THE INTEGRATED DENSITY OF STATES FOR SOME RANDOM OPERATORS WITH NONSIGN DEFINITE POTENTIALS

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Dedicated to Jean-Michel Combes on the Occasion of his 60^{th} *Birthday*

Abstract

We study the integrated density of states of random Anderson-type additive and multiplicative perturbations of deterministic background operators for which the single-site potential does not have a fixed sign. Our main result states that, under a suitable assumption on the regularity of the random variables, the integrated density of states of such random operators is locally Hölder continuous at energies below the bottom of the essential spectrum of the background operator for any nonzero disorder, and at energies in the unperturbed spectral gaps, provided the randomness is sufficiently small. The result is based on a proof of a Wegner estimate with the correct volume dependence. The proof relies upon the L^p-theory of the spectral shift function for $p \geq 1$ [9], and the vector field methods of [20]. We discuss the application of this result to Schrödinger operators with random magnetic fields and to band-edge localization.

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1 Introduction and Main Results

In this paper, we combine the results of [9] and of [20] to prove that, in certain energy regions, the integrated density of states (IDS) for additive and multiplicative perturbations of background operators by Anderson-type random potentials constructed with nonsign definite single-site potentials, is locally Hölder continuous. To our knowledge, this is the first, general result in dimensions $d \geq 1$ for such random potentials. For one-dimensional Anderson models, Stolz [35] proved localization at all energies with no sign restriction on the single-site potential. He does not, however, obtain a result on the IDS. Recently, Veselic [36] obtained results similar to ours for additive perturbations with a special class of nonsign definite potentials included in the class treated here.

Our result is based on a *Wegner estimate* with the correct volume behavior. A Wegner estimate is an upper bound on the probability that the spectrum of the local Hamiltonian H_Λ (i.e. H restricted to a box Λ with self-adjoint boundary conditions) lies within an η-neighborhood of a given energy E. A good Wegner estimate is one for which the upper bound depends linearly on the volume $|\Lambda|$, and vanishes as the size of the energy neighborhood η shrinks to zero. The linear dependence on the volume is essential for the proof of the regularity of the IDS. The rate of vanishing of the upper bound as $\eta \to 0$ determines the continuity of the IDS.

The models that can be treated by this method are described as follows. We can treat both multiplicative (M) and additive (A) perturbations of a background self-adjoint operator H_0^X, for $X = M$ or $X = A$. Multiplicatively perturbed operators describe the propagation of acoustic and electromagnetic waves in disordered media, and we refer to [10] for a discussion of their physical interpretation. Additive perturbations describe quantum propagation in disordered media. For the Wegner estimate, we are interested in perturbations V_Λ of a background operator H_0^X, that are local with respect to a bounded region $\Lambda \subset \mathbb{R}^d$. Multiplicatively perturbed operators H_Λ^M are of the form

$$H_\Lambda^M = A_\Lambda^{-1/2} H_0^M A_\Lambda^{-1/2}, \tag{1.1}$$

where $A_\Lambda = 1 + V_\Lambda$. We assume that $(1 + V_\Lambda)$ is invertible (cf. [10] for a discussion of this condition). Additively perturbed operators H_Λ^A are of the form

$$H_\Lambda^A = H_0^A + V_\Lambda. \tag{1.2}$$

The unperturbed, background medium in the multiplicative case is de-

scribed by a divergence form operator

$$H_0^M = -C_0 \rho_0^{1/2} \nabla \cdot \rho_0^{-1} \nabla \rho_0^{-1/2} C_0, \tag{1.3}$$

where ρ_0 and C_0 are positive functions that describe the unperturbed density and sound velocity. We assume that ρ_0 and C_0 are sufficiently regular so that $C_0^\infty(\mathbb{R}^d)$ is an operator core for H_0^M. The unperturbed, background medium in the additive case is described by a Schrödinger operator H_0 given by

$$H_0^A = (-i\nabla - A)^2 + W, \tag{1.4}$$

where A is a vector potential with $A \in L^2_{loc}(\mathbb{R}^d)$, and $W = W_+ - W_-$ is a background potential with $W_- \in K_d(\mathbb{R}^d)$ and $W_+ \in K_d^{loc}(\mathbb{R}^d)$.

The perturbations V_Λ that can be treated by the method of Klopp [20] are Anderson-type random potentials. Let $\tilde{\Lambda}$ denote the lattice points in the region Λ, so that $\tilde{\Lambda} \equiv \Lambda \cap \mathbb{Z}^d$. The local perturbation in the Anderson-type model is defined by:

$$V_\Lambda(x) = \sum_{i \in \tilde{\Lambda}} \lambda_i(\omega) u_i(x - i - \xi_i(\omega')), \tag{1.5}$$

provided the random variables $\xi_i(\omega')$, modeling thermal vibrations, are uniformly bounded in $i \in \mathbb{Z}^d$. The functions u_i are compactly supported in a neighborhood of the origin. They need not be of the form $u_i(x) = u(x)$, for some fixed u, since ergodicity plays no role in the Wegner estimate. However, the proof of the existence and deterministic nature of the IDS requires ergodicity, so we will make the assumption that $u_j(x) = u(x)$ for those results.

We will put conditions of the random variables $\lambda_i(\omega)$ and the single-site potentials u_i. We note that the Wegner estimate is a local estimate.

(H1a) The self-adjoint operator H_0^X is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, for $X = A$ and for $X = M$. The operator H_0^X is semi-bounded and has an open spectral gap. That is, there exist constants $-\infty < M_0 \leq C_0 \leq B_- < B_+ < C_1 \leq \infty$ so that $\sigma(H_0) \subset [M_0, \infty)$, and

$$\sigma(H_0) \cap (C_0, C_1) = (C_0, B_-] \cup [B_+, C_1).$$

(H1b) The self-adjoint operator H_0^X is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, and is semi-bounded.

(H2) The operator H_0^X is locally compact in the sense that for any $f \in L^\infty(\mathbb{R}^d)$ with compact support, the operator $f(H_0^X - M_1)^{-1}$ is compact for any $M_1 < M_0$.

(H3) The single-site potential $u_k \in C_0(\mathbb{R}^d)$. For each $k \in \mathbb{Z}^d$, there exists a nonempty open set B_k containing k so that the single-site potential $u_k \neq 0$ on B_k. Furthermore, we assume that

$$\sum_{j \in \mathbb{Z}^d} \left\{ \int_{\Lambda_1(0)} |u_j(x-j)|^p \right\}^{1/p} < \infty, \tag{1.6}$$

for $p \geq d$ when $d \geq 2$ and $p = 2$ when $d = 1$.

(H4) The conditional probability distribution of λ_0, conditioned on ${\lambda_0}^\perp \equiv \{\lambda_i \mid i \neq 0\}$, is absolutely continuous with respect to Lebesgue measure. The density h_0 has compact support contained, say, in $[m, M]$, for some constants (m, M) with $-\infty < m < M < \infty$. The density h_0 is assumed to be locally absolutely continuous.

We refer to the review article of Kirsch [17] for a proof of the fact that hypotheses (H3)–(H4) imply the essential self-adjointness of H_ω^A on $C_0^\infty(\mathbb{R}^d)$ (see [10] for the $X = M$ case). We note that condition (1.6) in (H3) is unnecessary if $u_j(x) = u(x)$, for some $u \in C_0(\mathbb{R}^d)$. Let us also note that we can take $m = 0$ in (H4) without any loss of generality. To see this, let us define a modified background operator for $X = A$ by $\tilde{H}_0 \equiv H_0 + \sum_j m u_j$, and a modified random potential by $\tilde{V}_\omega \equiv \sum_j (\lambda_j(\omega) - m) u_j$, so that $H_0 + V_\omega = \tilde{H}_0 + \tilde{V}_\omega$. If we define random variables $\tilde{\lambda}_j(\omega) \equiv (\lambda_j(\omega) - m)$, then these new random variables are distributed with a density $\tilde{h}_0(\lambda) = h_0(\lambda + m)$, having support $[0, M - m]$. For the case $X = M$, we can write the prefactor in (1.1) as $(1 + \sum_j \lambda_j u_j) = (1 + \sum_j m u_j)(1 + (1 + \sum_j m u_j)^{-1} \tilde{V}_\omega)$, with $\tilde{V}_\omega$ defined above, and absorb the deterministic factor $(1 + \sum_j m u_j)$ into the velocity function C_0 in (1.3). *Therefore, without loss of generality, we will assume that the random variables are independent, and identically distributed (iid), with a common density h_0 supported on $[0, M]$, for some $0 < M < \infty$.* We remark that the results hold in the correlated case (cf. [7]).

We remark that a semibounded operator H_0^X always has at least one open gap $(-\infty, \Sigma_0^X)$, where $\Sigma_0^X \equiv \inf \sigma(H_0^X)$. We also note that we always have $\Sigma_0^M = 0$.

The existence of the integrated density of states for additively perturbed, infinite-volume ergodic models like (1.2) is well-known. A textbook account

is found in the lecture notes of Kirsch [17]. The same proof applies to the ergodic, multiplicatively perturbed model (1.1), with minor modifications. Recently, Nakamura [27] showed the uniqueness of the IDS, in the sense that the IDS is independent of the Dirichlet or Neumann boundary conditions needed to define the local operators, in the case of Schrödinger operators with magnetic fields. The same proof applies to the multiplicatively perturbed models. It is interesting to note that the proof uses the L^1-theory of the spectral shift function.

We mention the recent papers of Kostrykin and Schrader [23, 24, 25] in which they define and construct a spectral shift density (SSD) for random Anderson-type Schrödinger operators with sign definite single-site potentials. Among other applications, the SSD gives an alternate proof of the existence of the IDS for sign definite models.

1.1 Below the Infimum of the Spectrum of H_0^A

The main result under these hypotheses on the unperturbed operator H_0^A and the local perturbation V_Λ, is the following theorem. We recall that for multiplicative perturbations, we have $\Sigma_0^M = \inf \Sigma^M = 0$, where $\Sigma^X \equiv \sigma(H_\omega^X)$ almost surely, so these results apply only to additive perturbations.

Theorem 1.1. *Assume (H1b), (H2), (H3), and (H4). For any $q > 1$, and for any $E_0 \in (-\infty, \Sigma_0^A)$, there exists a finite, positive constant C_{E_0}, depending only on $[\text{dist}\,(\sigma(H_\Lambda^A), E_0)]^{-1}$, the dimension d, and $q > 1$, so that for any $\eta < \text{dist}\,(\sigma(H_0^A), E_0)$, we have*

$$I\!P\left\{\text{dist}\,(E_0, \sigma(H_\Lambda^A)) \leq \eta\right\} \leq C_{E_0}\eta^{1/q}\,|\Lambda|\ . \tag{1.7}$$

There are several prior results on the Wegner estimate for multidimensional, continuous Schrödinger operators with Anderson-type random potentials provided the single-site potential u is sign definite. Kotani and Simon [26] proved a Wegner estimate with a $|\Lambda|$-dependence for Anderson models with overlapping single-site potentials. This condition was removed and extensions were made to the band-edge case in [6] and [1]. An extension to multiplicative perturbations was made in [10]. These methods require a spectral averaging theorem. Wegner's original proof [38] for Anderson models did not require spectral averaging. Following Wegner's argument, Kirsch gave a nice, short proof of the Wegner estimate in [18], but obtained a $|\Lambda|^2$-dependence. Recently, Stollmann [34] presented a short, elementary

proof of the Wegner estimate for Anderson-type models with singular single-site probability distributions, such as Hölder continuous measures. He also obtains a $|\Lambda|^2$-dependence. These proofs, and the proof in this paper, do not require spectral averaging. *It is not clear, however, how to extend the methods of this paper in order to prove a Wegner estimate for singular distributions with the correct volume dependence.*

As an immediate corollary of Theorem 1.1, and of the definition of the density of states, we obtain

Corollary 1.1. *Assume (H1b)-(H4), and that the model H_ω^A is ergodic. The integrated density of states is locally Hölder continuous of order $1/q$, for any $q > 1$, on the interval $(-\infty, \Sigma_0^A)$.*

1.2 The Case of a General Band Edge and Small Disorder

There may also be other open gaps in the spectrum of H_0^X. To study the regularity of the density of states in these gaps, we prove a Wegner estimate for energies in an unperturbed spectral gap under the additional condition that the disorder is small relative to the size of the gap $G \equiv (B_+ - B_-)$. In particular, we obtain a good Wegner estimate for additive perturbations $H_\Lambda^A(\lambda) = H_0^A + \lambda V_\omega$, and for multiplicative perturbations $H_\Lambda^M(\lambda) = (1 + \lambda V_\Lambda)^{-1/2} H_0^M (1 + \lambda V_\Lambda)^{-1/2}$, provided the random potential V_ω is bounded, and for all $|\lambda|$ sufficiently small.

