SPECTRAL THEORY FOR THE DIHEDRAL GROUP. [*†‡ §]

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Abstract

Let $H = -\Delta + V$ be a two-dimensional Schrödinger operator defined on a bounded domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary conditions on $\partial\Omega$. Suppose that H commutes with the actions of the dihedral group $\mathbb{D}_{2n}$, the group of the regular n-gone. We analyze completely the multiplicity of the groundstate eigenvalues associated to the different symmetry subspaces related to the irreducible representations of $\mathbb{D}_{2n}$. In particular we find that the multiplicities of these groundstate eigenvalues equal the degree of the corresponding irreducible representation. We also obtain an ordering of these eigenvalues. In addition we analyze the qualitative properties of the nodal sets of the corresponding ground state eigenfunctions.

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1 Introduction and main results.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^∞-boundary $\partial\Omega$ and $V \in C^\infty(\overline{\Omega})$ be real valued. We assume that Ω is either homeomorphic to the disk or the annulus. Let $a(x)$ be a real valued symmetric positive definite 2 by 2 matrix with entries $a_{ij} \in C^\infty(\overline{\Omega})$. We consider the elliptic operator

$$H = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + V(x) \tag{1.1}$$

on $L^2(\Omega)$ with Dirichlet boundary conditions. Its quadratic form is given with form domain $Q(H) = W_0^{1,2}(\Omega)$ by

$$\langle \varphi\ ,\ H\,\varphi \rangle = \int_\Omega \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial \overline{\varphi}}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\, dx + \int_\Omega V(x) |\varphi(x)|^2\, dx\ . \tag{1.2}$$

The operator domain of H is $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.
We consider the eigenvalue problem

$$Hu = \lambda u, \qquad \text{with } u = 0 \text{ in } \partial\Omega. \tag{1.3}$$

There is a sequence of eigenvalues $\lambda^1 < \lambda^2 \leq \dots \leq \lambda^k \leq \dots$ with corresponding eigenvectors u^k. By elliptic regularity the $u^k \in C^\infty(\overline{\Omega})$. Since H is real, for any complex valued eigenfunction eigenfunction u also $\Re u$ and $\Im u$ are eigenfunctions. We shall denote the restriction of $Q(H)$ to real valued functions by $Q_\Re(H)$. Of course, if $a_{ij} = \delta_{ij}$, then (1.3) becomes the standard eigenvalue problem for the Dirichlet realization of the Schrödinger operator :

$$(-\Delta + V)u = \lambda u\ . \tag{1.4}$$

In the following we assume that H and Ω are invariant with respect to the actions of $\mathbb{D}_{2n}$, the group of the regular n-gone.

$\mathbb{D}_{2n}$ has $2n$ elements: $\{h_1, \dots, h_{2n}\}$, n rotations $e, g, g^2, \dots, g^{n-1}$, where e denotes the identity, and n reflections, $T, Tg, \dots, Tg^{n-1}$. As usual, (see for instance [16]), we represent them by orthogonal 2×2 matrices acting on $\mathbb{R}^2$. The element g of the group is represented by a rotation by $2\pi/n$ and the entries of the corresponding matrix are

$$g_{11} = g_{22} = \cos(2\pi/n), \quad g_{12} = -g_{21} = \sin(2\pi/n). \tag{1.5}$$

The element T is represented by the reflection $(x_1, x_2) \mapsto (x_1, -x_2)$ with respect to the x_1-axis and given by

$$T_{11} = -T_{22} = 1, \quad T_{12} = T_{21} = 0. \tag{1.6}$$

The elements T and g generate the group and the multiplication law of $\mathbb{D}_{2n}$ can be worked out from the relations

$$g^k T = T g^{n-k}, \quad k = 0, \ldots, n \tag{1.7}$$
$$g^n = e \tag{1.8}$$
$$T^2 = e. \tag{1.9}$$

In polar coordinates g acts on a point $(x_1, x_2) = (r \cos \omega, r \sin \omega)$ in Ω by

$$g(r, \omega) = (r, \omega + 2\pi/n) ,$$

and T by

$$T(r, \omega) = (r, -\omega) .$$

We define, for $h \in \mathbb{D}_{2n}$ and $u \in L^2(\Omega)$, hu by :

$$hu(x_1, x_2) = u(h^{-1}(x_1, x_2)) ,$$

where the h are the orthogonal matrices defined above. By an abuse of notation we do not distinguish between the group element h and the associated orthogonal matrix.

$\mathbb{D}_{2n}$ can be considered as a semidirect product of $\mathbb{Z}_n$ and $\mathbb{Z}_2$. The rotations alone give $\mathbb{Z}_n$ and we can also generate $\mathbb{D}_{2n}$ by the reflections. The dimensions of the different irreducible representations of $\mathbb{D}_{2n}$ are given, (see [16]), by :
(i) Two one-dimensional irreducible representations and $(n-1)/2$ two - dimensional ones when **n is odd**.
(ii) Four one-dimensional irreducible representations and $(n-2)/2$ two-dimensional ones when **n is even**.
If H and Ω are invariant with respect to the actions of $\mathbb{D}_{2n}$, the quadratic form domain of H, $Q_{\Re}(H)$, can be decomposed into invariant subspaces $A_{(\cdot)}$ (defined below) associated to the irreducible representations of $\mathbb{D}_{2n}$ and we can write H as a direct sum. In the following the $A_{(\cdot)}$ will always denote invariant subspaces of real valued functions. This leads, (see for example [16]), to the following symmetry subspaces.

$$A_{0,s} = \{u \in Q_{\Re}(H) : gu = u \text{ and } Tu = u\} . \tag{1.10}$$

and

$$A_{0,a} = \{u \in Q_{\Re}(H) : gu = u \text{ and } Tu = -u\}. \tag{1.11}$$

For the case n even we have two additional symmetry subspaces. But we consider first the symmetry subspaces associated to the two-dimensional irreducible representations which are related to the rotations. In the complex case these symmetry spaces can be parametrized by ℓ with $0 < \ell < \frac{n}{2}$ and are defined by:

$$C_\ell = B_\ell \oplus B_{-\ell} \,, \tag{1.12}$$

where

$$B_\ell = \{w \in Q(H) \ : \ gw = e^{2\pi i\ell/n} w\} \,. \tag{1.13}$$

T maps B_ℓ onto $B_{-\ell}$ and so C_ℓ is a representation space of $\mathbb{D}_{2n}$. Complex conjugation maps B_ℓ onto $B_{-\ell}$ and so C_ℓ can be recognized as the complexification of the real space A_ℓ

$$A_\ell = \{u \in Q_{\Re}(H) : u - 2\cos(2\ell\pi/n) gu + g^2 u = 0\} \tag{1.14}$$

such that

$$C_\ell = A_\ell \otimes \mathbb{C} \tag{1.15}$$

where (1.14) follows from an easy computation based on (1.13), see also [16].

Finally for n even, $\ell = n/2$ plays a special role since $e^{i\pi} = -1$ and this leads to the two other invariant symmetry spaces defined by :

$$A_{n/2,s} = \{u \in Q_{\Re}(H) : gu = -u, \quad Tu = u\}, \tag{1.16}$$

and

$$A_{n/2,a} = \{u \in Q_{\Re}(H) : gu = -u, \quad Tu = -u\}. \tag{1.17}$$

The spaces $A_{(\cdot)}$ are invariant with respect to the actions of D_{2n}.

Let us now write H into a direct sum of operators where each $H_{(\cdot)}$ is the Friedrichs extension associated to the quadratic form one obtains regarding H with quadratic form domain $A_{(\cdot)}$.

Proposition 1.1 *Let H and Ω be invariant under the actions of $\mathbb{D}_{2n}$. Then*

$$H = H_{0,s} \oplus \Big(\bigoplus_{\ell=1}^{(n-1)/2} H_\ell \Big) \oplus H_{0,a}, \ \textit{for } \mathbf{n\ odd} \tag{1.18}$$

and

$$H = H_{0,s} \oplus (\bigoplus_{\ell=1}^{n/2-1} H_\ell) \oplus H_{n/2,s} \oplus H_{n/2,a} \oplus H_{0,a} \text{ for } \mathbf{n\ even}. \tag{1.19}$$

For each of these operators, we have a spectrum and we write for the corresponding eigenvalue problem

$$H_{(\cdot)} u^i_{(\cdot)} = \lambda^i_{(\cdot)} u^i_{(\cdot)}, \quad i \in \mathbb{N}, \quad \lambda^1_{(\cdot)} \le \lambda^2_{(\cdot)} \cdots \le \lambda^k_{(\cdot)} \le \dots . \tag{1.20}$$

For simplicity we write for the groundstate eigenvalues in the various symmetry subspaces $\lambda_{(\cdot)} = \lambda^1_{(\cdot)}$. Also the eigenfunctions and eigenspaces associated to the $\lambda_{(\cdot)}$ will be denoted by $u_{(\cdot)}$ and $U_{(\cdot)}$. We can now formulate our main results.

