_1^4(\lambda_2+i)\lambda_4^4(\lambda_4-i)^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} \quad (6.28)$$

The term $I_3^{(4)}$ given by (6.13) is also simple to treat

$$I_3^{(4)} = \frac{i}{2}\frac{(\lambda_1+i)^4\lambda_2^3(\lambda_3+i)(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3-i)} =$$

$$= -\frac{i}{2}\frac{(\lambda_1+i)^4\lambda_2^3(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)} + \frac{i}{2}\frac{(\lambda_1+i)^4\lambda_2^3(\lambda_4+i)\lambda_4^4}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3-i)} \sim$$

$$\sim \frac{i}{2}\frac{\lambda_1^4\lambda_2^3(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{i}{2}\frac{\lambda_1^4\lambda_2^3(\lambda_4+2i)(\lambda_4+i)^4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} \quad (6.29)$$

The fourth term $I_4^{(4)}$ (6.14) demands more work in order to reduce it to the form (6.26)

$$I_4^{(4)} \sim \frac{i}{2}\frac{(\lambda_1+i)(\lambda_2+i)^4(\lambda_3+i)\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3-i)} =$$

$$= -\frac{i}{2}\frac{(\lambda_1+i)(\lambda_2+i)^4\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)} + \frac{i}{2}\frac{(\lambda_1+i)(\lambda_2+i)^4\lambda_3^3\lambda_4^4}{(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3-i)} =$$

$$= \frac{i}{2}\frac{(\lambda_2+i)^4\lambda_3^3\lambda_4^3}{(\lambda_2-\lambda_1+i)} - \frac{i}{2}\frac{(\lambda_2+i)^4\lambda_3^4\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)} +$$

$$+\frac{i(\lambda_1+i)(\lambda_2+i)^4(\lambda_3^2+\lambda_3(\lambda_1+i)+(\lambda_1+i)^2)\lambda_4^4}{2(\lambda_2-\lambda_1+i)(\lambda_4-\lambda_3-i)} + \frac{i(\lambda_1+i)^4(\lambda_2+i)^4\lambda_4^4}{2(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3-i)} \sim$$

$$\sim -\frac{i}{2}\frac{\lambda_2^4(\lambda_3+i)^4\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{i}{2}\frac{(\lambda_1+i)\lambda_2^4(\lambda_3^2+\lambda_3(\lambda_1+i)+(\lambda_1+i)^2)(\lambda_4+i)^4}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} -$$

$$- \frac{i}{2}\frac{\lambda_1^4\lambda_2^4(\lambda_4+i)^4}{(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} \sim$$

$$\sim -\frac{i}{2}\frac{\lambda_2^4(\lambda_3+i)^4\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{i}{2}\frac{(\lambda_1+i)\lambda_2^4(\lambda_3^2+\lambda_3(\lambda_1+i)+(\lambda_1+i)^2)(\lambda_4+i)^4}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} +$$

$$+\frac{i(\lambda_1^3+\lambda_1^2(\lambda_2+i)+\lambda_1(\lambda_2+i)^2+(\lambda_2+i)^3)\lambda_2^4(\lambda_4+i)^4}{2(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} - \frac{i(\lambda_2+i)^4\lambda_2^4(\lambda_4+i)^4}{2(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} \sim$$

$$\sim\ -\frac{i}{2}\frac{\lambda_2^4(\lambda_3+i)^4\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\ +\ \frac{i}{2}\frac{(\lambda_1+i)\lambda_2^4(\lambda_3^2+\lambda_3(\lambda_1+i)+(\lambda_1+i)^2)(\lambda_4+i)^4}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)}\ -$$
$$-\ \frac{i}{2}\frac{(\lambda_2+i)^4\lambda_4^4(\lambda_3^3+\lambda_3^2(\lambda_4+i)+\lambda_3(\lambda_4+i)^2+(\lambda_4+i)^3)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\ +\ \frac{i}{2}\frac{\lambda_2^4(\lambda_2-i)^4(\lambda_4+i)^4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} \tag{6.30}$$