Theorem 1.2. *We assume that H_0^X and V_ω satisfy (H1a), (H2)-(H4), and let $H_\Lambda^A(\lambda) \equiv H_0^A + \lambda V_\Lambda$, and $H_\Lambda^M(\lambda) = (1 + \lambda V_\Lambda)^{-1/2} H_0^M (1 + \lambda V_\Lambda)^{-1/2}$. Let $E_0 \in (B_-, B_+)$ be any energy in the unperturbed spectral gap of H_0, and define $\delta_\pm(E_0) \equiv dist\,(E_0, B_\pm)$. We define a constant*

$$\lambda(E_0) \equiv min\, \left(\frac{(B_+ - B_-)}{4\|V_\Lambda\|}, \frac{1}{4\|V_\Lambda\|} \left(\frac{\delta_+(E_0)\delta_-(E_0)}{2} \right)^{1/2} \right).$$

Then, for any $q > 1$, there exists a finite constant C_{E_0}, depending on λ_0, the dimension d, the index $q > 1$, and $[dist(\sigma(H_0), I)]^{-1}$, so that for all $|\lambda| < \lambda(E_0)$, and for all $\eta < min\,(\delta_-(E_0), \delta_+(E_0))/32$, we have

$$I\!P\{\, dist(\sigma(H_\Lambda^X(\lambda)), E_0) \leq \eta\} \leq C_{E_0} \eta^{1/q} |\Lambda|. \tag{1.8}$$

With reference to the constant $\lambda(E_0)$, let us note that the band-edges of the almost sure spectrum of $H_\omega(\lambda)$ scale linearly in λ (cf. [1, 19]), and hence

are of $\mathcal{O}(\sqrt{\delta_\pm(E_0)})$ from the edges $B_\pm$. Hence, for small coupling constant $|\lambda|$, our results are valid for energies in bands of size $[B_- + C\lambda^2, B_- + C|\lambda|] \cup [B_+ - C|\lambda|, B_+ - C\lambda^2]$. Consequently, the deterministic spectrum is nonempty in the regions considered. As an immediate corollary of Theorem 1.2, and of the definition of the density of state, we obtain

Corollary 1.2. *We assume that H_0^X and V_ω satisfy (H1a), (H2)-(H4), and let $H_\omega^A(\lambda) \equiv H_0^A + \lambda V_\omega$, and $H_\omega^M(\lambda) = (1 + \lambda V_\omega)^{-1/2} H_0^M (1 + \lambda V_\omega)^{-1/2}$. We assume that the models are ergodic. For any proper, open interval $I \subset (B_-, B_+)$ in the resolvent set of H_0, we define $\delta_\pm(I) \equiv dist\ (I, B_\pm)$. In analogy to Theorem 1.2, we define a constant*

$$\lambda_0(I) \equiv min\ \left(\frac{(B_+ - B_-)}{4\|V_\Lambda\|}, \frac{1}{4\|V_\Lambda\|} \left(\frac{\delta_+(I)\delta_-(I)}{2} \right)^{1/2} \right),$$

so that, for $|\lambda| \leq \lambda_0(I)$, and for any $q > 1$, the integrated density of states for $H_\omega(\lambda)$ is locally Hölder continuous of order $1/q$, on the interval I.

We note that we could equally phrase Theorem 1.1 as follows. For any interval $I \subset (B_-, B_+)$, there exists a $\lambda_0(I) > 0$ so that for any $|\lambda| < \lambda_0(I)$, the models $H_\Lambda^X(\lambda)$ satisfy a Wegner estimate as in (1.8) with $\eta^{1/q}$ replaced by $|I|^{1/q}$. We also mention that the constants $\lambda(E_0)$ and $\lambda_0(I)$ are not optimal.

1.3 Contents of the Paper

The contents of this article are as follows. The L^p-theory of the spectral shift function for $p > 1$ is reviewed in section 2. We prove Wegner's estimate, Theorem 1.1, for energies below $\inf \sigma(H_0^A)$ in section 3 along the ideas of the original argument as appearing in [9], and incorporating the work of [20]. An application of the theory developed in section 2 to the single-site spectral shift function allows us to obtain the correct volume dependence. In section 4, we prove Theorem 1.2 by adopting the methods of section 3 using the Feshbach projection method. Some simple proofs of the trace class estimates used in the proof of Wegner's estimate in sections 3 and 4 are presented in section 5. Finally, in section 6, we discuss the application of these ideas to prove the local Hölder continuity of the IDS for a family of Schrödinger operators with random magnetic fields, and to prove band-edge localization.

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2 The L^p-Theory of the Spectral Shift Function, $p \geq 1$

The L^p-theory of the spectral shift function for $p \geq 1$ was developed in [9]. We briefly recall the essential aspects here. The L^1-theory can be found in the review paper of Birman and Yafaev [3], and the book of Yafaev [39]. Suppose that H_0 and H are two self-adjoint operators on a Hilbert space $\mathcal{H}$ having the property that $V \equiv H - H_0$ is in the trace class. Under these conditions, we can define the Krein spectral shift function (SSF) $\xi(\lambda; H, H_0)$ through the perturbation determinant. Let $R_0(z) = (H_0 - z)^{-1}$, for $Im\, z \neq 0$. We then have

$$\xi(\lambda; H, H_0) \equiv \frac{1}{\pi} \lim_{\epsilon \to 0^+} \arg \det\, (1 + V R_0(\lambda + i\epsilon)). \tag{2.1}$$

It is well-known that

$$\int_{I\!R} \xi(\lambda; H, H_0)\, d\lambda = Tr V, \tag{2.2}$$

and that the SSF satisfies the L^1-estimate:

$$\|\xi(\cdot\,; H, H_0)\|_{L^1} \;\leq\; \|V\|_1. \tag{2.3}$$

Let A be a compact operator on $\mathcal{H}$ and let $\mu_j(A)$ denote the j^{th} singular value of A. We say that $A \in \mathcal{I}_{1/p}$, for some $p > 0$, if

$$\sum_j \mu_j(A)^{1/p} < \infty. \tag{2.4}$$

For $p > 1$, this means that the singular values of A converge very rapidly to zero. We define a nonnegative functional on the ideal $\mathcal{I}_{1/p}$ by

$$\|A\|_{1/p} \equiv \left(\sum_j \mu_j(A)^{1/p} \right)^p. \tag{2.5}$$

For $p > 1$, this functional is not a norm but satisfies

$$\|A + B\|_{1/p}^{1/p} \;\leq \|A\|_{1/p}^{1/p} + \|B\|_{1/p}^{1/p}. \tag{2.6}$$

If we define a metric $\rho_{1/p}(A,B) \equiv \|A-B\|_{1/p}^{1/p}$ on $\mathcal{I}_{1/p}$, then the linear space $\mathcal{I}_{1/p}$ is a complete, separable linear metric space. The finite rank operators are dense in $\mathcal{I}_{1/p}$ (cf. [2]).

Since $\mathcal{I}_{1/p} \subset \mathcal{I}_1$, for all $p \geq 1$, we refer to $A \in \mathcal{I}_{1/p}$ as being super-trace class. Consequently, we can define the SSF for a pair of self-adjoint operators H_0 and H for which $V = H - H_0 \in \mathcal{I}_{1/p}$. The main theorem is the following and we refer to [9] for the proof. D. Hundertmark and B. Simon have recently proved an optimal L^p-bound on the spectral shift function [16].

Theorem 2.1. *Suppose that H_0 and H are self-adjoint operators so that $V = H - H_0 \in \mathcal{I}_{1/p}$, for some $p \geq 1$. Then, the SSF $\xi(\lambda; H, H_0) \in L^p(\mathbb{R})$, and satisfies the bound*

$$\|\xi(\cdot\,; H, H_0)\|_{L^p} \;\leq\; \|V\|_{1/p}^{1/p}. \tag{2.7}$$

3 A Proof of Wegner's Estimate for Energies below $\inf \sigma(H_0^A)$

We give the proof of Wegner's estimate for single-site potentials with non-definite sign at energies below $\inf \sigma(H_0^A)$. The proof is simpler than previous ones as it does not require spectral averaging, nor does it require the eigenfunction localization result of Kirsch, Stollmann, and Stolz [19]. Following [20], we formulate the Wegner estimate in terms of the resolvent of H_Λ^A. If $E_0 < \inf \sigma(H_0^A)$, we have that $(H_0^A - E_0) > 0$. Consequently, we can write the resolvent of H_Λ^A, at an energy E_0 in the resolvent set of H_Λ^A, as

$$R_\Lambda(E_0) = (H_\Lambda^A - E_0)^{-1} = (H_0^A - E_0)^{-1/2}(1 + \Gamma_\Lambda(E_0;\omega))^{-1}(H_0^A - E_0)^{-1/2}. \tag{3.1}$$

The operator $\Gamma_\Lambda(E_0;\omega)$ is defined by

$$\begin{aligned} \Gamma_\Lambda(E_0;\omega) &= (H_0^A - E_0)^{-1/2} V_\Lambda (H_0^A - E_0)^{-1/2} \\ &= \sum_{j \in \tilde{\Lambda}} \lambda_j(\omega)(H_0^A - E_0)^{-1/2} u_j (H_0^A - E_0)^{-1/2}. \end{aligned} \tag{3.2}$$

Since supp u_j is compact and the sum over $j \in \tilde{\Lambda}$ is finite, the operator $\Gamma(E_0;\omega_\Lambda)$ is compact, self-adjoint, and uniformly bounded. Let us write δ

for $\mathrm{dist}(E_0, \inf\sigma(H_0^A))$. It follows from (3.1) that

$$\|R_\Lambda(E_0)\| \;\leq\; \{\mathrm{dist}\,(\sigma(H_0^A), E_0)\}^{-1}\;\|(1+\Gamma_\Lambda(E_0;\omega))^{-1}\| \leq \delta^{-1}\|(1+\Gamma_\Lambda(E_0;\omega))^{-1}\|. \tag{3.3}$$

It follows from (3.3) that

$$I\!P\{\|R_\Lambda(E_0)\| \leq 1/\eta\} \geq I\!P\{\|(1+\Gamma_\Lambda(E_0;\omega))^{-1}\| \leq \delta/\eta\}. \tag{3.4}$$

Consequently, Wegner's estimate can be reformulated as

$$\begin{aligned} I\!P\{\mathrm{dist}\,(\sigma(H_\Lambda^A), E_0) < \eta\} &= I\!P\{\;\|R_\Lambda(E_0)\| > 1/\eta\} \\ &\leq I\!P\{\;\|(1+\Gamma_\Lambda(E_0;\omega))^{-1}\| \;>\; \delta/\eta\} \\ &= I\!P\{\;\mathrm{dist}\,(\sigma(\Gamma_\Lambda(E_0;\omega)), -1) <\; \eta/\delta\} \end{aligned} \tag{3.5}$$

Hence, it suffices to compute

$$I\!P\{\;\mathrm{dist}\,(\sigma(\Gamma_\Lambda(E_0;\omega)), -1) <\; \eta/\delta\}. \tag{3.6}$$

The key observation of [20] that takes the place of monotonicity and the eigenfunction localization theorem of Kirsch, Stollmann, and Stolz [19] is the following. We define a vector field A_Λ on $L^2([m,M]^{\tilde\Lambda}, \prod_{j\in\tilde\Lambda} h_0(\lambda_j)d\lambda_j)$ by

$$A_\Lambda \equiv \sum_{j\in\tilde\Lambda}\; \lambda_j(\omega)\frac{\partial}{\partial\lambda_j(\omega)}. \tag{3.7}$$

Then, the operator $\Gamma_\Lambda(E_0;\omega)$ is an eigenvector of A_Λ in that

$$A_\Lambda\Gamma_\Lambda(E_0;\omega) = \Gamma_\Lambda(E_0;\omega). \tag{3.8}$$

It is this relationship that replaces the positivity used in [9] since, if $\Gamma_\Lambda(E_0;\omega)$ is restricted to the spectral subspace where the operator is smaller than $(-1+3\kappa/2)$, we have that $-\Gamma_\Lambda(E_0;\omega)$ is strictly positive, and hence invertible. We will use this below.

The outline of the proof follows Wegner's original argument [38]. We follow the presentation in [9]. We work with the compact, self-adjoint operator $\Gamma_\Lambda(E_0;\omega)$, as follows from (3.6). As in [9], the key estimate on the number of eigenvalues created by the variation of one random variable is obtained by first expressing the quantity in terms of a spectral shift function corresponding to a perturbation by a single-site potential, and then by estimating the L^p-norm of this spectral shift function, for $p > 1$. The proof

below uses some of the modifications of Wegner's proof [38] introduced by Kirsch [18]. We note that this proof of the Wegner estimate does not require spectral averaging [8].

