Renormalisation of flows on the multidimensional torus

João Lopes Dias*
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge
Silver Street, Cambridge CB3 9EW, U.K.

April, 2001

Abstract

We use a renormalisation operator $\mathcal{R}$ acting on a space of vector fields on $\mathbb{T}^d$, $d \geq 2$, to prove the existence of a submanifold of vector fields equivalent to constant. The result comes from the existence of a fixed point ω of $\mathcal{R}$ which is hyperbolic. This is done for a certain class of constant vector fields ω. The transformation $\mathcal{R}$ is constructed using a time rescaling, a linear change of basis plus a periodic non-linear map isotopic to the identity, which we derive by a "homotopy trick".

1 Introduction

In this paper we consider a renormalisation transformation $\mathcal{R}$ that acts on a Banach space of analytic vector fields X on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $d \geq 2$. Its domain is an open ball around a non-zero constant vector field $\omega \in \mathbb{R}^d$ (that generates a linear flow). The operator $\mathcal{R}$ is a C^1 infinite-dimensional dynamical system with a fixed point at ω. We show that the stable and unstable manifolds, given by the local dynamics of $\mathcal{R}$ around ω, separate different equivalence classes of vector fields. The elements of the codimension-$(d-1)$ stable manifold are equivalent to ω, as $\mathcal{R}$ is designed to be an equivalence between flows. In other words, the renormalisation asymptotically contracts locally any X into a $(d-1)$-parameter family of constant vector fields that includes ω.

The renormalisation consists of linear and non-linear coordinate changes of $\mathbb{T}^d$, and a time rescaling. The existence of the linear part, based on a number theory result used in [6], depends on ω. In particular, for $d = 2$, the class of such vectors corresponds to the set of vectors with a "quadratic irrational" slope. Specifically for the two-dimensional case, it is defined in

*E-mail: j.lopes-dias@damtp.cam.ac.uk

[4] a family of renormalisation iterative schemes allowing a full Lebesgue measure set of diophantine vectors. The methods involved therein are not easily generalisable to higher dimensions, as it is required a suitable choice of a multidimensional continued fraction expansion algorithm.

The linear transformation is the main feature in $\mathcal{R}$. It is a change of the basis of $\mathbb{T}^d$ lifted to $\mathbb{R}^d$ made to enlarge the region around the orbits of the linear flow. By iterating it, we look closer at chosen regions, but periodicity and the whole torus are kept at each stage. We also need to rescale the time as orbits take longer to cross the "new" torus. So, if a vector field is a fixed point of this procedure, it means that its orbits exhibit self-similarity between the different scales. This is trivially deduced for the fixed point ω.

The non-linear part of the transformation $\mathcal{R}$ is obtained by a homotopy method to reduce the perturbation. We use a flow of coordinate changes to find one which fully eliminates some of the Fourier terms of X. Those are chosen to be the easiest to do so, since we keep the "resonant" terms associated with "small denominators" (as it appears in the usual KAM theory). We are not concerned with trying to cancel all the perturbation terms because the linear transformation shifts resonant to "non-resonant" terms. Eventually all perturbation terms are eliminated while iterating $\mathcal{R}$.

This idea of renormalising vector fields is due to MacKay [8], and we follow an approach partly inspired by the work of Koch [6] on renormalisation for d-degrees of freedom analytic Hamiltonian systems. The latter apply to the problem of stability of invariant tori associated with a frequency vector ω.

This paper is organised as follows. In Section 2 we recall the basic ingredients of equivalence of flows, and in Section 3 we introduce the renormalisation idea through an example. We rigourously construct the operator $\mathcal{R}$ in Sections 4, 5, 6 and 7, with proofs of the statements also given in Section 9. Finally, in Section 8 we present the main result using the spectral properties of the derivative at the fixed point, with further discussion in Section 10.

2 Equivalence of flows

Consider a continuous vector field X on $\mathbb{T}^d$ that generates the flow ϕ_t, $\dot\theta = X(\theta)$, $\theta \in \mathbb{T}^d$, with lift Φ_t to the universal cover $\mathbb{R}^d$. Choosing a norm $\|\cdot\|$ in $\mathbb{R}^d$, we define $w_X(\theta_0) = \lim_{t\to\infty} \Phi_t(\theta_0)/\|\Phi_t(\theta_0)\|$ to be the *winding ratio* of X for the orbit of $\theta_0 \in \mathbb{T}^d$, if the limit exists and $\lim_{t\to\infty} \|\Phi_t(\theta_0)\| = \infty$. Otherwise, if $\Phi_t(\theta_0)$ is bounded we put $w_X(\theta_0) = 0$. Also, if the limit does not exist or if $\|\Phi_t(\theta_0)\|$ is unbounded but does not tend to infinity, the winding ratio is not defined.

We say that two flows $\phi_t, \psi_t \colon \mathbb{T}^d \to \mathbb{T}^d$ are C^r*-equivalent* if there is a C^r-diffeomorphism $h\colon \mathbb{T}^d \to \mathbb{T}^d$ taking orbits of ϕ_t onto those of ψ_t, preserving orientation. We are allowing to have $h(\phi_t(\theta)) = \psi_{\tau(h(\theta),t)}(h(\theta))$, where $\tau(\theta,\cdot)$ is a homeomorphism of $\mathbb{R}$ for any $\theta \in \mathbb{T}^d$. This is the same to say that two vector fields X, Y on $\mathbb{T}^d$ are equivalent if $\tau' X \circ h = DhY$ where h is a flow equivalence as above and τ' the time derivative of τ. We emphasise that this relaxation of the usual requirement that t be preserved ($h \circ \phi_t = \psi_t \circ h$) provides more satisfactory equivalence classes for flows, since we are interested mainly in qualitative properties of the flow. Every C^r-equivalence is isotopic to a map with a lift to $\mathbb{R}^d$ in the group of the linear automorphisms of the lattice $\mathbb{Z}^d$ with determinant ± 1, $GL(d,\mathbb{Z})$. The isotopy is given by periodic homeomorphisms on $\mathbb{T}^d$. These kind of coordinate changes preserve the d-torus structure and volume.

The set of winding ratios of a flow on $\mathbb{T}^d$ generated by X is $w_X = \{w_X(\theta) \colon \theta \in \mathbb{T}^d\}$, which is called the *winding set*. Any automorphism $T \colon \mathbb{R}^d \to \mathbb{R}^d$ induces the map $\hat{T} \colon \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ given by $x \mapsto Tx/\|Tx\|$, where $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \colon \|x\| = 1\}$. The winding set of a flow on $\mathbb{T}^d$ generated by a vector field X is invariant up to the action of $\hat{T}$, with a C^r-equivalence h isotopic to a map with lift $T \in GL(d,\mathbb{Z})$. That is, if $X' = (Dh)^{-1} X \circ h$, then $w_{X'}(\theta') = \hat{T} w_X(\theta)$, where $\theta' = h^{-1}(\theta)$.

3 Motivating example for $d = 2$

We start by motivating the renormalisation procedure with a simple example. Consider the case $d = 2$ and the linear flow described by the differential equation $\dot{\theta} = \omega$, with $\omega \in \mathbb{R}^2$ and $\theta \in \mathbb{T}^2$. Given an initial condition $\theta_0 \in \mathbb{T}^2$ for $t = 0$, the solution of the flow is $\phi_t(\theta_0) = \theta_0 + \omega t \mod 1$, $t > 0$. The motion is simply a rotation with frequency vector ω and winding ratio $w_\omega = \omega/\|\omega\|$. If, for all non-zero integer vectors k, $k \cdot \omega := k_1\omega_1 + k_2\omega_2 \neq 0$ (i.e. the slope ω_2/ω_1 is irrational, assuming ω_1 does not vanish), then all the orbits are dense in the torus (the flow is minimal). Otherwise, they are closed curves (periodic orbits).

Considering a perturbation to a constant "irrational" vector field ω lifted to $\mathbb{R}^2$, we want to determine under which conditions there is still equivalence to ω. Let X be a vector field close to ω arising from a time-independent perturbation of ω. We choose e.g. $\omega = (1,\gamma)$ where $\gamma = \frac{1+\sqrt{5}}{2}$ is the golden ratio. The main idea is to perform a change of basis, from the canonical base to $\{(0,1),(1,1)\}$, enlarging the region of $\mathbb{R}^2$ around the orbits of ω (see Figure 1). This is achieved by the linear transformation: $\theta' = T^{-1}\theta$, where $T = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] \in GL(2,\mathbb{Z})$. The eigenvalues and eigenvectors of T are given by: $T\omega = \gamma\omega$ and $T\Omega = -\frac{1}{\gamma}\Omega$, where $\Omega \perp \omega$. We also rescale the time, $t' = \frac{1}{\gamma}t$, because the orbits take longer to cross the new torus. This transformation does not affect the unperturbed vector field ω, as it is given by $\gamma DT^{-1} \circ T(\theta')\, \omega \circ T(\theta') = \gamma T^{-1}\omega = \omega$.