Now we treat $I_6^{(4)}$ given by (6.16) as follows

$$I_6^{(4)}\ \sim\ i\frac{(\lambda_1+i)(\lambda_2+i)\lambda_2^3\lambda_3^3(\lambda_4+i)^4}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)}\ =$$
$$=\ i\frac{(\lambda_1+i)(\lambda_2+i)(\lambda_2^2+\lambda_2(\lambda_1+i)+(\lambda_1+i)^2)\lambda_3^3(\lambda_4+i)^4}{(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)}\ +$$
$$+\ i\frac{(\lambda_1+i)^4(\lambda_2+i)\lambda_3^3(\lambda_4+i)^4}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)}\ =$$
$$=\ -i\frac{(\lambda_2+i)(\lambda_2^2+\lambda_2(\lambda_1+i)+(\lambda_1+i)^2)\lambda_3^3(\lambda_4+i)^4}{\lambda_4-\lambda_3+i}\ +$$
$$+\ i\frac{(\lambda_2+i)(\lambda_2^2+\lambda_2(\lambda_1+i)+(\lambda_1+i)^2)\lambda_3^4(\lambda_4+i)^4}{(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)}\ +\ i\frac{(\lambda_1+i)^4(\lambda_2+i)\lambda_3^3(\lambda_4+i)^4}{(\lambda_2-\lambda_1-i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)}\ =$$
$$=\ i\frac{(\lambda_2+i)(\lambda_2^2+\lambda_2(\lambda_1+i)+(\lambda_1+i)^2)\lambda_3^3\lambda_4^4}{\lambda_4-\lambda_3}\ +$$
$$+\ i\frac{(\lambda_2+i)(\lambda_2^2+\lambda_2(\lambda_1+i)+(\lambda_1+i)^2)(\lambda_3+i)^4(\lambda_4+i)^4}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)}\ +\ i\frac{\lambda_1^4(\lambda_2+i)\lambda_3^3\lambda_4^4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)}\ \sim$$
$$\sim\ -i\frac{\lambda_1^3\lambda_2^4(\lambda_3+i)(\lambda_3^2+\lambda_3(\lambda_4+i)+(\lambda_4+i)^2)}{\lambda_2-\lambda_1}\ -$$
$$-\ i\frac{(\lambda_1+i)^4(\lambda_2+i)^4(\lambda_4^2+\lambda_4(\lambda_3+i)+(\lambda_3+i)^2)(\lambda_4+i)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\ +\ i\frac{\lambda_1^4(\lambda_2+i)\lambda_3^3\lambda_4^4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} \tag{6.31}$$

The last term $I_8^{(4)}$ (6.18) is more simple

$$I_8^{(4)}\ \sim\ -i\frac{(\lambda_1+i)(\lambda_2+i)^4\lambda_3^3(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)}\ =$$
$$=\ i\frac{(\lambda_2+i)^4\lambda_3^3(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_4-\lambda_3+i)}\ -\ i\frac{(\lambda_2+i)^4\lambda_3^4(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)}\ =$$
$$=\ i\frac{(\lambda_2+i)^4\lambda_3^3\lambda_4^3}{\lambda_2-\lambda_1+i}\ +\ i\frac{(\lambda_2+i)^4\lambda_3^4\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_4-\lambda_3+i)}\ -\ i\frac{(\lambda_2+i)^4\lambda_3^4(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1+i)(\lambda_3-\lambda_1-i)(\lambda_4-\lambda_3+i)}\ \sim$$
$$\sim\ i\frac{\lambda_2^4(\lambda_3+i)^4\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)}\ -\ i\frac{\lambda_2^4(\lambda_3+i)^4(\lambda_4+i)\lambda_4^3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} \tag{6.32}$$

Now we are prepared to perform the next our step. Namely, we will gather all the six results (6.27-6.32) into the form like (6.26)

$$I_1^{(4)}\ +\ I_2^{(4)}\ +\ I_3^{(4)}\ +\ I_4^{(4)}\ +\ I_6^{(4)}\ +\ I_8^{(4)}\ \sim$$

$$\sim \frac{A}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} + \frac{B}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{C}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} + \frac{D}{\lambda_2-\lambda_1} \quad (6.33)$$

where

$$A = -\frac{i}{2}(\lambda_1+i)\lambda_2^4(\lambda_3+i)^4(\lambda_4+i)^3 + \frac{i}{2}\lambda_1^4(\lambda_2+i)\lambda_4^4(\lambda_4-i)^3 + \frac{i}{2}\lambda_1^4\lambda_2^3(\lambda_4+2i)(\lambda_4+i)^4 +$$
$$+ \frac{i}{2}\lambda_2^4(\lambda_2-i)(\lambda_4-i)^4 + i\lambda_1^4(\lambda_2+i)\lambda_3^3\lambda_4^4 - i\lambda_2^4(\lambda_3+i)^4(\lambda_4+i)\lambda_4^3 \quad (6.34)$$

$$B = \frac{i}{2}(\lambda_1+i)^4(\lambda_2+i)^4(\lambda_3^2+\lambda_3(\lambda_4+i)+(\lambda_4+i)^2)(\lambda_4+i) - i\lambda_2^4(\lambda_3+i)^4(\lambda_4+i)^3 +$$
$$+ \frac{i}{2}\lambda_1^4(\lambda_2+i)(\lambda_3^2+\lambda_3(\lambda_4+i)+(\lambda_4+i)^2)(\lambda_4+i)\lambda_4^3 + i\lambda_1^4\lambda_2^3(\lambda_4+i)\lambda_4^3 +$$
$$+ i(\lambda_1+i)^4(\lambda_2+i)^4(\lambda_4^2+\lambda_4(\lambda_3+i)+(\lambda_3+i)^2)(\lambda_4+i) \quad (6.35)$$