Theorem 1.2 .
The multiplicities of the eigenvalues $m(\lambda_{(\cdot)})$ satisfy

$$m(\lambda_{0,s}) = 1, \tag{1.21}$$

$$m(\lambda_{0,a}) = 1 \tag{1.22}$$

and for $0 < \ell < n/2$

$$m(\lambda_\ell) = 2. \tag{1.23}$$

If n is even, then also

$$m(\lambda_{n/2,s}) = m(\lambda_{n/2,a}) = 1. \tag{1.24}$$

Theorem 1.3 .
The eigenvalues satisfy the following ordering:
When n is odd

$$\lambda_{0,s} < \lambda_1 < \cdots < \lambda_{(n-1)/2} < \lambda_{0,a} . \tag{1.25}$$

When n is even

$$\lambda_{0,s} < \lambda_1 < \cdots < \lambda_{n/2-1} < \min(\lambda_{n/2,s}, \lambda_{n/2,a}) \le \max(\lambda_{n/2,s}, \lambda_{n/2,a}) < \lambda_{0,a} . \tag{1.26}$$

Remarks 1.4 .

(i) *Below we will describe qualitatively the nodal sets of the groundstate eigenfunctions in the various symmetry subspaces $A_{(\cdot)}$ and this will be actually the key point for the proof of the theorems in combination with a comparison with a spherically symmetric problem and some perturbation arguments. These arguments apparently do not carry over in general to the case when we have just rotational symmetry, hence, if the operator just commutes with the actions of $\mathbb{Z}_n$. For $\mathbb{D}_{2n}$ the simple use of group-theoretical arguments seems hopeless when $n > 4$.*

(ii) *Note that for two-dimensional problems* (1.3) *there are universal bounds to $m(\lambda^k)$ [4, 5, 2, 14, 12, 11]. Thereby Courant's nodal theorem plays a crucial role, see e.g. [3]. The situation here is quite different. There could be, and this can be seen already for the radial case with a potential, k eigenvalues in the symmetry subspace $A_{0,s}$ below λ_1 for any given k.*

(iii) *We have chosen to treat in this paper the simplest non-trivial case. In a forthcoming paper [8] we shall investigate related problems in a more general setting, e.g. periodic Schrödinger operators and Aharonov-Bohm Hamiltonians.*

We finally indicate the structure of the proof of these two theorems. In the next section we collect some observations concerning the representation theory for $\mathbb{D}_{2n}$. In section 3 we describe some direct spectral consequences of $\mathbb{D}_{2n}$-symmetry. In sections 4 and 5 we reduce the problem to a radially symmetric domain and then set up a perturbation theory for which the radial case serves as the unperturbed case. In section 6 we collect some general observations on nodal sets of eigenfunctions. In section 7 we analyze the nodal structure of eigenfunctions of radial problems, introduce the notion of canonicity for the general problem and state in Theorem 7.2 properties of nodal sets. In section 8 we analyze the stability of the canonicity of nodal sets with respect to perturbations. In section 9 (this is the core of the proof) we prove the multiplicity results by analyzing how this canonicity can be broken and prove Theorem 7.2. Finally in section 10 we prove the ordering of the eigenvalues using the previous analysis of the nodal sets and some simple partial integration argument.

2 Representation Theory.

2.1 Basics.

We first come back to the link between C_ℓ, B_ℓ and A_ℓ appearing in the introduction.
The group $\mathbb{D}_{2n}$ contains a subgroup of order n, $\mathbb{Z}_n$ generated by e and g and it will be useful to play between the representation theory of $\mathbb{Z}_n$ and $\mathbb{D}_{2n}$.
Let $0 < \ell < \frac{n}{2}$. We recall that A_ℓ (resp. C_ℓ) is the real (resp. complex) invariant space corresponding to an irreducible representation of $\mathbb{D}_{2n}$ of degree 2 and B_ℓ corresponds to an irreducible representation of $\mathbb{Z}_n$ of degree 1. We have also mentioned the relation between these different spaces. Let us be a little more explicit. Given a normalized $u \in A_\ell$, there exists $v \in A_\ell$, such that $w = u + iv$ is in B_ℓ. Moreover v is unique since there are no purely imaginary elements in B_ℓ. Let us mention further useful properties of this decomposition. Observing that B_ℓ and $B_{-\ell}$ are orthogonal, we get

$$\int_{\Omega_0} w(x)^2\, dx = 0 \tag{2.1}$$

which is equivalent to

$$\int_{\Omega_0} u(x)^2\, dx = \int_{\Omega_0} v(x)^2\, dx \text{ and } \int_{\Omega_0} u(x)v(x)\, dx = 0. \tag{2.2}$$

In particular, u and v form an orthonormal basis for a two-dimensional space, $\mathcal{W}_\ell$. Using (1.13), we get indeed :

$$g\, u = \cos(2\pi\ell/n)u - \sin(2\pi\ell/n)v \tag{2.3}$$

and this shows that gu belongs to $\mathcal{W}_\ell$ and is not a multiple of u, due to the fact that $\sin(2\pi\ell/n) \neq 0$.
As a corollary, we get :

Lemma 2.1 *For $0 < \ell < n/2$ and for $u \in A_\ell$ ($u \neq 0$), u and gu are linearly independent.*

Note similarly, that we have, for any $k \in \mathbb{Z}$,

$$g^k\, u = \cos(2\pi k\ell/n)u - \sin(2\pi k\ell/n)v \tag{2.4}$$

and this gives :

Lemma 2.2 *For $0 < \ell < n/2$ and, for $u \in A_\ell$ ($u \neq 0$), the real vector space spanned by the $\mathbb{Z}_n$ orbit of u (i.e $\mathfrak{O}_{Z_n}(u) = \{g^j u, j = 0, \ldots, n-1\}$) is a two-dimensional vector space.*

Using (2.4) with $k = 1$ and -1, we get also an explicit formula for v :

$$v = \frac{g^{-1}u - gu}{2\sin(2\pi\ell/n)}$$

Moreover, this defines on A_ℓ a map $u \mapsto v = \mathcal{R}_\ell u$:

$$\mathcal{R}_\ell := \frac{1}{2\sin(2\pi\ell/n)}(g^{-1} - g) \,, \tag{2.5}$$

which according to (2.2) is unitary.

2.2 Further decomposition.

We have seen that T exchanges B_ℓ and $B_{-\ell}$ and that $\mathbb{Z}_n$ leaves B_ℓ invariant. We hence can decompose further the space A_ℓ into the direct sum :

$$A_\ell = A_{\ell,s} \oplus A_{\ell,a}, \tag{2.6}$$

where

$$\begin{array}{ll} A_{\ell,s} & := \{u \in A_\ell \mid Tu = u\} \\ A_{\ell,a} & := \{u \in A_\ell \mid Tu = -u\} \,. \end{array} \tag{2.7}$$

We emphasize that these two spaces are no more representation-spaces of the group, but that it is still meaningful to analyze the spectrum of the restriction of the operator H to $A_{\ell,s}$ or $A_{\ell,a}$ which will be denoted respectively by $H_{\ell,s}$ and $H_{\ell,a}$.

Moreover, we observe that $\mathcal{R}_\ell$ anticommutes with T :

$$T\mathcal{R}_\ell = -\mathcal{R}_\ell T \,. \tag{2.8}$$

So $\mathcal{R}_\ell$ permits to send isometrically $A_{\ell,s}$ onto $A_{\ell,a}$ and $A_{\ell,a}$ onto $A_{\ell,s}$.

As a consequence, we have

Proposition 2.3 *For $0 < \ell < \frac{n}{2}$, the operators $H_{\ell,s}$ and $H_{\ell,a}$ are isospectral. In particular the multiplicity of any eigenvalue λ_ℓ^k of H_ℓ is even and equals twice the multiplicity of λ_ℓ^k as eigenvalue (in this case denoted by $\lambda_{\ell,a}^k$) of $H_{\ell,a}$.*

This means in particular for showing (1.23), it suffices to prove

$$m(\lambda_{\ell,a}) = 1 \text{ for } 0 < \ell < n/2. \tag{2.9}$$

2.3 Lines of symmetry.

We analyze which part of the nodal set is determined directly by symmetry arguments. We define the nodal set of a solution u of (1.1) by

$$\mathcal{N}(u) = \overline{\{x \in \Omega : u(x) = 0\}}$$

A nodal domain of a real valued eigenfunction u is just a component of $\Omega \setminus \mathcal{N}(u)$. Nodal sets will be dicussed further in section 6. For example, the property that : $Tu = -u$, implies that $\{x_1 = 0\} \cap \Omega$ belongs to the zero set of $u \in A_{\ell,a}$. But we can replace T by another reflection. The natural question is then to determine, for which j, we have :

$$Tg^j u = -u \ , \ \forall u \in A_{\ell,a} \ .$$

It is easily seen from (2.4) that this corresponds to the condition :

$$\cos(2\pi j\ell/n) = 1 \ ,$$

which leads to

$$\frac{\ell j}{n} \in \{0, \cdots, 2\ell - 1\} \ . \tag{2.10}$$

The corresponding lines of invariance will be called "special antisymmetric lines". We shall denote the union of these special lines by :

$$\mathcal{N}_{\ell,a} = \{x \in \Omega \ : \ \omega = \pi \frac{j}{n} \text{ for } j \text{ satisfying (2.10)}\}. \tag{2.11}$$