We should now consider a coordinate change $h(\theta') = U_X \circ T(\theta')$ with U

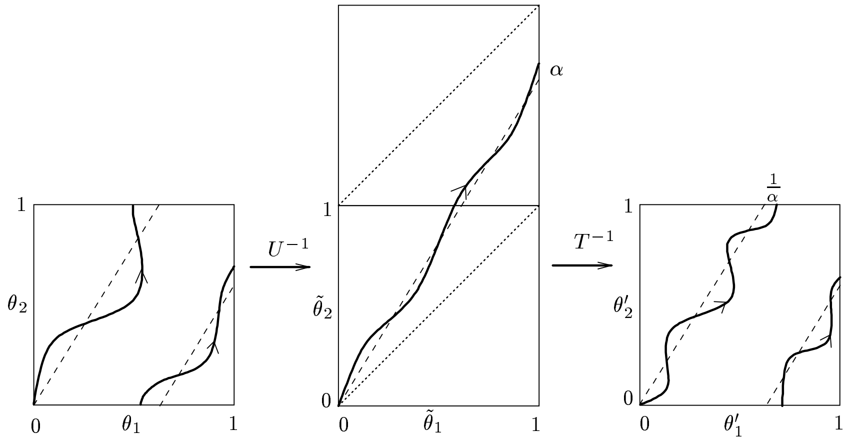

Figure 1: The action of T and U on an orbit of the lifted flow of X.

satisfying $U_\omega = \mathrm{Id}$. The fundamental requirement is that it has to cancel the growth of the wiggles in the orbits of X, which are enlarged by the above linear coordinate change (see orbit in Figure 1). The idea is to perform a non-linear, close to the identity periodic coordinate transformation U_X, in order to remove as many perturbation terms as possible.

There are some terms which are more relevant – "resonant". To understand what they are and how they appear, consider the vector field X in the form of its Fourier decomposition:

$$\dot{\theta} = \omega + \varepsilon \sum_{k \in \mathbb{Z}^2} f_k e^{2\pi i k \cdot \theta}$$

with $f_k \in \mathbb{C}^2$ and $\varepsilon > 0$ "small". We can approximately solve the above system using the unperturbed solution plus an order ε term. So, the solution $\theta(t)$ on the universal cover is

$$\theta(t) = \theta_0 + \omega t + \varepsilon f_0 \, t + \varepsilon \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{f_k}{2\pi i k \cdot \omega} e^{2\pi i k \cdot (\theta_0 + \omega t)} + \mathcal{O}(\varepsilon^2), \qquad (1)$$

for $t > 0$. The resonant terms in the solution above are the ones whose vector k is almost perpendicular to ω.

We need to add some conditions to ω in order to avoid having small denominators in the solution of the flow. This is done in the usual KAM type proofs by imposing a diophantine condition on ω, i.e. the denominators $|k \cdot \omega|$ in (1) admit a lower bound. We will show in the following that it is enough to find U such that it eliminates only "far from resonance terms". This is so because of the extra linear change of coordinates T described before, that is responsible for a "shift" of resonant into non-resonant modes. In some cases, by iterating this operator the orbits can be straightened.

We define U for any non-zero vector ω. Naturally, a restriction on ω will appear again in the procedure with T (see also [4]), since the result is not true for any vector with irrational slope, counter-examples are known for some non-diophantine vectors [1].

4 Space of Analytic Vector Fields

The following is valid for any d-dimensional torus $\mathbb{T}^d$, $d \geq 2$, for a class of frequencies ω to be defined. The vector fields considered are inside a ball around ω in some adequate space, and can be regarded as maps of $\mathbb{R}^d$ by lifting their domains. We make use of the analyticity to extend to the complex domain, so we deal with complex analytic vector fields. We construct the renormalisation operator $\mathcal{R}$ and we look at the spectral properties of its derivative at the fixed point ω, to relate to the local dynamics.

The use of analytic function spaces as the domain of the renormalisation functional operator $\mathcal{R}$ is justified by the usefulness of $\mathcal{R}$ being C^1. Otherwise the "picture" of $\mathcal{R}$ being a dynamical system with stable and unstable manifolds of the fixed points would vanish. The problem lies in the compositions that appear in $\mathcal{R}$, thus reducing the degree of differentiability of the image of the renormalisation operator. Considering C^∞ functions would also solve the problem, but complicate the technical parts of the method.

Let $r > 0$ and the domain lifted to a complex neighbourhood of $\mathbb{R}^d$:

$$\mathcal{D}(r) = \left\{ \theta \in \mathbb{C}^d : \|\mathrm{Im}\, \theta\| < \frac{r}{2\pi} \right\},$$

where $\|\cdot\|$ is the ℓ_1-norm on $\mathbb{C}^d$. That is, $\|z\| = \sum_{i=1}^d |z_i|$, with $|\cdot|$ the usual norm on $\mathbb{C}$. We will also be using the inner product: $z \cdot z' = \sum_{i=1}^d z_i z_i'$.

An analytic function $f\colon \mathcal{D}(r) \to \mathbb{C}^d$, 2π-periodic in each variable θ_i, is represented in Fourier series as

$$f(\theta) = \sum_{k \in \mathbb{Z}^d} f_k e^{2\pi i k \cdot \theta},$$

with coefficients $f_k \in \mathbb{C}^d$. The Banach spaces $(\mathcal{A}_d(r), \|\cdot\|_r)$ and $(\mathcal{A}_d'(r), \|\cdot\|_r')$ are the subsets of the set of these functions such that the respective norms

$$\|f\|_r = \sum_{k \in \mathbb{Z}^d} \|f_k\| e^{r\|k\|} \quad \text{and} \quad \|f\|_r' = \sum_{k \in \mathbb{Z}^d} (1 + 2\pi\|k\|) \|f_k\| e^{r\|k\|}$$

are finite.

Let $\rho > 0$ be a fixed value in the following. Consider the vector fields of the form $X(\cdot) = \omega + f(\cdot)$, where $\omega \in \mathbb{R}^d$ and $f \in \mathcal{A}_d'(\rho)$. Assume that X has no equilibrium points, $X(\theta) \neq 0$ for any $\theta \in \mathbb{T}^d$. A condition like $\|f\|_\rho' < \|\omega\|$ is enough to assure that there are no equilibria and, for any $\theta \in \mathbb{T}^d$, there is no $\theta' \in \mathbb{T}^d$ such that $X(\theta')/\|X(\theta')\| = -X(\theta)/\|X(\theta)\|$. So, we could not have two symmetric winding ratios for such a vector field, since it would have implied having at least two symmetric normalised tangent vectors. The set of such vector fields on $\mathbb{T}^2$ corresponds to the class of the Poincaré flows (fixed-point-free with only one possible winding ratio, see e.g. [2]). We study these systems in more detail in [4].

5 Elimination of the Far from Resonance Terms

We will be dealing with the vector fields considered above decomposed in Fourier series: $X(\theta) = \sum_{k\in\mathbb{Z}^d} X_k e^{2\pi i k\cdot\theta}$. As will be seen, it is important to distinguish different classes of terms according to their respective indices, as in the following

Definition 5.1 For a fixed value of $\sigma > 0$ and $\psi \in \mathbb{C}^d$, we define the *far from resonance terms* with respect to ψ to be the ones whose indices are in

$$I_\sigma^-(\psi) = \left\{ k \in \mathbb{Z}^d \colon |\psi \cdot k| > \sigma\|k\| \right\}.$$

Similarly, the *resonant terms* are $I_\sigma^+(\psi) = \mathbb{Z}^d \setminus I_\sigma^-(\psi)$. It is also useful to define the projections $\mathbb{I}_\sigma^+(\psi)$ and $\mathbb{I}_\sigma^-(\psi)$ for any vector field X by

$$[\mathbb{I}_\sigma^\pm(\psi)]X(\theta) = \sum_{k\in I_\sigma^\pm(\psi)} X_k e^{2\pi i k\cdot\theta}$$

and $\mathbb{I} = \mathbb{I}_\sigma^+(\psi) + \mathbb{I}_\sigma^-(\psi)$ is the identity operator.

The existence of a nonlinear change of coordinates U, close to the identity, that eliminates the far from resonance terms $I_\sigma^-(\omega)$ of a vector field X in a neighbourhood of a non-zero constant vector field, is given by

Theorem 5.2 *Let $0 < \rho' < \rho$. There exists $\mathcal{C} > 0$ for any $\omega \in \mathbb{C}^d \setminus \{0\}$ and $\sigma > 0$ such that: if X is in the open ball $\hat{B} \subset \mathcal{A}'_d(\rho)$ centred at ω with radius $\min\{\sigma/4, \mathcal{C}\sigma^2/\|\omega\|\}$, there is an invertible coordinate change $U\colon \mathcal{D}(\rho') \to \mathcal{D}(\rho)$ close to the identity, satisfying*

$$[\mathbb{I}_\sigma^-(\omega)](DU)^{-1} X \circ U = 0, \qquad \textit{and} \qquad U = \mathrm{Id} \textit{ if } [\mathbb{I}_\sigma^-(\omega)]X = 0.$$

The map $\mathcal{U}\colon \hat{B} \to [\mathbb{I}_\sigma^+(\omega)]\mathcal{A}_d(\rho')$ given by $X \mapsto (DU)^{-1} X \circ U$ is analytic, and the derivative at every constant vector field $\omega+\psi$ in $\hat{B}$ is equal to $\mathbb{I}_\sigma^+(\omega)$.

In Section 9 we include a proof of the above Theorem using the "homotopy method". It is worth noting here that since we are requiring to eliminate only the far from resonance terms outside $I_\sigma^+(\omega)$, we avoid the problem in the proof of dealing with small denominators (see Section 9.3).

6 Change of Basis

All the coordinate changes of $\mathbb{T}^d$ are isotopic to a transformation in $GL(d, \mathbb{Z})$. Such linear maps preserve the structure and volume of the d-torus and act on a vector field by shifting terms in the space of indices $\mathbb{Z}^d$ and changing coefficients.