$$C = \frac{i}{2}(\lambda_1+i)\lambda_2^4(\lambda_3^2+\lambda_3(\lambda_1+i)+(\lambda_1+i)^2)(\lambda_4+i)^4 + i\lambda_2^4(\lambda_3+i)^4\lambda_4^3 \quad (6.36)$$

$$D = \frac{i}{2}\lambda_1^3\lambda_2^4(\lambda_3^2+\lambda_3(\lambda_4+i)+(\lambda_4+i)^2)(\lambda_4+i) - i\lambda_2^4\lambda_3^3(\lambda_4+i)^3 -$$
$$- i\lambda_1^3\lambda_2^4(\lambda_3^2+\lambda_3(\lambda_4+i)+(\lambda_4+i)^2)(\lambda_3+i) \quad (6.37)$$

Now we want to get the "canonical" form (2.7) for the expression (6.33). To do this we actively used the program *MATHEMATICA* because the calculations are straightforward but become more cumbersome. Let us outline our further actions.

It is more convenient to start with the first term in the r.h.s. of the formula (6.33)

$$\frac{A}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} \quad (6.38)$$

We shall use the fact that the denominator is antisymmetric under the substitution

$$\lambda_1 \leftrightarrow \lambda_3$$
$$\lambda_2 \leftrightarrow \lambda_4$$

Since A is a polynomial given by (6.34) then the simplification procedure of the term (6.38) is as follows. If in the expression (6.34) one faces a monomial

$$\lambda_1^{i_1}\lambda_2^{i_2}\lambda_3^{i_3}\lambda_4^{i_4} \quad (6.39)$$

where without loss of generality $i_4 \geq i_2$ one can apply the evident identity

$$\lambda_4^{i_4} = \lambda_4^{i_2}\lambda_3^{i_4-i_2} + \begin{cases} (\lambda_4-\lambda_3)\sum_{k=0}^{i_4-i_2-1}\lambda_4^{i_2+k}\lambda_3^{i_4-i_2-1-k}, & i_4 > i_2; \\ 0 \quad , & i_4 = i_2 \end{cases} \quad (6.40)$$

Therefore if $i_4 > i_2$ then

$$\frac{\lambda_1^{i_1}\lambda_2^{i_2}\lambda_3^{i_3}\lambda_4^{i_4}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} = \frac{\lambda_1^{i_1}\lambda_2^{i_2}\lambda_3^{i_4-i_2+i_3}\lambda_4^{i_2}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} + \sum_{k=0}^{i_4-i_2-1}\frac{\lambda_1^{i_1}\lambda_2^{i_2}\lambda_3^{i_4-i_2+i_3-1-k}\lambda_4^{i_2+k}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} \quad (6.41)$$

Let us note that the second term in (6.41) gives rise into the second term "B" in the r.h.s. of the formula (6.33). If $i_4 = i_2$ then only the first term in (6.41) survives. The first term in (6.41) is symmetric under the transposition $\lambda_2 \leftrightarrow \lambda_4$. If $i_1 + i_2 = i_3 + i_4$ then it is also symmetric under the transposition $\lambda_1 \leftrightarrow \lambda_3$ and the first term is "weakly" equivalent to zero according to the formula (2.15). If $i_1 + i_2 < i_3 + i_4$ the following "weak" equality is valid

$$\frac{\lambda_1^{i_1}\lambda_2^{i_2}\lambda_3^{i_4-i_2+i_3}\lambda_4^{i_2}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} \sim \frac{1}{2}\frac{(\lambda_1^{i_1}\lambda_3^{i_3+i_4-i_2} - \lambda_3^{i_1}\lambda_1^{i_3+i_4-i_2})\lambda_2^{i_2}\lambda_4^{i_2}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3)} =$$

$$= \frac{1}{2}\sum_{k=0}^{i_3+i_4-i_2-i_1-1}\frac{\lambda_1^{i_1+k}\lambda_2^{i_2}\lambda_3^{i_3+i_4-i_2-1-k}\lambda_4^{i_2}}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} \tag{6.42}$$

For the case $i_1 + i_2 > i_3 + i_4$ the sum in (6.42) should be substituted by

$$-\frac{1}{2}\sum_{k=0}^{i_1+i_2-i_3-i_4-1}\frac{\lambda_1^{i_1-1-k}\lambda_2^{i_2}\lambda_3^{i_3+i_4-i_2+k}\lambda_4^{i_2}}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} \tag{6.43}$$

In both cases, namely, if $i_1 + i_2 \neq i_3 + i_4$ the sum in (6.42) or (6.43) gives rise into the third term "C" in the r.h.s. of (6.33).