Similarily, we can discuss the symmetric case. We have to determine, for which j

$$Tg^j u = -u \ , \ \forall u \in A_{\ell,s} \ .$$

It is easily seen from (2.4) that this corresponds to the condition :

$$\cos(2\pi j\ell/n) = -1 \ , \tag{2.12}$$

which leads to

$$\frac{\ell j}{n} - \frac{1}{2} \in \{0, \cdots, 2\ell - 1\} \ . \tag{2.13}$$

The corresponding lines of invariance will be called "special symmetric lines". Note that in the symmetric case we could have no special symmetric lines. We denote the union of these special lines by :

$$\mathcal{N}_{\ell,s} = \{x \in \Omega \ : \ \omega = \pi \frac{j}{n} \text{ for } j \text{ satisfying (2.13)}\}. \tag{2.14}$$

So we have proved :

Proposition 2.4 *A state $u \in A_{\ell,a}$ necessarily vanishes on $\mathcal{N}_{\ell,a}$. Similarily, $u \in A_{\ell,s}$ necessarily vanishes on $\mathcal{N}_{\ell,s}$.*

2.4 More on the span of an orbit.

Pick a point $x_0 \in \Omega$, then we can look at the orbit of x_0

$$\mathfrak{O}(x_0) = \{hx_0 : h \in \mathbb{D}_{2n}\}. \tag{2.15}$$

There are two cases:

$$\text{if } x_0 \text{ belongs to a line of symmetry of } \Omega, \text{ then } |\mathfrak{O}(x_0)| = n \tag{2.16}$$

and otherwise

$$|\mathfrak{O}(x_0)| = 2n. \tag{2.17}$$

The bars denote the cardinality. (Of course for the case of the disk the origin $\{0\}$ remains invariant under the actions of $\mathbb{D}_{2n}$.)

If $u \in A_\ell$, it is natural to ask about the linear space generated by the actions of $\mathbb{D}_{2n}$ on u, hence the orbit of u, $\mathfrak{O}(u)$.

Lemma 2.5 *Suppose $0 < \ell < n/2$ and $u \in A_\ell$. If*

$$Tu \in \mathit{span}\ (u, gu), \tag{2.18}$$

then

$$\mathit{dim\ span}\ \mathfrak{O}(u) = 2, \tag{2.19}$$

otherwise its dimension is four. In particular the dimension of the span of the orbit of any symmetric or antisymmetric function in A_ℓ is two .

Proof. Suppose first that $Tu \in$ span (u, gu). This implies also that $Tg^j u = g^{n-j}Tu$ belongs to span (u, gu) and hence by Lemma 2.2 the dimension of the span of $\mathfrak{O}(u)$ is two.

If $Tu \notin$ span (u, gu) then also gTu cannot be in this span and the dimension of the span must be greater than 3. On the other hand, it has to be even (a multiple of the degree of the irreducible representation) and less than four using again Lemma 2.2. □

Lemma 2.6 *If $u \in A_\ell$ and if the span of $\mathfrak{O}(u)$ is two-dimensional, then there exists $u_{\ell,a} \in A_{\ell,a}$ such that*

$$\mathit{span}\ \mathfrak{O}(u_{\ell,a}) = \mathit{span}\ \mathfrak{O}(u)\ .$$

Lemma 2.7 *Suppose again that $0 < \ell < n/2$. Let $u \in A_\ell$. Then*

$$N(gu) \cap \mathcal{N}(u) \text{ is invariant by the actions of } \mathbb{Z}_n \,. \tag{2.20}$$

Proof. This follows from Lemma 2.1.

Remark 2.8 *In the case when u is antisymmetric (or symmetric), we have the property that $\mathcal{N}(gu) \cap \mathcal{N}(u)$ is invariant by $\mathbb{D}_{2n}$. Note also, that*

$$\mathcal{N}(gu) \cap \mathcal{N}(u) = \mathcal{N}(\mathcal{R}u) \cap \mathcal{N}(u) = \mathcal{N}(w)\,, \tag{2.21}$$

where $w = u + iv$ with $v = \mathcal{R}u$, (see (2.5)).

Remark 2.9 *Without loss we have chosen a particular reflection axis.*

3 Direct spectral consequences of symmetry considerations.

First we note that $\lambda_{0,s}$ is the lowest eigenvalue of H which is simple and that the associated eigenfunction can be chosen to be strictly positive in Ω. For the eigenvalues $\lambda_{0,a}$ and the associated eigenfunctions and for n even for the eigenvalues $\lambda_{n/2,s}, \lambda_{n/2,a}$ and their associated eigenfunctions we can use directly the results of the preceeding section to prove (1.21),(1.22) and (1.24). We shall further show here a part of Theorem 1.3, namely,

$$\lambda_{0,a} > \max\{\lambda_{0,s}, \max_{0<\ell<n/2} \lambda_\ell\} \text{ and for n even } \lambda_{0,a} > \max\{\lambda_{n/2,s}, \lambda_{n/2,a}\}. \tag{3.1}$$

The other considerations will be based on the fact that the nodal sets of $u_{0,a}$ and when n is even of $u_{n/2,s}$, respectively, $u_{n/2,a}$ are enforced by symmetry.
Defining

$$\tau_j = \{x \in \overline{\Omega} : \omega = \pi j/n\}, \quad j = 0, \dots, 2n-1, \tag{3.2}$$

(1.11) implies that

$$\cup_{j=0}^{2n-1} \tau_j \subseteq \mathcal{N}(u_{0,a}). \tag{3.3}$$

Similarly for n even we obtain from (1.16) and (1.17)

$$\cup_{i=0}^{n-1} \tau_{2i} \subseteq \mathcal{N}(u_{n/2,a}) \text{ and } \cup_{i=1}^{n-1} \tau_{2i+1} \subseteq \mathcal{N}(u_{n/2,s}). \tag{3.4}$$

Now in general every real valued eigenfunction of H has the property that its restriction to a nodal domain D is the groundstate of H in this domain with Dirichlet boundary conditions. Furthermore if we have a domain $D' \subset D$ with meas $(D \backslash D') > 0$ then $\lambda' > \lambda$ where λ and λ' denote the the groundstate eigenvalues of H in D, respectively, D' with Dirichlet boundary conditions.

We first show that $\lambda_{0,a}, \lambda_{n/2,a} \lambda_{n/2,s}$ are simple. Let

$$\Omega^0 = \{x \in \Omega : 0 < \omega < \pi/n\}$$

and suppose that Φ is the associated groundstate of H in Ω^0 with Dirichlet boundary conditions, so that for some $\lambda^* \in \mathbb{R}$, $H\Phi = \lambda^* \Phi$ in Ω^0. Extending Φ by 0 in $\Omega \setminus \Omega^0$ we define

$$\widetilde{\Phi} = \cup_{j=0}^{n-1} g^j (\Phi - T\Phi). \tag{3.5}$$

It is easy to see that $\widetilde{\Phi} \in A_{0,a}$ and by construction Ω^0 is a nodal domain. The variational principle and the uniqueness of groundstates hence implies that $\lambda^* = \lambda_{0,a}$ and that $m(\lambda_{0,a}) = 1$.

For n even the arguments for $\lambda_{n/2,a}, \lambda_{n/2,s}$ are very similar. Indeed, let $\Omega^1 = \{x \in \Omega : 0 < \omega < 2\pi/n\}$, and define Φ to be the groundstate of H in Ω^1 with Dirichlet boundary conditions and extend it by 0 outside Ω^1. Then

$$\widetilde{\Phi} = \cup_{j=0}^{n-1} (-1)^j g^j \Phi \in A_{n/2,a}$$

and we can repeat the arguments from above to conclude that $m(\lambda_{n/2,a}) = 1$. The proof of the corresponding result for $\lambda_{n/2,s}$ is almost identical.

To conclude this section we want to show (3.1). Note first that $\Omega^0 \subset \Omega^1$. Hence by the monotonicity of groundstate eigenvalues with respect to domains $\lambda_{0,a} > \lambda_{n/2,a}$. The proof that $\lambda_{0,a} > \lambda_{n/2,s}$ is almost identical. Indeed, for $u_{n/2,s}$, $\Omega^* = \{x \in \Omega : \pi/n < \omega < 3\pi/n\}$ is a nodal domain and contains strictly the nodal domain $\{x \in \Omega : \pi/n < \omega < 2\pi\}$ of $u_{0,a}$.