Definition 6.1 For a fixed non-zero vector $\omega \in \mathbb{R}^d$ with rationally independent components, i.e. $k \cdot \omega \neq 0$ for any non-zero integer vector k, assume that $\omega = (1, \omega_2, \ldots, \omega_d)$ where its components are real algebraic numbers (roots of non-zero polynomials over $\mathbb{Q}$). The vectors $c\omega$, $c \in \mathbb{R} \setminus \{0\}$, are said to belong to the set Υ_d if the algebraic extension of $\mathbb{Q}$ by the numbers $\omega_2, \ldots, \omega_d$ (the smallest field containing $\omega_2, \ldots, \omega_d$ and $\mathbb{Q}$) is of degree d.

An important property, the existence of a linear change of basis T corresponding to the above class of vectors, follows from

Lemma 6.2 (Koch [6] – Lemma 4.1) *A vector ω is in Υ_d if and only if there is an integral $d \times d$ matrix T, such that $0 < |\lambda_d| \leq \cdots \leq |\lambda_2| < 1 < |\lambda_1|$ and $T\omega = \lambda_1 \omega$, where $\lambda_1, \ldots, \lambda_d$ are simple eigenvalues of T. Furthermore, some of the (integral) matrices satisfying these properties are in $GL(d, \mathbb{Z})$.*

Remark 6.3 For $d = 2$, the numbers ω_2 that produce a degree 2 algebraic extension of $\mathbb{Q}$ are the quadratic irrationals (roots of non-zero degree 2 polynomials over $\mathbb{Q}$). These are characterised by an eventually periodic continued fraction expansion [7]. For $d = 3$, the base for the algebraic extension of $\mathbb{Q}$ by α and β is $\{1, \alpha, \alpha^2\}$, where α has to be a cubic irrational (root of a non-zero degree 3 polynomial over $\mathbb{Q}$). Then $\beta = c_1 + c_2\alpha + c_3\alpha^2$, $c_i \in \mathbb{Q}$.

In what follows, a vector $\omega \in \Upsilon_d$ and a corresponding matrix $T \in GL(d, \mathbb{Z})$ are chosen according to Lemma 6.2. Also, $\bar{\omega} \in \mathbb{R}^d$ is chosen to be the (unstable) λ_1-eigenvector of T such that $\bar{\omega} \cdot \omega = 1$. Notice that Υ_d is a subset of the diophantine vectors $DC(\beta)$ with $\beta = -1 - \ln|\lambda_1| / \ln|\lambda_2|$ ([6] – Corollary 4.2), i.e. there is a constant $C > 0$ that satisfies $|\omega \cdot k| > C\|k\|^{-\beta-1}$ for every integer vector $k \neq 0$. In addition, $\prod_{j=1}^{d} |\lambda_j| = 1$, where λ_j are the eigenvalues of T, with $\lambda_1 \in \mathbb{R}$ since it is the only one outside the unit circle. Also, $\omega \cdot \omega^{(j)} = 0$, with $\omega^{(j)}$ being the eigenvector of T corresponding to λ_j, $2 \leq j \leq d$, because $\omega \cdot \omega^{(j)} = T^*\omega \cdot T^{-1}\omega^{(j)} = (\lambda_1/\lambda_j)\omega \cdot \omega^{(j)}$. We denote the transpose matrix of T by T^*.

From now on, we simplify the notation writing $I_\sigma^\pm$ and $\mathbb{I}_\sigma^\pm$ instead of the sets $I_\sigma^\pm(\omega)$ and the projections $\mathbb{I}_\sigma^\pm(\omega)$, respectively.

The next Proposition determines that the change of basis corresponding to the matrix T above, acting on the space of "resonant vector fields", is analyticity improving.

Proposition 6.4 *Let $0 < \kappa < 1$ and $0 < \rho' < \rho$. If $\kappa\rho < \rho'$, then, for some $0 < \sigma < \frac{1}{2}\|\bar{\omega}\|^{-1}$, any $X \in \mathbb{I}_\sigma^+ \mathcal{A}_d(\rho')$ has an analytic extension in $T\mathcal{D}(\rho)$. The linear map $\mathcal{T}(X) = T^{-1} X \circ T$ from $\mathbb{I}_\sigma^+ \mathcal{A}_d(\rho')$ to $\mathcal{A}'_d(\rho)$ is compact.*

The proof is given in Section 9.4. The values of σ have to be chosen such that

$$I_\sigma^+ \subset I^\kappa := \{x \in \mathbb{R}^d \colon \|T^* x\| \leq \kappa \|x\|\}$$

for a given κ. The proof of this Proposition only requires that the resonant terms are inside the set I^κ. Thus, we could have different definitions for $I_\cdot^+$. The different choices are restricted by the part of the proof of Theorem 5.2 in Section 9.3, that requires the denominator terms $|\omega \cdot k|$ to have at least a $\|k\|$-linear lower bound on $I_\cdot^-$.

It is also important to remark that one assumes that the orthogonal hypersurface in $\mathbb{R}^d$ with respect to ω is contained in I^κ. This condition is indeed verified by choosing appropriate norms on $\mathbb{R}^d$ or by considering a sufficiently large power of T instead. The indices of "small denominator" resonant terms are therefore shifted towards I_σ^-, eventually becoming far from resonance terms.

7 Renormalisation Operator

We are now in condition to construct a renormalisation operator $\mathcal{R}$ based on the coordinate transformations introduced before.

Fix $\omega \in \Upsilon_d$, T and λ_j, $1 \leq j \leq d$, as in Lemma 6.2. We denote by $\mathbb{E}(X)$ the average on $\mathbb{T}^d$ of a vector field X, i.e.

$$\mathbb{E}(X) = \int_{\mathbb{T}^d} X(\theta)\, d\theta,$$

where $d\theta$ is the normalised Lebesgue measure on $\mathbb{T}^d$.

Definition 7.1 Consider the maps $\mathcal{U}$ and $\mathcal{T}$ given by Theorem 5.2 and Proposition 6.4, respectively. The *renormalisation operator* $\mathcal{R}$ is defined to be

$$\mathcal{R}(X) = \frac{1}{\bar{\omega} \cdot \mathbb{E}(\tilde{X})} \tilde{X}, \quad \tilde{X} = \mathcal{T} \circ \mathcal{U}(X),$$

for vector fields X.

The rescaling of time is chosen to preserve the term ω. Otherwise, its iterates $\mathcal{R}^n(\omega) = \lambda_1^{-n}\omega$, $n \geq 0$, would go to zero. We can fix this by considering the product by λ_1 inside the calculation of $\tilde{X}$, but then all the elements of the one-parameter family $X_\mu = (1+\mu)\,\omega$, $\mu \in \mathbb{C}$, would be fixed points of the renormalisation. This "neutral" direction is contracted by $\mathcal{R}$ as in the definition, allowing ω to be an isolated fixed point.

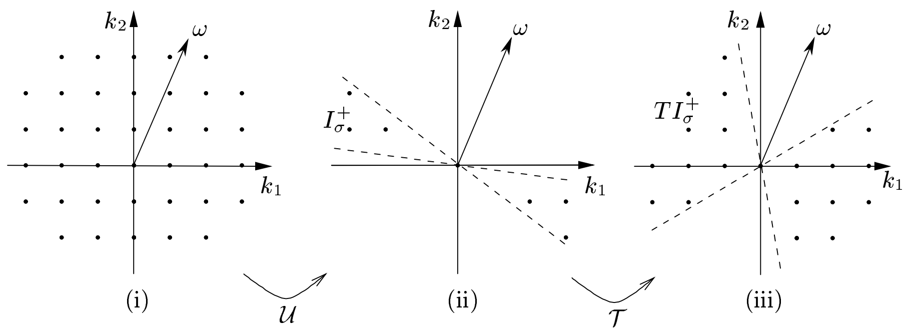

Figure 2: Steps of the renormalisation operator $\mathcal{R}$ for $d = 2$, on the lattice $\mathbb{Z}^2$, the indices of the Fourier coefficients (notice that each coefficient is a vector in $\mathbb{C}^d$).

This choice of rescaling has the disadvantage of reducing the domain of the operator (see Proposition 7.2 below), when comparing with the size of the ball $\hat{B}$ obtained in Theorem 5.2. In any case, we could always use the rescaling suggested first, and so working with a one-parameter family of fixed points and a larger domain. Note also that, by Proposition 6.4, $\sigma < \frac{1}{2}\|\omega\|^{-1}\|\omega\|_2^2 \leq \frac{1}{2}\|\omega\|$, where $\|\cdot\|_2$ represents the Euclidean norm. This means that the radius of an open ball given by Theorem 5.2 is in fact $\mathcal{C}\sigma^2\|\omega\|^{-1}$, assuming $\mathcal{C} < \frac{1}{2}$. Hence, the domain of $\mathcal{R}$ only contains equilibria-free vector fields.

The change of basis given by T turns some of the resonant terms into I_σ^- where they will be completely eliminated by the operator $\mathcal{U}$ of the next iteration of $\mathcal{R}$. The use of such $\mathcal{U}$ is not essential, it would be enough to "sufficiently" reduce the resonant terms in a way that analyticity would not be lost. As the elimination is complete, we get a faster convergence to the fixed point and a much simpler and clearer analysis starting from the fact that the derivative of $\mathcal{U}$ is straightforward. If we would only apply $\mathcal{U}$, all the neutral directions that are eliminated by the use of $\mathcal{T}$ would persist (no elimination of the resonant terms), and the analyticity domain would not be fully recovered. The right combination of the two coordinate changes is responsible for the usefulness of $\mathcal{R}$ (see Figure 2 for the $d = 2$ case).