Proof of Theorem 1.1.

1. It follows from the reduction given above that we need to estimate

$$I\!P\{ \text{ dist } (\sigma(\Gamma_\Lambda(E_0;\omega)), -1) < \eta/\delta\}, \tag{3.9}$$

where $\delta = \text{dist } (E_0, \inf \sigma(H_0^A))$. Let $G = (-\infty, \inf \sigma(H_0^A))$ be the unperturbed spectral gap. Since the local potential V_Λ is a relatively compact perturbation of H_0^A, the operator $\Gamma_\Lambda(E_0;\omega)$ has only discrete spectrum with zero the only possible accumulation point. Let us write $\kappa \equiv \eta/\delta$. We choose $\eta > 0$ small enough so that $[E_0 - \eta, E_0 + \eta] \subset G$, and so that $[-1 - 2\kappa, -1 + 2\kappa] \subset I\!R^-$. We denote by I_κ the interval $[-1 - \kappa, -1 + \kappa]$. The probability in (3.9) is expressible in term of the finite-rank spectral projector for the interval I_κ and $\Gamma_\Lambda(E_0;\omega)$, which we write as $E_\Lambda(I_\kappa)$. Like $\Gamma_\Lambda(E_0;\omega)$, this projection is a random variable, but we will suppress any reference to ω in the notation. We now apply Chebyshev's inequality to the random variable $Tr(E_\Lambda(I_\kappa))$ and obtain

$$\begin{aligned} I\!P\{ \text{ dist } (\sigma(\Gamma_\Lambda(E_0)), -1) < \kappa\} &= I\!P\{Tr(E_\Lambda(I_\kappa)) \geq 1\} \\ &\leq I\!E\{Tr(E_\Lambda(I_\kappa))\}. \end{aligned} \tag{3.10}$$

2. We now proceed to estimate the expectation of the trace, following the original argument of Wegner [38] as modified by Kirsch [18]. Let ρ be a nonnegative, smooth, monotone decreasing function such that $\rho(x) = 1$, for $x < -\kappa/2$, and $\rho(x) = 0$, for $x \geq \kappa/2$. We can assume that ρ has compact support since $\Gamma_\Lambda(E_0)$ is lower semibounded, independent of Λ. As in [9], we have

$$\begin{aligned} I\!E_\Lambda\{Tr(E_\Lambda(I_\kappa))\} &\leq I\!E_\Lambda\{Tr[\rho(\Gamma_\Lambda(E_0) + 1 - 3\kappa/2) - \rho(\Gamma_\Lambda(E_0) + 1 + 3\kappa/2)]\} \\ &\leq I\!E_\Lambda \left\{ Tr \left[\int_{-3\kappa/2}^{3\kappa/2} \frac{d}{dt} \rho(\Gamma_\Lambda(E_0) + 1 - t) \, dt \right] \right\}. \end{aligned} \tag{3.11}$$

In order to evaluate the ρ' term, we use the fact that $\Gamma_\Lambda(E_0)$ is an eigenfunction for the vector field A_Λ, as expressed in (3.8). We write ρ' as

$$\begin{aligned} A_\Lambda \rho(\Gamma_\Lambda(E_0) + 1 - t) &= \rho'(\Gamma_\Lambda(E_0) + 1 - t) \, A_\Lambda \Gamma_\Lambda(E_0) \\ &= \rho'(\Gamma_\Lambda(E_0) + 1 - t) \Gamma_\Lambda(E_0). \end{aligned} \tag{3.12}$$

We now note that $\rho' \leq 0$ (in the region of interest), and that on $\operatorname{supp}\rho'$, the operator $\Gamma_\Lambda(E_0) \leq (-1+2\kappa)$, so we obtain

$$-\rho'(\Gamma_\Lambda(E_0)+1-t) \leq -\frac{1}{(1-2\kappa)} \sum_{k\in\tilde{\Lambda}} \lambda_k \frac{\partial\rho}{\partial\lambda_k}(\Gamma_\Lambda(E_0)+1-t). \quad (3.13)$$

With this estimate, and the fact that $d\rho(x+1-t)/dt = -\rho'(x+1-t)$, the right side of (3.11) can be bounded above by

$$-\frac{1}{(1-2\kappa)} \sum_{k\in\tilde{\Lambda}} \int_{-3\kappa/2}^{3\kappa/2} I\!E\{\lambda_k \frac{\partial}{\partial\lambda_k} Tr[\rho(\Gamma_\Lambda(E_0)+1-t)]\}\, dt. \quad (3.14)$$

In order to evaluate the expectation, we select one random variable, say λ_k, with $k \in \tilde{\Lambda}$, and first integrate with respect to this variable using hypothesis (H4). The local absolute continuity property is necessary here because a single term in the sum of (3.13) is not necessarily positive. Let us suppose that there is a decomposition $[0,M] = \cup_{l=0}^{N-1}(M_l, M_{l+1})$ so that h_0 is absolutely continuous on each subinterval. We denote by $\tilde{h}_0$ the function $\tilde{h}_0(\lambda) \equiv \lambda h_0(\lambda)$. As $\tilde{h}_0$ is locally absolutely continuous, we can integrate by parts and obtain

$$\begin{aligned}
&\left| \int_0^M d\lambda_k \tilde{h}_0(\lambda_k) \frac{\partial}{\partial\lambda_k} Tr\{\rho(\Gamma_\Lambda(E_0)+1-t) - \rho(\Gamma_\Lambda(E_0)^{0,k}+1-t\} \right| \\
&= \left| \sum_{l=0}^{N-1} \int_{M_l}^{M_{l+1}} d\lambda_k \tilde{h}_0(\lambda_k) \frac{\partial}{\partial\lambda_k} Tr\{\rho(\lambda_k) - \rho(\lambda_k=0)\} \right| \\
&\leq \tilde{h}_0(M) |Tr\{\rho(\Gamma_\Lambda(E_0)^{M,k}+1-t) - \rho(\Gamma_\Lambda(E_0)^{0,k}+1-t)\}| \\
&\quad + \|\tilde{h}_0'\|_\infty \sup_{\lambda\in[0,M]} |Tr\{\rho(\Gamma_\Lambda(E_0)^{\lambda,k}+1-t) - \rho(\Gamma_\Lambda(E_0)^{0,k}+1-t)\}|,
\end{aligned} \quad (3.15)$$

where $\Gamma_\Lambda(E_0)^{\lambda,k}$ is the operator $\Gamma_\Lambda(E_0)$ with the coupling constant λ_k at the k^{th}-site fixed at the value $\lambda_k = \lambda$. Similarly, the value 0 or M denotes the coupling constant λ_k fixed at those values. Consequently, we are left with the task of estimating

$$\frac{\max\left(\|\tilde{h}_0'\|_\infty, \tilde{h}_0(M)\right)}{(1-2\kappa)} \sum_{k\in\tilde{\Lambda}} \int_{-3\kappa/2}^{3\kappa/2} dt \int_0^M \Pi_{l\neq k}\, h_0(\lambda_l)\, d\lambda_l\, |Tr\{D(k, E_0, 0, \lambda_k^+)\}|, \quad (3.16)$$

where $D(k, E_0, 0, \lambda_k^+)$ denotes the operator

$$D(k, E_0, 0, \lambda_k^+) \equiv \rho(\Gamma_\Lambda(E_0)^{0,k} + 1 - t) - \rho(\Gamma_\Lambda(E_0)^{\lambda_k^+,k} + 1 - t), \qquad (3.17)$$

and $\lambda_k^+ \in [0, M]$ denotes the value of the coupling constant λ_k where the maximum in (3.15) is obtained. We remark that each term in (3.16) is easily seen to be trace-class since the operator $\Gamma_\Lambda(E_0)$ has discrete spectrum with zero the only accumulation point, and the function $\rho(x+1-t)$ is supported in x in a compact interval away from 0 for $t \in [-3\kappa/2, 3\kappa/2]$.

3. The trace in (3.16) can be rewritten in terms of a spectral shift function as follows. We let $H_1 \equiv \Gamma_\Lambda(E_0)^{0,k}$ be the unperturbed operator, and write,

$$\begin{aligned} \Gamma_\Lambda(E_0)^{\lambda_k^+,k} &= H_1 + \lambda_k^+ (H_0^A - E_0)^{-1/2} u_k (H_0^A - E_0)^{-1/2} \\ &= H_1 + V. \end{aligned} \qquad (3.18)$$

Although the difference V is not trace class, the single-site potential u_k does have compact support. We show in section 6 that the difference of sufficiently large powers of the bounded operators $H_1 = \Gamma_\Lambda(E_0)^{0,k}$ and $H_1 + V = \Gamma_\Lambda(E_0)^{\lambda_k^+,k}$ is not only in the trace class, but is in the super-trace class $\mathcal{I}_{1/p}$, for all $p \geq 1$. Specifically, let us define the function $g(\lambda) = \lambda^k$. We prove that for $k > pd/2 + 1$, and $p > 1$,

$$g(H_1 + V) - g(H_1) \in \mathcal{I}_{1/p}. \qquad (3.19)$$

The spectral shift function $\xi(\lambda\,; H_1 + V, H_1)$ is defined for the pair $(H_1, H_1 + V)$ by

$$\xi(\lambda\,; H_1 + V, H_1) = sgn(g'(\lambda))\ \xi(g(\lambda)\,; g(H_1 + V), g(H_1)). \qquad (3.20)$$

Recall that both ρ and ρ' have compact support. Because of this, and the fact that the difference $\{g(H_1 + V) - g(H_1)\}$ is super-trace class, we can apply the Birman-Krein identity [3] to the trace in (3.16). This gives

$$\begin{aligned} & Tr\{\rho(\Gamma_\Lambda(E_0)^{\lambda_k^+,k} + 1 - t) - \rho(\Gamma_\Lambda(E_0)^{0,k} + 1 - t)\} \\ &= -\int_{I\!R} \frac{d}{d\lambda} \rho(\lambda + 1 - t)\ \xi(\lambda; H_1 + V, H_1)\ d\lambda \\ &= -\int_{I\!R} \frac{d}{d\lambda} \rho(\lambda + 1 - t)\ \xi(g(\lambda);\ g(H_1 + V), g(H_1)). \end{aligned} \qquad (3.21)$$

We estimate the integral using the Hölder inequality and the L^p-theory of section 2. Let $\tilde{\xi}(\lambda) = \xi(g(\lambda);\ g(H_1 + V), g(H_1))$, for notational convenience.

Let $\chi(x)$ be the characteristic function for the support of $\rho'(x)$ for $x > 0$, and we write $\tilde{\chi}(x) \equiv \chi(\lambda + 1 - t)$, so that the support of $\tilde{\chi}$ is contained in $[-1-2\kappa, -1+2\kappa]$. For any $p > 1$, and q such that $\frac{1}{p} + \frac{1}{q} = 1$, the right side of (3.21) can be bounded above by

$$\left\{ \int |\rho'|^q \right\}^{1/q} \left\{ \int |\tilde{\xi}(\lambda)\ \tilde{\chi}(\lambda)|^p \right\}^{1/p} \leq C_0 \kappa^{(1-q)/q}\ \|\tilde{\xi}\tilde{\chi}\|_{L^p}, \tag{3.22}$$

Here, we integrated one power of ρ', using the fact that $-\rho' > 0$ in the region of interest, and used the fact that $|\rho'| = \mathcal{O}(\kappa^{-1})$, to obtain

$$\left\{ \int |\rho'|^{q-1}\ |\rho'| \right\}^{1/q} \leq \kappa^{(1-q)/q} \left\{ -\int \rho' \right\}^{1/q} \leq C_0 \kappa^{(1-q)/q}. \tag{3.23}$$

By a simple change of variables, we find

$$\begin{aligned} \|\tilde{\xi}\tilde{\chi}\|_p &= \left\{ \int |\xi(g(\lambda);\ g(H_1+V), g(H_1))|^p\ \tilde{\chi}(\lambda)\ d\lambda \right\}^{1/p} \\ &\leq C_1 \left\{ \int_{I\!R} |\xi(\lambda; g(H_1+V), g(H_1))|^p\ d\lambda \right\}^{1/p} \\ &\leq C_1\ \|g(H_1+V) - g(H_1)\|_{1/p}^{1/p}. \end{aligned} \tag{3.24}$$