It remains to show

Lemma 3.1

$$\lambda_{0,a} > \lambda_\ell \textit{ for } 0 < \ell < n/2.$$

Proof. Let Φ be again the restriction of $u_{0,a}$ to Ω^0. Fix $0 < \ell < n/2$ and consider

$$\psi_\ell = \sum_{j=1}^{n} a_j \Phi(g^{j-1}x), \quad \forall x \in \Omega, \tag{3.6}$$

where the a_j are real constants. Note that $\Omega^0 \cap g^j \Omega^0 = \emptyset$ for $0 < j < n-1$. We observe that $\psi_\ell \in W_0^{1,2}(\Omega)$. We want to pick the a_j such that $\psi_\ell \in A_\ell$. We just have to take $a_j = \Re e^{2\pi i \ell j/n}$ and define

$$\psi_\ell(x) = \sum_{j=1}^{n} \cos(2\pi \ell j/n) \Phi(g^{j-1}x), \quad \forall x \in \Omega \, .$$

We apply now the variation principle and get

$$\lambda_\ell \leq \frac{\langle \psi_\ell \, , \, H\psi_\ell \rangle}{||\psi_\ell||^2} = \lambda_{0,a}.$$

We get strict inequality since by construction the set $\Omega \backslash \text{supp } \psi_\ell$ has positive measure. $\square$

With similar arguments we could have also shown that for n even $\lambda_{n/2,s}$ and $\lambda_{n/2,a}$ are both strictly greater than λ_l for $0 \leq \ell < n/2$. But this will be shown in section 9 in a more direct way.

4 Reduction to the annulus or the disk and construction of an interpolating family.

Without changing the divergence structure of H, we can reformulate the eigenvalue problem (1.3) in such a way, that we can replace the operator H with $\mathbb{D}_{2n}$ symmetry by an operator still commuting with the actions of $\mathbb{D}_{2n}$, but Ω becomes Ω_0 which is either the annulus $\Omega_0 = \{x \in \mathbb{R}^2 : |x| \in (R_1, R_2)\}$ with $0 < R_1 < R_2$ or the disk with $\Omega_0 = \{x \in \mathbb{R}^2 : |x| < R\}$ To do this we observe that by definition there is a smooth diffeormorphism Φ sending Ω onto Ω_0, which can be extended from $\overline{\Omega}$ onto $\overline{\Omega_0}$ and sending the exterior, respectively the interior boundary of Ω to the corresponding boundary of Ω_0. If Ω is simply connected, Φ is assumed to map Ω onto the disk. We may also require in the case of $\mathbb{D}_{2n}$ symmetry of our original problem that Φ commutes with the actions of $\mathbb{D}_{2n}$, hence $\Phi \circ h = h \circ \Phi$ for $h \in \mathbb{D}_{2n}$.

Associated with Φ, there is a canonical unitary tranformation Φ_* sending $L^2(\Omega)$ onto $L^2(\Omega_0)$ which is defined by :

$$(\Phi_* u)(y) = J(\Phi)^{1/2}(y) u(\Phi^{-1}(y)), \tag{4.1}$$

where $J(\Phi)$ is the Jacobian of Φ at $x = \Phi^{-1}(y)$. A computation shows that the unitarily equivalent operator

$$H_\Phi = \Phi_* \, H \, \Phi_*^{-1} \tag{4.2}$$

is still of the form as described by (1.1), but of course for a different matrix (a_{ij}). The new potential $\Phi_* V \Phi_*^{-1}$ turns out to be $V \circ \Phi^{-1}$. Hence it suffices to consider (1.3) on the annulus, respectively on the disk. By an abuse of notation we will henceforth talk about H instead of H_Φ. Let H be our operator which commutes with the actions of $\mathbb{D}_{2n}$ such that $hH = Hh$ for all $h \in \mathbb{D}_{2n}$ defined on $L^2(\Omega_0)$, Ω_0 the annulus or the disk, as defined in Section 1.

We introduce for α in a neighborhood of $[0,1]$ the following interpolating family of operators:

$$H(\alpha) = \alpha H + (1-\alpha)(-\Delta)\ , \tag{4.3}$$

whose domain is $W^{2,2}(\Omega_0) \cap W_0^{1,2}(\Omega_0)$. We consider groundstate eigenvalues in the real symmetry sectors as in Section 1 and denote them by $\lambda_\ell(\alpha)$, respectively, by $\lambda_{\ell,s}(\alpha)$ or $\lambda_{\ell,a}(\alpha)$. The corresponding eigenspaces are denoted by $U_\ell(\alpha)$. Note that by construction the operator H_Φ introduced above is in divergence form (see (1.1)), so that the corresponding matrix a is strictly positive definite in $\overline{\Omega}_0$. Hence $H(\alpha)$ is well defined for α in a neighborhood of the interval $[0,1]$, say $(-\epsilon_0, 1+\epsilon_0)$ with $\epsilon_0 > 0$ sufficiently small.

5 Perturbation theory for the interpolating family $H(\alpha)$.

We are interested in the multiplicity of the groundstate of $H(\alpha)$ restricted to A_ℓ for $\alpha = 1$. We first introduce, for $0 < \ell < n/2$:

$$I_\ell = \{\alpha \in [0, 1+\epsilon_0) : m(\lambda_\ell(\alpha)) \geq 3\}\ .$$

¿From now on, we use systematically Proposition 2.3 and shall work with $H_{\ell,a}$ instead of H_ℓ. It is then immediate that

$$I_\ell = \{\alpha \in [0, 1+\epsilon_0) : m(\lambda_{\ell,a}(\alpha)) \geq 2\}\ . \tag{5.1}$$

If $I_\ell \neq \emptyset$, let us define :

$$\alpha_0(\ell) = \inf\{\alpha \in (0, 1+\epsilon_0) : m(\lambda_{\ell,a}(\alpha)) \geq 2\}\ , \tag{5.2}$$

and, if $I_\ell = \emptyset$, we set:

$$\alpha_0(\ell) = 1 + \epsilon_0. \tag{5.3}$$

The reference to "antisymmetric" will not be systematically repeated in this section.

Lemma 5.1 *Let $0 < \ell < n/2$ and for $\alpha \in (-\epsilon_0, 1+\epsilon_0)$, let $P_\ell(\alpha)$ be the orthogonal projector on $U_\ell(\alpha)$. Then the map $\alpha \mapsto P_\ell(\alpha)$ is C^∞ for $\alpha \in [0, \alpha_0(\ell))$, in the sense that, for any $u \in L^2$, $\alpha \mapsto P_\ell(\alpha)u$ is C^∞ with value in $W^{k,2}(\Omega_0)$, for any $k \in \mathbb{N}$.*
Furthermore we have a decomposition

$$U_l(\alpha_0(\ell)) = U_\ell^1(\alpha_0(\ell)) \oplus U_\ell^2(\alpha_0(\ell)) \,,$$

where

$$\dim U_\ell^1(\alpha_0(\ell)) = 1 \,,$$

and

$$U_\ell^1(\alpha_0(\ell)) = \lim_{\alpha \uparrow \alpha_0(\ell)} U_\ell(\alpha) \,.$$

The above limit is considered in the sense that, for any $u \in L^2$, the vector $P_\ell(\alpha)u$ tends to $P_\ell^1(\alpha_0(\ell))u$ as $\alpha \uparrow \alpha_0(\ell)$ in any $W^{k,2}(\Omega_0)$ and hence in $C^\infty(\overline{\Omega_0})$. Here $P_\ell^1(\alpha_0(\ell))$ is the orthogonal projector on $U_\ell^1(\alpha_0(\ell))$.

Proof of Lemma 5.1.
We first note that $H(\alpha)$ can be extended as an analytic family of type A in the sense of Kato [13], see Chapter VII, Subsection 3.2. Indeed we observe by standard elliptic theory that, for α in a complex neighborhood of $[0,1]$, the operator domain $\mathcal{D}(H(\alpha))$ is $W^{2,2}(\Omega_0) \cap W_0^{1,2}(\Omega_0)$ and that $H^*(\alpha) = H(\overline{\alpha})$. The statements are then consequences of the standard Kato theory (see [13, 15]). The reduction to antisymmetric states has permitted us to work with simple eigenvalues. □

As immediate consequence of Lemma 5.1 and using what we explicitely know in the radial case (see section 7 below), we get :

Lemma 5.2 .
There exists an $\tilde{\alpha}_0 > 0$ such that $[0, \tilde{\alpha}_0] \cap I_\ell = \emptyset$ for $0 < \ell < n/2$.

6 Local and global structure of nodal sets.

All the material which is presented in this section is rather standard, but because we shall need to control the continuity of the nodal sets when varying the parameter α, it is helpful to recall what is needed in this context. We formulate everything for the domain Ω_0 which is either the annulus or the disk.

6.1 Local structure.

Let $u \in W_0^{1,2}(\Omega_0)$ be a real valued solution of

$$Lu = \sum_{i,j}^{2} a_{i,j}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u + \sum_{i=1}^{2} b_i(x) \frac{\partial}{\partial x_i} u + W u = 0, \tag{6.1}$$

where we assume that the matrix $a = (a_{i,j})$ is positive definite with entries $a_{ij} \in C^\infty(\overline{\Omega_0})$ and that the functions W and b_i $(i = 1, 2)$ are smooth in $\overline{\Omega_0}$ and real valued. This implies $u \in C^\infty(\overline{\Omega_0})$ For each $x_0 \in \overline{\Omega}_0$, there is a linear non-singular coordinate transformation T_{x_0} such that $\widetilde{u} = u \circ T_{x_0}$ satisfies $\widetilde{L}\widetilde{u} = 0$ where $\widetilde{L}$ is of the form given in (6.1) but with : $\widetilde{a}_{ij}(x_0) = \delta_{ij}$.