Proposition 7.2 *Let $\rho > 0$. The renormalisation operator $\mathcal{R}$ is a well-defined analytic map from an open neighbourhood $B \subset \mathcal{A}'_d(\rho)$ of ω, to $\mathcal{A}'_d(\rho)$.*

Proof: Consider the complex-valued continuous functional $F(\tilde{X}) = \bar{\omega} \cdot \mathbb{E}(\tilde{X})$, and restrict its domain to a neighbourhood $\tilde{B} \subset \mathcal{A}'_d(\rho)$ of ω such that $F(\tilde{B}) \subset \{z \in \mathbb{C}\colon |\lambda_1 z - 1| < 1/2\}$, i.e. F is bounded away from zero in $\tilde{B}$.

The domain B of $\mathcal{R}$ is a subset of $\hat{B}$ such that $\mathcal{T} \circ \mathcal{U}(B) \subset \tilde{B}$. That is satisfied for a small enough choice of the radius of B, or of $\mathcal{C}$, since $\mathcal{U}(X) =$

$\omega + \mathbb{I}_\sigma^+ f + \mathcal{O}(\|f\|_\rho'^2)$ and $\lambda_1 \bar{\omega} \cdot \mathbb{E}\mathcal{T} \circ \mathcal{U}(X) = \bar{\omega} \cdot \omega + \lambda_1 \bar{\omega} \cdot T^{-1}\, \mathbb{E} f + \mathcal{O}(\|f\|_\rho'^2)$, with $X = \omega + f$. It is then sufficient to have $|\bar{\omega} \cdot (\mathbb{E} f + \mathcal{O}(\|f\|_\rho'^2))| < \frac{1}{2}$. Hence, F is bounded and analytic in $\tilde{B}$. This, Theorem 5.2 and Proposition 6.4 prove the claim. $\square$

8 Hyperbolicity of the Fixed Point ω

The transformation $\mathcal{R}$ was constructed in a way to hold a fixed point at ω. The local behaviour of $\mathcal{R}$ around it is given by the derivative. In this way, we hope to characterise the vector fields which are close enough to ω, realising the existence or not of an analytic equivalence.

We can rewrite $\mathcal{R}$ by the expression $\mathcal{R} = \mathcal{F} \circ \mathcal{T} \circ \mathcal{U}$, where the time-rescaling step is given as $\mathcal{F}(X) = [\bar{\omega} \cdot \mathbb{E}(X)]^{-1} X$ and $D\mathcal{F}(\omega) f = f - [\bar{\omega} \cdot \mathbb{E}(f)]\, \omega$. The derivative of $\mathcal{R}$ at ω is then the linear map

$$D\mathcal{R}(\omega)\, f = \mathcal{L} f - [\bar{\omega} \cdot \mathbb{E}(\mathcal{L} f)]\, \omega, \quad f \in \mathcal{A}_d'(\rho), \quad \text{with} \quad \mathcal{L} f = \mathcal{T} \circ \mathbb{I}_\sigma^+ f.$$

This operator is compact by Proposition 6.4 and using the fact that $\mathbb{I}_\sigma^+$ is bounded. The eigenvalues of $\mathcal{L}$ are zero and those of T^{-1}. The vector ω gives the subspace corresponding to the eigenvalue λ_1^{-1}. As the renormalisation should not depend on the size of ω, this direction is fully eliminated in $D\mathcal{R}(\omega)$ by the rescaling of time: $D\mathcal{R}(\omega)\, \omega = \lambda_1^{-1} \omega - (\bar{\omega} \cdot \lambda_1^{-1} \omega)\, \omega = 0$. There are $(d-1)$ unstable directions of $D\mathcal{R}(\omega)$ given by the other eigenvectors of T^{-1}, with eigenvalues λ_j^{-1}, $2 \leq j \leq d$, of modulus greater than one (see Section 10).

The remaining eigenvalue of $D\mathcal{R}(\omega)$ is zero, meaning that the elimination of only the far from resonant terms in conjuction with the change of basis are sufficient to eliminate all the non-constant Fourier terms of a vector field close to ω. Therefore, there is a codimension-$(d-1)$ manifold inside a neighbourhood of ω in $\mathcal{A}_d'(\rho)$ being mapped by $\mathcal{R}$ into itself. Thus, we have proved

Theorem 8.1 *If $\omega \in \Upsilon_d$, then it is a hyperbolic fixed point of $\mathcal{R}$ with a local codimension-$(d-1)$ stable manifold $\mathcal{W}^s(\omega)$ and a local $(d-1)$-dimensional unstable manifold $\mathcal{W}^u(\omega)$.*

We have determined an analytic equivalence between the elements of $\mathcal{W}^s(\omega)$ and in particular with ω, that is given by the operator $\lim_{n \to +\infty} \mathcal{R}^n$. The local stable manifold $\mathcal{W}^s(\omega)$ is the set of all X in some neighbourhood B with winding set $w_X = \omega / \|\omega\|$ (see Section 10). On the other hand, the local unstable manifold $\mathcal{W}^u(\omega)$ is the affine space of the vector fields in the form $X = \omega + v \in B$, where $v \in \operatorname{span}\{\omega^{(2)}, \ldots, \omega^{(d)}\}$, as this spectral subspace is $\mathcal{R}$-invariant. A schematic representation of the renormalisation scheme described is in Figure 3.

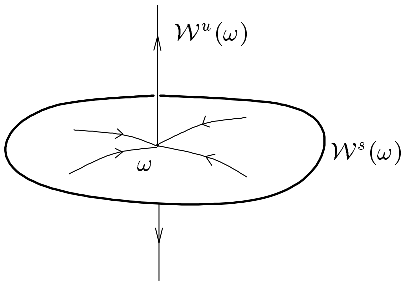

Figure 3: The renormalisation picture given by the action of $\mathcal{R}$ in a neighbourhood of ω.

9 Proofs of Theorem 5.2 and Proposition 6.4

9.1 Preliminaries

We include here some technical details useful for the proofs.

Firstly, note that writing $f = (f^1, \dots, f^d)$, one can rewrite the formulas of the norms on the spaces $\mathcal{A}_d(r)$ and $\mathcal{A}'_d(r)$. In fact,

$$\|f\|_r = \sum_{i=1}^d \|f^i\|_r, \quad f^i \in \mathcal{A}_1(r) \quad \text{and} \quad \|f\|'_r = \sum_{i=1}^d \|f^i\|'_r, \quad f^i \in \mathcal{A}'_1(r),$$

where

$$\|f^i\|'_r = \|f^i\|_r + \sum_{j=1}^d \|\partial_j f^i\|_r, \quad i = 1, \dots, d.$$

One has $\|f\|_r \le \|f\|'_r$, $\mathcal{A}'_d(r) \subset \mathcal{A}_d(r)$ and $\|f\|_{r'} \le \|f\|_r$, for $0 < r' < r$.

Define the inclusion maps $\mathcal{I}\colon \mathcal{A}_d(r) \to \mathcal{A}_d(r')$ and $\mathcal{I}'\colon \mathcal{A}'_d(r) \to \mathcal{A}'_d(r')$, $0 < r' < r$, by the restriction of the domain $\mathcal{D}(r)$ to $\mathcal{D}(r')$, i.e. $f \mapsto f|_{\mathcal{D}(r')}$. It is easy to check that $\mathcal{I}(\mathcal{A}_d(r)) \subset \mathcal{A}_d(r')$ and $\mathcal{I}'(\mathcal{A}'_d(r)) \subset \mathcal{A}'_d(r')$. The inclusion maps $\mathcal{I}$ and $\mathcal{I}'$ are compact linear operators.

Consider a bounded linear map $A\colon \mathcal{A}_d(r) \to \mathcal{A}_d(r)$ with $(Af)(\theta) = A(\theta)\, f(\theta)$, where $A(\theta)$ is a linear map on $\mathbb{C}^d$. Its operator norm satisfies $\|A\| \le \sum_{i,j=1}^d \|a_{ij}\|_r$, where $a_{ij} \in \mathcal{A}_1(r)$ such that, for each $\theta \in \mathcal{D}(r)$, $a_{ij}(\theta)$ are the coefficients of the matrix which represents $A(\theta)$ in the canonical base. In particular, if A is the derivative of a function $f \in \mathcal{A}_d(r)$, $(Df\, g)(\theta) = Df(\theta)\, g(\theta)$, $g \in \mathcal{A}_d(r)$ and $\theta \in \mathcal{D}(r)$, then $\|Df\| \le \|f\|'_r$.

If $\alpha \in \mathcal{A}_1(r)$ and $f, g \in \mathcal{A}_d(r)$, then $\alpha\, f \in \mathcal{A}_d(r)$, $f \cdot g \in \mathcal{A}_1(r)$, $\|\alpha\, f\|_r \le \|\alpha\|_r \|f\|_r$ and $\|f \cdot g\|_r \le \|f\|_r \|g\|_r$. Similarly, if $\alpha \in \mathcal{A}'_1(r)$ and $f, g \in \mathcal{A}'_d(r)$, then $\alpha\, f \in \mathcal{A}'_d(r)$, $f \cdot g \in \mathcal{A}'_1(r)$, $\|\alpha\, f\|'_r \le \|\alpha\|'_r \|f\|'_r$ and $\|f \cdot g\|'_r \le \|f\|'_r \|g\|'_r$.