We recall that

$$V = \lambda_k^+ (H_0^A - E_0)^{-1/2} u_k (H_0^A - E_0)^{-1/2}. \tag{3.25}$$

In particular, the volume of the support of V has order one, and is independent of $|\Lambda|$. We prove in section 5 that the constant $\|g(H_1+V) - g(H_1)\|_{1/p}^{1/p}$ depends only on the single-site potential u_k and $\mathrm{dist}(E_0, \inf \sigma(H_0^A))$, and is independent of $|\Lambda|$. Consequently, the right side of (3.24) is bounded above by $C_0 \kappa^{(1-q)/q}$, independent of $|\Lambda|$. This estimate, equations (3.16) and (3.21), lead us to the result

$$I\!P\{\mathrm{dist}\ (-1, \sigma(\Gamma_\Lambda(E_0))) < \kappa\} \leq C_W \kappa^{1/q} \|g\|_\infty |\Lambda|, \tag{3.26}$$

for any $q > 1$. □

4 Internal Gaps for $H_0^X, X = A,$ or $X = M$

Before proceeding with the proof of Theorem 1.2, let us show how to treat multiplicative perturbations in the same framework as additive ones. Recall

from (1.1) that the multiplicatively perturbed local operator $H_\Lambda^M(\lambda)$ has the form $H_\Lambda^M(\lambda) = A_\Lambda^{-1/2}(\lambda) H_0 A_\Lambda^{-1/2}(\lambda)$, where $A_\Lambda(\lambda) = 1 + \lambda V_\Lambda$ is boundedly invertible provided $0 \leq |\lambda| < \|V_\Lambda\|_\infty^{-1}$. For computational convenience, we always assume $|\lambda| < (2\|V_\Lambda\|_\infty)^{-1}$ for multiplicative models. For ergodic models, the norm $\|V_\Lambda\|_\infty$ is bounded independently of $|\Lambda|$. It follows as in the beginning of section 3 that

$$I\!P\{dist(\sigma(H_\Lambda^M(\lambda)), E_0) < \eta\} \leq I\!P\{\|(H_0 - \lambda E_0 V_\Lambda - E_0)^{-1}\| > 2/\eta\}. \quad (4.1)$$

Let us consider the E_0-dependent Schrödinger operator $H_0 + \lambda \tilde{V}_\Lambda$, where $\tilde{V}_\Lambda \equiv -E_0 V_\Lambda$. The Wegner estimate for $H_\Lambda^M(\lambda)$ follows by (4.1) from an estimate for

$$I\!P\{\ \|(H_0 + \lambda \tilde{V}_\Lambda - E_0)^{-1}\| > 2/\eta\}. \quad (4.2)$$

The proof given ahead for the additive case now applies to the Schrödinger operator $H_0 + \lambda \tilde{V}_\Lambda$ and, as a consequence, we obtain an estimate of the probability in (4.2). By (4.1), this allows us to prove a Wegner estimate for the multiplicative case. The limitations on the disorder strength $|\lambda|$ arise from two constraints, as discussed ahead. The first, the requirement that the gap remain open after the local perturbation, implies that $|\lambda| < (B_+ - B_-)/(4E_0\|V_\Lambda\|_\infty)$. So we must first work with $|\lambda| < \lambda_0^{(1)}$, where $\lambda_0^{(1)} \equiv \min\{1/(2\|V_\omega\|_\infty), (B_+ - B_-)/(4E_0\|V_\Lambda\|_\infty)\}$. Secondly, the positivity condition (4.19) must hold. This requires that we define $\lambda_0^{(2)} = (1/(4E_0\|V_\Lambda\|_\infty))(\delta_+(E_0)\delta_-(E_0)/2)^{1/2}$. The constant in Theorem 1.2, $\lambda(E_0)$, is defined by $\lambda(E_0) = \min(\lambda_0^{(1)}, \lambda_0^{(2)})$. With this change, the proof given ahead holds for the multiplicative model.

We now turn to the proof of Theorem 1.2 and, as explained above, we will give the proof for the additive model $H_\omega(\lambda) = H_0 + \lambda V_\Lambda$.

Proof of Theorem 1.2.

1. Let $P_\pm$ denote the spectral projectors for H_0 corresponding to the components of the spectrum $[B_+, \infty)$ and $(-\infty, B_-]$, respectively, so that $P_+ + P_- = 1$, and $P_+ P_- = 0$. We use the Feshbach method to decompose the problem relative to these two orthogonal projectors. Let $H_0^\pm \equiv P_\pm H_0$, and denote by $H_\pm(\lambda) \equiv H_0^\pm + \lambda P_\pm V P_\pm$. We will need the various projections of the potential between the subspaces $P_\pm L^2(I\!R^d)$, and we denote them by $V_\pm \equiv P_\pm V P_\pm$, and $V_{+-} \equiv P_+ V P_-$, with $V_{-+} = V_{+-}^* = P_- V P_+$. Let $z \in \mathbb{C}$, with $\mathrm{Im} z \neq 0$. We can write the resolvent $R_\Lambda(z) = (H_\Lambda(\lambda) - z)^{-1}$ in terms of the resolvents of the projected operators $H_\pm(\lambda)$ as follows. In order to

write a formula valid for either P_+ or for P_-, we let $P = P_\pm$, $Q = 1 - P_\pm$, and write $R_P(z) = (PH_0 + \lambda PV_\Lambda P - zP)^{-1}$. We then have

$$R_\Lambda(z) = PR_P(z)P + \{Q - \lambda PR_P(z)PV_\Lambda Q\}\mathcal{G}(z)\{Q - \lambda QV_\Lambda PR_P(z)^*P\}, \tag{4.3}$$

where the operator $\mathcal{G}(z)$ is given by

$$\mathcal{G}(z) = \{QH_0 + \lambda QV_\Lambda Q - zQ - \lambda^2 QV_\Lambda PR_P(z)PV_\Lambda Q\}^{-1}. \tag{4.4}$$

2. The choice of $P_\pm$ depends upon whether E_0 is located near $B_\pm$, respectively. Let us suppose that $E_0 = \mathrm{Re}z$ is close to B_+ in that $E_0 > (B_+ + B_-)/2$. In this case, we use formulae (4.3)–(4.4) with $Q = P_+$ and $P = P_-$. We define distances $\delta_\pm(E_0) \equiv \mathrm{dist}\,(\sigma(H_0^\pm), E_0)$, in analogy with δ of section 3. If we take $|\lambda| < \delta_-(E_0)/2\|V_\Lambda\|_\infty$, for example, then the first term on the right in (4.3) satisfies the bound

$$\|P_-(H_0^- + \lambda V_- - E_0P_-)^{-1}P_-\| \;\leq\; 2/\delta_-(E_0). \tag{4.5}$$

Let us note that according to our choice of E_0 near B_+, we have $\delta_-(E_0) > (B_+ - B_-)/2$. Consequently, we define a constant $\lambda_0^{(1)} = (B_+ - B_-)/4\|V_\Lambda\|_\infty$. The bound in (4.5), and formulae (4.3)–(4.4), show that the resolvent of $H_\Lambda(\lambda)$ has large norm at energies near E_0 provided $\mathcal{G}(E_0)$ be large and provided $|\lambda| < \lambda_0^{(1)}$. It follows from (4.5) that for $|\lambda| < \lambda_0^{(1)}$,

$$\|(P_+ - \lambda P_-(H_-(\lambda) - E_0P_-)^{-1}P_-V_\Lambda P_+)\| \;\leq\; 2.$$

Following the analysis of the proof of Theorem 1.1, we find that

$$\begin{aligned} &I\!P\{\ \|R_\Lambda(E_0)\| \;\leq\; 1/\eta\} \\ &\quad\geq\; I\!P\{\ \|(H_-(\lambda) - E_0P_-)^{-1}\| \;\leq 1/(2\eta) \text{ and } \|\mathcal{G}(E_0)\| \;\leq 1/(8\eta)\}. \end{aligned} \tag{4.6}$$

Consequently, the probability that H_Λ has spectrum in an η-neighborhood of E_0, where $E_0 > (B_+ + B_-)/2$, is bounded above by

$$\begin{aligned} I\!P\{\ \|R_\Lambda(E_0)\| \;>\; 1/\eta\} \;\leq\; &I\!P\{\ \|(H_-(\lambda) - E_0P_-)^{-1}\| \;> 1/(2\eta)\} \\ &+I\!P\{\ \|\mathcal{G}(E_0)\| \;> 1/(8\eta)\}. \end{aligned} \tag{4.7}$$

In light of (4.5) and (4.7), and the fact that $\eta < \mathrm{dist}\,(E_0, B_+) < \delta_-(E_0)$, we see that for $|\lambda| < \lambda_0^{(1)}$,

$$\begin{aligned} I\!P\{\mathrm{dist}(\sigma(H_\Lambda), E_0) < \eta\} \;&=\; I\!P\{\ \|R_\Lambda(E_0)\| > 1/\eta\} \\ &\leq\; I\!P\{\ \|\mathcal{G}(E_0)\| \;> 1/(8\eta)\}. \end{aligned} \tag{4.8}$$

3. We next reduce the estimate of $I\!P\{\ \|\mathcal{G}(E_0)\| > 1/(8\eta)\}$ to an equivalent spectral formulation for a certain self-adjoint, compact operator. Let $R_0^+(z) = (H_0^+ - z)^{-1}$. Since $(H_0^+ - E_0) > 0$, the square root $R_0^+(E_0)^{1/2}$ is well-defined. In analogy with (3.1)–(3.2), we can write $\mathcal{G}(E_0)$ as

$$\mathcal{G}(E_0) = R_0^+(E_0)^{1/2}(1 + \tilde{\Gamma}_+(E_0))^{-1}R_0^+(E_0)^{1/2}, \tag{4.9}$$

where we define $\tilde{\Gamma}_+(E_0)$ by

$$\begin{aligned}\tilde{\Gamma}_+(E_0) \quad \equiv \quad & \lambda R_0^+(E_0)^{1/2}V_+R_0^+(E_0)^{1/2} \\ & +\lambda^2 R_0^+(E_0)^{1/2}V_{+-}(E_0P_- - H_-(\lambda))^{-1}V_{-+}R_0^+(E_0)^{1/2}.\end{aligned} \tag{4.10}$$

Because of the compactness of the support of the local potential, and hypothesis (H2), the operator $\tilde{\Gamma}_+(E_0)$ is self-adjoint and compact. Exactly as in the proof of Theorem 1.1, we show that if $E_0 > (B_- + B_+)/2$,

$$\begin{aligned}I\!P\{\ \mathrm{dist}(\sigma(H_\Lambda, E_0) < \eta\} \quad &= \quad I\!P\{\|R_\Lambda(E_0)\| \ > 1/\eta\} \\ &\leq \quad I\!P\{\ \|(1 + \tilde{\Gamma}_+(E_0))^{-1}\| \ > \delta_+(E_0)/(8\eta)\} \\ &= \quad I\!P\{\ \mathrm{dist}(\sigma(\tilde{\Gamma}_+(E_0)), -1) < 8\eta/\delta_+(E_0)\}.\end{aligned} \tag{4.11}$$

4. To estimate the last probability on the right in (4.11), we proceed as in (3.10)–(3.11) of the proof of Theorem 1.1. Let $\rho \geq 0$ be the function defined in part 2 of the proof of Theorem 1.1 with $\kappa = 8\eta/\delta_+(E_0)$. In analogy with (3.11), we must estimate

$$I\!E_\Lambda\left\{Tr\left[\int_{-3\kappa/2}^{3\kappa/2} \frac{d}{dt}\rho(\tilde{\Gamma}_+(E_0) + 1 - t)dt\right]\right\}. \tag{4.12}$$

We do this using the operator A_Λ introduced in (3.7),

$$A_\Lambda = \sum_{j\in\tilde{\Lambda}} \lambda_j(\omega)\frac{\partial}{\partial\lambda_j(\omega)}. \tag{4.13}$$

However, unlike (3.8), the operator $\tilde{\Gamma}_+(E_0)$ is no longer an eigenvector of A_Λ. A straightforward calculation yields instead

$$A_\Lambda\tilde{\Gamma}_+(E_0) = \tilde{\Gamma}_+(E_0) + \lambda^2 W(E_0). \tag{4.14}$$

The remainder term $W(E_0)$ is given by

$$W(E_0) = R_0^+(E_0)^{1/2} V_{+-} R_-(E_0)(E_0 P_- - H_0^-) R_-(E_0) V_{-+} R_0^+(E_0)^{1/2}, \tag{4.15}$$

where $R_-(E_0)$ is the reduced resolvent $(E_0 P_- - H_-(\lambda))^{-1}$. For $|\lambda| < \lambda_0^{(1)}$, we easily compute the bound,

$$\|W(E_0)\| \;\leq\; \left(\frac{4}{\delta_-(E_0)\delta_+(E_0)}\right) \|V_\Lambda\|_\infty^2. \tag{4.16}$$

We replace the calculation (3.12) by the Gohberg-Krein formula (cf. [32]) that states

$$Tr\{A_\Lambda \rho(\tilde{\Gamma}_+(E_0) + 1 - t)\} = Tr\{\rho'(\tilde{\Gamma}_+(E_0) + 1 - t) A_\Lambda \tilde{\Gamma}_+(E_0)\}. \tag{4.17}$$