Performing this procedure for all monomials of the form (6.39) participating in the polynomial A given by the formula (6.34) one can arrive at the formula

$$I_1^{(4)} + I_2^{(4)} + I_3^{(4)} + I_4^{(4)} + I_6^{(4)} + I_8^{(4)} \sim \frac{B'}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{C'}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} + \frac{D}{\lambda_2-\lambda_1} \tag{6.44}$$

with some other polynomials B' and C'.

Due to the $2 \leftrightarrow 3$ symmetry of the denominator of the first term in (6.44) one can treat any monomial $\lambda_1^{j_1}\lambda_2^{j_2}\lambda_3^{j_3}\lambda_4^{j_4}$ participating in B' as follows

$$\frac{\lambda_1^{j_1}\lambda_2^{j_2}\lambda_3^{j_3}\lambda_4^{j_4}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} \sim \sum_{k=0}^{j_3-j_2-1}\frac{\lambda_1^{j_1}\lambda_2^{j_2+k}\lambda_3^{j_3-k-1}\lambda_4^{j_4}}{\lambda_2-\lambda_1} \tag{6.45}$$

where without loss of generality it is implied that $j_3 > j_2$ because if $j_2 = j_3$ then due to the $2 \leftrightarrow 3$ symmetry and the formula (2.15) this term would make zero contribution. So the r.h.s. of the (6.45) gives rise into the third term in (6.44). Using this one can treat the whole first term in (6.44).

For any monomial $\lambda_1^{k_1}\lambda_2^{k_2}\lambda_3^{k_3}\lambda_4^{k_4}$ participating in C' of the second term of the expression (6.44) one can write

$$\frac{\lambda_1^{k_1}\lambda_2^{k_2}\lambda_3^{k_3}\lambda_4^{k_4}}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} = \sum_{l=0}^{k_2-1}\frac{\lambda_1^{k_1+l}\lambda_2^{k_2}\lambda_3^{k_3-1-l}\lambda_4^{k_4}}{\lambda_4-\lambda_3} + \sum_{l=0}^{k_4-1}\frac{\lambda_1^{k_1+k_2}\lambda_3^{k_3+l}\lambda_4^{k_4-1-l}}{\lambda_2-\lambda_1} + \frac{\lambda_1^{k_1+k_2}\lambda_3^{k_3+k_4}}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} \sim \tag{6.46}$$

$$\sim \sum_{l=0}^{k_2-1}\frac{\lambda_1^{k_1}\lambda_2^{k_4}\lambda_3^{k_1+l}\lambda_4^{k_2-1-l}}{\lambda_2-\lambda_1} + \sum_{l=0}^{k_4-1}\frac{\lambda_1^{k_1+k_2}\lambda_3^{k_3+l}\lambda_4^{k_4-1-l}}{\lambda_2-\lambda_1} + \frac{\lambda_1^{k_1+k_2}\lambda_3^{k_3+k_4}}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} \tag{6.47}$$

where for the first sum of (6.46) we have applied transformation

$$\lambda_1 \leftrightarrow \lambda_3$$
$$\lambda_2 \leftrightarrow \lambda_4$$

So the both the first term and the second term in (6.47) give rise into the third "D" term in (6.44) while the third term in (6.47) gives rise to the second term "C" in (6.44). Proceeding this way one can treat the whole expression

$$\frac{C'}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)}.$$

As a result of performing this scheme one can arrive at the expression

$$I_1^{(4)} + I_2^{(4)} + I_3^{(4)} + I_4^{(4)} + I_6^{(4)} + I_8^{(4)} \sim$$
$$\sim \frac{C''}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} + \frac{D'}{\lambda_2-\lambda_1} \tag{6.48}$$

where C'' is a polynomial of two variable λ_1 and λ_3.

Now with the help of the identity

$$\lambda_1^{i_1}\lambda_2^{i_2} = \lambda_1^{i_1+i_2} + \begin{cases} (\lambda_2-\lambda_1)\sum_{k=0}^{i_2-1}\lambda_1^{i_1+k}\lambda_2^{i_2-1-k}, & i_2>0; \\ 0 \quad , & i_2=0 \end{cases} \tag{6.49}$$

one can reduce the second term in (6.48) to the form

$$\frac{D'}{\lambda_2-\lambda_1} \sim \frac{D''}{\lambda_2-\lambda_1} + E$$

where D'' is a polynomial of λ_1, λ_3 and λ_4 and E is some polynomial.