Lemma 9.1 *Let $0 < r' < r$ and $f \in \mathcal{A}'_d(r)$. If $u \in \mathcal{A}_d(r')$ and $\|u\|_{r'} < (r - r')/4\pi$, then*

1. $\|f(\mathrm{Id}+u)\|_{r'} \leq \|f\|_{(r+r')/2}$,

2. $\|Df(\mathrm{Id}+u)\| \leq \|f\|'_{(r+r')/2}$,

3. $\|f(\mathrm{Id}+u)-f\|_{r'} \leq \|f\|'_{(r+r')/2}\,\|u\|_{r'}$,

4. $\|Df(\mathrm{Id}+u)-Df\| \leq \frac{4\pi}{r-r'}\|f\|'_r\,\|u\|_{r'}$.

Proof: Knowing that $\|e^{2\pi i k\cdot u}\|_{r'} \leq e^{2\pi\|k\|\|u\|_{r'}}$ one gets 1 and 2. The mean value theorem gives 3. To prove 4 apply again the mean value theorem now to $\|\partial_j f^i(\mathrm{Id}+u)-\partial_j f^i\|_{r'}$ and also

$$\begin{aligned}\|\partial_j f^i\|'_{(r+r')/2} &= \sum_{k\in\mathbb{Z}^d} 2\pi|k_j|(1+\|k\|)|f^i_k|e^{r\|k\|}e^{-(r-r')\|k\|/2}\\ &\leq \frac{4\pi}{r-r'}\sum_{k\in\mathbb{Z}^d}(1+\|k\|)|f^i_k|e^{r\|k\|} = \frac{4\pi}{r-r'}\|f^i\|'_r,\end{aligned}$$

where we have used the inequality: $\sup_{t\geq 0} te^{-\beta t} \leq 1/\beta$ for $\beta>0$. □

9.2 Homotopy Method

The sometimes called "homotopy trick" has been used in different problems, such as by J. Moser to show that all smooth volume forms on a compact orientable manifold are equivalent up to a diffeomorphism ([5] – Section 5.1e). Other examples of its application are proofs of the Darboux Theorem or the Poincaré Lemma ([5] – Sections 5.5.9 and A.3.11, respectively), and Roussarie's proof of Morse's Lemma. This procedure is also known as "deformation method", and has been used in KAM theory as a flow of symplectomorphisms that reduce in each iterate the size of the perturbation of an integrable Hamiltonian [9, 3].

In the following fix $\sigma>0$, thus dropping the index σ for the sets $I^{\pm}$ and corresponding projections $\mathbb{I}^{\pm}$. Choose also ρ and ρ' such that $0<\rho'<\rho$. Let $\delta>0$ and ε be a positive constant which will be determined along the proof and contains the restrictions on the size of the perturbation of ω depending on σ and the norm of ω, as will be seen in Section 9.3.

For vector fields in the form $X=\omega+f$, consider the open neighbourhood $\mathcal{E}$ of the term f:

$$\mathcal{E} = \{f \in \mathcal{A}'_d(\rho)\colon \|f\|'_\rho < \varepsilon\}.$$

The coordinate transformation U is written as $U=\mathrm{Id}+u$, with Id as the identity transformation and u in $\mathcal{B}$ defined as the open ball of radius δ in $\mathbb{I}^-\mathcal{A}'_d(\rho')$, i.e.

$$\mathcal{B} = \left\{u \in \mathbb{I}^-\mathcal{A}'_d(\rho')\colon \|u\|'_{\rho'} < \delta\right\}.$$

For a given function $f\in\mathcal{E}$ consider the operator $F\colon\mathcal{B}\to\mathbb{I}^-\mathcal{A}_d(\rho')$,

$$F(u) = \mathbb{I}^-(I+Du)^{-1}\left[\omega+f(\mathrm{Id}+u)\right],$$

where I is the identity operator. This is simply the transformed vector field $\mathbb{I}^- DU^{-1} \circ U\, X \circ U$, with $X = \omega + f$.

Lemma 9.2 *The derivative of F at $u \in \mathcal{B}$ is the linear map from $\mathbb{I}^-\mathcal{A}'_d(\rho')$ to $\mathbb{I}^-\mathcal{A}_d(\rho')$,*

$$\begin{aligned} DF(u)\, h = \; & \mathbb{I}^-(I + Du)^{-1} \left[Df \circ (\mathrm{Id} + u)\, h \right. \\ & \left. - Dh\, (I + Du)^{-1} \left(\omega + f \circ (\mathrm{Id} + u)\right)\right], \end{aligned}$$

with h chosen such that $u + h \in \mathcal{B}$.

Proof: We need to compute the linear term on h of $F(u + h) - F(u)$. As $u \in \mathcal{B}$, we have that the bounded linear operator Du, from $\mathcal{A}_d(\rho')$ into itself, satisfies $\|Du\| \leq \|u\|'_{\rho'} < 1$. Hence, $(I + Du)^{-1} = \sum_{n \geq 0} (-Du)^n$. Using the following formulas:

$$\begin{aligned} & (Du + Dh)^n = Du^n + \sum_{j=0}^{n-1} Du^j\, Dh\, Du^{n-1-j} + \mathcal{O}(\|h\|'^{\,2}_{\rho'}), \quad n \geq 1, \\ & \sum_{n=1}^{+\infty} \sum_{j=0}^{n-1} (-1)^n\, Du^j\, Dh\, Du^{n-1-j} = -(I + Du)^{-1}\, Dh\, (I + Du)^{-1}, \end{aligned}$$

and the Taylor expansion of f around $\theta + u(\theta)$, that gives

$$f \circ (\mathrm{Id} + u + h) = f \circ (\mathrm{Id} + u) + Df \circ (\mathrm{Id} + u)\, h + \mathcal{O}(\|h\|'^{\,2}_{\rho'}),$$

we get

$$\begin{aligned} F(u + h) - F(u) = \; & \mathbb{I}^- \sum_{n=0}^{+\infty} (-1)^n \left[(Du + Dh)^n (\omega + f) \circ (\mathrm{Id} + u + h) \right. \\ & \left. - Du^n (\omega + f) \circ (\mathrm{Id} + u) \right] \\ = \; & \mathbb{I}^-(I + Du)^{-1} \left[(\omega + f) \circ (\mathrm{Id} + u + h) \right. \\ & - (\omega + f) \circ (\mathrm{Id} + u) \\ & \left. - Dh\, (I + Du)^{-1} (\omega + f) \circ (\mathrm{Id} + u + h) \right] \\ & + \mathcal{O}(\|h\|'^{\,2}_{\rho'}) \\ = \; & \mathbb{I}^-(I + Du)^{-1} \left[Df \circ (\mathrm{Id} + u)\, h \right. \\ & \left. - Dh\, (I + Du)^{-1} (\omega + f) \circ (\mathrm{Id} + u) \right] \\ & + \mathcal{O}(\|h\|'^{\,2}_{\rho'}). \end{aligned}$$

That completes the proof. $\square$

We want to find a solution for the equation

$$F(u) = 0. \tag{2}$$

For that, consider a continuous one-parameter family of maps: $U_\lambda = \mathrm{Id} + u_\lambda$, $\lambda \in [0, 1]$, with "initial" condition $U_0 = \mathrm{Id}$, i.e. $u_0 = 0$, such that

$$F(u_\lambda) = (1 - \lambda) F(u_0).$$

Differentiating the above equation in respect to λ, we get

$$DF(u_\lambda)\frac{du_\lambda}{d\lambda} = -F(0). \tag{3}$$

Remark 9.3 The derivative of F at u can be rewritten in the form

$$DF(u)\,h = \mathbb{I}^-(DU)^{-1}[DX\circ U\,h - Dh\,(DU)^{-1}\,X\circ U].$$

Evaluating on $v\circ U$, where v is the vector field that generates U, i.e. $dU/d\lambda = v\circ U$ and $v\circ U = DU\,v$, one gets

$$\begin{aligned} DF(u)\,v\circ U &= \mathbb{I}^-(DU)^{-1}\,[v,X]\circ U \\ &= \mathbb{I}^-[v,(DU)^{-1}\,X\circ U], \end{aligned}$$

where $[v,w] = Dw\,v - Dv\,w$ is the commutator for vector fields v, w. Thus, writing $\tilde{X} = (DU)^{-1}\,X\circ U$, it suffices to solve the commutator equation

$$\mathbb{I}^-[v,\tilde{X}] = -\mathbb{I}^- X$$

with respect to v satisfying $\mathbb{I}^+ v = 0$, or, equivalently, invert $DF(u)$. Note also that we allow to have $v = v_\lambda$.

Proposition 9.4 *If $u\in\mathcal{B}$, then $DF(u)^{-1}$ is a bounded linear operator from $\mathbb{I}^-\mathcal{A}_d(\rho')$ to $\mathbb{I}^-\mathcal{A}_d'(\rho')$ and*

$$\|DF(u)^{-1}\| < \frac{\delta}{\varepsilon}.$$

From the above Proposition (to be proved in Section 9.3) we know that $DF(u)$ is invertible for $u\in\mathcal{B}$, thus we may integrate (3) with respect to λ, obtaining

$$u_\lambda = -\int_0^\lambda DF(u_\mu)^{-1}\,F(0)\,d\mu. \tag{4}$$

In order to check that $u_\lambda\in\mathcal{B}$ for any $\lambda\in[0,1]$, we estimate its norm:

$$\begin{aligned} \|u_\lambda\|'_{\rho'} &\le \sup_{v\in\mathcal{B}}\|DF(v)^{-1}F(0)\|'_{\rho'} \\ &\le \sup_{v\in\mathcal{B}}\|DF(v)^{-1}\|.\|\mathbb{I}^- f\|_{\rho'} \\ &< \tfrac{\delta\varepsilon}{\varepsilon} = \delta. \end{aligned}$$

Therefore, the solution of (2) exists in $\mathcal{B}$ and is given by (4) when $\lambda = 1$.

