SELF-ADJOINT EXTENSIONS BY ADDITIVE PERTURBATIONS

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ABSTRACT. Let $A_{\mathcal{N}}$ be the restriction of A to $\mathcal{N}$, where $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator and $\mathcal{N} \subsetneq D(A)$ is a linear dense set which is closed with respect to the graph norm on $D(A)$. We show how to define any relatively prime self-adjoint extension $A_\Theta : D(\Theta) \subseteq \mathcal{H} \to \mathcal{H}$ of $A_{\mathcal{N}}$ by $A_\Theta := \bar{A} + T_\Theta$, where both the operators $\bar{A}$ and T_Θ take values in the strong dual of $D(A)$. The operator $\bar{A}$ is the closed extension of A to the whole $\mathcal{H}$ whereas T_Θ is explicitly written in terms of a (abstract) boundary condition depending on $\mathcal{N}$ and on the extension parameter Θ, a self-adjoint operator on an auxiliary Hilbert space isomorphic (as a set) to the deficiency spaces of $A_{\mathcal{N}}$. The explicit connection with both Kreĭn's resolvent formula and von Neumann's theory of self-adjoint extensions is given.

1. INTRODUCTION

Given a self-adjoint operator $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$, let $A_{\mathcal{N}}$ be the restriction of A to $\mathcal{N}$, where $\mathcal{N} \subsetneq D(A)$ is a dense linear subspace which is closed with respect to the graph norm. Then $A_{\mathcal{N}}$ is a closed, densely defined, symmetric operator. Since $\mathcal{N} \neq D(A)$, $A_{\mathcal{N}}$ is not essentially self-adjoint, as A is a non-trivial extension of $A_{\mathcal{N}}$, and, by the famed von Neumann's formulae [14], we know that $A_{\mathcal{N}}$ has an infinite family of self-adjoint extensions A_{U} parametrized by the unitary maps U from K_+ onto K_-, where $K_\pm :=\mathrm{Kernel}\,(-A_{\mathcal{N}}^* \pm i)$ denotes the deficiency space.

In section 2 we define a family A_Θ of extensions of $A_{\mathcal{N}}$ by means of a Kreĭn-like formula i.e. by explicitly giving its resolvent $(-A_\Theta + z)^{-1}$. We follow the approach developed in [15], thus describing the domain of A_Θ in terms of the boundary condition $\tau\phi_\star = \Theta\, Q_\phi$, where $\tau : D(A) \to \mathfrak{h}$ is a surjective continuos linear mapping with $\mathrm{Kernel}\,\tau = \mathcal{N}$, $\Theta : D(\Theta) \subseteq \mathfrak{h} \to \mathfrak{h}$ is self-adjoint and $\mathfrak{h}$ is a Hilbert space isomorphic (as a set) to $K_\pm$.

In section 3 we use theorem 2.1 to re-write A_Θ in a more appealing way as a sum $\bar{A} + T_\Theta$ where both $\bar{A}$ and T_Θ take values in the strong dual (with respect to the graph norm) of $D(A)$; $\bar{A}$ is nothing else that the closed extension of A to the whole Hilbert space $\mathcal{H}$ and T_Θ is explicitly related to the maps τ and Θ giving the boundary conditions. By remark 3.4, theorem 3.3 can be interpreted as an extension to the general case of the results obtained in [2], in §2, [8], [10], where $-A$ was strictly positive, and, as regards boundary conditions, in [9], where $-A = -\Delta + \lambda$, $\lambda > 0$, $\mathcal{N}$ the kernel of the evaluation map along a regular submanifold. For an extension of such results to arbitrary injective self-adjoint operators the reader is also refered to [16].

Successively, is section 4, we study the connection of the self-adjoint extensions defined in theorems 2.1 and 3.3 with the ones given by von Neumann's theory [14]. In theorem 4.1 we prove that the operator $\widetilde{A} = \bar{A} + T$ defined in theorem 3.3, of which the self-adjoint $A_\Theta = \bar{A} + T_\Theta$ is a restriction, coincides with $A_{\mathcal{N}}^*$ and moreover for any self-adjoint operator $\Theta : D(\Theta) \subseteq \mathfrak{h} \to \mathfrak{h}$ we define a unitary operator $\mathrm{U} : K_+ \to K_-$ such that $A_\Theta = A_\mathrm{U}$, where A_U denotes the von Neumann's extension corresponding to U. The converse of theorem 4.1 is given in theorem 4.3. This also shows (see corollary 4.4) that $\widetilde{A} = \bar{A} + T$ coincides with a relatively prime self-adjoint extension of $A_\mathcal{N}$ if and only if the boundary condition $\tau\phi_\star = \Theta\, Q_\phi$ holds for some self-adjoint operator Θ. Let us recall that A_Θ and A are said to be relatively prime (with respect to $A_\mathcal{N}$) when $D(A_\Theta) \cap D(A) = \mathcal{N}$. If an extension A_U is not relatively prime then theorem 3.3 still can be used by replacing $\mathcal{N}$ with $D(A_\mathrm{U}) \cap D(A)$ (see remark 4.5); however in this case it gives no practical advantage since A_U has to be known in advance.

In section 5 we conclude with some examples both in the case of finite and infinite deficiency indices. Example 5.1 (also see remark 4.2) shows that, in the case $\dim K_\pm < +\infty$, theorem 3.3 reproduces the theory of finite rank perturbations as given in [3], §3.1, and thus it can be viewed as an extension of such a theory to the infinite rank case. In example 5.1 we give two examples in the infinite rank case: infinitely many point interaction in three dimensions and singular perturbations, supported on d-sets with $0 < n - d < 2s$, of traslation invariant pseudo-differential operators with domain $H^s(\mathbb{R}^n)$.

Notations and definitions

- Given a Banach space $\mathcal{X}$ we denote by $\mathcal{X}'$ its strong dual.
- $\mathsf{L}(\mathcal{X}, \mathcal{Y})$ denotes the space of linear operators from the Banach space $\mathcal{X}$ to the Banach space $\mathcal{Y}$; $\mathsf{L}(\mathcal{X}) \equiv \mathsf{L}(\mathcal{X}, \mathcal{X})$.
- $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ denotes the Banach space of bounded, everywhere defined, linear operators on the Banach space $\mathcal{X}$ to the Banach space $\mathcal{Y}$; $\mathcal{B}(\mathcal{X}) \equiv \mathcal{B}(\mathcal{X}, \mathcal{X})$.
- Given $A \in \mathsf{L}(\mathcal{X}, \mathcal{Y})$ densely defined, the closed operator $A' \in \mathsf{L}(\mathcal{Y}', \mathcal{X}')$ is the adjoint of A i.e.
$$\forall\, \phi \in D(A) \subseteq \mathcal{X}, \quad \forall \lambda \in D(A') \subseteq \mathcal{Y}', \qquad (A'\lambda)(\phi) = \lambda(A\phi)\,.$$
- If $\mathcal{H}$ is a complex Hilbert space with scalar product (conjugate-linear with respect to the first variable) $\langle \cdot, \cdot \rangle$, then $C_\mathcal{H} : \mathcal{H} \to \mathcal{H}'$ denotes the conjugate-linear isomorphism defined by
$$(C_\mathcal{H}\, \psi)(\phi) := \langle \psi, \phi \rangle\,.$$
- The Hilbert adjoint $A^* \in \mathsf{L}(\mathcal{H}_2, \mathcal{H}_1)$ of the densely defined linear operator $A \in \mathsf{L}(\mathcal{H}_1, \mathcal{H}_2)$ is defined as
$$A^* := C_{\mathcal{H}_1}^{-1} \cdot A' \cdot C_{\mathcal{H}_2}\,.$$
- For any self-adjoint $A \in \mathsf{L}(\mathcal{H})$ we define
$$\gamma(A) := \inf \{ \langle \phi, A\phi \rangle,\ \phi \in D(A),\ \|\phi\| = 1 \}\,.$$
- $\mathcal{F}$ and $*$ denote Fourier transform and convolution respectively.
- $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is the usual scale of Sobolev-Hilbert spaces, i.e. $H^s(\mathbb{R}^n)$ is the space of tempered distributions with a Fourier transform which is square integrable with respect to the measure with density $(1 + |x|^2)^s$.