Proposition 6.1 Local behaviour.
Let $u \not\equiv 0$ satisfy (6.1) *and $x_0 \in \overline{\Omega}_0$. Then there exists an $m \in \mathbb{N}$ such that :*

$$u(x) = p_m(x - x_0) + \mathcal{O}(|x - x_0|^{m+1}), \tag{6.2}$$

where p_m is a homogeneous polynomial of degree m which does not vanish identically and which satisfies the osculating equation

$$\sum_{i,j}^{2} a_{i,j}(x_0) \frac{\partial^2}{\partial x_i \partial x_j} p_m(x - x_0) = 0$$

Moreover, under the coordinate tranformation, T_{x_0}, $p_m(x - x_0)$ becomes a harmonic homogeneous polynomial $P_m(x - x_0)$ of degree m.

If $x_0 \in \partial\Omega_0$ and u satisfies Dirichlet boundary conditions, then $p_m(x - x_0)$ must have a zero line through x_0 which is tangent to $\partial\Omega_0$ at x_0. In particular, $m > 0$ for boundary points.

Suppose $0 \in \Omega_0$, hence Ω_0 is the disk, then for each $u \in A_\ell$, $u \not\equiv 0$, we have in polar coordinates

$$u(r, \omega) = p_k(r, \omega) + \mathcal{O}(r^{k+1}) \tag{6.3}$$

where

$$k \in J_\ell := \{|\ell + n\mathbb{Z}|\}. \tag{6.4}$$

In particular,

$$k \geq \ell \ , \ \text{if } 0 < \ell < n/2 \ . \tag{6.5}$$

This follows from the considerations which lead to Lemma 2.1.

Note that in polar coordinates the harmonic homogeneous polynomials P_m have the simple representation:

$$P_m(r,\omega) = r^m(a\cos m\omega + b\sin m\omega).$$

Proof of Proposition 6.1. This follows for our case just from a Taylor expansion with remainder (see [1]). $\square$

Recall that the nodal set $\mathcal{N}(u)$ is defined by

$$\mathcal{N}(u) = \overline{\{x \in \Omega_0 : u(x) = 0\}}$$

and a nodal domain of u is a connected component of $\Omega \backslash \mathcal{N}(u)$.

Proposition 6.2 *(see e.g. [12])*
(i) $\mathcal{N}(u)$ is a union of smoothly immersed circles in Ω_0 and immersed arcs connecting points of $\partial\Omega_0$. Each of these is called a nodal line. *Note that self-intersections are allowed. Maximal embedded pieces of nodal lines will be called* nodal arcs. *In addition intersections of these immersed arcs, circles and self-intersections are transversal. The immersed arcs hit $\partial\Omega_0$ transversally.*
(ii) If for some $x \in \Omega_0$, $u(x) = 0$, but $(\nabla u)(x) \neq 0$, then x lies on exactly one nodal line and is not a point of self-intersection. On the other hand if $x_0 \in \Omega_0$ is not a point of self-intersection and lies on a single nodal line of u, then $(\nabla u)(x_0) \neq 0$. Also if $x_0 \in \partial\Omega_0 \setminus \mathcal{N}(u)$, then also $(\nabla u)(x_0) \neq 0$.
(iii) If for some $x \in \Omega_0$ and for some $p \in \mathbb{N}$, $u(x) = 0, (\partial^\beta u)(x) = 0$ for $|\beta| \leq p$ but $(\partial^\alpha u)(x) \neq 0$ for some multiindex α of length $p+1$, then exactly $p+1$ distinct nodal lines go through x whose tangents meet transversally at x.
(iv) If x is a zero of u as in (iii) but lies in $\partial\Omega_0$, then the same property is true with the convention that one of the nodal lines is a component of $\partial\Omega_0$.

The proof of these results is given in [12] and follows also partially from Proposition 6.1.

Following [12] we can introduce some definitions:

Definition 6.3 *An* abstract nodal set *is a set $\mathcal{N} \subset \overline{\Omega}_0$ satisfying 6.2(i), where we do not require that it is actually the nodal set of an eigenfunction. An* isotopy *of nodal sets is a curve of nodal sets such that each immersed circle or arc moves along a smooth isotopy which respects nodal arcs.*
A nodal pattern *is an isotopy class of nodal sets.*

7 The radial case and canonicity.

7.1 Radial case.

We first analyze the radial case. Here we take Ω_0 to be the annulus or the disk, and $H = -\Delta$ with Dirichlet boundary conditions.
By a standard reduction to ordinary differential equations, we find that any $u_\ell^k \in A_\ell$ which solves the equation $-\Delta u_\ell^k = \lambda_\ell^k u_\ell^k$, can be written in polar coordinates as

$$u_\ell^k = f_\ell^k(r)(a \sin \ell\omega + b \cos \ell\omega) \tag{7.1}$$

where $(a, b) \neq (0, 0)$ and f_ℓ^k satisfies for $\ell \in \mathbb{N}$

$$(-d^2/dr^2 - r^{-1}d/dr + r^{-2}\ell^2) f_\ell^k(r) = \lambda_\ell^k f_\ell^k$$

with $f_\ell^k(R_i) = 0$, for $i = 1, 2$.
If we split A_ℓ into $A_{\ell,s}$ and $A_{\ell,a}$, then the cosine part, respectively the sine part in (7.1), corresponds to $A_{\ell,s}$, respectively $A_{\ell,a}$. (The f_ℓ^k are of course Bessel functions, but we will not use this.)
Taking $k = 1$ and writing for simplicity $\lambda_\ell = \lambda_\ell^1, u_\ell = u_\ell^1$, these observations imply that

$$m(\lambda_\ell) = 2, \text{ for } \ell > 0 \tag{7.2}$$

and that, by the monotonicity of the coefficient $r^{-2}\ell^2$ with respect to ℓ,

$$\lambda_\ell < \lambda_{\ell'}, \text{ for } \ell < \ell'. \tag{7.3}$$

Obviously we can write in the radial case

$$H = \bigoplus_{\ell=0}^{\infty} H_\ell$$

as an infinite direct Hilbertian sum.

¿From now on we will always denote the antisymmetric functions in $A_{\ell,a}$ by u_ℓ and the corresponding symmetric ones in $A_{\ell,s}$ by $v_\ell = \mathcal{R}_\ell\, u_\ell$, where $\mathcal{R}_\ell$ is defined in (2.5) (note that $\mathcal{R}_\ell \sin \ell\omega = \cos \ell\omega$).

We consider for illustration the spherical case with respect to the symmetries of an equilateral triangle and of the square, hence of $\mathbb{D}_6$ and $\mathbb{D}_8$.

For $\mathbb{D}_6$ we have $u_0 \in A_{0,s}$. There are two independent functions $u_1 = f_1(r)\sin\omega$ and $v_1 = f_1(r)\cos\omega$ in A_1. Finally $u_3 = f_3(r)\sin(3\omega) \in A_{0,a}$. Note that also $u_2 \in A_{1,a}$, but is not a groundstate in this symmetry subspace.

Now consider $\mathbb{D}_8$. u_1 and v_1 belong as before to $A_{1,a}$, respectively $A_{1,s}$. $u_2 = f_2(r)\sin(2\omega)$ belongs to $A_{2,a}$ whereas $v_2 = f_2(r)\cos(2\omega)$ belongs to $A_{2,s}$. $u_4 = f_4(r)\sin(4\omega)$ is the groundstate in $A_{0,a}$. In our case here u_2 and v_2 yield the same eigenvalue, but this will not be necessarily the case once we do not have radial symmetry.

7.2 Nodal patterns in the radial case.

We proceed by analyzing the nodal patterns in greater detail.
We fix $0 < \ell < n/2$ and define as in section 3

$$\tau_j = \{x \in \overline{\Omega_0} : \omega = \pi j/n\}, \quad j = 0, \dots, 2n-1\ , \tag{7.4}$$

and

$$\Omega_{0j} = \{x \in \overline{\Omega_0} : \omega \in (\pi j/n, \pi(j+1)/n)\}, \quad j = 0, \dots, 2n-1\ . \tag{7.5}$$

The τ_j, τ_{j+1} are just the boundaries of Ω_{0j} without $\partial\Omega_0$.