Now, the open ball $\hat{B}$ as claimed is simply given by

$$\hat{B} = \{X\in\mathcal{A}'(\rho)\colon X = \omega + f, f\in\mathcal{E}\}.$$

If for $X = \omega + f \in \hat{B}$ we have $\|f\|'_\rho = \mathcal{O}(\nu)$ with $\nu > 0$ small enough, then $\|u\|'_{\rho'} = \mathcal{O}(\nu)$. Thus, $\|\mathcal{U}(X) - \omega - \mathbb{I}^+ f\|_{\rho'} = \mathcal{O}(\nu^2)$. This means that the derivative at ω of the map $\mathcal{U}$ is $\mathbb{I}^+$. Now, assume $\|f - \psi\|'_\rho = \mathcal{O}(\nu)$, $\omega + \psi \in \hat{B} \cap \mathbb{C}^d$, and write $\mathcal{U}(X) = \mathcal{U}(\omega + \psi + (f - \psi))$. So,

$$\|\mathcal{U}(\omega + \psi + (f - \psi)) - (\omega + \psi) - \mathbb{I}^+(f - \psi)\|_{\rho'} = \mathcal{O}(\nu^2),$$

i.e. $\mathbb{I}^+$ is the derivative of $\mathcal{U}$ at $\omega + \psi$. That completes the proof of Theorem 5.2.

9.3 Proof of Proposition 9.4

To prove Proposition 9.4 we start by inverting the derivative of F at $u = 0$.

Lemma 9.5 *If $\varepsilon < \sigma/4$, the bounded linear operator $DF(0)^{-1}$, from $\mathbb{I}^-\mathcal{A}_d(\rho')$ to $\mathbb{I}^-\mathcal{A}'_d(\rho')$, satisfies:*

$$\|DF(0)^{-1}\| < \frac{2}{\sigma - 4\varepsilon}.$$

Proof: From Lemma 9.2 one has

$$\begin{aligned} DF(0)\, h &= \mathbb{I}^-(\hat{f}\, h - Dh\, \omega) \\ &= \mathbb{I}^- \left[\hat{f} - (D \cdot \omega)\right]\, h \\ &= -\left[\mathbb{I} - \mathbb{I}^- \hat{f}\,(D \cdot \omega)^{-1}\right] (D \cdot \omega)\, h, \end{aligned}$$

where $\hat{f}\, h = Df\, h - Dh\, f$. Thus, the inverse of this operator, if it exists, is given by

$$DF(0)^{-1} = -(D \cdot \omega)^{-1} \left[\mathbb{I} - \mathbb{I}^- \hat{f}\,(D \cdot \omega)^{-1}\right]^{-1}.$$

The inverse of $(D \cdot \omega)$ is the linear map from $\mathbb{I}^-\mathcal{A}_d(\rho')$ to $\mathbb{I}^-\mathcal{A}'_d(\rho')$:

$$(D \cdot \omega)^{-1}\, g(\theta) = \sum_{k \in I^-} \frac{g_k}{2\pi i k \cdot \omega} e^{2\pi i k \cdot \theta}.$$

So,

$$\begin{aligned} \|(D \cdot \omega)^{-1}\, g\|'_{\rho'} &= \sum_{k \in I^-} \tfrac{1 + 2\pi\|k\|}{2\pi|k \cdot \omega|} \|g_k\| e^{\rho'\|k\|} \\ &< \sum_{k \in I^-} \tfrac{1 + 2\pi\|k\|}{2\pi\sigma\|k\|} \|g_k\| e^{\rho'\|k\|} \\ &\leq \tfrac{2}{\sigma}\|g\|_{\rho'}, \end{aligned}$$

where the use of the definition of I^- (Definition 5.1) was crucial to avoid dealing with arbitrarily small denominators. Hence, $\|(D \cdot \omega)^{-1}\| < \frac{2}{\sigma}$.

Similarly, for $\hat{f} \colon \mathbb{I}^-\mathcal{A}'_d(\rho') \to \mathcal{A}_d(\rho')$,

$$\|\hat{f}\, h\|_{\rho'} \leq \|f\|'_{\rho'}\|h\|_{\rho'} + \|f\|_{\rho'}\|h\|'_{\rho'}$$

Hence it is a bounded operator with $\|\hat{f}\| \leq 2\|f\|'_{\rho'}$. Therefore,

$$\|\mathbb{I}^- \hat{f}(D \cdot \omega)^{-1}\| < \frac{4}{\sigma}\|f\|'_{\rho'} < 1,$$

assuming $\varepsilon < \sigma/4$. So, $\left[\mathbb{I} - \mathbb{I}^- \hat{f}(D \cdot \omega)^{-1}\right]^{-1}$ is a bounded linear map in $\mathbb{I}^-\mathcal{A}_d(\rho')$ such that:

$$\left\| \left[\mathbb{I} - \mathbb{I}^- \hat{f}(D \cdot \omega)^{-1}\right]^{-1} \right\| < \frac{\sigma}{\sigma - 4\|f\|'_{\rho'}},$$

completing the proof of the Lemma. □

It remains now to estimate the variation of DF along $u \in \mathcal{B}$.

Lemma 9.6 *Let $0 < \delta < \frac{\rho - \rho'}{4\pi}$. Given $u \in \mathcal{B}$, the linear operator $DF(u) - DF(0)$, mapping $\mathbb{I}^-\mathcal{A}'_d(\rho')$ into $\mathbb{I}^-\mathcal{A}_d(\rho')$, is bounded and*

$$\|DF(u) - DF(0)\| < \frac{\delta}{1-\delta}\left[\left(\frac{4\pi}{\rho - \rho'} + \frac{4 - 2\delta}{1 - \delta}\right)\varepsilon + \frac{2-\delta}{1-\delta}\|\omega\|\right].$$

Proof: Lemma 9.2 gives us that

$$\begin{aligned}
[DF(u) - DF(0)]\, h = \; & \mathbb{I}^-(I + Du)^{-1}\left[Df \circ (\mathrm{Id} + u)\, h - (I + Du) Df\, h\right. \\
& -Dh\,(I + Du)^{-1}(\omega + f) \circ (\mathrm{Id} + u) \\
& \left. +(I + Du) Dh\,(\omega + f)\right] \\
= \; & \mathbb{I}^-(I + Du)^{-1}\{[Df \circ (\mathrm{Id} + u) - Df - Du\, Df]\, h \\
& +Du\, Dh\,(\omega + f) \\
& -Dh\,(I + Du)^{-1}\left[f \circ (\mathrm{Id} + u) - f - Du\,(\omega + f)\right]\} \\
= \; & \mathbb{I}^-(I + Du)^{-1}\{A + B + C\},
\end{aligned}$$

where A, B and C are each of the respective three terms in the previous sum. Thus, $\|(I + Du)^{-1}\| \leq (1 - \|Du\|)^{-1}$. Using Lemma 9.1, one obtains:

$$\|A\|_{\rho'} \leq \left(\frac{4\pi}{\rho - \rho'}\|f\|'_{\rho}\|u\|_{\rho'} + \|f\|'_{\rho'}\|u\|'_{\rho'}\right)\|h\|_{\rho'}$$

and $\|B\|_{\rho'} \leq \left(\|\omega\| + \|f\|_{\rho'}\right)\|u\|'_{\rho'}\|h\|'_{\rho'}$. Finally, noting that

$$\|Du\,(I + Du)^{-1}\| \leq \|Du\|.(1 - \|Du\|)^{-1},$$

and from Lemma 9.1 again, one gets the bound for the third term:

$$\|C\|_{\rho'} \leq \frac{1}{1 - \|u\|'_{\rho'}}\left[\|f\|'_{(\rho+\rho')/2}\|u\|_{\rho'} + \|u\|'_{\rho'}\left(\|\omega\| + \|f\|_{\rho'}\right)\right]\|h\|'_{\rho'}.$$

□

To conclude the proof of Proposition 9.4, notice that, for $u \in \mathcal{B}$ and $f \in \mathcal{E}$:

$$\begin{aligned} \|DF(u)^{-1}\| &\leq \left(\|DF(0)^{-1}\|^{-1} - \|DF(u) - DF(0)\|\right)^{-1} \\ &< \left\{\tfrac{\sigma}{2} - 2\varepsilon - \tfrac{\delta}{1-\delta}\left[\left(\tfrac{4\pi}{\rho-\rho'} + \tfrac{4-2\delta}{1-\delta}\right)\varepsilon + \tfrac{2-\delta}{1-\delta}\|\omega\|\right]\right\}^{-1} \\ &< \tfrac{\delta}{\varepsilon}. \end{aligned}$$

The last inequality is true if we choose ε to be

$$\varepsilon < \delta\left[\frac{\sigma}{2} - \frac{\delta(2-\delta)}{(1-\delta)^2}\|\omega\|\right]\left[1 + 2\delta + \frac{\delta^2}{1-\delta}\left(\frac{4\pi}{\rho-\rho'} + \frac{4-2\delta}{1-\delta}\right)\right]^{-1}, \tag{5}$$

and having also

$$\delta < \frac{1}{2} - \frac{1}{2}\left[1 + \frac{\sigma}{2\|\omega\|}\right]^{-\frac{1}{2}} < 1.$$

An effective value of ε for each σ can then be determined by the minimum value of the upper bounds given in Lemma 9.5 and (5). A more simple – although rougher– bound for ε can be obtained from a specific choice of δ given by

$$\hat{\delta} = \min\left\{\frac{\rho-\rho'}{4\pi}, C\frac{\sigma}{\|\omega\|}\right\} < \frac{1}{2} - \frac{1}{2}\left[1 + \frac{\sigma}{2\|\omega\|}\right]^{-\frac{1}{2}},$$

as it exists a positive constant C that verifies the condition above. Therefore, for a suitable constant $\mathcal{C} > 0$, we can have

$$\varepsilon < \frac{\mathcal{C}\sigma^2}{\|\omega\|}, \tag{6}$$

where it was used for (5) the inequalities:

$$\frac{1}{1-\delta} < \sqrt{1 + \frac{\sigma}{2\|\omega\|}} < 1 + \frac{\sigma}{4\|\omega\|}, \quad \frac{\delta(2-\delta)}{(1-\delta)^2}\|\omega\| < \frac{\sigma}{4},$$

and knowing that σ can be chosen to be smaller than $\|\omega\|$.