2. Extensions by a Kreĭn-like formula

Given the Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle$ (we denote by $\|\cdot\|$ the corresponding norm), let $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator and let $\mathcal{N} \subsetneq D(A)$ be a linear dense set which is closed with respect to the graph norm on $D(A)$. We denote by $\mathcal{H}_+$ the Hilbert space given by the set $D(A)$ equipped with the scalar product $\langle\cdot,\cdot\rangle_+$ leading to the graph norm, i.e.

$$\langle\phi_1,\phi_2\rangle_+ := \langle(A^2+1)^{1/2}\phi_1,(A^2+1)^{1/2}\phi_2\rangle\,.$$

Being $\mathcal{N}$ closed we have $\mathcal{H}_+ = \mathcal{N}\oplus\mathcal{N}^\perp$ and we can then consider the orthogonal projection $\pi : \mathcal{H}_+ \to \mathcal{N}^\perp$. From now on, since this gives advantages in concrete applications where usually (a variant of) π is what is known in advance, we will simply consider a linear map

$$\tau : \mathcal{H}_+ \to \mathfrak{h}\,,\qquad \tau\in\mathcal{B}(\mathcal{H}_+,\mathfrak{h})\,,$$

where $\mathfrak{h}$ is a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$, such that

$$\operatorname{Range}\tau = \mathfrak{h} \tag{2.1}$$

and

$$\operatorname{Range}\tau'\cap\mathcal{H}' = \{0\}\,. \tag{2.2}$$

The set $\mathcal{N}$ will then be recovered by defining

$$\mathcal{N} := \operatorname{Kernel}\tau\,.$$

By (2.1) one has $\mathfrak{h}\simeq\mathcal{H}_+/\operatorname{Kernel}\tau\simeq(\operatorname{Kernel}\tau)^\perp$ so that

$$\mathcal{H}_+\simeq\operatorname{Kernel}\tau\oplus\mathfrak{h}$$

whereas hypothesis (2.2) is equivalent to the denseness of $\mathcal{N}$. Indeed, by Riesz lemma, there exists $Q\in\mathfrak{h}$ such that $\tau'\cdot C_{\mathfrak{h}}Q\in\mathcal{H}'$ if and only if there exists $\psi\in\mathcal{H}$ (necessarily orthogonal to N) such that for any $\phi\in D(A)$ one has $\langle\psi,\phi\rangle=\langle Q,\tau\phi\rangle_{\mathfrak{h}}$. Here we used the embedding of $\mathcal{H}'$ into $D(A)'\supseteq\operatorname{Range}\tau'$ given by $\varphi\mapsto\langle C_{\mathcal{H}}^{-1}\varphi,\cdot\rangle$.

Being $\rho(A)$ the resolvent set of A, we define $R(z)\in\mathcal{B}(\mathcal{H},\mathcal{H}_+)$, $z\in\rho(A)$, by

$$R(z) := (-A+z)^{-1}$$

and we then introduce, for any $z\in\rho(A)$, the bounded linear operators $\breve G(z)\in\mathcal{B}(\mathcal{H},\mathfrak{h})$ and $G(z)\in\mathcal{B}(\mathfrak{h},\mathcal{H})$ by

$$\breve G(z) := \tau\cdot R(z)\,,\qquad G(z) := \breve G(\bar z)^*\,.$$

By (2.2) one has

$$\operatorname{Range}G(z)\cap D(A)=\{0\}\,, \tag{2.3}$$

and, as an immediate consequence of the first resolvent identity for $R(z)$ (see [15], lemma 2.1)

$$(z-w)\,R(w)\cdot G(z) = G(w)-G(z)\,. \tag{2.4}$$

These relations imply

$$\operatorname{Range}(G(w)-G(z))\subseteq D(A) \tag{2.5}$$

and

$$\operatorname{Range}(G(w)+G(z))\cap D(A)=\{0\}\,.$$

By [15] (combining thm. 2.1, prop. 2.1, lemma 2.2, remarks 2.10, 2.12 and 2.13) one then obtains the following

Theorem 2.1. *Given* $z_0 \in \mathbb{C}\backslash\mathbb{R}$ *define*

$$G_\star := \frac{1}{2}\,(G(z_0)+G(\bar z_0))$$

and, given then any self-adjoint operator $\Theta : D(\Theta) \subseteq \mathfrak{h} \to \mathfrak{h}$*, define*

$$R_\Theta(z) := R(z) + G(z)\cdot(\Theta+\Gamma(z))^{-1}\cdot \breve G(z)\,, \qquad z \in W_\Theta^- \cup W_\Theta^+ \cup \mathbb{C}\backslash\mathbb{R}\,,$$

where

$$\Gamma(z) := \tau\cdot(G_\star - G(z))$$

and

$$W_\Theta^\pm := \{\, \lambda \in \mathbb{R}\cap\rho(A)\ :\ \gamma(\pm\Gamma(\lambda)) > -\gamma(\pm\Theta)\,\}\ .$$

Then R_Θ *is the resolvent of the self-adjoint extension of* $A_\mathcal{N}$ *defined by*

$$D(A_\Theta) := \{\phi \in \mathcal{H}\ :\ \phi = \phi_\star + G_\star Q_\phi,\ \phi_\star \in D(A),\ Q_\phi \in D(\Theta),\ \tau\phi_\star = \Theta\, Q_\phi\}\ ,$$

$$A_\Theta\,\phi := A\,\phi_\star + \mathrm{Re}(z_0)\,G_\star Q_\phi + \mathrm{Im}(z_0)\,(G(z_0)-G(\bar z_0))Q_\phi\,.$$