What is left now is to reduce the power of the polynomials C'', D'' and E with the help of the formulae (2.20), (2.21), (2.22) or (6.1-6.4). We should also use the fact that if we do the substitution $\lambda_j \to x_j - i/2$ there is a restriction that the function should be even under the transformation $\{x_1, x_2, x_3, x_4\} \to \{-x_1, -x_2, -x_3, -x_4\}$ according to the formulae (2.9), (2.10) and (2.11).

As a result of all these actions described above we get the "canonical" form for the sum

$$I_1^{(4)} + I_2^{(4)} + I_3^{(4)} + I_4^{(4)} + I_6^{(4)} + I_8^{(4)} \sim \frac{C'''}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} + \frac{D'''}{\lambda_2-\lambda_1} + E' \tag{6.50}$$

where

$$C''' = 2\lambda_1^2\lambda_3^2 + 4i\lambda_1\lambda_3^2 - \frac{3}{2}\lambda_3^2 - \frac{3}{2}\lambda_1\lambda_3 - i\lambda_3 + \frac{1}{5} \tag{6.51}$$

$$D''' = \lambda_1^2(22\lambda_3^2\lambda_4^3 + 22i\lambda_3\lambda_4^3 - \frac{29}{2}\lambda_4^3 + 19\lambda_3\lambda_4^2 - \frac{5}{4}i\lambda_4^2) +$$
$$+ \lambda_1(22i\lambda_3^2\lambda_4^3 - \frac{47}{2}\lambda_3\lambda_4^3 - \frac{61}{4}i\lambda_4^3 + \frac{67}{4}i\lambda_3\lambda_4^2 - \frac{3}{2}\lambda_4^2 + \frac{23}{4}i\lambda_4) -$$
$$- \frac{88}{5}\lambda_3^2\lambda_4^3 - \frac{367}{20}i\lambda_3\lambda_4^3 + \frac{427}{40}\lambda_4^3 - \frac{303}{40}\lambda_3\lambda_4^2 + \frac{37}{40}i\lambda_4^2 - \frac{97}{20}\lambda_4) \tag{6.52}$$

$$E' = -\frac{57}{10}\lambda_2\lambda_3^2\lambda_4^3 \tag{6.53}$$

Now we have to sum up the four contributions (6.10), (6.24), (6.25) and (6.50) and get the "canonical" form for $T_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$

$$T_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \;\sim\; P_0^{(4)} \;+\; \frac{P_1^{(4)}}{\lambda_2 - \lambda_1} \;+\; \frac{P_2^{(4)}}{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)} \tag{6.54}$$

where

$$P_0^{(4)} \;=\; -\frac{34}{5}\lambda_2\lambda_3^2\lambda_4^3 \tag{6.55}$$

$$\begin{aligned} P_1^{(4)} \;=\; & \lambda_1^2\,(30\,\lambda_3^2\lambda_4^3 \;+\; 30\,i\lambda_3\lambda_4^3 \;-\; 16\,\lambda_4^3 \;+\; 18\,\lambda_3\lambda_4^2 \;+\; 8\,\lambda_4) \;+ \\ & +\; \lambda_1\,(30\,i\lambda_3^2\lambda_4^3 \;+\; 30\,\lambda_3\lambda_4^3 \;-\; 16\,i\lambda_4^3 \;+\; 18\,i\lambda_3\lambda_4^2 \;-\; 4\,\lambda_4^2 \;+\; 4\,i\lambda_4) \;- \\ & -\; 20\,\lambda_3^2\lambda_4^3 \;-\; 20\,i\lambda_3\lambda_4^3 \;+\; \frac{54}{5}\lambda_4^3 \;-\; \frac{42}{5}\lambda_3\lambda_4^2 \;-\; \frac{43}{10}i\lambda_4 \end{aligned} \tag{6.56}$$

$$P_2^{(4)} \;=\; 2\lambda_1^2\lambda_3^2 \;+\; 4i\lambda_1\lambda_3^2 \;-\; \frac{3}{2}\lambda_3^2 \;-\; \frac{3}{2}\lambda_1\lambda_3 \;-\; i\lambda_3 \;+\; \frac{1}{5} \tag{6.57}$$