9.4 Analyticity Improvement

We want to prove Proposition 6.4, i.e. that $X \circ T$ is analytic in $\mathcal{D}(\rho)$ and has bounded derivative. Let $\sigma > 0$ such that

$$\max\left\{\frac{\|T^* x\|}{\|x\|} : \|x\| \neq 0, |\omega \cdot x| \leq \sigma\|x\|\right\} < \kappa.$$

Using $X \circ T(\theta) = \sum_{k \in I_\sigma^+} X_k e^{2\pi i T^* k \cdot \theta}$,

$$\begin{aligned} \|X \circ T\|_\rho &\leq \sum_{k \in I_\sigma^+} \|X_k\| e^{\rho\|T^* k\|} \\ &\leq \sum_{k \in I_\sigma^+} \|X_k\| e^{(\rho\kappa - \rho')\|k\|} e^{\rho'\|k\|} \leq \|X\|_{\rho'}, \end{aligned}$$

and for the derivative we have

$$\begin{aligned}\|D(X\circ T)\|_\rho &\le 2\pi \sum_{k\in I_\sigma^+} \|T^*k\| e^{-\beta\|k\|}\|X_k\| e^{(\rho\kappa+\beta-\rho')\|k\|} e^{\rho'\|k\|} \\ &\le \tfrac{2\pi}{\beta}\kappa \|X\|_{\rho'},\end{aligned}$$

by choosing $0<\beta<\rho'-\kappa\rho$, and using the relation $\sup_{t\ge 0} te^{-\xi t} \le 1/\xi$ for $\xi>0$. These bounds imply that $\|\mathcal{T}(X)\|'_\rho \le (1+2\pi\kappa/\beta)\|T^{-1}\|\,\|X\|_{\rho'}$.

Let $r>\rho$ such that $\rho'>r\kappa$. What was done above for $\mathcal{D}(\rho)$ applies as well to $\mathcal{D}(r)$. Therefore, one can decompose $\mathcal{T}=\mathcal{I}\circ\mathcal{J}$, where $\mathcal{J}\colon \mathbb{I}^+\mathcal{A}(\rho')\to \mathcal{A}'(r)$ as before, and $\mathcal{I}\colon \mathcal{A}'(r)\to\mathcal{A}'(\rho)$ is the inclusion map $\mathcal{I}(X)=X|_{\mathcal{D}(\rho)}$. Note that $\mathcal{J}$ is bounded and $\mathcal{I}$ is compact, thus completing the proof.

10 Spectral Properties of $D\mathcal{R}(\omega)$ and Invariant Manifolds

As σ remains fixed, we continue to drop the index of the sets $I^\pm$ and the projections $\mathbb{I}^\pm$.

First we study the behaviour of the linear map $\mathcal{L}$ whose spectral properties are closely related to the ones of $D\mathcal{R}(\omega)$. Notice that $\mathbb{E}\mathcal{A}'(\rho)=\mathbb{C}^d$ and $(\mathbb{I}-\mathbb{E})\mathcal{A}'(\rho)$ are invariant subspaces of $\mathcal{L}$ as $\mathcal{L}\circ\mathbb{E}=\mathbb{E}\circ\mathcal{L}$.

Lemma 10.1 *If $\omega\in DC(\beta)$, for any $\beta>0$, and $\sigma\|\bar\omega\|<\frac12$, one finds constants $a>0$ and $b,c>1$ such that*

$$\|\mathcal{L}^{n+1}(\mathbb{I}-\mathbb{E})\| \le b\lambda_1 e^{-ac^n}\|\mathcal{L}^n(\mathbb{I}-\mathbb{E})\|, \quad n\ge 0$$

Proof: For every $n\ge 0$, we define I_n^+ as the set of indices k that verify $(T^*)^n k\in I^+\setminus\{0\}$. Hence,

$$\sigma\|(T^*)^n k\| \ge |\omega\cdot (T^*)^n k| = |T^n\omega\cdot k| = |\lambda_1|^n|\omega\cdot k|, \quad \text{for } k\in I_n^+.$$

This inequality implies that

$$|\omega\cdot k| \le \sigma\|(\omega\cdot k)\,\bar\omega\| + \sigma\left\|\left(\lambda_1^{-1}T^*\right)^n[k-(\omega\cdot k)\,\bar\omega]\right\|.$$

Thus, assuming $\sigma\|\bar\omega\|<\frac12$ and knowing that $k-(\omega\cdot k)\,\bar\omega$ is the component of $k\in I_n^+$ on the spectral directions of T^* corresponding to the eigenvalues λ_j, $j=2,\ldots,d$,

$$|\omega\cdot k| \le \frac{\sigma}{1-\sigma\|\bar\omega\|}\left|\frac{\lambda_2}{\lambda_1}\right|^n \|k-(\omega\cdot k)\,\bar\omega\|$$

As ω is a diophantine vector of degree $\beta>0$, there is a constant $C>0$ such that

$$|\omega\cdot k| > C\|k\|^{-\beta-1},$$

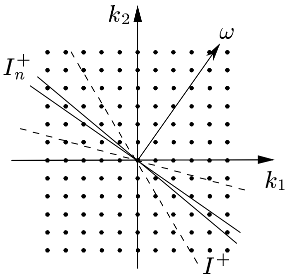

Figure 4: The set $I_n^+ \subset I^+$ for the two-dimensional case.

and we get a lower bound for the norm of $k \in I_n^+$,

$$\|k\|^{2+\beta} > C\frac{1-\sigma\|\bar{\omega}\|}{2\sigma}\left|\frac{\lambda_1}{\lambda_2}\right|^n. \tag{7}$$

Let us now define the operator $\mathbb{I}_n^+ \colon \mathcal{A}'(r) \to \mathcal{A}'(\rho)$, where we have chosen $r > \rho$. This is a projection for the indices in I_n^+ together with an analytic inclusion. Making use of the bound (7), the operator norm of $\mathbb{I}_n^+$ follows from

$$\|\mathbb{I}_n^+ f\|'_\rho = \sum_{I_n^+}(1+2\pi\|k\|)\|f_k\|e^{r\|k\|}e^{-(r-\rho)\|k\|} \leq e^{-ac^n}\|f\|'_r,$$

with $a = (r-\rho)[C(2\sigma)^{-1}(1-\sigma\|\bar{\omega}\|)]^{1/(2+\beta)} > 0$ and $c = |\lambda_1\lambda_2^{-1}|^{1/(2+\beta)} > 1$.

We can write $\mathcal{L}^{n+1}$ with respect to $\mathcal{L}^n$, including the operator $\mathbb{I}_n^+$, in the form

$$\mathcal{L}^{n+1}(\mathbb{I}-\mathbb{E})f = \mathcal{L}^n(\mathbb{I}-\mathbb{E})\mathbb{I}_n^+\bar{\mathcal{L}}f,$$

where $\bar{\mathcal{L}} \colon \mathcal{A}'(\rho) \to \mathcal{A}'(r)$ is $\mathcal{L}$ followed by an analytic extension. That is, $\bar{\mathcal{L}} = \lambda_1\bar{\mathcal{T}} \circ \mathbb{I}^+$ where $\bar{\mathcal{T}} \colon \mathbb{I}^+\mathcal{A}'(\rho) \subset \mathbb{I}^+\mathcal{A}(\rho) \to \mathcal{A}'(r)$ given as in Proposition 6.4. The norm $\|\bar{\mathcal{L}}\| \leq \|\bar{\mathcal{T}}\|$ is determined by

$$\|\bar{\mathcal{T}}f\|'_r \leq b\|T^{-1}\|\|f\|'_\rho,$$

for some $r > \rho$ and a constant $b > 1$ from the proof of Proposition 6.4 in Section 9.4. So, the claim follows from

$$\|\mathcal{L}^{n+1}(\mathbb{I}-\mathbb{E})f\|'_\rho \leq \|\mathcal{L}^n(\mathbb{I}-\mathbb{E})\|\,\|\mathbb{I}_n^+\|\,\|\bar{\mathcal{L}}\|\,\|f\|'_\rho,$$

and $\|T^{-1}\| \leq |\lambda_1|$. $\square$

So, the direction given by the non-constant terms $\mathbb{I}-\mathbb{E}$ is a stable eigenspace with eigenvalue zero. The spectrum of $\mathcal{L}\circ\mathbb{E}$ is simply the one of T^{-1}. That is, the eigenvalues are λ_j^{-1}, $j = 2,\ldots,d$, and λ_1^{-1} corresponding to the eigenvector ω. Hence, there is a stable direction given by ω whereas the remaining ones corresponding to $\omega^{(2)}, \dots, \omega^{(d)}$ are all unstable.