Proof. Here we just give the main steps of the proof refering to [15], §2, for the details. Using the suggestion given by the resolvent formula for a bounded perturbation, one is lead to write the presumed resolvent of an extension $\widetilde A$ of $A_\mathcal{N}$ as

$$\widetilde R(z) = R(z) + B(z)\cdot\tau\cdot R(z) \equiv R(z) + B(z)\cdot\breve G(z)\,,$$

where $B(z) \in \mathcal{B}(\mathfrak{h},\mathcal{H})$ has to be determined. Self-adjointness requires $\widetilde R(z)^* = \widetilde R(\bar z)$ or, equivalently,

$$G(\bar z)\cdot B(z)^* = B(\bar z)\cdot \breve G(\bar z)\ . \tag{2.6}$$

Therefore posing $B(z) = G(z)\cdot\Lambda(z)$, where $\Lambda(z) \in \mathcal{B}(\mathfrak{h})$, (2.6) is equivalent to

$$\Lambda(z)^* = \Lambda(\bar z)\ . \tag{2.7}$$

The resolvent identity

$$(z-w)\,\widetilde R(w)\widetilde R(z) = \widetilde R(w) - \widetilde R(z) \tag{2.8}$$

is then equivalent to

$$\Lambda(w) - \Lambda(z) = (z-w)\,\Lambda(w)\cdot\breve G(w)\cdot G(z)\cdot\Lambda(z)\ . \tag{2.9}$$

Suppose now that there exist a (necessarily closed) operator

$$\Gamma(z) : D \subseteq \mathfrak{h} \to \mathfrak{h}$$

and an open set $Z \subseteq \rho(A)$, invariant with respect to complex conjugation, such that

$$\forall\, z \in Z, \qquad \Gamma(z)^{-1} = \Lambda(z)\ .$$

Then (2.9) forces $\Gamma(z)$ to satisfy the relation

$$\Gamma(z) - \Gamma(w) = (z-w)\,\breve G(w)\cdot G(z)\,, \tag{2.10}$$

whereas (2.7), at least in the case $\Gamma(z)$ is densely defined, and has a bounded inverse given by $\Lambda(z)$ as we are pretending, is equivalent to

$$\Gamma(z)^* = \Gamma(\bar z)\,. \tag{2.11}$$

By [15], lemma 2.2, for any self-adjoint Θ, the linear operator

$$\Theta + \tau\cdot(G_\star - G(z))$$

satisfies (2.10), (2.11) and, by [15], proposition 2.1, has a bounded inverse for any $z \in W_\Theta^- \cup W_\Theta^+ \cup \mathbb{C}\backslash\mathbb{R}$ (at this point hypothesis (2.1) is used). Therefore (see the proof of theorem 2.1 in [15])

$$R_\Theta(z) := R(z) + G(z) \cdot (\Theta + \Gamma(z))^{-1} \cdot \breve{G}(z)$$

is the resolvent of a self-adjoint operator A_Θ (here hypotheses (2.2) is needed). For any $z \in W_\Theta^- \cup W_\Theta^+ \cup \mathbb{C}\backslash\mathbb{R}$ one has

$$D(A_\Theta) = \left\{ \phi \in \mathcal{H} \ : \ \phi = \phi_z + G(z) \cdot (\Gamma + \Theta(z))^{-1} \cdot \tau\, \phi_z, \quad \phi_z \in D(A) \right\} ,$$

$$(-A_\Theta + z)\phi = (-A + z)\phi_z \,,$$

the definition of A_Θ being z-independent thanks to resolvent identity (2.8). Being $G(z)$ injective, (2.3) and (2.5) imply

$$\phi_w + G(w)Q_1 = \phi_z + G(z)Q_2 \quad \Rightarrow \quad Q_1 = Q_2$$

and so the definition

$$Q_\phi := (\Theta + \Gamma(z))^{-1} \cdot \tau\, \phi_z$$

is z-independent. Therefore any $\phi \in D(A_\Theta)$ can be equivalently re-written as

$$\phi = \phi_z + G(z)Q_\phi \,,$$

where $Q_\phi \in D(\Theta)$ and

$$\tau\phi_z = \Theta\, Q_\phi + \Gamma(z)Q_\phi \,.$$

This implies, for any $\phi \in D(A_\Theta)$,

$$\begin{aligned}
\phi =& \frac{1}{2}\,(\phi_{z_0} + G(z_0)Q_\phi + \phi_{\bar{z}_0} + G(\bar{z}_0)Q_\phi) \equiv \phi_\star + G_\star Q_\phi \,,\\
\tau\phi_\star \equiv& \frac{1}{2}\,\tau(\phi_{z_0} + \phi_{\bar{z}_0}) = \Theta\, Q_\phi + \frac{1}{2}\,(\Gamma(z_0) + \Gamma(\bar{z}_0)Q_\phi) = \Theta\, Q_\phi \,,\\
A_\Theta\phi =& \frac{1}{2}\,(A\phi_{z_0} + z_0 G(z_0)Q_\phi + A\phi_{\bar{z}_0} + \bar{z}_0 G(\bar{z}_0)Q_\phi)\\
\equiv& A\phi_\star + \mathrm{Re}(z_0)\, G_\star Q_\phi + \mathrm{Im}(z_0)\,(G(z_0) - G(\bar{z}_0))Q_\phi \,.
\end{aligned}$$

□

Remark 2.2. The results quoted in the previous theorem are consequences of an alternative version of Kreĭn's resolvent formula. The original one was obtained in [11], [12], [17] for the cases where $\dim K_\pm = 1$, $\dim K_\pm < +\infty$, $\dim K_\pm = +\infty$ respectively; also see [4], [6], [13] for more recent formulations. In standard Kreĭn's formula (usually written with $z_0 = i \equiv \sqrt{-1}$) the main ingredient is the orthogonal projection $P : \mathcal{H} \to K_+$ whereas we used, exploiting the a priori knowledge of the extension A, the map τ, which plays the role of the orthogonal projection $\pi : \mathcal{H}_+ \to \mathcal{N}^\perp$. The version given in [15] allows τ to be not surjective and $\mathfrak{h}$ can be a Banach space; the use of the map τ simplifies the exposition and makes easier to work out concrete applications. Indeed, as we already said, frequently what is explicitely known is the map τ and $\mathcal{N}$ is then simply defined as its kernel: see the many examples in [15] where τ is the trace (restriction) map along some null subset of $\mathbb{R}^n$ and A is a (pseudo-)differential operator. Moreover this approach allows a natural formulation in terms of the boundary condition $\tau\phi_\star = \Theta\, Q_\phi$.

Remark 2.3. The one given in the theorem above is not the only possible definition of the operator $\Gamma(z)$. Any other not necessarily bounded, densely defined operator satisfying

$$\Gamma(z)-\Gamma(w)=(z-w)\,\breve G(w)\cdot G(z)\,,$$
$$\Gamma(\bar z)\subseteq\Gamma(z)^*$$

and such that $\Theta+\Gamma(z)$ is boundedly invertible would suffice (see [15], thm. 2.1); note that, once Θ is given, $\Gamma(z)$ univocally defines $(-A_\Theta+z)^{-1}$ and hence A_Θ itself. For alternative choices we refer to [15]; also see [16] where it is shown how, under the hypotheses Kernel $A=\{0\}$ and $\|\tau\phi\|_{\mathfrak h}\le c\,\|A\phi\|$, it is always possible to take $z_0=0$ in theorem 1.1 (at the expense of having then $\phi_\star$ in the completion of $D(A)$ with respect to the norm $\phi\mapsto\|A\phi\|$). However we remark that any different choice (either of z_0 or of the operator $\Gamma(z)$ itself) does not change the family of extensions as a whole.

Remark 2.4. In the case A has a non-empty real resolvent set, by [15], remark 2.7, if in theorem 1.1 one consider only the sub-family of extensions in which the Θ's have bounded inverses, then one can take $z_0\in\mathbb R\cap\rho(A)$. More generally one can take $z_0\in W^-_\Theta\cup W^+_\Theta$ independently of the inveritility of Θ; however this could give rise to implicit conditions (related to the location of the spectrum of A_Θ) on the choice of z_0.