Let us note that in terms of the real variables x_j the polynomials $\tilde{P}_j^{(4)}$ (see (2.11)) look a little bit simpler

$$\tilde{P}_0^{(4)} \;=\; -\frac{34}{5}x_2x_3^2x_4^3 \tag{6.58}$$

$$\begin{aligned} \tilde{P}_1^{(4)} \;=\; & x_1^2\,(30\,x_3^2x_4^3 \;-\; \frac{17}{2}x_4^3 \;+\; \frac{81}{2}x_3x_4^2 \;+\; \frac{79}{8}x_4) \;-\; 4\,x_1x_4^2 \;- \\ & -\; \frac{25}{2}x_3^2x_4^3 \;+\; \frac{147}{40}x_4^3 \;-\; \frac{531}{40}x_3x_4^2 \;-\; \frac{653}{160}x_4 \end{aligned} \tag{6.59}$$

$$\tilde{P}_2^{(4)} \;=\; 2\,x_1^2x_3^2 \;+\; \frac{1}{2}x_1x_3 \;-\; \frac{1}{2}x_3^2 \;+\; \frac{3}{40} \tag{6.60}$$

So, the function

$$\tilde{T}_4^c(x_1, x_2, x_3, x_4) \;=\; \tilde{P}_0^{(4)} \;+\; \frac{\tilde{P}_1^{(4)}}{x_2 - x_1} \;+\; \frac{\tilde{P}_2^{(4)}}{(x_2 - x_1)(x_4 - x_3)} \tag{6.61}$$

is even i.e.

$$\tilde{T}_4^c(-x_1, -x_2, -x_3, -x_4) \;=\; \tilde{T}_4^c(x_1, x_2, x_3, x_4)$$

as it should be according to the formula (2.13) with $n = 4$.

Now let us start the second step of our general scheme, namely, the performing of the integration of the "canonical" form (6.54)

$$P(4) \;=\; \prod_{j=1}^{4} \int_{C_{-1/2}} \frac{d\lambda_j}{2\pi i}\, U_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\, T_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \;=\; J_0^{(4)} \;+\; J_1^{(4)} \;+\; J_2^{(4)} \tag{6.62}$$

where

$$J_0^{(4)} = \prod_{j=1}^{4} \int_{C_{-1/2}} \frac{d\lambda_j}{2\pi i}\, U_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\, P_0^{(4)} \tag{6.63}$$

$$J_1^{(4)} = \prod_{j=1}^{4} \int_{C_{-1/2}} \frac{d\lambda_j}{2\pi i}\, U_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\, \frac{P_1^{(4)}}{\lambda_2-\lambda_1} \tag{6.64}$$

$$J_1^{(4)} = \prod_{j=1}^{4} \int_{C_{-1/2}} \frac{d\lambda_j}{2\pi i}\, U_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\, \frac{P_2^{(4)}}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} \tag{6.65}$$

Using the formulae (6.63-6.65), (6.55-6.57),(2.30),(2.31), (2.35-2.38) we get

$$J_0^{(4)} \;=\; \prod_{j=1}^{4} \int_{C_{-1/2}} \frac{d\lambda_j}{2\pi i}\, U_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\, (-\frac{34}{5})\lambda_2\lambda_3^2\lambda_4^3 \;=$$

$$=\; D^{(4)} \sum_{l_1=0}^{\infty}(-1)^{l_1} \sum_{l_2=0}^{\infty}(-1)^{l_2} \sum_{l_3=0}^{\infty}(-1)^{l_3} \sum_{l_4=0}^{\infty}(-1)^{l_4}(-\frac{34}{5})(il_2+\epsilon_2)(il_3+\epsilon_3)^2(il_4+\epsilon_4)^2 \;=$$

$$=\; D^{(4)} \sum_{l_1=0}^{\infty}(-1)^{l_1} \sum_{l_2=0}^{\infty}(-1)^{l_2} \sum_{l_3=0}^{\infty}(-1)^{l_3} \sum_{l_4=0}^{\infty}(-1)^{l_4}(-\frac{34}{5})\epsilon_2\epsilon_3^2\epsilon_4^3 \;=$$

$$=\; \frac{1}{12} \prod_{0\le k<j\le 4} \Big(\frac{\partial}{\partial\epsilon_k} - \frac{\partial}{\partial\epsilon_j}\Big)_{\vec{\epsilon}\to 0} (-\frac{34}{5})\epsilon_2\epsilon_3^2\epsilon_4^3 \;=\; -\frac{17}{40} \tag{6.66}$$

Restoring for convenience the dependence of the polynomials $P_1^{(4)}$ and $P_2^{(4)}$ on λ-s as in the formula (2.8) we can get for the term $J_1^{(4)}$

$$J_1^{(4)} \;=\; \prod_{j=1}^{4} \int_{C_{-1/2}} \frac{d\lambda_j}{2\pi i}\, U_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\, \frac{P_1^{(4)}(\lambda_1|\lambda_3,\lambda_4)}{\lambda_2-\lambda_1} \;=$$