The spectral properties of $D\mathcal{R}(\omega)$ are easily related to those of $\mathcal{L}$ because

$$D\mathcal{R}(\omega) = (\mathbb{I} - \mathbb{P}_\omega \mathbb{E})\mathcal{L},$$

where $\mathbb{P}_\omega$ is the projection on the subspace spanned by ω, i.e. $\mathbb{P}_\omega f_0 = (\omega \cdot f_0)\,\bar{\omega}$. Note also that $\mathcal{L}$ commutes with the projection $\mathbb{P}_\omega \mathbb{E}$ because ω is a eigenvector of T, thus of $\mathcal{L}$. So,

$$(\mathbb{I} - \mathbb{P}_\omega \mathbb{E})\mathcal{L} = \mathcal{L}(\mathbb{I} - \mathbb{P}_\omega \mathbb{E}).$$

Therefore, the projection $\mathbb{P}^u = (\mathbb{I} - \mathbb{P}_\omega)\mathbb{E}$ determines the linear space $\mathbb{P}^u \mathcal{A}'(\rho)$ spanned by $\{\omega^{(2)}, \dots \omega^{(j)}\}$, which is the unstable eigenspace of the linearised map at ω. In a similar way, $\mathbb{P}^s = \mathbb{I} - \mathbb{P}^u$ is the stable part.

We are now in a position to define and determine the stable and unstable invariant local manifolds of $\mathcal{R}$ at the fixed point ω. The stable one, $\mathcal{W}^s(\omega)$, corresponds to all the vector fields X in some neighbourhood B such that $\mathcal{R}^n(X) \in B$, for any $n \geq 0$. Similarly, considering the inverse iterates of $\mathcal{R}$, we define the unstable local manifold $\mathcal{W}^u(\omega)$.

If $X \in \mathcal{W}^s(\omega)$, the winding set is equal to that of ω, since it is preserved by $\mathcal{R}$. This is in fact true because ω is an eigenvector of the matrix $T^* \in GL(d, \mathbb{Z})$, the linear part of the transformation induced by $\mathcal{R}$ on the domain of X. The winding set is preserved up to the action of $\hat{T}$, related to T, by iterating $\mathcal{R}$. If the winding set of a vector $X \in B$ is $w_X = \omega / \|\omega\|$, then it is also in $\mathcal{W}^s(\omega)$, otherwise would approximate the unstable direction and w_X would be different.

The set of constant vector fields is invariant under $\mathcal{R}$, i.e. $\mathcal{R}(\mathbb{C}^d \cap B) \subset \mathbb{C}^d$, as can be seen from the way the non-linear coordinate change $\mathcal{U}$ was constructed in Theorem 5.2. This allows us to conclude that the invariant unstable local manifold is $\mathcal{W}^u(\omega) = \mathbb{P}^u(B)$.

Lemma 10.2 *We can find a ball $B' \subset \mathcal{A}'_d(\rho)$ around ω such that, for any $X \in \mathcal{W}^s(\omega) \cap B'$, there are constants $K > 0$ and $\theta \in]0, 1[$ yielding*

$$\|\mathcal{R}^m(X) - \omega\|'_\rho < K\theta^m \|X - \omega\|'_\rho, \quad m > 0.$$

Moreover, there exists $a > 0$, $c > 1$ and $N > 0$ satisfying

$$\|(\mathbb{I} - \mathbb{E})\mathcal{R}^n(X)\|'_\rho < e^{-ac^n} \|X - \omega\|'_\rho + \theta^{2n} \|X - \omega\|'^{\,2}_\rho, \quad n > N.$$

Proof: Fixing $0 < t' < t < \theta < 1$ and choosing an appropriate norm $\|\cdot\|$ on $\mathcal{A}'(\rho)$ equivalent to $\|\cdot\|'_\rho$, it is possible to have $\|D\mathcal{R}(\omega)\, f\| < t' \|f\|$, $f \in \mathbb{P}^s(\mathcal{A}'(\rho))$, since $D\mathcal{R}(\omega)$ is compact.

We can write $X = \omega + f + g$ where $f = \mathbb{P}^s(X - \omega)$ and $g = \mathbb{P}^u(X - \omega)$. By the stable manifold Theorem, there is a constant $A > 0$ for which

$$\|g\| < A\,\|f + g\|^2, \quad \text{and} \quad \|f\| < \|f + g\| + A\|f + g\|^2. \tag{8}$$

The image of X is then $X' = \mathcal{R}(X) = \omega + f' + g'$ with

$$
\begin{aligned}
f' &= D\mathcal{R}(\omega)\, f, \\
g' &= D\mathcal{R}(\omega)\, g + \mathcal{O}(\|f+g\|^2).
\end{aligned}
\tag{9}
$$

The analyticity of $\mathcal{R}$ and the formula of the second-order Taylor remaining imply that there exists constants $B, C > 0$ satisfying

$$
\begin{aligned}
\|f'\| &< t'\|f\|, \\
\|g'\| &< C\,\|g\| + B\,\|f+g\|^2.
\end{aligned}
\tag{10}
$$

By combining this with (8) we obtain

$$\|f'+g'\| < t'\|f+g\| + (At' + AC + B)\|f+g\|^2.$$

So, there is a radius $r > 0$ such that

$$\|X' - \omega\| < t\|X - \omega\|$$

for $\|X - \omega\| < r$. Iterating this inequality m times we obtain

$$\|\mathcal{R}^m(X) - \omega\|'_\rho < K\theta^m \|X - \omega\|'_\rho,$$

if $B' = \{X \in \mathcal{A}'_d(\rho) \colon \|X - \omega\| < r\}$, that is the first of our claims.

From Lemma 10.1 we can find $a > 0$ and $c > 1$ and a sequence of positive integers $k_n < \varepsilon n$, $\varepsilon > 0$, such that

$$\|(\mathbb{I} - \mathbb{E})[D\mathcal{R}(\omega)]^{k_n}\| < e^{-ac^n},$$

whenever n is sufficiently large. Given such n and $X \in \mathcal{W}^s(\omega) \cap B'$, the first part of the proof showed that $\|\mathcal{R}^m(X) - \omega\| < \delta = t^m r$, with $m = n - k_n$. Consider a map $F \colon (f, g) \mapsto (f', g')$ defined by (9), and let $f_0 = \mathbb{P}^s(\mathcal{R}^m(X) - \omega)$ and $g_0 = \mathbb{P}^u(\mathcal{R}^m(X) - \omega)$. The pair (f_0, g_0) satisfies

$$
\begin{aligned}
\|f_0\| &< 2\delta, \\
\|g_0\| &< A\delta^2.
\end{aligned}
$$

Applying the inequalities (10), if $1 < A < 1/\delta$, then $\|g_1\| < (C + 9B)A\delta^2$, using the notation $(f_i, g_i) = F^i(f_0, g_0)$, $i \geq 0$. The fact that $\|f_i\|$ is bounded by at least 2δ allows us to iterate F for k_n times, obtaining:

$$
\begin{aligned}
f_{k_n} &= [D\mathcal{R}(\omega)]^{k_n} f_0, \\
\|g_{k_n}\| &< (C + 9B)^{k_n} A\delta^2,
\end{aligned}
$$

as long as $(C + 9B)^{k_n} A\delta < 1$. This is in fact verified if we choose $\varepsilon > 0$ such that $(C + 9B)^{k_n} t^{2m} < \theta^{2n}$, or, more strongly,

$$[t^{-2}(C + 9B)]^\varepsilon < (\theta/t)^2.$$

Now, $\mathcal{R}^n(X) = \omega + f_{k_n} + g_{k_n}$ and $(\mathbb{I} - \mathbb{E})\mathcal{R}^n(X) = (\mathbb{I} - \mathbb{E})(f_{k_n} + g_{k_n})$. Hence,

$$\|(\mathbb{I} - \mathbb{E})\mathcal{R}^n(X)\| < 2e^{-ac^n} t^m r + A\theta^{2n} r^2,$$

which completes the proof of the Theorem, if n is chosen sufficiently large. □

Acknowledgements

I would like to express my gratitude to Professor R. S. MacKay for the orientation and the support given. I also wish to thank Cristel Chandre and Tim Hunt for useful conversations. The author is funded by Fundação para a Ciência e a Tecnologia, under the research grant BD/11230/97.

References

[1] V. I. Arnol'd. Small denominators I, mappings of the circumference onto itself. *Transl. AMS 2nd Series*, 46:213–284, 1961.

[2] C. Baesens, J. Guckenheimer, S. Kim, and R. S. MacKay. Three coupled oscillators: mode-locking, global bifurcations and toroidal chaos. *Physica D*, 49:387–475, 1991.

[3] A. Delshams and R. de la Llave. KAM theory and a partial justification of Greene's criterion for non-twist maps. *SIAM J. Math. Anal.*, 31(6):1235–1269, 2000.

[4] J. Lopes Dias. Renormalisation of vector fields for a generic frequency vector. *to be submitted*, 2001.

[5] A. Katok and B. Hasselblatt. *Introduction to the modern theory of Dynamical Systems.* Cambridge University Press, 1995.

[6] H. Koch. A renormalization group for Hamiltonians, with applications to KAM tori. *Erg. Theor. Dyn. Syst.*, 19:475–521, 1999.

[7] S. Lang. *Introduction to diophantine approximations.* Springer-Verlag, 2nd edition, 1995.

[8] R. S. MacKay. Three topics in Hamiltonian dynamics. In Y. Aizawa, S. Saito, and K. Shiraiwa, editors, *Dynamical Systems and Chaos*, volume 2. World Scientific, 1995.

[9] J. Pöschel. On elliptic lower dimensional tori in Hamiltonian systems. *Math. Z.*, 202:559–608, 1989.