3. Extensions by Additive Perturbations

We define the pre-Hilbert space $\tilde{\mathcal H}_-$ as the set $\mathcal H$ equipped with the scalar product

$$\langle\phi_1,\phi_2\rangle_-:=\langle(A^2+1)^{-1/2}\phi_1,(A^2+1)^{-1/2}\phi_2\rangle\,.$$

We denote then by $\mathcal H_-$ the Hilbert space given by the completion of $\tilde{\mathcal H}_-$. We will avoid to identify $\mathcal H_+$ and $\mathcal H_-$ with their duals; indeed, see lemma 3.1 below, we will identify $\mathcal H'_+$ with $\mathcal H_-$.

As usual $\mathcal H$ will be treated as a (dense) subspace of $\mathcal H_-$ by means of the canonical embedding

$$\mathcal I_-:\mathcal H\to\mathcal H_-$$

which associates to ϕ the set of all the Cauchy sequences converging to ϕ. Considering also the canonical embedding (with dense range)

$$\mathcal I_+:\mathcal H'\to\mathcal H'_+\,,\qquad \mathcal I_+\ell(\phi):=\langle C_{\mathcal H}^{-1}\ell,\phi\rangle\,,$$

we can then define

$$\mathcal J_-:\mathcal H'_+\to\mathcal H_-\,,\qquad \mathcal J\in\mathcal B(\mathcal H'_+,\mathcal H_-)\,,$$

as the unique bounded extension of

$$\mathcal I_-\cdot C_{\mathcal H}^{-1}\cdot\mathcal I_+^{-1}:\mathcal I_+(\mathcal H')\subseteq\mathcal H'_+\to\mathcal H_-\,.$$

Analogously we define

$$\mathcal J_+:\mathcal H_-\to\mathcal H'_+\,,\qquad \mathcal J\in\mathcal B(\mathcal H_-,\mathcal H'_+)\,,$$

as the unique bounded extension of

$$\mathcal I_+\cdot C_{\mathcal H}\cdot\mathcal I_-^{-1}:\mathcal I_-(\mathcal H)\subseteq\mathcal H_-\to\mathcal H'_+\,.$$

These definitions immediately lead to the following

Lemma 3.1. *One has*

$$\mathcal{J}_+ = \mathcal{J}_-^{-1}\,, \qquad \mathcal{J}_- = \mathcal{J}_+^{-1}\,,$$

so that

$$\mathcal{H}_+' \simeq \mathcal{H}_-\,.$$

We will denote by

$$(\cdot,\cdot) : \mathcal{H}_- \times \mathcal{H}_+ \to \mathbb{C}\,, \qquad (\varphi, \phi) := \mathcal{J}_+\varphi(\phi)$$

the pairing between $\mathcal{H}_-$ and $\mathcal{H}_+$. It is nothing else that the extension of the scalar product of $\mathcal{H}$, being

$$(\mathcal{I}_-\phi_1, \phi_2) = \langle \phi_1, \phi_2 \rangle\,.$$

We consider now the linear operator

$$\mathcal{I}_- \cdot A : \mathcal{H}_+ \subseteq \mathcal{H} \to \mathcal{H}_-\,.$$

Since

$$\|(A^2+1)^{-1/2}A\phi\| \leq \|\phi\|\,,$$

the operator $\mathcal{I}_- \cdot A$ has an unique extension

$$\bar{A} : \mathcal{H} \to \mathcal{H}_-\,, \qquad \bar{A} \in \mathcal{B}(\mathcal{H}, \mathcal{H}_-)\,.$$

Lemma 3.2. *Let $A' : \mathcal{H}' \to \mathcal{H}_+'$ be the adjoint of the linear operator A viewed as an element of $\mathcal{B}(\mathcal{H}_+, \mathcal{H})$. Then one has*

$$\bar{A} = \mathcal{J}_- \cdot A' \cdot C_{\mathcal{H}}\,.$$

Proof. Being $\mathcal{I}_-$ injective, by continuity and density the thesis follows from the identity

$$A = A^* \equiv C_{\mathcal{H}}^{-1} \cdot A' \cdot C_{\mathcal{H}}\,.$$

□

Now we can reformulate theorem 1.1 in terms of additive perturbations:

Theorem 3.3. *Define*

$$D(\widetilde{A}) := \{\, \phi \in \mathcal{H} \,:\, \phi = \phi_\star + G_\star Q_\phi,\ \phi_\star \in D(A),\ Q_\phi \in \mathfrak{h} \,\}\,,$$

$$\widetilde{A} : D(\widetilde{A}) \to \mathcal{H}_-\,, \qquad \widetilde{A} := \bar{A} + T\,,$$

where

$$T : D(\widetilde{A}) \to \mathcal{H}_-\,, \qquad T\phi := \mathcal{J}_- \cdot \tau' \cdot C_{\mathfrak{h}}\, Q_\phi\,.$$

Then the linear operator $\widetilde{A}$ is $\mathcal{H}$-valued and coincides with A_Θ when restricted to $D(A_\Theta)$, i.e. when a boundary condition of the kind $\tau\phi_\star = \Theta\, Q_\phi$ holds for some self-adjoint operator Θ. Therefore, posing $T_\Theta := T_{\,|D(A_\Theta)}$, one has

$$A_\Theta : D(A_\Theta) \to \mathcal{H}\,, \qquad A_\Theta\phi := \bar{A} + T_\Theta\,,$$

and, in the case Θ has a bounded inverse,

$$A_\Theta : D(A_\Theta) \to \mathcal{H}\,, \qquad A_\Theta\phi = \bar{A}\phi + V_\Theta\phi_\star\,,$$

where

$$V_\Theta : \mathcal{H}_+ \to \mathcal{H}_-\,, \qquad (V_\Theta\phi_1, \phi_2) := \langle \Theta^{-1}\tau\phi_1, \tau\phi_2 \rangle_{\mathfrak{h}}\,.$$

Proof. By the definition of $\bar A$ and $G_\star$ one has, for any $\phi\in D(\widetilde A)$,

$$\begin{aligned}\bar A\phi =&\mathcal{I}_-\cdot A\phi_\star+\mathcal{J}_-\cdot A'\cdot C_{\mathcal{H}}\cdot G_\star Q_\phi\\ =&\mathcal{I}_-\cdot A\phi_\star+\frac12\mathcal{J}_-\cdot A'\cdot R(\bar z_0)'\cdot\tau'\cdot C_{\mathfrak{h}}Q_\phi+\frac12\mathcal{J}_-\cdot A'\cdot R(z_0)'\cdot\tau'\cdot C_{\mathfrak{h}}Q_\phi\\ =&\mathcal{I}_-\cdot(A\phi_\star+\mathrm{Re}(z_0)\,G_\star Q_\phi+\mathrm{Im}(z_0)\,(G(z_0)-G(\bar z_0))Q_\phi)-TQ_\phi\,.\end{aligned}$$