The nodal set of a $\phi_{\ell,rad} = a\, u_{\ell,rad} + b\, v_{\ell,rad} = f_\ell(r)(a\cos\ell\omega + b\sin\ell\omega)$. consists of 2ℓ components given by

$$\mathcal{N}(\phi_{\ell,rad}) = \{x \in \overline{\Omega_0}\ :\ a\sin\ell\omega + b\cos\ell\omega = 0\} \tag{7.6}$$

Thereby we assume without loss that $f_\ell(r) > 0$ in Ω_0. In particular we have for $u_{\ell,rad}$, the antisymmetric function

$$\mathcal{N}(u_{\ell,rad}) = \{x \in \overline{\Omega_0} : \omega = k\pi/\ell, \quad k = 0, \dots, 2\ell - 1\} \tag{7.7}$$

whereas for $v_{\ell,rad} = f_\ell(r)\cos\ell\omega$, the symmetric function,

$$\mathcal{N}(v_{\ell,rad}) = \{x \in \overline{\Omega_0} : \omega = (2k+1)\pi/2\ell \quad k = 0, \dots, 2\ell - 1.\} \tag{7.8}$$

Of course, we recover in this particular case the general results of Proposition 2.4.

7.3 Nodal pattern for groundstates in $A_{\ell,a}$.

We assume $0 < \ell < \frac{n}{2}$ and consider now a groundstate eigenfunction $u_\ell \in A_{\ell,a}$ where we assume that we have transformed the problem to Ω_0. We recall that we can associate to u_ℓ a symmetric eigenstate by : $v_\ell = \mathcal{R}_\ell u_\ell$. We also recall that by Proposition 2.4

$$\mathbf{C0}(\ell) \qquad \mathcal{N}_{\ell,a} \subseteq \mathcal{N}(u_\ell). \tag{7.9}$$

In addition to this property, we have observed above that in the radial case the following properties were satisfied:

C1(ℓ)
$\mathcal{N}(u_\ell)$, consists of 2ℓ non-intersecting nodal arcs σ_j, $j = 0, \dots 2\ell - 1$, where each of the σ_j connects the outer and the inner boundary in the case of the annulus. In the case of the disk we just shrink the inner boundary to the origin $\{0\}$.

C2(ℓ)

$$\begin{aligned} &\mathcal{N}(u_\ell) \cap \mathcal{N}(\mathcal{R}_\ell u_\ell) = \emptyset \text{ for the annulus.} \\ &\text{For the disk } \mathcal{N}(u_\ell) \cap \mathcal{N}(\mathcal{R}_l u_\ell) = \{0\}. \end{aligned} \tag{7.10}$$

C3(ℓ)
For the annulus and the disk the components of $\mathcal{N}(u_\ell) \setminus \mathcal{N}_{\ell,a}$ are in the same Ω_{0j} as the corresponding radial antisymmetric solution $u_{\ell,rad}$.

Our aim is to show that all these properties (together with **C0**(ℓ) which is automatically satisfied) remain true for a general antisymmetric ground state in A_ℓ.

Definition 7.1 (Canonicity).
We say that $u_\ell \in A_{\ell,a}$ is **canonical** *if the properties* **C1**(ℓ), **C2**(ℓ) *and* **C3**(ℓ) *are satisfied.*

We are now able to state our result for the nodal sets of eigenfunctions u_ℓ associated to the symmetry subspaces of an operator H as described in (1.1).

Theorem 7.2 *Fix $n \geq 1$, then*

$$\mathcal{N}(u_{0,s}) = \emptyset \tag{7.11}$$

and

$$\mathcal{N}(u_{0,a}) = \cup_{j=0}^{2n-1} \tau_j. \tag{7.12}$$

For $0 < \ell < n/2$

$$u_\ell \quad \textit{is } \textbf{canonical.} \tag{7.13}$$

For n *even*

$$\mathcal{N}(u_{n/2,s}) = \cup_{j=0}^{n-1} \tau_{2j} \text{ and } \mathcal{N}(u_{n/2,a}) = \cup_{j=0}^{n-1} \tau_{2j+1}. \tag{7.14}$$

Remarks 7.3 .

(i) (7.11),(7.12) *as well as* (7.14) *have been shown in section 3.*

(ii) We could have defined in an analogous way also the canonicity of any function in the span $\mathfrak{O}(u_\ell)$ *which would amount to saying that all* $u \in$ *span* $\mathfrak{O}(u_\ell)$ *have the same nodal pattern and some additional localization of the nodal lines. But for the proof of our results we just need the canonicity of* u_ℓ*.*

8 Stability of the canonicity.

The next lemma will show the stability of canonicity with respect to the parameter α.

Lemma 8.1 *Let* $\alpha_0(\ell)$ *be given according to* (5.2)*. The set of* α*'s in* $(-\epsilon_0, \alpha_0(\ell))$*, such that* $u_\ell(\alpha)$ *is canonical is open.*

Since $\dim U_{\ell,a}(\alpha) = 1$, $u_\ell(\alpha)$ is well defined up to multiplication by a scalar. $u_\ell(0) = u_{\ell,rad}$ is canonical, see section 7.2, and this implies directly

Lemma 8.2 *For any* $0 < \ell < n/2$*, there exists an* $\tilde{\alpha}_1(\ell) > 0$ *such that* $u_\ell(\alpha)$ *is* **canonical** *for* $\alpha \in [0, \tilde{\alpha}_1(\ell))$*.*

Proof of Lemma 8.1:
Let $\beta_0 \in (-\epsilon_0, \alpha_0(\ell))$ such that $u_\ell(\beta_0)$ is canonical and denote by $U_{\ell,a}(\beta_0)$ (or more simply U_ℓ) the vector space generated by $u_\ell(\beta_0)$.

For $u \in U_\ell(\beta_0) \setminus \{0\}$, let us define $u(\cdot;\alpha)$ by :

$$u(\cdot;\alpha) = P_\ell(\alpha) u \,.$$

($P_\ell(\alpha)$ was defined in Lemma 5.1). In particular $u(\cdot;\beta_0) = u$.
We want to show the existence of $\epsilon > 0$, such that $u(\cdot;\alpha)$ keeps the same nodal pattern for $|\alpha - \beta_0| < \epsilon$.

The case of the annulus.

We first observe, using Lemma 5.1, that $(x,\alpha) \mapsto u(x;\alpha)$ is C^∞ on $\overline{\Omega_0} \times (\beta_0 - \epsilon, \beta_0 + \epsilon)$. So by continuity $\mathcal{N}(u(\cdot;\alpha)) \cap \partial\Omega_0$ stays for $|\alpha - \beta_0| < \epsilon$ ($\epsilon > 0$ sufficiently small) in a neighborhood of $\mathcal{N}(u) \cap \partial\Omega_0$.
Now, around each point (x_0, β_0) with $x_0 \in \mathcal{N}(u) \cap \Omega_0$, we can apply the standard implicit function theorem to the function $(x,\alpha) \mapsto u(x;\alpha)$. We observe indeed that $u(x_0;\beta_0) = 0$ and that $\nabla u(x_0;\beta_0) \neq 0$ (here we use Proposition 6.2, assertions (ii) and (iii)). Then in a small neighborhood of x_0 the zero set of $u(\cdot;\alpha)$ consists of a nodal line (when $|\alpha-\beta_0|$ is small enough).
Now, for $x_0 \in \partial\Omega_0 \setminus (\mathcal{N}(u) \cap \partial\Omega_0)$, we observe that $(\partial u/\partial n)(x_0,\beta_0) \neq 0$ and this property is stable under small perturbations. So no nodal line of $u(\cdot;\alpha)$ can hit the boundary outside a given neighborhood of $\mathcal{N}(u)\cap\partial\Omega_0$, if $|\alpha-\beta_0|$ is small enough.

It is a bit more difficult to control the nodal set of $u(\cdot;\alpha)$ near $\mathcal{N}(u)\cap\partial\Omega_0$. Let us consider a point $x_0 \in (\partial\Omega_0 \cap \mathcal{N}(u;\beta_0))$ where one nodal arc hits the boundary. We use local coordinates (y_1, y_2) such that $x_0 = (0,0)$ and Ω_0 is locally defined by $\{y \mid y_2 > 0\}$. Let $u(x;\alpha) = P_\ell(\alpha)u(x,\beta_0)$. In the new coordinates y, we write : $\tilde{u}(y;\alpha) = u(x;\alpha)$. This function is C^∞ in α and y (up to the boundary), and we observe that $\tilde{u}(y_1, 0;\alpha) = 0$ in a neighborhood of b_0 and $\{y = (0,0)\}$. Using Taylor's formula, there exists locally a C^∞ function g on $\overline{\Omega_0} \times (\beta_0 - \epsilon, \beta_0 + \epsilon)$ such that :

$$\tilde{u}(y_1, y_2;\alpha) = y_2\, g(y_1, y_2;\alpha)\ .$$

We note that in Ω_0 near $(0,0)$, $\mathcal{N}(u(\cdot;\alpha)) = \mathcal{N}(\tilde{u}(\cdot;\alpha)) = \mathcal{N}(g(\cdot;\alpha))$.
But let us compute $\nabla_y g(y_1,y_2;\alpha)$ at $(0,0;\beta_0)$. The local description of $u(\cdot;\beta_0)$ near $(0,0)$ shows (because $u(x)$ vanishes of first order at x_0) that $|\nabla_y g(0,0;\beta_0)| \neq 0$. More precisely, we can see that $(\partial_{y_1} g)(0,0;\beta_0) \neq 0$.
We can now apply the implicit function theorem. In order to apply it in the standard form, we first extend g outside Ω_0 and in a neighborhood of $(0,0;\beta_0)$ as a C^∞ function of all the arguments. The implicit function theorem implies then that the zeros of g near $(0,0;\beta_0)$ are described by :

$$\mathcal{N}(g(\cdot;\alpha)) = \{(y_1,y_2) \mid y_1 = \phi(y_2;\alpha) \text{ with } y_2 > 0\}\ .$$

Here ϕ is a C^∞ function on $[0,\epsilon) \times (\beta_0 - \epsilon, \beta_0 + \epsilon)$ with $\epsilon > 0$ sufficiently small. This shows the local continuity of the nodal pattern near the boundary for $\epsilon > 0$ small enough.