In order to evaluate the right side of (4.17), we recall that $\rho'(x + 1 - t)$ has compact support in $[-1-2\kappa, -1+2\kappa]$, for any $t \in [-3\kappa/2, 3\kappa/2]$. We expand the trace using the eigenfunctions ϕ_k of $\tilde{\Gamma}_+(E_0)$ satisfying $\tilde{\Gamma}_+(E_0)\phi_k = E_k \phi_k$, $\|\phi_k\| = 1$, and $E_k \in [-1 - 2\kappa, -1 + 2\kappa]$. This gives

$$\begin{aligned} & Tr\{\rho'(\tilde{\Gamma}_+(E_0) + 1 - t) A_\Lambda \tilde{\Gamma}_+(E_0)\} \\ &= \sum_k \rho'(E_k + 1 - t)\, \langle \phi_k, A_\Lambda \tilde{\Gamma}_+(E_0)\phi_k \rangle \\ &= \sum_k \rho'(E_k + 1 - t)\, \langle \phi_k, (E_k + \lambda^2 W(E_0))\phi_k \rangle \\ &\geq \sum_k -\rho'(E_k + 1 - t)(1 - 2\kappa - \lambda^2 \|W(E_0)\|). \end{aligned} \tag{4.18}$$

The second constraint on $|\lambda|$ arises from this expression. We have the lower bound on the last term in (4.18):

$$(1 - 2\kappa - \lambda^2 \|W(E_0)\|) \geq \left(1 - \frac{16\eta}{\delta_+(E_0)} - \lambda^2 \|V_\Lambda\|^2 \frac{4}{\delta_-(E_0)\delta_+(E_0)}\right). \tag{4.19}$$

We define $\lambda(E_0)$ of Theorem 1.2 as

$$\lambda(E_0) \equiv \min\left(\lambda_0^{(1)} = \frac{(B_+ - B_-)}{4\|V_\Lambda\|}, \frac{1}{4\|V_\Lambda\|}\left(\frac{\delta_+(E_0)\delta_-(E_0)}{2}\right)^{1/2}\right). \tag{4.20}$$

So for all $|\lambda| < \lambda(E_0)$, we have a lower bound for (4.18),

$$\begin{aligned} & Tr\{\rho'(\tilde{\Gamma}_+(E_0) + 1 - t) A_\Lambda \tilde{\Gamma}_+(E_0)\} \\ &\geq -C_1 Tr\{\rho'(\tilde{\Gamma}_+(E_0) + 1 - t)\}, \end{aligned} \tag{4.21}$$

for a finite constant $C_1 > 0$. This estimate replaces (3.13). We are then left with estimating a term similar to the one in (3.14) where $\Gamma_\Lambda(E_0)$ has been replaced by $\tilde{\Gamma}_+(E_0)$.

Using the notation of the proof of Theorem 1.1, a variation in the k^{th} coupling constant results in the operator difference similar to (3.18),

$$\begin{aligned}\tilde{\Gamma}_+^{\lambda_k^+,k}(E_0) - \tilde{\Gamma}_+^{0,k}(E_0) &= \lambda_k^+ \lambda R_0^+(E_0)^{1/2} P_+ u_k P_+ R_0^+(E_0)^{1/2} \\ &\quad + \lambda^2 (\lambda_k^+)^2 R_0^+(E_0)^{1/2} P_+ u_k P_- R_-(E_0) P_- \\ &\quad \times u_k P_+ R_0^+(E_0)^{1/2}, \end{aligned} \tag{4.22}$$

where $R_-(E_0) \equiv (E_0 P_- - H_-(\lambda))^{-1}$. The operator on the right side of (4.22) is compact. We show in section 5 how to modify the proof in [9] to prove that the difference of the l^{th}-powers of $\tilde{\Gamma}_+^{\lambda_k^+,k}(E_0)$ and $\tilde{\Gamma}_+^{0,k}(E_0)$ is in the super-trace class $\mathcal{I}_{1/p}$, for any $p > 1$, provided $l > pd/2 + 1$. With this, the proof of Theorem 1.2 continues exactly as in section 3. □

5 A Trace Estimate

We present the estimates on additive perturbations needed in sections 3 and 4. The calculations for multiplicative perturbations can be reduced to those for additive ones as discussed at the beginning of section 4. We let $K_d(\mathbb{R}^d)$ denote the Kato class of potentials, and we refer the reader to Simon's article [31] for a complete description. We recall the main estimate from [9], and then show how to apply it to the present case. We let H_0 be the Schrödinger operator

$$H_0 = (-i\nabla - A)^2 + W, \tag{5.1}$$

where A is a vector potential with $A \in L^2_{loc}(\mathbb{R}^d)$, and $W = W_+ - W_-$ is a background potential with $W_- \in K_d(\mathbb{R}^d)$ and $W_+ \in K_d^{loc}(\mathbb{R}^d)$. We denote by $H = H_0 + V$, for suitable real-valued functions V. We are interested in a bounded potential V with compact support. The main result on the trace proved in [9] is the following.

Proposition 5.1. *Let H_0 be as in (5.1), and let V_1 be a Kato-class potential such that $\|V_1\|_{K_d} \leq M_1$. Let $H_1 \equiv H_0 + V_1$, and let $M > 0$ be a sufficiently large constant given in the proof. Let V be a real-valued, Kato-class function supported on $B(R)$, the ball of radius $R > 0$ with center at the origin. Then, for any $p > 0$, we have*

$$V_{eff} \equiv (H_1 + V + M)^{-l} - (H_1 + M)^{-l} \in \mathcal{I}_{1/p}, \tag{5.2}$$

provided $l > dp/2 + 2$. Under these conditions, there exists a constant $C_0 > 0$, depending on $p, k, H_0, M_1, \|V_1\|_{K_d}$, and R, so that

$$\|V_{eff}\|_{1/p} \leq C_0. \tag{5.3}$$

We sketch the proof of a modifications of this theorem needed in the proofs of Theorems 1.1 and 1.2. We first consider the condition required in the proof of Theorem 1.1. We replace the resolvent $(H_1 + V + M)^{-l}$ by $\Gamma_\Lambda^{b,k}(E_0)^l$, and we replace $(H_1 + M)^{-l}$ by $\Gamma_\Lambda^{a,k}(E_0)^l$, where the superscript (b, k) (respectively, (a, k)) denotes the operator $\Gamma_\Lambda(E_0)$ with the k^{th} coupling constant λ_k fixed at the value b, respectively, a, for any two values $a, b \in [0, M]$. We then have,

$$\Gamma_\Lambda^{b,k}(E_0)^l - \Gamma_\Lambda^{a,k}(E_0)^l = \sum_{j=0}^{l-1} \Gamma_\Lambda^{b,k}(E_0)^{l-j-1} V_k \Gamma_\Lambda^{a,k}(E_0)^j, \tag{5.4}$$

where V_k is defined in (3.18),

$$V_k = (b-a)(H_0 - E_0)^{-1/2} u_k (H_0 - E_0)^{-1/2}. \tag{5.5}$$

Let us call the difference on the left in (5.4) the effective potential V_{eff}. Let $J_k \in C_0^\infty(\mathbb{R}^d)$ be chosen so that $J_k u_k = u_k$, with supp J_k being slightly larger than supp u_k. Following the proof in [27], we first write V_{eff} as

$$V_{eff} = (b-a) \sum_{j=0}^{l-1} [J_k^{l-j-1} R_0(E_0)^{1/2} \Gamma_\Lambda^{b,k}(E_0)^{l-j-1}]^* u_k [J_k^j R_0(E_0)^{1/2} \Gamma_\Lambda^{a,k}(E)^j]. \tag{5.6}$$

Now for any $p \in \text{supp}\, h_0$, and for any $r \in \mathbb{N}$, we have

$$R_0(E_0)^{1/2} \Gamma_\Lambda^{p,k}(E_0)^r = (R_0(E_0) V_\Lambda^{p,k})^r R_0(E_0))^{1/2}. \tag{5.7}$$

Consequently, we can write the terms in square brackets in (5.6) as

$$J_k^r R_0(E_0)^{1/2} \Gamma_\Lambda^{p,k}(E_0)^r = J_k^r (R_0(E_0) V_\Lambda^{p,k})^r R_0(E_0)^{1/2}. \tag{5.8}$$

As in [27], we can commute powers of J_k to the right and express the term as

$$J_k^r (R_0(E_0) V_\Lambda^{p,k})^r R_0(E_0)^{1/2} = \sum_{\alpha=1}^{N} \Pi_{\beta=1}^r J_{\alpha\beta} R_0(E_0) B_{\alpha\beta}. \tag{5.9}$$

Here, the bounded operators $J_{\alpha\beta}$ are combinations of the derivatives of J_k, and hence have the support contained in the support of J_k, and the operators $B_{\alpha,\beta}$ are uniformly bounded independently of $|\Lambda|$. Notice that $J_k V_\Lambda^{p,k} = pu_k + v_k$, where $v_k \equiv J_k(V_\Lambda^{p,k}) - pu_k$ has support in a bounded neighborhood of u_k, depending only on the choice of J_k and the overlap of the supports of u_k and u_n, for $n \neq k$. Consequently, the Lebesgue measure $|J_k V_\Lambda^{p,k}|$ is bounded independent of $|\Lambda|$. We use the basic fact that $J_k R_0(E_0) \in \mathcal{I}_{2q}$, provided $q > d/4$ (cf. [27]). The $\mathcal{I}_{2q}$-norm depends only on $|\text{supp } J_k|$, and is thus independent of $|\Lambda|$. It follows from this, standard trace ideal estimates, and the expansions (5.9) and (5.6), that each term of the sum on the right side of (5.6) is in the super-trace ideal $\mathcal{I}_{1/p}$, for $p > 1$, provided l is chosen to satisfy $l > pd/2 + 1$. This lower bound on l differs slightly from Proposition 5.1 due to an extra resolvent factor coming from the definition of $\Gamma_\Lambda(E_0)$. Because of the support properties of $J_{\alpha\beta}$ mentioned above, the $\mathcal{I}_{1/p}$-norm is independent of $|\Lambda|$.

We now mention the modifications needed for the proof of Theorem 1.2. Instead of working with the operators $\Gamma_\Lambda^{p,k}(E_0)$, we must use the operators $\tilde{\Gamma}_+^{p,k}(E_0)$ defined in (4.10) and (4.22). An equation analogous to (5.4) holds for the difference of the l^{th} power of these operators. The potential V_k, appearing in the right side of (5.4), is replaced by the difference given in (4.22):

$$\begin{aligned} V_k &= \tilde{\Gamma}_+^{\lambda_k^+,k}(E_0) - \tilde{\Gamma}_+^{0,k}(E_0) \\ &= \lambda_k^+ \lambda R_0^+(E_0)^{1/2} P_+ u_k P_+ R_0^+(E_0)^{1/2} \\ &\quad + (\lambda_k^+)^2 \lambda^2 R_0^+(E_0)^{1/2} P_+ u_k P_- R_-(E_0) P_- u_k P_+ R_0^+(E_0)^{1/2}. \end{aligned} \tag{5.10}$$

Thus, there are two terms that enter into the analog of the right side of (5.4), so we write

$$\tilde{\Gamma}_+^{\lambda_k^+,k}(E_0)^l - \tilde{\Gamma}_+^{0,k}(E_0)^l = V_{eff}^{(1)} + V_{eff}^{(2)}. \tag{5.11}$$

The first term is identical in form to (5.6), and we can write it as

$$V_{eff}^{(1)} = \lambda_k^+ \lambda \sum_{j=0}^{l-1} \tilde{\Gamma}_+^{\lambda_k^+,k}(E_0)^{l-j-1} \{R_0^+(E_0)^{1/2} u_k R_0(E_0)^{1/2}\} \tilde{\Gamma}_+^{0,k}(E_0)^j. \tag{5.12}$$

As above, let $J_k \in C_0^\infty(\mathbb{R}^d)$ be a smooth function satisfying $J_k u_k = u_k$, and having slightly larger support. Each term in the sum (5.12) can be written

as

$$[J_k^{l-j-1} R_0^+(E_0)^{1/2} \tilde{\Gamma}_+^{\lambda_k^+,k}(E_0)^{l-j-1}]^* u_k [J_k^j R_0^+(E_0)^{1/2} \tilde{\Gamma}_+^{0,k}(E_0)^j]. \tag{5.13}$$

Each of the terms in the square brackets has the form, for $r \in \mathbb{N}$, and $p \in \text{supp } h_0$,

$$\begin{aligned} J_k^r (R_0^+(E_0)^{1/2} \tilde{\Gamma}_+^{p,k}(E_0)^r) &= \sum_{s=0}^{r} C(r,s) \lambda^{r+s} (-1)^s J_k^r (R_0^+(E_0) V_+)^{r-s} \\ &\quad \times (R_0^+(E_0) V_{+-} R_-(E_0) V_{-+})^s R_0^+(E_0)^{1/2} \end{aligned} \tag{5.14}$$

where $R_-(E_0) = (H_-(\lambda) - E_0 P_-)^{-1}$.