The proof is then concluded by theorem 1.1. □

Remark 3.4. In the case $0\in\rho(A)$ and Θ is boundedly invertible, by theorem 2.1 and remark 2.4 (taking $z_0=0$) one can define A_Θ either by $A_\Theta\phi:=A\phi_\star$ or, equivalently, by

$$A_\Theta^{-1}=A^{-1}+G\cdot\Theta^{-1}\cdot\breve G\,,$$

where $G:=G(0)$, $\breve G:=\breve G(0)$. Since, for any $\phi_1,\phi_2\in\mathcal{H}$, one has

$$\begin{aligned}&\langle\bar A^{-1}\cdot V_\Theta\cdot A^{-1}\phi_1,\phi_2\rangle=(V_\Theta A^{-1}\phi_1,A^{-1}\phi_2)\\ =&\langle\Theta^{-1}\tau\cdot A^{-1}\phi_1,\tau\cdot A^{-1}\phi_2\rangle_{\mathfrak{h}}=\langle\Theta^{-1}\breve G\phi_1,\breve G\phi_2\rangle_{\mathfrak{h}}\\ =&\langle G\cdot\Theta^{-1}\cdot\breve G\phi_1,\phi_2\rangle\,,\end{aligned}$$

the self-adjoint extension A_Θ could be defined directly in terms of V_Θ by

$$A_\Theta^{-1}=A^{-1}+\bar A^{-1}\cdot V_\Theta\cdot A^{-1}\,.$$

4. The connection with von Neumann's Theory

In this section we explore the connection between the results given in the previous sections and von Neumann's theory of self-adjoint extensions. Such a theory (see e.g. [5], §13, for a very compact exposition) tells us that $D(A_{\mathcal{N}}^*)=\mathcal{N}\oplus K_+\oplus K_-$ and that any self-adjoint extension A_{U} of $A_{\mathcal{N}}$ can be obtained by restricting $A_{\mathcal{N}}^*$ to a subspace of the kind $\mathcal{N}\oplus\mathrm{Graph}\,\mathrm{U}$, where $\mathrm{U}:K_+\to K_-$ is unitary.

For simplicity in the next theorem we will consider only the case $z_0=i$ and we will pose $G_\pm:=G(\pm i)$ and $\Gamma:=\Gamma(i)$.

Theorem 4.1. *Let $\widetilde A=\bar A+T$ as defined in theorem 3.3. Then*

$$\widetilde A=A_{\mathcal{N}}^*\,.$$

The linear operator

$$G_\pm:\mathfrak{h}\to K_\pm$$

is a continuos bijection which becomes unitary when one puts on $\mathfrak{h}$ the scalar product

$$\langle Q_1,Q_2\rangle_\Gamma:=\langle\sqrt{-i\Gamma}\,Q_1,\sqrt{-i\Gamma}\,Q_2\rangle_{\mathfrak{h}}\,.$$

The linear operator

$$\mathrm{U}:K_+\to K_-\,,\qquad \mathrm{U}:=-G_-\cdot(1+2(\Theta-\Gamma)^{-1}\cdot\Gamma)\cdot G_+^{-1}$$

is unitary and the corresponding von Neumann's extension A_{U} coincides with the self-adjoint operator A_Θ defined in theorems 2.1 and 3.3.

Proof. By the definition of $\breve G_\pm\equiv\breve G(\pm i)$ one has

$$\mathrm{Range}\,(-A_{\mathcal{N}}\pm i)=\mathrm{Kernel}\,\breve G_\pm$$

and so, since

$$K_\pm=\mathrm{Range}\,(-A_{\mathcal{N}}\mp i)^\perp$$

and

$$\operatorname{Range} G_{\pm}{}^{\perp} = \operatorname{Kernel} \breve{G}_{\mp} ,$$

in conclusion there follows

$$\operatorname{Range} G_{\pm} = K_{\pm}$$

if and only if $\operatorname{Range} G_{\pm}$ is closed. By the closed range theorem $\operatorname{Range} G_{\pm}$ is closed if and only if the $\operatorname{Range} \breve{G}_{\pm}$ is closed, and this is equivalent to the range of τ being closed. Being τ surjective, $G_{\pm}$ is injective with a closed range and so

$$G_{\pm} : \mathfrak{h} \to K_{\pm}$$

is a bijection.

By von Neumann's theory we know that any $\phi \in D(A^*_{\mathcal{N}})$ can be univocally decomposed as

$$\phi = \phi_0 + \phi_+ + \phi_- , \qquad \phi_0 \in \mathcal{N},\ \phi_{\pm} \in K_{\pm} ,$$

i.e.

$$\phi = \phi_0 + G_+ Q_+ + G_- Q_- , \qquad \phi_0 \in \mathcal{N},\ Q_{\pm} \in \mathfrak{h} .$$

The above decomposition can be then rearranged as

$$\begin{aligned}\phi =&\phi_0 + \frac{1}{2}\,(G_+ - G_-)Q_+ + \frac{1}{2}\,(G_+ + G_-)Q_+ \\ &+ \frac{1}{2}\,(G_- - G_+)Q_- + \frac{1}{2}\,(G_- + G_+)Q_- \\ =&\phi_0 + \frac{1}{2}\,(G_- - G_+)(Q_- - Q_+) + G_\star(Q_- + Q_+) .\end{aligned}$$

By (2.4) one has

$$G_{\mp} - G_{\pm} = \pm 2i\, R(\mp i) \cdot G_{\pm} . \tag{4.1}$$

Since the scalar product of $\mathcal{H}_+$ can be equivalently written as

$$\langle \phi_1, \phi_2 \rangle_+ = \langle (-A + i)\phi_1, (-A + i)\phi_2 \rangle ,$$

one has

$$G_- - G_+ = 2i\, R(-i) \cdot R(-i)^* \cdot \tau^* = 2i\, \tau^* .$$

This implies, since $\operatorname{Range} G_+$ is closed,

$$\operatorname{Range}(G_- - G_+) = \operatorname{Range} \tau^* = \operatorname{Kernel} \tau^{\perp} .$$

Thus, being $\mathcal{H}_+ = \mathcal{N} \oplus \mathcal{N}^{\perp}$, the vector

$$\phi_0 + \frac{1}{2}\,(G_- - G_+)(Q_- - Q_+)$$

is a generic element of $D(A)$ and we have shown that $D(\widetilde{A}) = D(A^*_{\mathcal{N}})$. It is then straighforward to check that $\widetilde{A} = A^*_{\mathcal{N}}$.