The case of the disk.
We proceed as in the case of the annulus away from the origin. In a neighborhood of the origin, we can write, using (6.3) and (6.5) : $u(x_1, x_2; \alpha) = r^\ell v(r, \theta; \alpha)$ where v is C^∞ on $[0, r_0) \times S^1 \times (\beta_0 - \epsilon_0, \beta_0 + \epsilon_0)$ and (r, θ) correspond to the polar coordinates.
More precisely we know from (6.5), that u vanishes at least to the order ℓ at the origin. Using Taylor's Formula (with integral remainder) for the function $[0,1] \ni t \mapsto f(t) = u(tx_1, tx_2)$, we get for $f(1) = u(x_1, x_2)$ the expansion :

$$u(x_1, x_2) = \sum_{i+j=l} c_{i,j}(x_1, x_2; \alpha) x_1^i x_2^j \; .$$

We can then take :

$$v(r, \theta; \alpha) = \sum_{i+j=l} c_{i,j}(r\cos\theta, r\sin\theta; \alpha)(\cos\theta)^i(\sin\theta)^j \; .$$

Using the local description of $u(\cdot; \beta_0)$ and its canonicity, we observe that $\partial_\theta v(0, \theta_0; \beta_0) \neq 0$ where θ_0 is a given angle such that $v(0, \theta_0; \beta_0) = 0$. The implicit function theorem gives locally the existence of a function $(r, \alpha) \mapsto \theta(r, \alpha)$ such that $\theta(0, \beta_0) = \theta_0$ and $v(r, \theta(r, \alpha); \alpha) = 0$. Moreover this map describes all the zeros in the neighborhood of the origin belonging to a sector : $\{r \exp i\theta_0 \mid r > 0\}$.

As mentioned above, we were dealing with a fixed canonical $u \in U_\ell(\beta_0) \setminus \{0\}$. We have shown that for $|\alpha - \beta_0| < \epsilon$ with ϵ sufficiently small the nodal pattern of $u(x; \alpha)$ does not change. Thus, since by assumption $\mathbf{C1}(\ell)$, $\mathbf{C3}(\ell)$ hold for $u(x; \beta_0)$ these properties hold also in a neighborhood of β_0. The proof of the stability of property $\mathbf{C2}(\ell)$ also follows readily from these observations. $\square$

9 Proof of Theorem 1.2 and of Theorem 7.2.

In section 3 we have already shown that that $\lambda_{0,s}, \lambda_{0,a}$ are simple as well as $\lambda_{n/2,s}, \lambda_{n/2,s}$. Thus it remains to prove (1.23) of Theorem 1.2 and (7.13) of Theorem 7.2.

9.1 On the critical values.

We look at $H_{\ell,a}(\alpha)$ and the corresponding lowest eigenvalue $\lambda_{\ell,a}(\alpha)$. ¿From the analysis of the radial case and from Lemma 8.2, we know that $m(\lambda_{\ell,a}(\alpha)) =$

1 for $-\epsilon < \alpha < \epsilon$ for sufficiently small ϵ. We recall that $\alpha_0(\ell)$ was defined in (5.2)-(5.3) and we now introduce another critical value corresponding to the breaking of canonicity. We note that for $\alpha \in [0, \alpha_0(\ell))$, the antisymmetric eigenfunction $u_\ell(\alpha)$ can be uniquely defined, keeping the C^∞ dependence with respect to α, if one adds the properties :

$$||u_\ell(\cdot;\alpha)|| = 1 \text{ and } u_\ell(\cdot,\alpha) > 0 \text{ in } \Omega_{00} .$$

where Ω_{00} has been defined in (7.5). We consequently take this choice.

Definition 9.1 *If there is some $\alpha > 0$, for which $u_\ell(\alpha)$ is not canonical, we define*

$$\alpha_1(\ell) = \inf\{0 < \alpha < \alpha_0(\ell) \;:\; u_\ell \textit{ is not } \textbf{canonical}\}. \tag{9.1}$$

Otherwise we simply define it by :

$$\alpha_1(\ell) = \alpha_0(\ell) .$$

Proposition 9.2 *Fix $0 < \ell < n/2$. Then*

$$\alpha_1(\ell) = \alpha_0(\ell). \tag{9.2}$$

Proof. For the proof we shall investigate how a nodal line, which is not a special antisymmetric line, see section 2.3, can be deformed in a continuous way such that the canonicity is violated.

Assume for contradiction $\alpha_1(\ell) < \alpha_0(\ell)$.

¿From the definition of $\alpha_1(\ell)$, we know that $u_\ell(\alpha)$ is canonical 'in the past' (i.e. for $\alpha < \alpha_1(\ell)$). This will imply some localization of the nodal lines for $\alpha = \alpha_1(\ell)$.
$u_\ell(\alpha_1(\ell))$ still satisfies, by 'former' canonicity (property **C3**(ℓ)), the weaker property
C3'(ℓ):
The components of $\mathcal{N}\,(u_\ell(\alpha_1(\ell)))\backslash\mathcal{N}_{\ell,a}$ are in the same $\overline{\Omega_{0j}}$ as the corresponding radial antisymmetric solution $u_{\ell,rad}$. $\mathcal{N}_{\ell,a}$ has been defined in (2.11).

Due to property **C3**(ℓ) we have :

$$\text{Two distinct nodal lines cannot belong to two neighboring } \Omega_{0j} . \tag{9.3}$$

This is an immediate consequence of the assumption that $\ell < \frac{n}{2}$. But this remains true under the weaker property **C3'**(ℓ), so we get

C3"(ℓ):
Two distinct nodal lines of $u_\ell(\alpha_1(\ell))$ cannot belong to two neighboring $\overline{\Omega_{0j}}$.

A similar argument shows that σ_1, (defined in section 7) cannot belong to $\overline{\Omega_{00}} \cap \overline{\Omega_{01}}$. This implies :

$$\mathcal{N}(u_\ell) \cap \{x \in \overline{\Omega_0} \mid \omega \in (0, 2\pi/n)\} = \emptyset \,. \tag{9.4}$$

Once these preliminary consequences of the 'former' canonicity are observed, there are three possible breakings of canonicity.

Case 1. Possible breaking of C1(ℓ).
It is enough to consider nodal lines contained in some $\overline{\Omega_{0j}}$. In order to conclude that $\mathbf{C1}(\ell)$ is still satisfied for $u_\ell(\alpha_1(\ell))$, it remains to show that two "accidents" cannot occur.

Case 1a : Self-intersection of a nodal line.
In this case there exists at least a nodal domain included in $\Omega_{0,j}$ contradicting, as we have shown in section 3, that $\lambda_{0,a} > \lambda_\ell$.

Case 1b : double (or higher) touching of a nodal line at the same component of the boundary.
Again, a nodal domain has to be contained in an $\overline{\Omega_{0,j}}$, and we get a similar contradiction.

In the case of the disk a 'double touching' of the origin means that the order of vanishing of u at 0 must be $\geq \ell + 1$, see (6.4). But as above two nodal lines cannot emanate from the origin and stay within one Ω_{0j} since this would again imply that there is a nodal domain inside Ω_{0j}.

Case 2. Possible breaking of C2(l).
We now analyze the case when $\mathcal{N}\,(u_\ell(\alpha_1(\ell))) \cap \mathcal{N}\,(\mathcal{R}_\ell u_\ell(\alpha_1(\ell)))$ is not empty.
We recall that we have introduced $w_\ell = u_\ell + iv_\ell$ with $v_\ell = \mathcal{R}_\ell u_\ell$.
We now consider $x_0 \neq 0$ such that (for short we write $u = u_\ell(\alpha_1(\ell))$)

$$u(x_0) = (gu)(x_0) = 0 \,.$$

One can distinguish two subcases.

Case 2a x_0 lies on a line of reflection.
In this case, we have n points obtained by successive rotations of $2\pi/n$ which are contained in $\mathcal{N}(w_\ell)$.

Case 2aa
If $x_0 = (r_0, \omega_0)$ and ω_0 is an odd multiple of $\frac{\pi}{n}$, we get a contradiction to (9.4): there is some point $g^k x_0$ such that the corresponding $g^k\omega_0$ is $\frac{\pi}{n}$.