The second term $V_{eff}^{(2)}$ has the form

$$\begin{aligned} V_{eff}^{(2)} &= (\lambda_K^+)^2 \lambda^2 \sum_{j=0}^{l-1} \tilde{\Gamma}_+^{\lambda_k^+,k}(E_0)^{l-j-1} \\ &\quad \times \{R_0^+(E_0)^{1/2} u_k R_-(E_0) u_k R_0^+(E_0)^{1/2}\} \tilde{\Gamma}_+^{0,k}(E_0)^j. \end{aligned} \tag{5.15}$$

Each term in the sum on the right in (5.15) can be expanded as in (5.13)–(5.14). The expression corresponding to (5.13) is obtained by replacing the u_k appearing there by $u_k R_-(E_0) u_k$. The corresponding terms in the square brackets are the same as in (5.14).

We now turn to the computation of the the super-trace class norms of the effective potentials in (5.12) and (5.15). Because of the spectral projectors appearing in these terms, we cannot prove a representation for each of these terms as in (5.9). Instead, we use the exponential decay of the projectors and resolvents appearing in (5.14) in a manner similar to that used in [1]. We begin by summarizing the decay estimates that we need. Let $a, b \in \mathbb{R}^d$ be two distinct points and let $\chi_a, \chi_b \in C_0^\infty(\mathbb{R}^d)$ be two functions localized near a and b, respectively, with disjoint supports. By hypothesis (H1), the operator H_0 is assumed to be semibounded from below with an open spectral gap $G = (B_-, B_-)$. It follows from the contour integral representation of the spectral projection, and the Combes-Thomas estimate on the resolvent (cf. [11] or [1]), that for any $\delta > 0$, there exist two constants $0 < C_\delta, \sigma_\delta < \infty$, uniform in $a, b \in \mathbb{R}^d$, so that

$$\|\chi_a P_- \chi_b\| \leq C_\delta e^{-\sigma_\delta \|a-b\|}. \tag{5.16}$$

This estimate implies that

$$\|\chi_a P_+ \chi_b\| \;\leq\; C_\delta e^{-\sigma_\delta \|a-b\|}, \tag{5.17}$$

since $P_+ = 1 - P_-$, and the supports of χ_a and χ_b are disjoint. Of course, when the supports are not disjoint, the bound on the right side is simply a constant. Since $E_0 \in G$, the resolvent $R_0(E_0)$ decays exponentially when localized between χ_a and χ_b. It follows from the argument below, that the operator norms of $\chi_a R_0^+(E_0)\chi_b$ and $\chi_a R_-(E_0)\chi_b$ both exhibit exponential decay when localized between χ_a and χ_b.

We now prove that for any $q > d/4$, there exist constant $0 < C_q, \sigma_q < \infty$, so that, uniformly in $a, b \in I\!R^d$, we have

$$\|\chi_a R_0^+(E_0)\chi_b\|_{2q} \;\leq\; C_q e^{-\sigma_q \|a-b\|}. \tag{5.18}$$

A similar bound holds for $\chi_a R_-(E_0)\chi_b$. As mentioned above, for any χ of compact support, the operator $\chi R_0(E_0) \in \mathcal{I}_{2q}$, for $q > d/4$. Let $\{\chi_l \mid l \in Z\!\!\!Z^d\}$ be a partition on unity on $I\!R^d$ so that χ_l is supported in a unit cube centered at $l \in Z\!\!\!Z^d$. We then have

$$\begin{aligned}
\|\chi_a R_0^+(E_0)\chi_b\|_{2q} &\leq \sum_{l\in Z\!\!\!Z^d} \|\chi_a P_+ \chi_l\| \, \|\chi_l R_0(E_0)\chi_b\|_{2q} \\
&\leq \sum_{l\in Z\!\!\!Z^d} C_\delta e^{-\sigma_\delta\|a-l\|} \, \|\chi_l R_0(E_0)\chi_b\|_{2q} \\
&\leq \sum_{l\cap b\neq\emptyset} C_\delta e^{-\sigma_\delta\|a-l\|} \, \|\chi_l R_0(E_0)\chi_b\|_{2q} \\
&\quad + \sum_{l\cap b=\emptyset} C_\delta e^{-\sigma_\delta\|a-l\|} \, \|\chi_l R_0(E_0)\chi_b\|_{2q}.
\end{aligned} \tag{5.19}$$

The notation $l \cap b \neq \emptyset$ means that $\chi_l \chi_b \neq 0$. The first sum on the right side of the last term in (5.19) is finite and, with a possible change in weight depending only on the size of the support of χ_b, it satisfies the bound (5.18). As for the second sum in (5.19), we compute the norm $\|\chi_l R_0(E_0)\chi_b\|_{2q}$, for $\chi_l\chi_b = 0$, as follows. Let $\tilde{\chi}_l^{(1)} \in C_0^1(I\!R^d)$ be a function with slightly larger support than χ_l and satisfying $\tilde{\chi}_l^{(1)}\chi_l = \chi_l$. Let $W(\tilde{\chi}_l^{(1)})$ be the commutator $[H_0, \tilde{\chi}_l^{(1)}]$. This is a first-order operator and relatively-H_0 bounded. Finally, let $\tilde{\chi}_l^{(2)} \in C_0^1(I\!R^d)$ satisfy $W(\tilde{\chi}_l^{(1)})\tilde{\chi}_l^{(2)} = W(\tilde{\chi}_l^{(1)})$. We then have,

$$\begin{aligned}
\|\chi_l R_0(E_0)\chi_b\|_{2q} &\leq \|\chi_l R_0(E_0) W(\chi_l^{(1)}) R_0(E_0)\chi_b\|_{2q} \\
&\leq \|\chi_l R_0(E_0) W(\chi_l^{(1)}))\|_{2q} \, \|\chi_l^{(2)} R_0(E_0)\chi_b\| \\
&\leq C_0 e^{-\sigma(E_0)\|l-b\|},
\end{aligned} \tag{5.20}$$

where $\sigma(E_0)$ is the exponent in the Combes-Thomas estimate.

With these estimates in hand, we now turn to the $\mathcal{I}_{1/p}$ estimates on $V_{eff}^{(i)}, i = 1, 2$. We begin with a term of $V_{eff}^{(1)}$, given in (5.12). We write,

$$\|V_{eff}^{(1)}\|_{1/p} \;\leq\; C_0 \|J_k^{l-j-1} R_0^+(E_0)^{1/2} \tilde{\Gamma}_+^{\lambda_k^+,k}(E_0)^{l-j-1}\|_o \; \|J_k^j R_0^+(E_0)^{1/2} \tilde{\Gamma}_+^{0,k}(E_0)^j\|_n, \tag{5.21}$$

where $p = 1/o + 1/n$. Each term is now expanded according to (5.14). The first factor on the right in (5.21) is bounded above by

$$C \sum_{s_1=0}^{l-j-1} \|J_k^{l-j-1} (R_0^+(E_0)V_+)^{l-j-1-s_1} (R_0^+(E_0)V_{+-}R_-(E_0)V_{-+})^{s_1}\|_o, \tag{5.22}$$

for $0 \leq s_1 \leq l-j-1$. We expand each local potential in (5.22) using the definition (1.5). In this way, we obtain a sum over $(l-j-1+s_1)$ variables, each running over the points of $\tilde{\Lambda}$. Let J stand for the $(l-j-1-s_1)$-tuple of indices $J = (J_1, \ldots, J_{l-j-1-s_1})$, and let K and L stand for s_1-tuples of indices, all taking values in $\tilde{\Lambda}$. We also write λ_J for the product $\lambda_{j_1} \ldots \lambda_{j_{l-j-1-s_1}}$, and similarly for the other index sets. A typical element in this sum has the form,

$$\begin{aligned} &\sum_{J,K,L} |\lambda_J \lambda_K \lambda_L| \; \|J_k R_0^+(E_0) u_{J_1} \ldots R_0^+(E_0) u_{J_{l-j-1-s_1}} \\ &\times R_0^+(E_0) u_{L_1} R_-(E_0) u_{K_1} \ldots R_0^+(E_0) u_{L_s} R_-(E_0) u_{K_s}\|_o. \end{aligned} \tag{5.23}$$

To compute the $\mathcal{I}_o$ norm of this term, we bound each random variable by M, and use the Hölder inequality for trace norms repeatedly based on the bound (5.18). For each index set $X = J, K, L$, and single-site potential u_{X_i}, let χ_{X_i} be a function of compact support in a region slightly larger than supp u_{X_i}, and satisfying $\chi_{X_i} u_{X_i} = u_{X_i}$. In this way, we obtain an upper bound on (5.23),

$$\begin{aligned} &M^{l-j-1+s_1} \sum_{J,K,L \in \tilde{\Lambda}} \|J_k R_0^+(E_0) u_{J_1}\|_{2q} \ldots \|\chi_{J_{l-j-2-s_1}} R_0^+(E_0) u_{J_{l-j-1-s_1}}\|_{2q} \\ &\times \|\chi_{J_{l-j-1-s_1}} R_0^+(E_0) u_{K_1}\|_{2q} \; \|\chi_{K_1} R_-(E_0) u_{L_1}\|_{2q} \cdots \|\chi_{K_{s-1}i} R_-(E_0) u_{L_s}\|_{2q'}, \end{aligned} \tag{5.24}$$

where $2q' = 2qo/(2q - (l-j-2+s_1)o)$. According to (5.18), we require that $2q' \geq 2q$ in order for the norms to be finite, that is,

$$o > 2q - (l-j-2+s_1)o, \tag{5.25}$$

where $0 \leq s_1 \leq l - j - 1$. Similarly, the second factor in (5.21) can be bounded as in (5.24), provided n satisfies

$$n > 2q - (j - s_2 - 1)n, \tag{5.26}$$

where $0 \leq s_2 \leq j$ is the index from the expansion as in (5.14). Recalling that $p = 1/o + 1/n$, conditions (5.25)–(5.26) require that $l > pd/2 + 1$. Finally, the sum over all the indices (J, K, L) is controlled by the exponential decay of each term as given in (5.18). It follows that the sums are bounded independently of $|\Lambda|$. This proves that (5.21) is uniformly bounded in $|\Lambda|$ provided $l > pd/2 + 1$. The proof for $V_{eff}^{(2)}$ is similar. This implies that the operators $V_{eff}^{(i)} \in \mathcal{I}_{1/p}, i = 1, 2$, for any $p > 1$, and are bounded uniformly in $|\Lambda|$, provided we choose $l > pd/2 + 1$.

6 Extensions and Comments on Localization

6.1 Generalizations: Schrödinger operators with random magnetic fields

The methods of this paper can be used to treat a more general family of random perturbations that includes Schrödinger operators with random magnetic potentials. We will show that we can treat families of random operators of the form

$$H_\omega(\lambda) = H_0 + \lambda H_{1,\omega} + \lambda^2 H_{2,\omega}, \tag{6.1}$$

provided $|\lambda|$ is sufficiently small, where H_0, a second-order, self-adjoint, partial differential operator, is a deterministic background operator, and the perturbations $H_{j,\omega}, j = 1, 2$ are symmetric, relatively-H_0 bounded, first-order differential operators. For the Wegner estimate, it suffices to consider the operators $H_{j,\omega}$ localized to finite volume regions $\Lambda \subset \mathbb{R}^d$. We say that a operator B_Λ is localized in Λ if there exists a constant $0 \leq R < \infty$ so that, with $\Lambda_R \equiv \cup_{x \in \Lambda} B_R(x)$, and for any element $\phi \in C_0^\infty(\mathbb{R}^d \backslash \Lambda_R)$, we have $B_\Lambda \phi = 0$. We make the following assumptions on the random operators $H_{j,\omega}^\Lambda, j = 1, 2$:

(H5) The operator $H_{1,\omega}^\Lambda$ is localized in Λ, linear in the random variables $\lambda_j(\omega)$, and has the form

$$H_{1,\omega}^\Lambda = \sum_{j \in \tilde{\Lambda}} \lambda_j(\omega) B_j, \tag{6.2}$$

where the deterministic operators B_j are symmetric, uniformly (in $|\Lambda|$) relatively-H_0 bounded, first-order partial differential operators with coefficients supported in regions with volumes independent of $|\Lambda|$.