By (4.1) one has

$$\Gamma = \pm\frac{1}{2}\,\tau \cdot (G_{\mp} - G_{\pm}) = i\,\tau \cdot R(\mp i) \cdot G_{\pm} = i\,\breve{G}_{\mp} \cdot G_{\pm} = i\, G^*_{\pm} \cdot G_{\pm} .$$

This implies

$$\|G_{\pm} Q\| = \|\sqrt{-i\Gamma}\, Q\|_{\mathfrak{h}} ,$$

thus $\mathrm{U} = -\,G_- \cdot (1 + 2(\Theta - \Gamma)^{-1} \cdot \Gamma) \cdot G_+^{-1}$ is isometric if and only if

$$\forall\, Q \in \mathfrak{h} , \qquad \|\sqrt{-i\Gamma} \cdot \widetilde{\mathrm{U}}\, Q\|_{\mathfrak{h}} = \|\sqrt{-i\Gamma}\, Q\|_{\mathfrak{h}} ,$$

where $\widetilde{\mathrm{U}} := G_-^{-1} \cdot \mathrm{U} \cdot G_+$. By using the identities $\Gamma^* = -\Gamma$ and

$$(\Theta - \Gamma)^{-1} - (\Theta + \Gamma)^{-1} = 2\,(\Theta + \Gamma)^{-1} \cdot \Gamma \cdot (\Theta - \Gamma)^{-1}\,, \tag{4.2}$$

one has

$$\begin{aligned}
&i\,\|\sqrt{-i\Gamma} \cdot (1 + 2(\Theta - \Gamma)^{-1} \cdot \Gamma)\,Q\|_{\mathfrak{h}} \\
=&\langle \Gamma\,Q + 2\Gamma \cdot (\Theta - \Gamma)^{-1} \cdot \Gamma\,Q, Q + 2(\Theta - \Gamma)^{-1} \cdot \Gamma\,Q\rangle \\
=&\langle \Gamma Q, Q\rangle + 2\langle \Gamma\,Q, (\Theta - \Gamma)^{-1} \cdot \Gamma\,Q\rangle + 2\langle \Gamma \cdot (\Theta - \Gamma)^{-1} \cdot \Gamma\,Q, Q\rangle \\
&+ 4\langle \Gamma \cdot (\Theta - \Gamma)^{-1} \cdot \Gamma\,Q, (\Theta - \Gamma)^{-1} \cdot \Gamma\,Q\rangle \\
=&\langle \Gamma Q, Q\rangle + 2\langle \Gamma\,Q, ((\Theta - \Gamma)^{-1} - (\Theta + \Gamma)^{-1}) \cdot \Gamma\,Q\rangle \\
&- 4\langle \Gamma\,Q, (\Theta + \Gamma)^{-1} \cdot \Gamma \cdot (\Theta - \Gamma)^{-1} \cdot \Gamma\,Q\rangle \\
=&\langle \Gamma Q, Q\rangle = i\,\|\sqrt{-i\Gamma}\,Q\|_{\mathfrak{h}}\,,
\end{aligned}$$

and so U is an isometry. By again using identity (4.2) one can check that U has an inverse defined by

$$\mathrm{U}^{-1} := -\,G_+ \cdot (1 - 2(\Theta + \Gamma)^{-1} \cdot \Gamma) \cdot G_-^{-1}\,.$$

Thus U is unitary. Let us now take $G_-Q_- = \mathrm{U}G_+Q_+$. Then

$$-2(\Theta - \Gamma)^{-1} \cdot \Gamma Q_+ = Q_- + Q_+$$

and so $Q_- + Q_+ \in D(\Theta)$ and

$$\tau\left(\phi_0 + \frac{1}{2}\,(G_- - G_+)(Q_- - Q_+)\right) \equiv \Gamma(Q_- - Q_+) = \Theta(Q_- + Q_+)\,.$$

□

Remark 4.2. Note that when Θ is bounded, in the previuos theorem one can rewrite the unitary U as

$$\mathrm{U} = -G_- \cdot (\Theta - \Gamma)^{-1} \cdot (\Theta + \Gamma) \cdot G_+^{-1}\,.$$

Being Θ always bounded when $\dim K_\pm = n$, the previous theorem gives an analogue of theorem 3.1.2 in [3] avoiding however the use of an admissible matrix R (see [3], definition 3.1.2).

The previous theorem has the following converse:

Theorem 4.3. *Let A_{U} be a relatively prime self-adjoint extension of $A_{\mathcal{N}}$ as given by von Neumann's theory and let $\mathcal{C}_A := (-A+i)\cdot(-A-i)^{-1}$ be the Cayley transform of A. Then the set*

$$D(\Theta) := \text{Range } G_-^{-1} \cdot (\mathrm{U} + \mathcal{C}_A)$$

is dense,

$$\Theta : D(\Theta) \subseteq \mathfrak{h} \to \mathfrak{h}\,, \qquad \Theta := i\,\breve{G}_+ \cdot (\mathrm{U} - \mathcal{C}_A) \cdot (\mathrm{U} + \mathcal{C}_A)^{-1} \cdot G_-\,,$$

is self-adjoint and the corresponding self-adjoint operator A_Θ, defined as either in theorem 2.1 or 3.3, coincides with A_{U}.

Proof. By (4.1) one has

$$G_- \cdot G_+^{-1} = 1 + 2i\,R(-i) = \mathcal{C}_A\,.$$

Thus, by inverting the relation $\mathrm{U} = -G_- \cdot (1 + 2(\Theta - \Gamma)^{-1} \cdot \Gamma) \cdot G_+^{-1}$ given in the previous theorem, one obtains

$$\begin{aligned}\Theta =&\Gamma \cdot (G_-^{-1} \cdot \mathrm{U} \cdot G_+ - 1) \cdot (G_-^{-1} \cdot \mathrm{U} \cdot G_+ + 1)^{-1}\\ =&\Gamma \cdot G_-^{-1} \cdot (\mathrm{U} - G_- \cdot G_+^{-1}) \cdot (\mathrm{U} + G_- \cdot G_+^{-1})^{-1} \cdot G_-\\ =&\Gamma \cdot G_-^{-1} \cdot (\mathrm{U} - \mathcal{C}_A) \cdot (\mathrm{U} + \mathcal{C}_A)^{-1} \cdot G_- \,.\end{aligned}$$

Since $\mathrm{U} = -\mathcal{C}_{A_\mathrm{U}}$ and $1 \notin \sigma_p(\mathcal{C}_A \cdot \mathcal{C}_{A_\mathrm{U}}^{-1})$ if and only if A_U is a relative prime extension (see e.g. [6], lemma 1), the range of $U + \mathcal{C}_A$ is dense and thus Θ is densely defined as G_- is a continuos bijection. By (4.1) one has

$$\Gamma \cdot G_-^{-1} = i\tau \cdot R(i) \equiv i\breve{G}_+$$

and so, since $\breve{G}_+^* = G_-$ and $G_-^* = \breve{G}_+$, Θ is self-adjoint if and only if

$$(\mathrm{U}^* + \mathcal{C}_A^*) \cdot (\mathrm{U} - \mathcal{C}_A) = -(\mathrm{U}^* - \mathcal{C}_A^*) \cdot (\mathrm{U} + \mathcal{C}_A)\,.$$

Such an equality is then an immediate conseguence of the unitarity of both U and $\mathcal{C}_A$. □

Corollary 4.4. *$\widetilde{A} = \bar{A} + T$ as defined in theorem 3.3 coincides with a relatively prime self-adjoint extension of $A_\mathcal{N}$ if and only if the boundary condition $\tau\phi_\star = \Theta\, Q_\phi$ holds for some self-adjoint operator $\Theta : D(\Theta) \subseteq \mathfrak{h} \to \mathfrak{h}$.*

Remark 4.5. When A_U is not relatively prime it is still possible to use theorem 3.3 and to write such an extension as an additive perturbation of A. It sufficies to take $\tau = \pi$, where π is the orthogonal projection $\pi : \mathcal{H}_+ \to \mathcal{M}^\perp$, $\mathcal{M} := D(A_\mathrm{U}) \cap D(A)$. However in this case theorem 3.3 is of no practical use since it is necessary to know A_U in advance.