$$=\; D^{(4)} \sum_{l_1=1}^{\infty}(-1)^{l_1} \sum_{l_2=0}^{\infty}(-1)^{l_2} \sum_{l_3=1}^{\infty}(-1)^{l_3} \sum_{l_4=0}^{\infty}(-1)^{l_4} \frac{P_1^{(4)}(-il_1+\epsilon_1|-il_3+\epsilon_3, il_4+\epsilon_4)}{il_2+il_1+\epsilon_2-\epsilon_1} \;=$$

$$=\; -\,iD^{(4)} \int_0^1 \frac{ds}{s} \sum_{l_1=1}^{\infty}(-1)^{l_1} \sum_{l_2=0}^{\infty}(-1)^{l_2} \sum_{l_3=1}^{\infty}(-1)^{l_3} \sum_{l_4=0}^{\infty}(-1)^{l_4}$$

$$P_1^{(4)}(-il_1+\epsilon_1|-il_3+\epsilon_3, il_4+\epsilon_4)s^{l_1+l_2+i(\epsilon_1-\epsilon_2)} \;=$$

$$=\; \int_0^1 ds\Big(-\frac{15}{2}\frac{(s-1)}{(1+s)^3} \;+\; \frac{3}{4}\frac{(7-26s+7s^2)}{(1+s)^4}\ln s \;+\; \frac{21}{8}\frac{(s-1)}{(1+s)^3}\ln^2 s \;-\; \frac{7}{10}\frac{(2-s+2s^2)}{(1+s)^4}\ln^3 s \;+$$

$$+\,\frac{5}{48}\frac{(s-1)}{(1+s)^3}\ln^4 s \;+\; \frac{1}{240}\frac{(1+22s+s^2)}{(1+s)^4}\ln^5 s\Big) \;=\; \frac{5}{8} \;-\; 2\ln 2 \;+\; \frac{61}{20}\zeta(3) \;-\; \frac{65}{32}\zeta(5) \tag{6.67}$$

The last term $J_2^{(4)}$ can be calculated as follows

$$J_2^{(4)} \;=\; \prod_{j=1}^{4} \int_{C_{-1/2}} \frac{d\lambda_j}{2\pi i}\, U_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\, \frac{P_2^{(4)}(\lambda_1,\lambda_3)}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)} \;=$$

$$=\; D^{(4)} \sum_{l_1=1}^{\infty}(-1)^{l_1} \sum_{l_2=0}^{\infty}(-1)^{l_2} \sum_{l_3=1}^{\infty}(-1)^{l_3} \sum_{l_4=0}^{\infty}(-1)^{l_4} \frac{P_2^{(4)}(-il_1+\epsilon_1, -il_3+\epsilon_3)}{(il_2+il_1+\epsilon_2-\epsilon_1)(il_4+il_3+\epsilon_4-\epsilon_3)} \;=$$

$$= -D^{(4)} \int_0^1 \frac{ds}{s} \int_0^1 \frac{dt}{t} \sum_{l_1=1}^{\infty} (-1)^{l_1} \sum_{l_2=0}^{\infty} (-1)^{l_2} \sum_{l_3=1}^{\infty} (-1)^{l_3} \sum_{l_4=0}^{\infty} (-1)^{l_4}$$

$$P_2^{(4)}(-il_1+\epsilon_1, -il_3+\epsilon_3) s^{l_1+l_2+i(\epsilon_1-\epsilon_2)} t^{l_3+l_4+i(\epsilon_3-\epsilon_4)} =$$

$$= \int_0^1 ds \int_0^1 dt \Big\{ 4\frac{\ln s(\ln s - \ln t)}{(1+s)^2(1+t)^2} + (-\frac{4}{3}(-3-s+t+3st)\ln^3 s + 4(-1+3s-3t+st)\ln^2 s \ln t -$$

$$- \frac{3}{4}(3-13s+3t+3st)\ln^2 s \ln^2 t + \frac{5}{12}(3-5s-5t+3st)\ln^4 s) \frac{1}{(1+s)^3(1+t)^3} -$$

$$- \frac{1}{6}\frac{(-1+5s-2s^2+2t-5st+s^2t)}{(1+s)^4(1+t)^3} \ln s(\ln^2 s - \ln^2 t)(\ln^2 s - 5\ln^2 t) +$$

$$+ (\frac{1}{3}(3+30s-5s^2-34t-4st-34s^2t-5t^2+30st^2+3s^2t^2)\ln^3 s \ln t +$$

$$+ \frac{1}{30}(4-17s+9s^2-17t+66st-17s^2t+9t^2-17st^2+4s^2t^2)\ln^5 s \ln t -$$

$$- \frac{1}{15}(2-31s+7s^2+14t+33st-21s^2t+2t^2+4st^2+2s^2t^2)\ln^3 s \ln^3 t) \frac{1}{(1+s)^4(1+t)^4} \Big\}$$