(H6) The operator $H^\Lambda_{2,\omega}$ is localized in Λ, quadratic in the random variables $\lambda_j(\omega)$, and has the form

$$H^\Lambda_{2,\omega} = \sum_{j,k\in\tilde{\Lambda}} \lambda_j(\omega)\lambda_k(\omega)C_{jk}, \tag{6.3}$$

where the deterministic operators $C_{jk} = C_{kj}$ are symmetric, uniformly (in $|\Lambda|$) relatively-H_0 bounded, first-order partial differential operators with coefficients supported in regions with volumes independent of $|\Lambda|$, and such that the support of the coefficients of $\sum_{j\in\tilde{\Lambda}} C_{jk}$, for each $k \in \tilde{\Lambda}$, are independent of $|\Lambda|$.

As the method shows, one can consider operators $H^\Lambda_{2,\omega}$ that depend polynomially on the random variables.

Our primary example is the following. We consider a vector potential $A^\Lambda_\omega(\lambda) \equiv A_0 + \lambda A^\Lambda_\omega$, where A_0 is a deterministic vector potential, and A^Λ_ω is local with respect to $|\Lambda|$. The background operator $H_0 = (-i\nabla - A_0)^2$ is assumed to be essentially self-adjoint on $C_0^\infty(I\!R^d)$. The corresponding magnetic Schrödinger operator can be written as

$$\begin{aligned} H^\Lambda_\omega(\lambda) &= (-i\nabla - A^\Lambda_\omega(\lambda))^2 \\ &= (-i\nabla - A_0)^2 - \lambda\{(-i\nabla - A_0)\cdot A^\Lambda_\omega + A^\Lambda_\omega \cdot (-i\nabla - A_0)\} \\ &\quad +\lambda^2 A^\Lambda_\omega \cdot A^\Lambda_\omega. \end{aligned} \tag{6.4}$$

Comparing with (6.1), we have

$$\begin{aligned} H^\Lambda_{1,\omega} &\equiv -\{(-i\nabla - A_0)\cdot A^\Lambda_\omega + A^\Lambda_\omega \cdot (-i\nabla - A_0)\}, \\ H^\Lambda_{2,\omega} &\equiv A^\Lambda_\omega \cdot A^\Lambda_\omega. \end{aligned} \tag{6.5}$$

The local, random vector potential A^Λ_ω has the Anderson-type form (1.5) with vector-valued, single-site potentials u_k. We assume that the single-site potentials and random variables $\lambda_k(\omega)$ satisfy hypotheses (H3) and (H4). For this choice, the operators $B_j = -(-i\nabla - A_0)\cdot u_j - u_j \cdot (-i\nabla - A_0)$, and $C_{jk} = u_j \cdot u_k$. It is clear that hypothesis (H3) on the u_j imply the locality conditions of hypotheses (H5)–(H6).

As in sections 3 and 4, we will consider two cases: 1) the Wegner estimate and the IDS near inf Σ, and 2) the Wegner estimate and the IDS near the internal gaps.

As a concrete example for both of these cases, we consider a perturbation of the Landau Hamiltonian H_0 on $\mathbb{R}^2$. The background vector potential A_0 can be chosen to be $A_0(x_1, x_2) = (B_0/2)(0, x_1)$. The spectrum of H_0 consists of a discrete family of $E_n(B_0) = (2n+1)B_0, n = 0, 1, \ldots$ of infinitely degenerate eigenvalues. We now consider a perturbation A_ω of Anderson-type, obtained from (1.5) by summing over all lattice points $\mathbb{Z}^d$. It is clear that for small $|\lambda|$ the deterministic spectrum Σ lies in nonoverlapping bands of width $\mathcal{O}(|\lambda|M)$ about the Landau levels, where supp $h_0 = [0, M]$, as above (cf. [4] for related results on the spectrum of magnetic Schrödinger operators). Hence, we can consider the Wegner estimate and the IDS at energies near the bottom of the first band and at higher band edges.

Concerning the first case, inf Σ, Nakamura [28] recently proved an upper bound on the IDS $N(E)$, exhibiting the Lifshitz tail behavior, near the bottom of the spectrum for a general family of Schrödinger operators with random magnetic fields. Nakamura considered the case of $\lambda = 1$ and $A_0 = 0$ in (6.4), and $d \geq 2$. He assumed that the magnetic field is a random and metrically transitive, bounded, closed, two-form on $\mathbb{R}^d$, that is asymptotically clustering in the sense that for any $f, g \in C_0(\mathbb{R}^d)$, the expectation $\mathbb{E}(fg)$ approaches $\mathbb{E}(f)\mathbb{E}(g)$, as the supports of f and g separate. Furthermore, the expectation of the average of the magnetic field over a unit cell is assumed to be strictly positive. Under these conditions, Nakamura proved that

$$\limsup_{E \to 0^+} \frac{\log\left(-\log N(E)\right)}{\log E} \leq -d/2. \tag{6.6}$$

For the special case of a random vector potential described above, we prove a Wegner estimate and the Hölder continuity of the IDS near inf Σ. *It follows from the comments below on localization that Nakamura's estimate (6.6), and the Wegner estimate, Theorem 6.1, prove Anderson localization for a class of random magnetic Schrödinger operators near the bottom of the deterministic spectrum provided inf* $\Sigma = 0$.

As for the situation of internal gaps, we can construct examples of families of random Schrödinger operators with random vector potentials starting with three types of background operators with open internal gaps. These internal gaps can be proved to remain open after a perturbation by random vector potential with weak disorder. First, for $d = 2$, we can con-

sider the Landau Hamiltonian discussed above. Secondly, pure magnetic Schrödinger operators with periodic vector potentials have been studied by Hempel and Herbst [15], and by Nakamura [29]. These authors prove that there may exist open spectral gaps for Schrödinger operators with strong, periodic magnetic fields. They give nontrivial examples for which there are open gaps in the spectrum. Finally, we consider the perturbation of a periodic Schrödinger operator $H_{00} = -\Delta + V_{per}$ by a small vector potential $\lambda_0 A_0$. It follows from the results of Briet and Cornean [4] that the operator $H_0(\lambda) = -(-i\nabla - \lambda A_0)^2 + V_{per}$ has open internal gaps provided $|\lambda|$ is taken sufficiently small.

We begin with the Wegner estimate for the general family of random operators (6.1) satisfying (H5)–(H6) near the bottom of the deterministic spectrum and near the band edges.

Theorem 6.1.
a.) Bottom of Σ. *Suppose that the deterministic background operator H_0 satisfies hypotheses (H1b), with $\Sigma_0 = inf\, \sigma(H_0) > 0$, and (H2), and that the random processes $H^{\Lambda}_{j,\omega}, j = 1, 2$, satisfy (H5)-(H6), with the random variables satisfying hypothesis (H4). Let $E_0 < \Sigma_0$, and choose $\eta > 0$ such that $I_\eta \equiv [E_0 - 2\eta, E_0 + 2\eta] \subset (-\infty, \Sigma_0)$. Then, there exists a constant $\lambda_0 > 0$, and, for any $q > 1$, a finite constant $C_W = C_W(\lambda_0, [dist\,(\Sigma_0, E_0)]^{-1}, q) > 0$, such that for all $|\lambda| < \lambda_0$, we have*

$$I\!P\{\, dist\,(\sigma(H_{\Lambda,\omega}(\lambda)), E_0) \leq \eta\} \;\leq\; C_W\, \eta^{1/q}\, |\Lambda|. \tag{6.7}$$

b.) Internal Gaps. *Suppose that the deterministic background operator H_0 satisfies hypotheses (H1a) and (H2), and that the random processes $H^{\Lambda}_{j,\omega}, j = 1, 2$, satisfy (H5)-(H6), with the random variables satisfying hypothesis (H4). Suppose $G = (B_-, B_+)$ is an open gap in the spectrum of H_0. For any $E_0 \in G$, choose $\eta > 0$ so that the interval $I_\eta = [E_0 - 2\eta, E_0 + 2\eta] \subset G$. Then, there exists a constant $\lambda_0 > 0$, and, for any $q > 1$, a finite constant $C_W = C_W(\lambda, [dist\,(E_0, \sigma(H_0))]^{-1}, q) > 0$, such that for all $|\lambda| < \lambda_0$, we have*

$$I\!P\{\, dist\,(\sigma(H_{\Lambda,\omega}(\lambda)), E_0) \leq \eta\} \;\leq\; C_W\, \eta^{1/q}\, |\Lambda|. \tag{6.8}$$

Proof. We follow the proofs of sections 3 and 4.
1. **Bottom of Σ.** The proof proceeds effectively as in section 3. The operator that replaces $\Gamma_\Lambda(E_0; \omega)$ in (3.2) of section 3 is

$$\Gamma^{\lambda}_{\Lambda}(E_0; \omega) \equiv R_0(E_0)^{1/2}(\lambda H^{\Lambda}_{1,\omega} + \lambda^2 H^{\Lambda}_{2,\omega}) R_0(E_0)^{1/2}, \tag{6.9}$$

where $H^{\Lambda}_{j,\omega}$ are defined in (6.2) and (6.3), respectively. Because of hypotheses (H5)–(H6), we easily find that

$$A_\Lambda H^{\Lambda}_{1,\omega} = H^{\Lambda}_{1,\omega}, \tag{6.10}$$

and

$$A_\Lambda H^{\Lambda}_{2,\omega} = 2H^{\Lambda}_{2,\omega}, \tag{6.11}$$

so that

$$A_\Lambda \Gamma^{\lambda}_{\Lambda}(E_0) = \Gamma_\Lambda(E_0) + \lambda^2 K_\Lambda(E_0). \tag{6.12}$$

The bounded operator $K_\Lambda(E_0)$ is defined by

$$K_\Lambda(E_0) \equiv R_0(E_0)^{1/2} H^{\Lambda}_{2,\omega} R_0(E_0)^{1/2}. \tag{6.13}$$

Let us write $\nu \equiv \|K_\Lambda(E_0)\|$. Because of the support of ρ', we need to invert the right side of (6.12) on the spectral subspace for which $\Gamma^{\lambda}_{\Lambda}(E_0) \leq (-1+2\kappa)$, where, as in section 3, $\kappa = \eta/\delta$, for $\delta = \text{dist}\,(E_0, \sigma(H_0))$. We fix λ_0 by the requirement that $\lambda_0^2 \nu = (1-2\kappa)/2$. Thus, for any $|\lambda| < \lambda_0$, we have

$$\|A_\Lambda \Gamma^{\lambda}_{\Lambda}(E_0) \rho'(\Gamma^{\lambda}_{\Lambda}(E_0) - t + 1)\| > (1-2\kappa)/2. \tag{6.14}$$

With these modifications, we arrive at the analogs of (3.15)–(3.16). In order to apply the results on the spectral shift function, we let $H_1 \equiv \Gamma^{\lambda}_{\Lambda}(E_0)^{0,k}$ and compute the analog of (3.18),

$$\begin{aligned} \Gamma^{\lambda}_{\Lambda}(E_0)^{\lambda^+_k,k} &= H_1 + \lambda \lambda^+_k R_0(E_0)^{1/2} B_k R_0(E_0)^{1/2} \\ &\quad + 2\lambda^2 \lambda^+_k R_0(E_0)^{1/2} \Big(\sum_{j \in \tilde{\Lambda}} \lambda_j C_{jk}\Big) R_0(E_0)^{1/2}, \end{aligned} \tag{6.15}$$

where we write $V_\Lambda = \sum_{j \in \tilde{\Lambda}} \lambda_j u_j$. This is similar to the form of the perturbation caused by varying a single coupling constant appearing in (3.18). The essential point is that the first-order operators B_k and $\sum_{j \in \tilde{\Lambda}} \lambda_j C_{jk}$, appearing in each term, are local operators whose supports are independent of Λ. This insures that the L^p-estimate on the corresponding spectral shift function is independent of $|\Lambda|$. Consequently, the proof concludes as in section 3.