5. Examples

Example 5.1. *Finite rank perturbations.* Suppose $\dim K_\pm = n$, so that $\mathfrak{h} \simeq \mathbb{C}^n$ and $\tau \in \mathcal{B}(\mathcal{H}_+, \mathbb{C}^n)$. Then necessarily

$$\tau : \mathcal{H}_+ \to \mathbb{C}^n\,, \qquad \tau\phi = \{(\varphi_j, \phi)\}_1^n\,,$$

with $\varphi_1, \dots, \varphi_n \in \mathcal{H}_-$. Hypotheses (2.1) and (2.2) correspond to

$$\exists\, \phi_1, \dots, \phi_n \in \mathcal{H}_+ \quad \text{s.t.} \quad (\varphi_i, \phi_j) = \delta_{ij}\,,$$

and

$$\sum_{j=1}^n c_j\, \varphi_j \in \mathcal{H} \quad \text{iff} \quad c_1 = \cdots = c_n = 0\,.$$

Considering then an Hermitean invertible matrix $\Theta = (\theta_{ij})$ with inverse $\Theta^{-1} = (t_{ij})$, by theorem 3.3 one can define the self-adjoint operator

$$A_\Theta\phi := \bar{A}\phi + \sum_{i,j=1}^n t_{ij}(\varphi_i, \phi_\star)\varphi_j$$

with

$$D(A_\Theta) :=$$
$$\left\{ \phi \in \mathcal{H} \,:\, \phi = \phi_\star + \sum_{j=1}^n Q_j R_\star \varphi_j,\ \phi_\star \in D(A),\ Q \in \mathbb{C}^n,\ (\varphi_i, \phi_\star) = \sum_{j=1}^n \theta_{ij} Q_j \right\}\,,$$

where

$$R_\star := \frac{1}{2}\,(\hat R(z_0) + \hat R(\bar z_0))\,,$$

$$\hat R(z) : \mathcal{H}_- \to \mathcal{H}\,, \qquad \langle \hat R(z)\varphi, \phi\rangle := (\varphi, R(\bar z)\phi)\,.$$

According to theorem 1.1 its resolvent is given by

$$(-A_\Theta + z)^{-1} = (-A+z)^{-1} + \sum_{i,j=1}^{n} (\Theta + \Gamma(z))^{-1}_{ij}\,\hat R(z)\varphi_i\,\hat R(\bar z)\varphi_j\,,$$

where

$$\Gamma(z)_{ij} = \frac{1}{2}\,(\varphi_i, (\hat R(z_0) + \hat R(z_0) - 2\hat R(z))\varphi_j)\,.$$

The operator A_Θ above coincides with a generic finite rank perturbation of the self-adjoint operator A as defined in [3], §3.1. In order to realize that the resolvent written above (in the case $z_0 = i$) is the same given there, the identity

$$\frac{1}{2}\,(R(i) + R(-i) - 2R(z)) = (1 + zA)\cdot(A-z)^{-1}\cdot(A^2+1)^{-1}$$

has to be used.

The previous construction can be applied to the case of so-called point interactions in three dimensions (see [1] and references therein). Since in example 5.2 below we will consider the case of infinitely many point interactions, here we just treat the simplest situation in which only one point interection (placed at the origin) is present. In this case we take $A = \Delta$, $\mathcal{H} = L^2(\mathbb{R}^3)$, $\mathcal{H}_+ = H^2(\mathbb{R}^3)$, $\mathcal{H}_- = H^{-2}(\mathbb{R}^3)$, and $\varphi = \delta_0$. Therefore τ is simply the evaluation map at the origin

$$\tau : H^2(\mathbb{R}^3) \to \mathbb{C}\,, \qquad \tau\phi = \phi(0)\,,$$

and we have the family of self-adjoint operators Δ_θ, $\theta \in \mathbb{R}\backslash\,\{0\}$, defined as (we take $z_0 = i$)

$$\Delta_\theta \phi := \Delta\phi + \theta^{-1}\phi_\star(0)\,\delta_0$$

on the domain

$$D(\Delta_\theta) := \left\{\phi \in L^2(\mathbb{R}^3) \,:\, \phi = \phi_\star + Q\,\mathcal{G}_\star,\ \phi_\star \in H^2(\mathbb{R}^3), Q \in \mathbb{C},\ \phi_\star(0) = \theta\, Q\right\}\,,$$

where

$$\mathcal{G}_\star(x) = \cos\frac{|x|}{\sqrt 2}\quad \frac{e^{-|x|/\sqrt 2}}{4\pi|x|}\,.$$

This reproduces the family given in [3], §1.5.1, and coincides with the family Δ_α given in [1], §I.1.1, when one takes $\alpha = \theta - (4\pi\sqrt 2\,)^{-1}$. The case $\alpha = -(4\pi\sqrt 2\,)^{-1}$ can be then recovered by directly using theorem 3.3 in the case $\theta = 0$.

Example 5.2. *Infinite rank perturbations.* Suppose $\dim K_\pm = +\infty$. Then (we suppose $\mathcal{H}$ is separable) $\mathfrak{h} \simeq \ell^2(\mathbb{N})$, $\tau \in \mathcal{B}(\mathcal{H}_+, \ell^2(\mathbb{N}))$ and necessarily

$$\tau : \mathcal{H}_+ \to \ell^2(\mathbb{N})\,, \qquad \tau\phi = \{(\varphi_j, \phi)\}_1^\infty\,,$$

with $\{\varphi_j\}_1^\infty \subset \mathcal{H}_-$. The generalization of the finite rank case to this situation is then evident. As concrete example one can consider infinitely many point interactions in three dimensions by taking $A = \Delta$, $\mathcal{H} = L^2(\mathbb{R}^3)$, $\mathcal{H}_+ = H^2(\mathbb{R}^3)$, $\mathcal{H}_- = H^{-2}(\mathbb{R}^3)$ as before and an infinite and countable set $Y \subset \mathbb{R}^3$ such that

$$\inf_{y \neq \tilde y} |y - \tilde y| = d > 0\,.$$

Defining then $\varphi_y := \delta_y$, by [1] (see page 172) one has $\tau \in \mathcal{B}(H^2(\mathbb{R}^3), \ell^2(Y))$, where

$$\tau : H^2(\mathbb{R}^3) \to \ell^2(Y)\,, \qquad \tau\phi = \{\phi(y)\}_{y\in Y}\ ,$$

and hypotheses (2.1) and (2.2) are an immediate conseguence of the discreteness of Y (see [15], example 3.4). By theoren 3.3, given any invertible infinite Hermitean matrix $\Theta = (\theta_{y\tilde{y}})$ with a bounded inverse $\Theta^{-1} = (t_{y\tilde{y}})$, one can then define the family of self-adjoint operators

$$\Delta_\Theta\phi := \Delta\phi + \sum_{y,\tilde{y}\in Y} t_{y\tilde{y}}\,\phi_\star(y)\,\delta_{\tilde{y}}$$

on the domain

$$D(\Delta_\Theta) :=$$

$$\left\{\phi \in L^2(\mathbb{R}^3)\ :\ \phi = \phi_\star + \sum_{y\in Y} Q_y \mathcal{G}^y_\star,\ \phi_\star \in H^2(\mathbb{R}^3), Q \in D(\Theta), \phi_\star(y) = \sum_{\tilde{y}\in Y}\theta_{y\tilde{y}}Q_{\tilde{y}}\right\},$$

where $\mathcal{G}^y_\star(x) := \mathcal{G}_\star(x-y)$. When

$$\theta_{yy} = \alpha + \frac{1}{4\pi\sqrt{2}}\,, \qquad \theta_{y\tilde{y}} = \mathcal{G}_\star(y-\tilde{y})\,, \quad y \neq \tilde{y}\,,$$

the self-adjoint extension Δ_Θ coincides with the operator $\Delta_{\alpha,Y}$ given in [1], §III.1.1 (also see [15], example 3.4).