Taking the integral we come to an answer for $J_2^{(4)}$

$$J_2^{(4)} = -\frac{1}{6}\zeta(3) - \frac{11}{6}\zeta(3)\ln 2 - \frac{51}{80}\zeta(3)^2 - \frac{25}{96}\zeta(5) + \frac{85}{24}\zeta(5)\ln 2 \qquad (6.68)$$

Summing up all the three results (6.66), (6.67) and (6.68) we finally get our main formula (1.9)

$$P(4) = J_0^{(4)} + J_1^{(4)} + J_2^{(4)} =$$

$$\frac{1}{5} - 2\ln 2 + \frac{173}{60}\zeta(3) - \frac{11}{6}\zeta(3)\ln 2 - \frac{51}{80}\zeta^2(3) - \frac{55}{24}\zeta(5) + \frac{85}{24}\zeta(5)\ln 2 \qquad (6.69)$$

References

[1] D. Kreimer, *"Knots and Feynman Diagrams"*, Cambridge University Press, 2000.

[2] H.E. Boos, V.E. Korepin *"Quantum Spin Chains and Riemann Zeta Function with Odd Arguments*, hep-th/0104008

[3] W. Heisenberg, Zeitschrift für Physik, **vol. 49** (9-10), 619, (1928)

[4] H. Bethe, Zeitschrift für Physik, **76**, 205 (1931)

[5] L. Hulthén, Ark. Mat. Astron. Fysik **A 26**, 1 (1939). *"Thermodynamics of one-dimensional solvable models"*,
Cambridge University Press, 1999.

[6] V.E. Korepin, A.G. Izergin and N.M. Bogoliubov, *"Quantum inverse scattering method and correlation functions"*, Cambridge Univ. Press, Cambridge, 1993.

[7] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, *Phys. Lett.* **A 166**, 256, (1992); hep-th/9205055.

[8] V.E. Korepin, A.G. Izergin, F. Essler, D.B. Uglov, *Phys. Lett.* **A 190**, 182-184, (1994).

[9] M. Takahashi, J. Phys. **C 10**, 1289, (1977); cond-mat/9708087.

[10] J. Dittrich and V.I. Inozemtsev, "On the second-neighbor correlator in 1D XXX quantum anti- ferromagnetic spin chain", cond-mat/9706263.

[11] F. Berruto, G. Grignani, G.W. Semenoff and P. Sodano, hep-th/9901142, preprint DFUPG-190-98, UBC/GS-6-98.

[12] J. de Gier and V.E. Korepin, math-ph/0101036.

[13] A.V. Razumov and Yu.G. Stroganov, cond-mat/0012141; cond-mat/0102247.

[14] E.C. Titchmarch, *"The Theory of the Riemann Zeta- Function "*, Clarendon Pr, 1987.

[15] B.Riemann, *"Über die Anzahl der Primzahlen unter einer gegebenen Grösse"*, Monat. der Koenigl. Preuss. Akad. der Wissen. zu Berlin aus dem Jahre 1859 (1860), 671-680; also Gesammelte mat. Werke und wissensch. Nachlass, 2 Aufl. 1892, 145-155

[16] Tanguy Rivoal, C. R. Acad. Sci. Paris Sr. I Math. 331 (2000), no. 4, 267–270 and http://front.math.ucdavis.edu/math.NT/0104221

[17] Don Zagier, *"Values of Zeta Functions and their Applications"*, First European Congress of Mathematics, Vo.II (Paris, 1992) Prog.Math., Birkhauser, Basel-Boston, page 497, 1994

[18] J.M.Borwein, D. M.Bradley, D.J.Broadhurst and P. Lisonek, *"Special values of multiple polylogarithms"*, math.CA/9910045

[19] A. Knauf, Reviews in Mathematical Physics, vol. 11, 1027-1060 (1999)

[20] N.Kitanine, J.M. Maillet and V.Terras, *"Correlation functions of the XXZ Heisenberg spin 1/2 chain in a magnetic field"*, math-ph/9907019

[21] P.J. Forrester and A.M. Odlyzko, Phys. Rev. E vol. 54, p 4493, 1996

[22] C. A. Tracy and H. Widom, *"Fredholm determinants, differential equations and matrix models"*, Commun. Math. Phys. 163, p 33-72 (1994)

Toeplitz Determinants

[23] A. R. Its, A. G. Izergin, V. E. Korepin, N. A. Slavnov, Int. Jour. Mod. Phys. B 4 page 1003, 1990

[24] A. M. Odlyzko, *"The* 10^{20} *zero of the Riemann zeta function and* 70 *million of its neighbors"*, AT&T Bell Lab preprint (1989).

[25] H. L. Montgomery, Proc. Symp. Pure Math. **24**, 181–193 (1973)