2. **Internal Gaps.** As in section 4, the projectors $P_\pm$ are the spectral projectors for H_0 corresponding to the spectral subspaces $[B_+, \infty)$ and $(-\infty, B_-]$, respectively. We consider the case when $E_0 \in G$ and $E_0 > (B_+ + B_-)/2$. The formulas for the Feshbach projection method are obtained from (4.3)–(4.4) by replacing the potential λV_Λ by $(\lambda H^{\Lambda}_{1,\omega} + \lambda^2 H^{\Lambda}_{2,\omega})$. Let the

free, reduced resolvent of $P_\pm H_0$ be denoted by $R_0^\pm(z) = (P_\pm H_0 - P_\pm z)^{-1}$. The resulting formula for $R_{P_-}(E_0) \equiv R_-(E_0)$, the first term on the right in (4.3), is

$$\begin{aligned} R_-(E_0) &= R_0^-(E_0)^{1/2}\{1 + R_0^-(E_0)^{1/2}P_-(\lambda H_{1,\omega}^\Lambda \\ &\quad +\lambda^2 H_{2,\omega}^\Lambda)P_- R_0^-(E_0)^{1/2}\}^{-1} R_0^-(E_0)^{1/2}, \end{aligned} \tag{6.16}$$

provided the inverse exists. We set $\delta_- = \text{dist}\,(\sigma(H_0^-), E_0)$. The first factor on the right in (6.16) exists provided $|\lambda| < \lambda_0^{(1)}$, where $\lambda_0^{(1)}$ is fixed by the requirement that

$$\lambda_0^{(1)}\delta_-^{-1/2}\{\|H_{1,\omega}^\Lambda R_0^-(E_0)^{1/2}\| + \lambda_0^{(1)}\|H_{2,\omega}^\Lambda R_0^-(E_0)^{1/2}\|\} < 1. \tag{6.17}$$

Similarly, the operator $\mathcal{G}(E_0)$ can be written, in analogy with (4.4), (4.9), and (4.10), as

$$\mathcal{G}(E_0) = R_0^+(E_0)^{1/2}\{1 + \tilde{\Gamma}_+(E_0)\}^{-1} R_0^+(E_0)^{1/2}. \tag{6.18}$$

The compact, self-adjoint operator $\tilde{\Gamma}_+(E_0)$ has an expansion in λ given by

$$\tilde{\Gamma}_+(E_0) = \sum_{j=1}^4 \lambda^j M_j(E_0), \tag{6.19}$$

where the coefficients are given by

$$\begin{aligned} M_1(E_0) &= R_0^+(E_0)^{1/2} P_+ H_{1,\omega}^\Lambda P_+ R_0^+(E_0)^{1/2}, \\ M_2(E_0) &= R_0^+(E_0)^{1/2}\{P_+ H_{2,\omega}^\Lambda P_+ \\ &\quad -P_+ H_{1,\omega}^\Lambda P_- R_-(E_0) P_- H_{1,\omega}^\Lambda P_+\} R_0^+(E_0)^{1/2}, \\ M_3(E_0) &= -R_0^+(E_0)^{1/2}\{P_+ H_{1,\omega}^\Lambda P_- R_-(E_0) P_- H_{2,\omega}^\Lambda P_+ \\ &\quad +P_+ H_{2,\omega}^\Lambda P_- R_-(E_0) P_- H_{1,\omega}^\Lambda P_+\} R_0^+(E_0)^{1/2}, \\ M_4(E_0) &= -R_0^+(E_0)^{1/2}\{P_+ H_{2,\omega}^\Lambda P_- R_-(E_0) P_- H_{2,\omega}^\Lambda P_+\} R_0^+(E_0)^{1/2}. \end{aligned} \tag{6.20}$$

We now compute the action of the vector field A_Λ, defined in (4.13), on the operator $\tilde{\Gamma}_+(E_0)$. Formulas (6.20) indicate that we need to compute the action of A_Λ on the local perturbations $H_{j,\omega}^\Lambda$, $j = 1, 2$, and on the resolvent $R_-(E_0)$. According to hypotheses (H5)–(H6), the action of A_Λ on these operators is the same as given in (6.10)–(6.11), and

$$A_\Lambda R_-(E_0) = -R_-(E_0)\{\lambda H_{1,\omega}^\Lambda + 2\lambda^2 H_{2,\omega}^\Lambda\} R_-(E_0). \tag{6.21}$$

Using these results, we obtain

$$A_\Lambda \tilde{\Gamma}_+(E_0) = \tilde{\Gamma}_+(E_0) + \sum_{j=2}^{6} \lambda^j K_j(E_0). \tag{6.22}$$

The remainder terms $K_j(E_0)$ are given by:

$$\begin{aligned}
K_2(E_0) &= M_2(E_0), \\
K_3(E_0) &= 2M_3(E_0) \\
&\quad + R_0^+(E_0)^{1/2}\{P_+ H_{1,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda P_+\} R_0^+(E_0)^{1/2}, \\
K_4(E_0) &= 3M_4(E_0) + R_0^+(E_0)^{1/2}\{2P_+ H_{1,\omega}^\Lambda R_-(E_0) H_{2,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda P_+ \\
&\quad + P_+ H_{1,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda R_-(E_0) H_{2,\omega}^\Lambda P_+ \\
&\quad + P_+ H_{2,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda P_+\} R_0^+(E_0)^{1/2}, \\
K_5(E_0) &= R_0^+(E_0)^{1/2}\{2P_+ H_{1,\omega}^\Lambda R_-(E_0) H_{2,\omega}^\Lambda R_-(E_0) H_{2,\omega}^\Lambda P_+ \\
&\quad + 2P_+ H_{2,\omega}^\Lambda R_-(E_0) H_{2,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda P_+ \\
&\quad + P_+ H_{2,\omega}^\Lambda R_-(E_0) H_{1,\omega}^\Lambda R_-(E_0) H_{2,\omega}^\Lambda P_+\} R_0^+(E_0)^{1/2}, \\
K_6(E_0) &= R_0^+(E_0)^{1/2}\{2P_+ H_{2,\omega}^\Lambda R_-(E_0) H_{2,\omega}^\Lambda R_-(E_0) H_{2,\omega}^\Lambda P_+\} R_0^+(E_0)^{1/2}.
\end{aligned} \tag{6.23}$$

As in part 1 of the proof, we need to compute $\|A_\Lambda \tilde{\Gamma}_+(E_0)\rho'(\tilde{\Gamma}_+(E_0) - t + 1)\|$. As in the first part of the proof, this requires that we choose λ sufficiently small so that

$$\sum_{j=2}^{6} \lambda^j \, \|K_j(E_0)\| \; < \; (1 - 2\kappa)/2. \tag{6.24}$$

Let $\lambda_0^{(2)} > 0$ be chosen so that $|\lambda| < \lambda^{(2)}$ guarantees that (6.24) holds. It is clear that $\lambda_0^{(2)}$ depends on the gap size, the location of E_0 relative to the gap edges $B_\pm$, and on the relative H_0-bounds of $H_{j,\omega}^\Lambda$. We now choose $\lambda_0 = \min(\lambda_0^{(1)}, \lambda_0^{(2)})$. We this choice, we can continue the proof as in section 4 and arrive at the analog of (4.19). An examination of (6.19)–(6.20) shows that the effective perturbation obtained by varying the k^{th} coupling constant has the correct form so that the trace estimate result of section 5 applies. □

Corollary 6.2. *Let $H_\omega(\lambda) = H_0 + \lambda H_{1,\omega} + \lambda^2 H_{2,\omega}$ be a random family of operators satisfying either 1) hypotheses (H1b), (H2), (H4)–(H6), or 2) hypotheses (H1a), (H2), (H4)–(H6). Suppose further that the family is ergodic. Then, for any closed interval $I \subset \mathbb{R} \backslash \sigma(H_0)$, there exists a constant*

$0 < \lambda_0(I)$ *such that for any* $|\lambda| < \lambda_0(I)$*, the integrated density of states for* $H_\omega(\lambda)$ *on* I *is Hölder continuous of order* $1/q$*, for any* $q > 1$.

6.2 Localization

The Wegner estimate plays a key role in the proof of localization for families of random operators. The Wegner estimate for nonsign definite potentials proved here can be used to prove band-edge localization as, for example in [1] and [10], under some additional assumptions. As the theory is not yet in optimal form, we indicate the lines of the proof and will return to this in a another paper. In order to prove localization, we need to establish an initial length scale estimate for the resolvent of the local Hamiltonian H_Λ. At present, only the method of Lifshitz tails appears to provide this estimate for the case of nonsign definite single-site potentials. The standard method (cf. [1, 19, 37]) depends on the monotonic variation of the eigenvalues of H_Λ^X with respect to the coupling constants that does not hold in the nonsign definite case. However, there is no satisfactory result for Lifshitz tails, either at the bottom of the spectrum or at internal band edges, for the case of nonsign definite single-site potentials.

1. Bottom of the spectrum Σ.

Let us suppose that the IDS exhibits a weak Lifshitz tail near $\inf \Sigma$, in the sense that

$$\lim_{E \to \inf \Sigma} (E - \inf \Sigma)^{-N} N(E) = 0, \tag{6.25}$$

for any $N \in I\!N$, for the models described in this paper, for which a Wegner estimate holds. Then a standard argument, as in [20], proves localization below $\inf \sigma(H_0^A)$. Indeed, let $N_\Lambda(E)$ be the number of eigenvalues of H_Λ in the interval $[\inf \Sigma, E] \subset [\inf \Sigma, \inf \sigma(H_0^A)]$. The standard argument uses the following estimate on the finite volume counting function by the IDS:

$$\frac{N_\Lambda(E)}{|\Lambda|} \leq N(E). \tag{6.26}$$

¿From this, we conclude that the probability that H_Λ has no eigenvalues in a small interval of size ϵ near $\inf \Sigma$ is less than $C_N |\Lambda| \epsilon^N$, for any $N > 0$. This is sufficient to prove an initial length scale estimate using the Combes-Thomas argument, upon taking ϵ to depend on the initial length scale.

2. Internal gaps.

The case of internal band-edges is more complicated. We need two results. First, we need weak internal Lifshitz tails (6.25) at the edges of the internal bands. Secondly, we need an analog of (6.26) in order to recover information about the finite-volume counting function near a band edge from the IDS.

Concerning the first point, Klopp [21] recently proved the following. Let H_0 be a periodic Schrödinger operator and V_ω an Anderson-type potential with single-site potentials $u_j(x) = u(x-j)$, and the single-site potential $u \geq 0$ is bounded with compact support. We assume that H_0 has an open spectral gap $G = (B_-, B_+)$. The common density h_0 of the random variables is assumed to be supported in $[0, M]$, for some $M > 0$, and h_0 vanishes more slowly than an exponential as $\lambda \to 0^+$. We assume that M is sufficiently small so that the deterministic spectrum Σ of $H_\omega = H_0 + V_\omega$ has an open gap $\tilde{G} = (\tilde{B}_-, B_+)$, for some $B_- \leq \tilde{B}_- < B_+$. Klopp proves that the IDS $N(E)$ of H_ω satisfies

$$\lim_{E \to B_+} \frac{log|log(N(E) - N(B_+))|}{logE} = -\frac{d}{2}, \tag{6.27}$$

where the limit is taken for $E \geq B_+$, if and only if the IDS $n(E)$ for the periodic operator H_0 is nondegenerate in the sense that

$$\lim_{E \to B_+} \frac{log(n(E) - n(B_+))}{logE} = \frac{d}{2}. \tag{6.28}$$

This result is, of course, stronger than the required weak Lifshitz tails behavior (6.25) near B_+.

This result can be extended to a special case of nonsign definite single-site potentials u as follows. The periodic Schrödinger operator H_0 admits a direct integral decomposition over the flat d-torus T^d, with fibre operators $H_0(\theta), \theta \in T^d$. For each $\theta \in T^d$, the operator $H_0(\theta)$ has a compact resolvent and hence discrete spectrum with eigenvalues $E_j(\theta)$. Let B be a band edge of $\sigma(H_0)$. At most finitely-many Floquet eigenvalues satisfy $E_j(\theta) = B, \theta \in T^d$. Let Π_B be the projector for the subspace generated by the corresponding Wannier functions in $L^2(I\!R^d)$. It follows from [21] that, in the small coupling constant limit, positivity of the potential V_ω, and thus of the single-site potential u, is sufficient, but not necessary, for the proof. Rather, one requires that $\Pi_B u \Pi_B \geq 0$. Hence, the proof of [21] can be modified to accommodate nonsign definite potentials $u = u_+ - u_-$ provided the negative part u_- is small in the sense that $\Pi_B u_+ \Pi_B \neq 0$ and $\Pi_B(u_+ - u_-)\Pi_B \geq \epsilon \Pi_B u_+ \Pi_B$. This is a condition on u and the back-

ground operator H_0. We remark that one can also apply this argument to Schrödinger operators with random magnetic fields, cf. [14].

To address the second problem, we refer to the recent article of Klopp and Wolff [22]. They prove a general result, valid for all dimensions $d \geq 1$, which provides the analog of (6.26) if Lifshitz tails are known to exist. Let H_Λ^P be the operator $H_0 + V_\omega$ restricted to a cube of side length L with periodic boundary conditions. The Wegner estimates, Theorems 1.1 and 1.2, proved for the local Hamiltonian $H_0 + V_\Lambda$, also hold for the operator H_Λ^P. A version of Proposition 7.1 of Klopp and Wolff, that holds even when the single-site potential u is nonsign definite, implies that for any $\nu > 0$,

$$I\!P\{\mathrm{dist}(\sigma(H_\Lambda^P), B_\pm) \leq L^{-1}\} \leq L^{-\nu}. \tag{6.29}$$

This is sufficient to establish the initial length scale hypothesis using a Combes-Thomas argument.

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