Case 2ab
If x_0 and the corresponding ω_0 is an even multiple of $\frac{\pi}{n}$, (9.4) does not lead to an immediate contradiction. The orbit of x_0,assume $x_0 \neq 0$ for the disk, is a subset of $\mathcal{N}(w_\ell)$ and we can assume that $x_0 \in \sigma_0$. By the assumption of 'former' canonicity there is a strictly increasing sequence j_t $(t = 0, \cdots, 2\ell - 1)$ such that $j_0 = 0$, $j_t < 2n$ and such that σ_k is contained in $\overline{\Omega_{0j_k}}$. Moreover, we have (due to (9.3) $j_{k+1} \geq j_k + 2$. This means that the $\overline{\Omega_{0j_k}} \setminus \{0\}$ are disjoint. We have to show a contradiction to the property that $\mathfrak{O}(x_0)$ is in this set. But each $g^m x_0$, $m = 0, .., n-1$, can only belong to one of the sets $\overline{\Omega_{0,j_k}}$ and there are only 2ℓ sets for n different points. This gives the contradiction.

Case 2b x_0 does not lie on a line of reflection.
Then we get easily a contradiction to (9.4).

Case 3. Possible breaking of C3(ℓ).
We consider the property **C3"**$(\ell 1)$. In order to violate this property first a " normal " nodal line σ_k must touch a τ_j. Here we mean by normal that σ_k is not a special antisymmetric line, that is that $kn/\ell \notin \mathbb{Z}$. Let x_0 be the point where some normal nodal line touches some reflection line τ_j. Then

$$T g^j x_0 = x_0 \ . \tag{9.5}$$

But u is antisymmetric, so $Tx_0 = g^j x_0$ belongs to $\mathcal{N}(u)$. It remains to show that this implies that $\mathfrak{O}(x_0) \subset \mathcal{N}(u)$.
But once we have shown $\mathfrak{O}(\mathbf{x_0}) \subset \mathcal{N}(\mathbf{u})$ we are back to case 2.
We have not used so far that x_0 is not on a special nodal line of u. This means that

$$(n-j)\ell/n \notin \mathbb{Z} \text{ or } j\ell/n \notin \mathbb{Z} \ . \tag{9.6}$$

In particular $j \neq 0$. But we have by (2.4) :

$$g^{-j} u = \sin(2\pi\ell j/n)\, v + \cos(2\pi\ell j/n)\, u \ .$$

If $j\ell/n \notin \mathbb{Z}$ we have to consider two cases.

For

$$2\ell j/n \notin \mathbb{Z}$$

$v(x_0) = 0$ and $x_0 \in \mathcal{N}(w)$.
For

$$\ell j/n \in \frac{1}{2} + \mathbb{Z}$$

x_0 belongs to a special ''symmetric" nodal line of v and we can apply Proposition 2.4. Hence we are again back to case 2.

In conclusion, we have shown for case 3, that the touching of a 'normal' nodal line of an antisymmetric eigenfunction must occur at a point $x_0 \in \mathcal{N}(w)$. Hence $\mathfrak{O}(x_0) \subset \mathcal{N}(u)$ and we are back to case 2.

In the three cases, we have shown that no breaking of canonicity can occur in contradiction to the assumption that $\alpha_1(\ell) < \alpha_0(\ell)$.

Remark 9.3 *We have only used the fact that at $\alpha = \alpha_1(\ell)$ the considered eigenfunction was canonical for $\alpha < \alpha_1(\ell)$.*

Remark 9.4 *The case when $x_0 \in \partial\Omega_0$ does not lead to any new problem.*

9.2 End of the proof of the results for $0 < \ell < n/2$.

In order to conclude the proofs of (1.23) and (7.13), it remains to prove

Lemma 9.5

$$\alpha_0(\ell) = 1 + \epsilon_0 \ .$$

We proceed via contradiction and assume that :

$$\alpha_0(\ell) < 1 + \epsilon_0 \ .$$

According to Lemma 5.1, we have

$$m\left(\lambda_{\ell,a}(\alpha_0(\ell))\right) = 2 \ , \tag{9.7}$$

and

$$U = U^1 \oplus U^2 = U_{\ell,a}(\alpha_0(\ell)) = U^1_{\ell,a}(\alpha_0(\ell)) \oplus U^2_{\ell,a}(\alpha_0(\ell)) \tag{9.8}$$

where U^1 is one-dimensional.

We now introduce :

$$u = \lim_{\alpha \uparrow \alpha_0(\ell)} u_\ell(\alpha) \ . \tag{9.9}$$

We first show that Proposition 9.2 implies also that u is canonical as a continuous limit of a family of antisymmetric canonical eigenfunctions.

If U has dimension ≥ 2, it is immediate to find a non-canonical $\phi \in U$. Pick just $f \in U^2$ $(f \neq 0)$ and take $x_0 \in \Omega_{00}$. Then, for some $(a,b) \neq (0,0)$,

$\phi(x_0) = (a\,u + b\,f)(x_0) = 0$. This contradicts $\mathbf{C3}(\ell)$.
We can now introduce, for $\theta \in [0, \frac{\pi}{2}]$:

$$w_\theta = \cos\theta\, u + \sin\theta\, \phi\ . \tag{9.10}$$

Let θ_0 be the smallest positive $\theta < \frac{\pi}{2}$ such that w_{θ_0} is not canonical. Again, w_{θ_0} appears as a limit (as $\theta \uparrow \theta_0$) of canonical antisymmetric states and we get a contradiction as above.

Proposition 9.2 and Lemma 9.5 imply that $m(\lambda_\ell) = 2$ and that u_ℓ is canonical completing the proofs of Theorem 1.2 and Theorem 7.2. □

10 Proof of Theorem 1.3.

Part of Theorem 1.3 has been already shown in section 3. We still have to show the ordering of the λ_ℓ for $0 < \ell < n/2$ and the inequality between these λ_ℓ and $\lambda_{n/2,s}$ and $\lambda_{n/2,a}$ for n even. We consider first the inequalities amongst the λ_ℓ for $0 < \ell < n/2$. Hence we take $0 < \ell < \ell' < n/2$ and consider the corresponding antisymmetric groundstate eigenfunctions u_ℓ and $u_{\ell'}$. We know that for $\alpha = 0$ the ordering holds and hence by section 5 also for sufficiently small positive α.

We again proceed via contradiction and assume that $\lambda_\ell(\alpha) = \lambda_{\ell'}(\alpha)$ for some $\alpha > 0$.
Now consider $\mathcal{N}(u_\ell)$ and $\mathcal{N}(u_{\ell'})$ (we supress the α) and remember that we already know that they are both canonical. Both functions are identically zero on τ_0.

We use now the following almost trivial

Lemma 10.1 *Let $\varphi_i, i = 1, 2$ two solutions of :*

$$Hf = \lambda f\ ,\ \ in\ \Omega_0\ , \tag{10.1}$$

and let D be a simply connected subset of Ω_0 such that $\varphi_1 = 0$ for $x \in \partial D$. We split ∂D in two connected subsets ∂D_j , $j = 1, 2$ and assume that $\varphi_2(x) = 0$ for $x \in \partial D_1$ and that ∂D_2 is a subset of the boundary of a nodal domain of φ_1. Then either $\varphi_2(x) = 0$ for $x \in \partial D_2$ or $\varphi_2(x)$ has both signs for $x \in \partial D_2$.

Proof. For simplicity we consider just the case of $H = -\Delta + V$. Then from the fact that, by assumption the functions φ_j solve the same equation (10.1), we obtain by Green's formula :

$$\int_D (\varphi_1 \Delta\varphi_2 - \varphi_2 \Delta\varphi_1) dx = -\int_{\partial D_2} \varphi_2 \frac{\partial \varphi_1}{\partial n} d\sigma = 0 \tag{10.2}$$

where $\partial/\partial n$ means normal derivative in the outward direction, and $d\sigma$ is the induced measure on $\partial\Omega$. Now, we observe that, by Hopf's boundary point Lemma and the assumption on φ_1, $\frac{\partial\varphi_1}{\partial n}$ keeps the same sign on ∂D_2. Hence as asserted φ_2 has to have both signs in ∂D_2 or vanishes identically there. $\square$

Using this observation we easily prove the ordering of the eigenvalues. We take $\varphi_1 = u_{\ell'}$ and $\varphi_2 = u_\ell$. Since the zero sets of φ_1 and φ_2 are canonical and τ_0 is in both zero sets, we just look at a D whose boundary inside Ω_0 is given by τ_0 and a nodal arc σ_{φ_1} of φ_1 such that

$$\mathcal{N}(\varphi_2) \cap \sigma_{\varphi_1} = \emptyset \, . \tag{10.3}$$

To see that this is possible we just note that there are 2ℓ nodal arcs of φ_2 and $2\ell'$ nodal arcs of φ_1 and that, due to canonicity, one given nodal arc of φ_1 (outside $\partial\Omega_0$) can cross at most one nodal arc of φ_2.
Now according to the lemma above φ_2 must vanish at a nodal arc of φ_1 which is impossible according to (10.3). This is the desired contradiction to the assumption that $\lambda_\ell = \lambda_{\ell'}$.

The proof that min $(\lambda_{n/2,s}, \lambda_{n/2,a}) > \lambda_\ell$ for n even is identical. $\square$

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