In more general situations where the set Y is not discrete the use of the unitary isomorphism $\mathfrak{h} \simeq \ell^2(\mathbb{N})$ given no advantages and, how the following example shows, it is better to work with $\mathfrak{h}$ itself.

Let $A = \Psi$, $\mathcal{H} = L^2(\mathbb{R}^n)$, $\mathcal{H}_+ = H^s(\mathbb{R}^n)$, $\mathcal{H}_- = H^{-s}(\mathbb{R}^n)$, where the self-adjoint pseudo-differential operator Ψ is defined by

$$\Psi : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,, \qquad \Psi\phi := \mathcal{F}^{-1}(\psi\,\mathcal{F}\phi)\ ,$$

with ψ is a real-valued function such that

$$\frac{1}{c}\,(1+|x|^2)^{s/2} \le 1 + |\psi(x)| \le c\,(1+|x|^2)^{s/2}\,, \quad c > 0\,.$$

We want now to define the self-adjoint extensions of the restriction of Ψ to functions vanishing on a d-set, with $0 < n-d < 2s$. A Borel set $F \subset \mathbb{R}^n$ is called a d-set, $d \in (0,n]$, if

$$\exists\, c_1,\, c_2 > 0\ :\ \forall\, x \in F,\ \forall\, r \in (0,1), \qquad c_1 r^d \le \mu_d(B_r(x)\cap F) \le c_2 r^d\ ,$$

where μ_d is the d-dimensional Hausdorff measure and $B_r(x)$ is the closed n- dimensional ball of radius r centered at the point x (see [7], §1.1, chap. VIII). Examples of d-sets are d-dimensional Lipschitz submanifolds and (when d is not an integer) self-similar fractals of Hausdorff dimension d (see [7], chap. II, example 2). We take as the linear operator τ the unique continuous surjective (thus (2.1) holds true) map

$$\tau_F : H^s(\mathbb{R}^n) \to B^{2,2}_\alpha(F)\,, \qquad \alpha = s - \frac{n-d}{2}$$

such that, for μ_d-a.e. $x \in F$,

$$\tau_F\phi(x) \equiv \left\{\phi^{(j)}_F(x)\right\}_{|j|<\alpha} = \left\{\lim_{r\downarrow 0}\,\frac{1}{\lambda_n(r)}\int_{B_r(x)} dy\, D^j\phi(y)\right\}_{|j|<\alpha}\,,$$

where $j \in \mathbb{Z}^n_+$, $|j| := j_1 + \cdots + j_n$, $D^j := \partial_{j_1} \cdots \partial_{j_n}$ and $\lambda_n(r)$ denotes the n-dimensional Lebesgue measure of $B_r(x)$. We refer to [7], thms. 1 and 3, chap. VII, for the existence of the map τ_F; obviously it coincides with the usual evaluation along F when restricted to smooth functions. The definition of the Besov-like space $B^{2,2}_\alpha(F)$ is quite involved and we will not reproduce it here (see [7], §2.1, chap. V). However, in the case $0 < \alpha < 1$ (i.e. $2(s-1) < n-d < 2s$), $B^{2,2}_\alpha(F)$ can be alternatively defined (see [7], §1.1, chap. V) as the Hilbert space of $f \in L^2(F;\mu_F)$ having finite norm

$$\|f\|^2_{B^{2,2}_\alpha(F)} := \|f\|^2_{L^2(F)} + \int_{|x-y|<1} d\mu_F(x)\, d\mu_F(y)\, \frac{|f(x)-f(y)|^2}{|x-y|^{d+2\alpha}}\ ,$$

where μ_F denotes the restriction of the d-dimensional Hausdorff measure μ_d to the set F.

The adjoint map τ'_F gives rive to the signed measure $Q\,T_F \in H^{-s}(\mathbb{R}^n)$, $Q \in B^{2,2}_\alpha(F)$, defined by

$$(Q\,T_F, \phi) = \langle Q, \tau_\mu \phi\rangle_{B^{2,2}_\alpha(F)}\,.$$

Since $Q\,T_F$ has support given by the closure of F, hypothesis (2.2) is always verified when the closure of F has zero Lebesgue measure. Defining then

$$\mathcal{K}^\psi_\star := \operatorname{Re}\mathcal{F}^{-1}\frac{1}{-\psi + z_0}\,,$$

one has

$$G_\star : B^{2,2}_\alpha(F) \to L^2(\mathbb{R}^n)\,, \qquad G_\star Q := \mathcal{K}^\psi_\star * Q\,T_F\,.$$

Therefore, given any self-adjoint $\Theta : D(\Theta) \subseteq B^{2,2}_\alpha(F) \to B^{2,2}_\alpha(F)$, one has the family of self-adjoint extensions

$$D(\Psi_\Theta) :=$$
$$\left\{\phi \in L^2(\mathbb{R}^n)\,:\, \phi = \phi_\star + \mathcal{K}^\psi_\star * Q_\phi T_F,\, \phi_\star \in H^s(\mathbb{R}^n),\, Q_\phi \in D(\Theta),\, \tau_F\phi_\star = \Theta\, Q_\phi\right\},$$

$$\Psi_\Theta\phi := \mathcal{F}^{-1}(\psi\mathcal{F}\phi) + Q_\phi T_F$$

(see [15], example 3.6, [16], §4, for alternative definitions).

When $F = M$, M a compact Riemannian manifold, Δ_{LB} the Laplace-Beltrami operator, one has

$$B^{2,2}_\alpha(F) = H^\alpha(M) = \left\{Q \in L^2(M)\,:\,(-\Delta_{LB})^{\alpha/2}Q \in L^2(M)\right\}$$

and

$$Q\,T_F = ((-\Delta_{LB})^\alpha Q)\,\delta_M\,,$$

where, for any $\tilde Q \in H^{-\alpha}(M) \equiv H^\alpha(M)'$,

$$(\tilde Q\,\delta_M, \phi) := \int_M dv\,(-\Delta_{LB})^{-\alpha/2}\bar{\tilde{Q}}\ (-\Delta_{LB})^{\alpha/2}\tau_M\phi\,,$$

dv denoting the volume element of M. Therefore in this case, when $\alpha \geq 1$ (i.e. $0 < n-d \leq 2(s-1)$), taking $\psi(k) = |k|^2$, $\Theta = (-\Delta_{LB})^{\alpha-1}$, one can define the self-adjoint extension

$$-\Delta_M\phi := -\Delta\phi - \Delta_{LB}\cdot\tau_M\phi_\star\,\delta_M\,,$$

and so the construction given here generalizes the examples given in [8] and [9]. Also see [16], example 14, for an alternative definition.

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